Generalizing 'z-less' mixing length for stable boundary layers

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Current research shows, recent and former research suggests, that the nature and evolution of the stable atmospheric boundary layer(s) (SABL) is still understood and modelled inadequately. The 'classical' SABL, almost always stratified weakly (i.e. gradient Richardson number $Ri \ll \infty$, typically $0 < Ri \leq 1$), has been modelled reasonably well during the last few decades or so, but the very stable case, i.e. the VSABL ($Ri \gg 1$), is generally not well understood. Excessively diffusive and much too deep VSABL flows, as often appearing in numerical models, were recently addressed; the over-diffusion was alleviated by improving the local turbulent mixing length. This demands an explicit inclusion of the vertical shear of horizontal wind, S, in the mixing length, besides the previously known role of buoyancy frequency, N. A generalization of this recent work is given here by a simplified turbulent kinetic energy (TKE) equation and a set of subsequent parametrizations for the eddy diffusivity and conductivity, i.e. K-parametrizations, in terms of a generalized 'z-less' mixing length, Λ. The aim is to produce a parametrization that is uniformly valid for all $Ri \geq 0$. It is shown that $\Lambda \sim (TKE)^{1/2}/|S| \cdot f(Ri, Pr)$, uniformly valid for $0 \leq Ri$, regardless of the other parametrization details (the details appear as corrections); Pr is the turbulent Prandtl number and $f(Ri, Pr)$ is a simple set of derived functions depending on the parametrization properties.

A couple of important shortcomings of the current turbulence parametrizations for the SABL, as modelled in numerical weather prediction, air-chemistry and climate models, will be remedied by using this new generalized 'z-less' mixing length. This approach also recommends that it should be better if various mixing length scales were derived from simplified main principles, instead of only being guessed from plausible reasoning or dimensional analysis. In particular, it has often been assumed that K and Λ profiles could be chosen for modelling purposes more or less independently from each other. It is shown, based on a simple renormalization for Λ, that this is not the case if one wishes to use a more consistent parametrization scheme.

Key Words: diffusion; parametrization; Prandtl number; Richardson gradient number; stably-stratified turbulence; very stable boundary layers

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1. Introduction

Turbulent structures in the stable and very stable boundary layers are not well understood yet (Kim and Mahrt, 1992; Mahrt, 1998, 2007, 2008; Grisogono and Oerlemans, 2001, 2002; King *et al.*, 2001; Mauritsen *et al.*, 2007). Current numerical mesoscale models often represent stable atmospheric boundary layers (SABLs) poorly in terms of the SABL depth, near-surface inversion characteristics (its position and strength), low-level wind profiles and overall mixing properties (e.g. King *et al.*, 2001; Jeričević and Grisogono, 2006; Söderberg and Parmhed, 2006; Banta, 2008; Zilitinkevich *et al.*, 2008). While modelling the SABL, typical numerical weather prediction (NWP) models generate either too much mixing due to exaggerated vertical diffusion (needed for various reasons), or almost on the contrary, produce frictional decoupling and runaway cooling (e.g. Steeneveld *et al.*, 2007). These few important reasons already ensure that stably stratified turbulence continues to remain at the focus of scientific interest (Hunt *et al.*, 1988; Van der Avoird and Duynkerke, 1999; Fernando, 2003; Parmhed *et al.*, 2004; Renfrew and Anderson, 2006; Cuxart and Jiménez, 2007; Grisogono *et al.*, 2007; Mahrt, 2007, 2008; Mauritsen *et al.*, 2007; Zilitinkevich and Esau, 2007; Grisogono and Belušić, 2008 (henceforth, GB08); Princevac *et al.*, 2008; Zilitinkevich *et al.*, 2008). The diversity and number of the various mechanisms acting in and on the SABL dictate the presence of many different approaches in order to observe and reveal its nature and theory, and formulate modelling of the SABL (e.g. Baklanov and Grisogono, 2007).

The 'classical' SABL is almost always stratified weakly (i.e. gradient Richardson number $Ri \ll \infty$, typically $0 < Ri \leq 1$); hence, it has been modelled reasonably well during the last few decades or so. Meanwhile, the strongly or very stable case, i.e. the VSABL (where typically $Ri \gg 1$), conforming to a weak mixing regime, is generally not understood well (Kim and Mahrt, 1992; Mahrt, 1998, 2008; Fernando, 2003; Banta, 2008; GB08; Zilitinkevich *et al.*, 2008). There even appear to be different types of VSABL flow, as well as different approaches and techniques to tackle the corresponding turbulent structures (e.g. Baklanov and Grisogono, 2007; Mahrt, 2007, 2008). For example, Esau and Byrkjedal (2007) make their contribution by optimizing current first-order closures via large-eddy simulations (LES). Mauritsen *et al.* (2007) and Zilitinkevich *et al.* (2008) propose and use successfully the concept of total turbulent energy. Meanwhile, GB08 obtained a VSABL flow regime just by improving the local 'z-less' mixing length (Nieuwstadt, 1984a, 1984b) deployed in a mesoscale model with a detailed but still standard higher-order turbulence (HOT) closure scheme (Andrén, 1990).

It is somewhat surprising that besides a multitude of various types and modifications of HOT closure schemes, relatively little has been done on significant improvements to the corresponding SABL set of mixing length scales. The overall problem has been addressed by e.g. Mellor and Yamada (1974, 1982), Nieuwstadt (1984a, 1984b), Hunt *et al.* (1988), Andrén (1990), Enger (1990), Schumann and Gerz (1995), Fernando (2003), Weng and Taylor (2003) etc. The significance and sensitivity of formulating the mixing length scale set is well explained in Mellor and Yamada (1982) who stated the following: 'The major weakness of all the models probably relates to the turbulent master length scale (or turbulent macro-scale, or turbulent inertial scale), and, most important, to the fact that one sets all process scales proportional to a single [master] scale.' Valuable exemptions to this statement do however exist, such as that of van de Wiel *et al.* (2008). In their study, there is a nicely combined analogy between the stable surface layer and the rest of the SABL above it. One of the main differences between van de Wiel *et al.* (2008) and this study is that they keep the Prandtl number constant and equal to one; moreover, they rely on the existence of a relevant surface layer. This study makes a contribution by re-deriving and further extending a 'z-less' mixing length scale for the SABL and VSABL that is recently proposed by Grisogono (2009). Furthermore, this paper generalizes the study of GB08; they, by introducing explicitly the vertical wind shear in the 'z-less' mixing length, largely answered a question by Van der Avoird and Duynkerke (1999): whether the turbulence in katabatic flow could be modelled as that in stable boundary layers over flat surfaces (the answer is: yes, but additional parametrization improvements are deemed necessary). The goal here is to offer a 'z-less' mixing length parametrization that is uniformly valid for $Ri \geq 0$.

Closing this overview and background, and briefly restating the motivation ('zoom-out' perspective), the paper's content will be given. Without improvements in various turbulence parametrizations, as well as other important modelling details, it will be impossible to obtain better weather forecasts (e.g. wintertime minimum temperatures), more reliable air-pollution and dispersion calculations (e.g. near-surface night-time concentrations) and more meaningful future climate scenarios (e.g. ice melting and permafrost area distributions). Further reasons for much-needed improvements of HOT closure schemes related to e.g. the region of Antarctica, are in Renfrew (2004) and Renfrew and Anderson (2006). Hence, there are plenty of space and reasons for further improvements of NWP, air-chemistry and climate models. This study presents and discusses a new parametrization for the mixing length scale in the VSABL where the wind shear plays a prominent role. After certain problems connected to numerical modelling of the VSABL have been discussed, a new generalized 'z-less' mixing length parametrization will be developed and applied. The consequences of the traditional over-diffusivity modelling of the SABL will be re-illustrated briefly and the results will be compared with those from GB08; furthermore, a new generalized turbulent mixing length scale parametrization will be introduced. Some very preliminary outlooks relating to buoyancy waves, turbulent transport/redistribution effects, etc., which might be sensed by using the new generalized mixing length, will be mentioned in passing, thus hinting at future research, before concluding remarks are made.

2. Simplified TKE equation and 'z-less' mixing length

We begin with a simplified turbulent kinetic energy (*TKE*) equation per unit mass in order to obtain a new 'z-less' length scale, Λ. In other words, Λ will not be formulated heuristically from e.g. scaling or dimensional arguments, but more rigorously, based on the governing equations, Reynolds averaging, and further systematic reasoning, i.e. using generally accepted procedures for turbulent flows (e.g. Mellor and Yamada, 1982; Andrén, 1990; Enger, 1990; Zilitinkevich *et al.*, 2008). The prognostic equation for *TKE* under the typical simplifying conditions, i.e. assuming horizontal homogeneity, alignment of the mean flow with the downflow x-axis, validity of the Boussinesq approximation, and the absence of mean vertical motions, is

$$\frac{\partial(TKE)}{\partial t} = -\overline{u'w'}\frac{\partial\overline{u}}{\partial z} + \frac{g}{\theta}\overline{w'\theta'} - \frac{\partial}{\partial z}\left\{\overline{w'(\frac{p'}{\rho_0} + TKE)}\right\} - \varepsilon, \quad (1)$$

where all the terms, symbols, primes and bars have their very usual meaning (e.g. Mellor and Yamada, 1982; Pielke, 1984; Stull, 1988). The local rate of change of *TKE* on the l.h.s. of (1) is balanced by the shear production, buoyant destruction in the SABL, transport and redistribution due to pressure- and turbulence-correlations and viscous dissipation, respectively. Assuming nearly steady state, and neglecting transport terms in the curly brackets, three terms remain in (1). We parametrize the turbulent momentum and heat fluxes, i.e. the first two terms on the r.h.s. of (1), as $K_m|S|^2$ and $K_h N^2$, where K_m and K_h are eddy diffusivity and conductivity (relating the turbulent fluxes to the corresponding mean vertical gradients), $|S| = |\frac{\partial\overline{u}}{\partial z}|$ and N are shearing and buoyancy frequency, respectively. The last term in (1) is parametrized in accordance with Kolmogorov's hypothesis of nearly-isotropic local, relatively small, dissipating eddies as $\varepsilon = b(TKE)^{3/2}/\Lambda$, where b is an empirical constant (e.g. Mellor and Yamada, 1974, 1982; Andrén, 1990) while the Ozmidov scale is implicitly invoked (see below). After these reasonable simplifications, that are somewhat similar to those in the so-called closure level 2, (1) yields:

$$K_m|S|^2 = K_h N^2 + \frac{b}{\Lambda}(TKE)^{3/2}. \quad (2)$$

The buoyant destruction and viscous dissipation on the r.h.s. of (2) compete for spending *TKE* after its mechanical/shear production on the l.h.s. of (2) since the other possible (non)local sources of *TKE* are excluded by the previous assumptions (non-local ones would be admissible if the third term on the r.h.s. of (1) were not neglected).

A simpler, first-order closure scheme usually assumes, based on the absolute shear $|S|$, that $K_m = a_1\Lambda^2|S|$ and $K_h = a_1\Lambda^2|S|/Pr$; where a_1 is a known model constant and Pr is the turbulent Prandtl number; typically $Pr \geq 1$ in the SABL (Kondo *et al.*, 1978; Kim and Mahrt, 1992; Monti *et al.*, 2002; Zilitinkevich *et al.*, 2008). An advanced and often better parametrization is a HOT closure scheme, simply because it tackles some of the higher-order turbulence moments; remember that the more statistical moments of a random field that are treated, generally the better the knowledge of the random field (e.g. Mellor and Yamada, 1974; Stull, 1988; Bougeault and Lacarrère, 1989; Andrén, 1990). Meanwhile, particular choices among many turbulence parametrization schemes that could be deployed in NWP and other models (e.g. for air-pollution studies, wind-energy resources, etc.) depend on a number of issues ranging from spatial resolution and time step, computing resources, overall complexity of the model used, degree of air-flow complexity needed to be calculated and post-processed, to explicit dynamical reasoning and scientific intuition.

A HOT parametrization closure scheme may take its basic form as $K_m = a_2\Lambda(TKE)^{1/2}$ and $K_h = a_2\Lambda(TKE)^{1/2}/Pr$; where a_2 is a model constant. A particular realization of this HOT closure scheme, for $N > 0$ and especially for $Ri \geq 1$, is of type $K_m = a_3(TKE/N)$ and $K_h = a_3 TKE/(PrN)$, which makes sense for the VSABL. The latter HOT scheme is obtained from the former, more general scheme, by invoking the Ozmidov scale, $(\varepsilon/N^3)^{1/2}$, for Λ and re-deploying the assumption about ε (e.g. Nieuwstadt, 1984a, 1984b; Lesieur, 1997; GB08); such $\Lambda \sim (TKE)^{1/2}/N$ formulation becomes singular for $N \to 0$. Physically, this means that stratification has a negligible effect on turbulence if actual eddies are much smaller than $(TKE)^{1/2}/N$; on the contrary, stratification becomes important dynamically for those eddies of the order of or greater than the Ozmidov scale. In the intermediate range, say, $0 \leq Ri \sim O(1)$, it is the wind shear which dominates the eddies' lifetime and size (e.g. Hunt *et al.* 1988; GB08). Various refinements and especially combinations of these basic formulations are available (e.g. Andrén, 1990; Enger, 1990). We proceed with the closure schemes mentioned in order to obtain uniformly valid parametrizations for the SABL mixing length. When either the first-order or HOT closure K-formulations are deployed for parametrizing turbulent fluxes in (2), the corresponding Λ, for the three particular forms $(K_m)_{1,2,3}$ considered, becomes:

$$\Lambda_{1,2,3} = \Lambda_0 f_{1,2,3}(Ri, Pr)$$
$$\Lambda_0 = \frac{(TKE)^{1/2}}{|S|}, \quad (3a)$$

where $f_{1,2,3}$, corresponding to the first-order and two HOT closure schemes respectively, are straightforwardly found as

$$\left.\begin{array}{l} f_1 = (b/a_1)^{1/3}(1 - Ri/Pr)^{-1/3} \Leftarrow K_{m1} = a_1\Lambda^2|S|, \\ f_2 = (b/a_2)^{1/2}(1 - Ri/Pr)^{-1/2} \Leftarrow K_{m2} = a_2\Lambda(TKE)^{1/2}, \\ f_3 = (b/a_3)Ri^{1/2}(1 - Ri/Pr)^{-1} \Leftarrow K_{m3} = a_3(TKE/N). \end{array}\right\} \quad (3b)$$

For the moment, it is assumed that $Pr > Ri$ in the SABL so that $f_{1,2,3}$ are well behaved; later on this will be substantiated by other studies (see below). While with first-order schemes *TKE* may be only diagnosed, *TKE* is forecasted regularly with HOT closures, and this is one of the main advantages of current HOT schemes in NWP models. Note from (3) that Λ in the first-order closure case (subscript 1) is relatively less sensitive to both dimensionless numbers (which are, in turn, also functions of the flow) than the HOT closures (subscript 2 and 3); this depends on particular value of the negative exponent in (3b). If a subscript to Λ is not given, one quietly assumes all forms of Λ generalized in this way. There are no new unknown coefficients involved in (3), which is important from technical and modelling points of view; typically, $0 < a_i < 1$, $0 < b < 0.1$. The third closure, subscript 3, is the HOT closure which presumably dictates the parametrization behaviour at large Ri. Let us rewrite from (3a), $\Lambda_3 = \Lambda_0 f_3$, in order to understand the role of shear and buoyancy in Λ under strongly stratified conditions:

$$\Lambda_3 = (b/a_3)\frac{TKE^{1/2}}{N}Ri(1 - Ri/Pr)^{-1}, \quad (3c)$$

where the governing dimensional factor, $\Lambda_N = (TKE)^{1/2}/N$, is proportional to a typical 'z-less' mixing length (e.g. Enger, 1990; GB08). This Λ_N under the previously introduced hypothesis of Kolmogorov, relates to Ozmidov length (e.g. Lesieur, 1997; GB08), via the assumption $\varepsilon \sim (TKE)^{3/2}/\Lambda$ that is already deployed above (2). The length as such (i.e. without its corrective factor depending on Ri and Pr), if taken as the master mixing length in a HOT closure scheme, yields too much vertical mixing for $0 < Ri \le 1$, as shown by GB08. Next, Λ_N becomes invalid for $N \approx +0$; this is now fixed by (3c). We shall return to this fact when comparing the old, recent and new mixing length formulations in the next section. However, the corrective factor, in terms of Ri and Pr will rescale the Ozmidov length to still be, in the limit, a proper mixing length (3c) even for $0 < Ri \le 1$ where, in fact, it is the wind shear which dominates turbulent processes in terms of the eddy generation and limiting size (Hunt *et al.*, 1988; Fernando, 2003; Cuxart and Jiménez, 2007; GB08).

Likewise, the first (main) HOT closure scheme yielding $\Lambda_2 = \Lambda_0 f_2$ can be also rewritten in similar manner as (3c) to appear as

$$\left.\begin{aligned}\Lambda_2 &= (b/a_2)^{1/2}\,\Lambda_0(1-Ri/Pr)^{-1/2}\\ &= (b/a_2)^{1/2}\cdot\frac{TKE^{1/2}}{N}Ri^{1/2}\cdot(1-Ri/Pr)^{-1/2}\\ &= (b/a_2)^{1/2}\cdot\Lambda_N\cdot Ri^{1/2}(1-Ri/Pr)^{-1/2}\end{aligned}\right\}, \qquad (3d)$$

and the same reasoning applies again. Of course, the first-order closure from (3a) and (3b) can be rewritten in similar fashion too. (We do not dwell here on how to estimate TKE in first-order closure models.) To summarize, the transformation from (3a) and (3b) to (3c) and (3d) is equivalent to the change: $\Lambda_0 f_{1,2,3}(Ri,\ Pr) = \Lambda_N\ Ri^{1/2}\ f_{1,2,3}(Ri,\ Pr)$, where the l.h.s., i.e. (3a) and (3b), is applicable over $0 \le Ri < \infty$, but the r.h.s., i.e. (3c) and (3d) is strictly valid 'only' for $0 < Ri < \infty$, in order to avoid division by zero in Λ_N. Depending on the parametrization choice of $(K_{\rm m})_{1,2,3}$, the corresponding $\Lambda_{1,2,3}$ is obtained based on the closure (2); therefore, it also follows from (2) that

$$(K_{\rm m})_{1,2,3}\Lambda_{1,2,3} = \frac{b}{|S|^2}\frac{TKE^{3/2}}{(1-Ri/Pr)}. \qquad (4)$$

In other words, the closure level 2 guarantees that the product between the chosen K-parametrization and the corresponding mixing length formulation must be constant in a consistently chosen parametrization closure scheme. Based on the above, the most general parametrization scheme considered here should rely on Λ_0.

What we have gained by the successive rewriting within (3), involving Λ_0 (preferably), or Λ_N, is the 'z-less' turbulent mixing length that is uniformly valid for the (V)SABL, i.e. $0 < Ri$, provided that $Pr > Ri$ (see below). The renormalization of the mixing length takes place for the modelled flow as the flow varies its energy scale at which turbulent processes occur. This rescaling of the dominant eddy sizes is continuously provided, given TKE and $|S|$, in the parameter subspace of Ri and Pr. By the same token, it is dynamically inconsistent to deploy ad hoc e.g. the Blackadar length parametrization type (e.g. Stull, 1988) in SABL modelling, which is *a priori* a prescribed mixing length that is insensitive to flow properties.

Our next step is accounting for a possible singularity due to $Ri \to Pr$ in (3), reformulated also in (4). The fundamental work of Zilitinkevich *et al.* (2008), previously by Zilitinkevich and Esau (2007), going all the way back to Kondo *et al.* (1978), Kim and Mahrt (1992), Monti *et al.* (2002) etc., is essential to resolve this possible singularity in the most simple and basic way. Namely, incorporating into (3) the very important recent finding about the SABL that

$$Pr \approx 0.8 + 5\ Ri \qquad (5)$$

(Zilitinkevich *et al.*, 2008) allows the denominators in (3b) to be justifiably expanded into binomial series because for the SABL (5) yields max(Ri/Pr) $\le$0.2, which is a sufficiently small number thus allowing the series expansion of (3). To add a point, Kim and Mahrt (1992) found that $Pr = 1 + 3.8\ Ri$. Thus, (5) could be generalized as e.g. $Pr = A + B\ Ri$, where $0.7 \le A \le 1$ and $3 < B \le 5$, as in e.g. Grisogono and Zovko Rajak (2009). In this way, one broadens the range of acceptable linear relationships between Pr and Ri. The crucial point should still remain the same, i.e. $Pr > Ri$ for the SABL is a valid assumption.

For this study it suffices that in the SABL, and especially VSABL, $Ri < Pr$, or more likely even $Ri \ll Pr$; hence, no substitution of (5) or the like is necessary to proceed at this point. From (5) it follows that the turbulent momentum flux becomes progressively more efficient than that for the heat as Ri increases. In the limit of very large Ri, the ratio of these fluxes starts to behave as that due to buoyancy waves (transporting momentum but not heat). Almost accidentally, the same fact that Pr is (very) large in the SABL was deployed in a qualitative study of wave-drag effects in the SABL by Grisogono (1994). The simplifying expansion of (3), provided by (5), is given in Table I. This Table contains a distilled parametrization recipe for the modellers and it is one of the main results of this study.

To sum up, all the three parametrizations considered in (3) are simplified by qualitatively using (5) and are shown in Table I. This recommendation to modellers, which will be compared to previous and recent results in the next section, is rewritten as

$$\Lambda = const\frac{(TKE)^{1/2}}{|S|}\ f\ (Ri, Pr), \qquad (6)$$

with $0 < const < 1$, multiplying $\Lambda_0 = (TKE)^{1/2}/|S|$ and $f(Ri,\ Pr)$ as a simple function. For first-order and two HOT closure schemes, the respective single coefficient on the r.h.s. of (6) is an *a priori* known number from the respective definitions of eddy diffusivities in each particular NWP or climate model deployed. Remember that codes for HOT schemes are often written for double TKE, i.e. using $q2 = 2TKE$, which should be borne in mind when recoding the related lines in suitable parametrization routines.

Mesoscale models with advanced HOT closure schemes, as e.g. the Meteorological Institute of Uppsala University (MIUU) model (Enger, 1990) that is used here, possess a multiple choice for calculating appropriate eddy diffusivity and conductivity under stable conditions; meanwhile, a suitable set of options and entering coefficients is already accommodated here implicitly with the proposed Λ. Any combination of the SABL parametrizations discussed end up now with (6), i.e. $\Lambda \sim \Lambda_0 = (TKE)^{1/2}/|S|$. This is provided by the systematic reduction of TKE, (2) to (3), which yields a consistent balance of the three terms deployed in Λ. Note that this procedure, a type of renormalization for the mixing length scale, is not necessarily linked only to the SABL turbulence parametrization, but may be extended to the rest of the troposphere, above e.g. a convective ABL.

Table I. Generalized 'z-less' mixing length Λ.

K_m	$\Lambda(\Lambda_0)$	$\Lambda(\Lambda_N)$
$a_1\Lambda^2\|S\|$	$\Lambda_0(b/a_1)^{1/3}\{1+Ri/(3Pr)\}$	$\Lambda_N\,(b/a_1)^{1/3}\,Ri^{1/2}\{1+Ri/(3Pr)\}$
$a_2\Lambda(TKE)^{1/2}$	$\Lambda_0(b/a_2)^{1/2}\{1+Ri/(2Pr)\}$	$\Lambda_N(b/a_2)^{1/2}Ri^{1/2}\{1+Ri/(2Pr)\}$
$a_3(TKE)/N$	$\Lambda_0(b/a_3)Ri^{1/2}\{1+Ri/Pr\}$	$\Lambda_N(b/a_3)Ri\{1+Ri/Pr\}$

For three typical types of eddy diffusivity, K_m, Λ is derived (3) in terms of either Λ_0 or Λ_N, including explicitly either shearing or buoyancy frequency, respectively, i.e. $\Lambda_0=(TKE)^{1/2}/|S|$ or $\Lambda_N=(TKE)^{1/2}/N$. Richardson gradient and turbulent Prandtl number are Ri and Pr, $0<a_i<1$, $0<b<0.1$ are constants.

3. Preliminary results

Numerical simulations using the new generalized mixing length Λ will first be compared to those from GB08 in the next subsection. Since it is very difficult or impossible to compare and test all relevant, meaningful numerical simulations related to various stratified flows and the VSABL in particular, the subsequent subsections will have more of a qualitative nature. In the second subsection, two heuristic snapshots about further Λ investigations will be offered, one relating Λ to buoyancy wave scaling, another suggesting a possible correction to Λ due to transport and redistribution effects (thus the reader may easily skip subsection 3.2).

3.1. *Comparison with recent results*

Some of the results about the generalized 'z-less' mixing length derivation are illustrated in this section. After the method has been explained, the results based on (6), in particular, using $\Lambda=\Lambda_0(b/a_2)^{1/2}\{1+Ri/(2Pr)\}$ from Table I, are compared to the recent result from GB08; to aid the comparison, Figure 1 is organized in a same manner as their Fig. 1 but now adding the rightmost column with two panels pertaining to the new, generalized result. Otherwise, the very same mesoscale numerical model is used there and here, i.e. the MIUU model (e.g. Andrén, 1990; Enger, 1990; Enger and Grisogono, 1998; Abiodun and Enger, 2002; Grisogono and Enger, 2004).

The main point in Figure 1, besides that the traditionally modelled SABL is excessively diffusive (Figure 1(a) and (d)), is that the result using the new generalized mixing length is indistinguishable from the recent result in GB08. They focused on parametrizing the 'z-less' mixing length for $0<Ri\le 1$ (their equation (1)), shown in Figure 1(b) and (e); the new generalizing approach provided here treats, in a monotonic way, the mixing length for $0\le Ri<\infty$ and fully allows for $Ri\gg 1$, Figure 1(c) and (f). The reason one trusts the simulations in Figure 1(b) and (c), and Figure 1(e) and (f), for the potential temperature, Θ, and the main (downslope) wind component, U, respectively, is that these correspond to the analytic model of Prandtl. The latter model combines one-dimensional dynamics and thermodynamics in a simple and elegant way; moreover, the basic assumptions behind the model are well understood (e.g. Nappo and Rao, 1987; Egger, 1990; Parmhed *et al.*, 2004; Kavčič and Grisogono, 2007; Stiperski *et al.*, 2007; GB08; Axelsen and Van Dop, 2009). While Parmhed *et al.* (2004) compared an improved Prandtl model with measurements in Iceland, Axelsen and Van Dop (2009) tested their LES results against the Prandtl model. The flow simulated in Figure 1 concurs with that due to a calm, constantly stratified atmosphere over an inclined, cooled surface, pertaining to the essence of the Prandtl model. A low-level jet, Figure 1(e) and (f), is imbedded in a sharp and strong near-surface inversion, Figure 1(b) and (c). The traditional SABL parametrization, using for the 'z-less' mixing only $\Lambda=\Lambda_N\sim TKE^{1/2}/N$, is unable to simulate sharp gradients that are clearly displayed in the simulations deploying recent and new parametrizations; further details are in GB08. This is a general improvement presented because the parametrization (2)–(6) does not treat differently inclined and horizontal surfaces; in other words, HOT closure schemes deployed do not sense any terrain slope assigned. Besides these main qualitative differences between the traditional and recent/new SABL treatments, there are also certain bulk quantitative variations. For instance, the range of values of U in Figure 1(d) is $-0.06\ \mathrm{m\,s^{-1}}\le U\le 7.85\ \mathrm{m\,s^{-1}}$, while in Figure 1(e) and (f) is: $-0.5\ \mathrm{m\,s^{-1}}\le U\le 8.28\ \mathrm{m\,s^{-1}}$, within $\pm 2.7\%$ for the latter two. Note that this type of VSABL may contain an extremely thin surface layer, where even the concept of a surface layer existence can be questioned (e.g. Grisogono and Oerlemans, 2001).

Figure 2 corroborates the results in Figure 1, now showing the across-slope wind speed, V, Figure 2(a)–(c), and the turbulent mixing length, Figure 2(d)–(f), for the same runs as in Figure 1. Again, the results are obtained using the MIUU model where a single change in the HOT closure scheme is made: the left panels show the traditional, over-diffusive SABL, the middle panels show the recent result corresponding to that in GB08 and the right panels relate to the new result using (6), the expression as in the middle of Table I. The old, traditional mixing length parametrization, Figure 2(a) and (d), is unable to gradually diffuse V vertically in time, which it should do within the realm of the Prandtl model, similar to that in e.g. Kavčič and Grisogono (2007), Stiperski *et al.* (2007) and Shapiro and Fedorovich (2008). This behaviour of V, shown in Figure 2(b) and (c), can be important in simulating the long-lasting SABL over polar regions. Again, the differences between the old and recent/new fields (Θ, U, V), are the consequences of a single change in the model, i.e. due to the generalization of mixing length now explicitly using the absolute wind shear, $|S|$, Figure 2(d)–(f).

The range of values of the traditional mixing length and its vertical distribution on one side, Figure 2(d), and its relative shortening and significant redistribution with the recent formulation, Figure 2(e), as well as with the generalization, Figure 2(f), is what matters the most. The old length scale, Figure 2(d), is too large and too smooth, thus allowing for excessive diffusion of the SABL. On the contrary, the recent and newly generalized mixing length, Figure 2(e) and (f), are (1) shortened appreciably, (2) contain sharper gradients and (3) are relatively more variable than in the old formulation. The limited diffusion through the low-level jet at the height of ~15 to 20 m, Figure 1(e) and (f), Figure 2(e) and (f), provides all the sharp profiles for this VSABL. Again, this

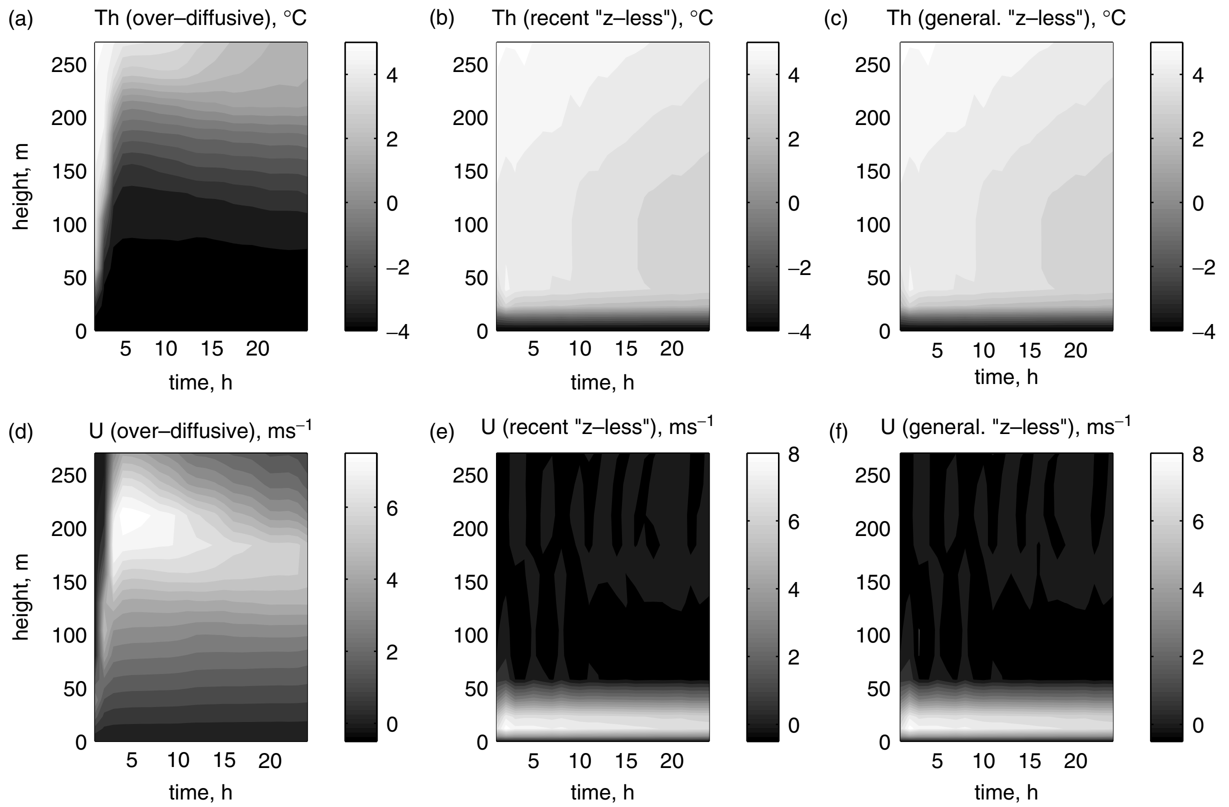

Figure 1. The old or traditional (left), recently modelled (middle) and new generalized (right) SABL. Upper plots (a), (b), (c) show the potential temperature, Θ, lower plots (d), (e), (f) show the corresponding main wind speed, U (m s^{-1}). The only difference among the three simulations is in the 'z-less' mixing length. The model basic input: stratification $\Delta\Theta/\Delta z = 5$ K/km $=$ *const*, constant slope of 2.2° and surface potential temperature deficit 6.5 K. The over-diffusive SABL (a), (d), is relaxed to a thin and sharp SABL (b), (e), which can be hardly distinguished from the new result (c), (f).

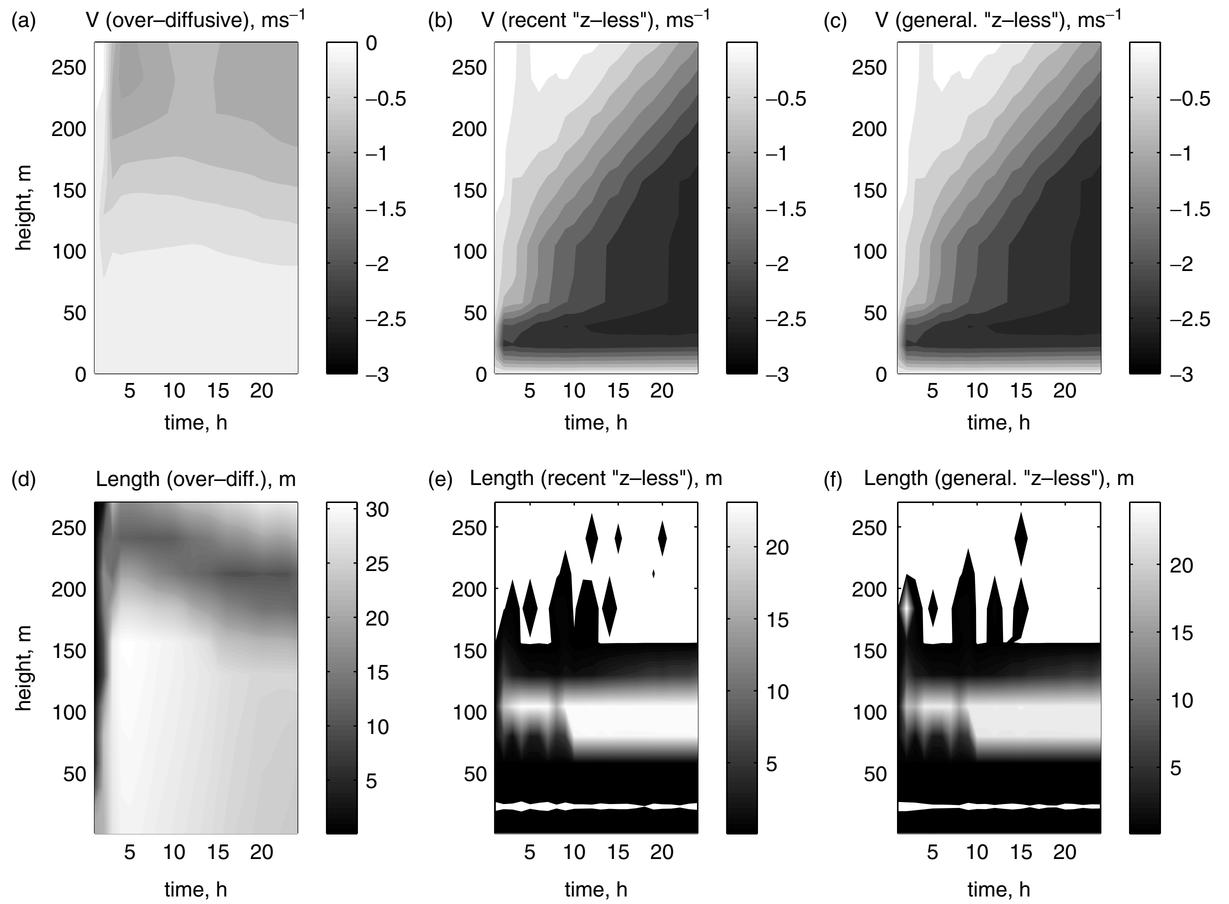

Figure 2. Same as Figure 1 but for two other flow fields: the across-slope wind speed, V (upper), and the mixing length, Λ (lower).

behaviour is absent in the traditional, over-diffusive SABL (the left panels).

3.2. *Further remarks*

Heuristic reasoning is provided in this subsection in order to further demonstrate robustness and generality of the proposed Λ in two different aspects. One is Λ's susceptibility to wave-like processes, another is Λ's possible applicability to transport and redistribution effects on *TKE*. Consider a stationary hydrostatic buoyancy wave and a related scaling similar to (6). Wave kinetic energy per unit mass, *WKE*, due to the wave components (u, w) in the (x, z) plane scales as

$$WKE = u^2 + w^2 \sim {U_0}^2 H^2 (m^2 + k^2), \qquad (7a)$$

where U_0, H, m and k are the background wind speed, maximum terrain height, vertical and horizontal wave number, respectively (e.g. Nappo, 2002). The horizontal wave-induced absolute vertical shear is

$$|S|_w = |\partial u/\partial z| = |\partial (U_0 Hm \ \exp\{\mathrm{i}(kx + mz)\})/\partial z| = |U_0 H m^2|. \qquad (7b)$$

Now formulate a length scale similar to (6), based on (7), recalling that $m \gg k$:

$$\Lambda_w \sim A \frac{(WKE)^{1/2}}{|S|_w} \sim \frac{1}{m} \sim \frac{U_0}{N} = \frac{\lambda_z}{2\pi}, \qquad (8)$$

with λ_z being the vertical wavelength ($= 2\pi U_0/N$). If the wave will transfer some of its *WKE* to *TKE*, $0 < A < 1$, and most of the shear will be due to the wave, then the proposed generalized mixing length (6) would be able to scale the dominant process properly because (8) has the same form as Λ_0. While typical mixing lengths for stratified flows in use today, e.g. the Blackadar type, or even Λ_N (unless the correction factor is included, Table I), do not scale buoyancy waves in a proper way, the newly proposed length apparently allows for inclusion of wave effects. For modelling more complex flows, expansions to higher-order terms in (3) are needed.

A correction due to transport and redistribution effects is plausibly assessed next. Reconsider (1) but treating the third term on the r.h.s. as a small, corrective term. To reword, since transport and redistribution terms are poorly understood in the VSABL and notoriously difficult to handle in the *TKE* equation, we treat them here with caution allowing them to appear in Λ only as a correction. The aim of this exercise is to estimate a plausible higher-order correction to Λ, pertaining to transport and redistribution terms lumped together, which is not accommodated by $f(Ri, Pr)$ in (6) and Table I. Thus, (2) extends to

$$K_{\mathrm{m}} |\ S\ |^2 = K_{\mathrm{h}} N^2 + \delta TR + \frac{b}{\Lambda}(TKE)^{3/2}, \qquad (9)$$

where δ is a small parameter multiplying the transport and redistribution term, *TR*; δ will be easily set to unity later because (9) and (10) will require only a simple regular perturbation approach to view a plausibly corrected Λ. We proceed with the previous parametrization $K_{\mathrm{m}} = a_2 \Lambda (TKE)^{1/2}$ which yields in (9) a straightforward quadratic equation for Λ (in terms of Λ_2 as in (3d), middle). The reasoning and procedure for using the other two parametrizations considered in (3) is the same as below and it is thus not pursued here. Now (9) is rewritten as

$$a_2 \Lambda^2 (TKE)^{1/2} |\ S\ |^2 \left(1 - \frac{Ri}{Pr}\right) - \Lambda \delta TR - b(TKE)^{3/2} = 0. \qquad (10)$$

Assuming that neither *TKE* nor $|S|$ is zero, (10) can be sorted out:

$$\left.\begin{array}{l} \Lambda^2 - 2\Lambda\delta B - C = 0, \\ B \equiv TR/\left\{2a_2(TKE)^{1/2} |\ S\ |^2 \left\{1 - (Ri/Pr)\right\}\right\}, \\ C \equiv \frac{bTKE}{a_2 |\ S\ |^2 \{1 - (Ri/Pr)\}} \end{array}\right\} \qquad (11)$$

which allows for a simple, compact and asymptotic solution, based on the starting assumption that $|B| \ll C$, now for $\Lambda \rightarrow \Lambda_{TR}$ (the exact solution is straightforward, but clumsy and less revealing):

$$\left.\begin{array}{l} \Lambda_{TR} = \delta B + C^{1/2}\left(1 + \frac{(\delta B)^2}{C}\right)^{1/2}, \\ \Lambda_{TR} = C^{1/2} + \delta B + \delta^2 B^2/2C^{1/2} + \ldots \\ \Lambda_{TR} \approx \Lambda_2 \left[1 + \frac{\delta TR}{2TKE |\ S\ | \{b a_2 \{1 - (Ri/Pr)\}\}^{1/2}} + \ldots\right], \end{array}\right\} \qquad (12)$$

where only the positive solution, taken up to the linear term in δ, $\delta \rightarrow 1$, is the physically relevant solution. Note in (12) that the perturbative solution for $\Lambda \approx \Lambda_{TR}$ is similar to that from before, i.e. Λ_2, but now also contains a small corrective term in the bracket due to *TR*, which could be positive or negative. Hence, it appears that this type of generalized mixing length would also be able to treat non-local flow features produced by wave effects and by *TKE* transport and redistribution effects. Depending on a particular type of assumptions and further parametrization choice involved in treating the *TKE* transport and redistribution, i.e. *TR* in (12), our asymptotic generalized mixing length, Λ_{TR}, includes the latter features in a perturbative fashion.

4. Conclusions

Modelling of the SABL was recently assessed using two models of different complexities in GB08. The mixing length parametrization for the SABL, pertaining to the so-called 'z-less' regime, was improved by including wind shear effects, *S*, explicitly. The improvement in the MIUU mesoscale numerical model was verified against a calibrated Prandtl model, thus simulating properly a type of VSABL flow. Here, this work is extended to a larger class of stably stratified regimes in the SABL flows, i.e. $Ri \geq 0$, $Pr < Ri$, in accordance with Zilitinkevich *et al.* (2008) and motivated a long time ago (Mellor and Yamada, 1982; Nieuwstadt, 1984a, 1984b). Overall need for these sort of studies dealing with stably stratified turbulence in boundary layers has been becoming increasingly important (e.g. Hunt *et al.*, 1988; Kim and Mahrt, 1992; Mahrt, 1998, 2007, 2008; King *et al.*, 2001; Fernando, 2003; Baklanov and Grisogono, 2007; Mauritsen *et al.*, 2007; van de Wiel *et al.*, 2008).

The generalization of the mixing length, Λ, is provided here in terms of a simplified *TKE* equation and a set of subsequent parametrizations for the eddy diffusivity and conductivity related through *Pr* and *Ri*. It is shown that $\Lambda \sim (TKE)^{1/2}/|S| \cdot f(Ri,\ Pr)$, uniformly valid for

$0 \leq Ri < \infty$, and $f(Ri, Pr)$ is a simple function depending on the parametrization used. The preliminary results compare favourably with those from GB08. A type of renormalization procedure for obtaining the 'z-less' mixing length provides the corresponding generalized Λ for SABL and VSABL. This is the main result of this study. Additional, very plausible reasoning is provided in order to further demonstrate robustness and generality of the proposed Λ in two different aspects that will be tackled in future works. One is Λ's susceptibility to wave-like processes, another is Λ's possible applicability to transport and redistribution effects on *TKE*. Hopefully, this study will be of some use for further simulating SABL flows with other numerical models, such as WRF, HIRLAM, EMEP, AROME etc., dealing with NWP, air-pollution, wind energy issues, regional cold climate scenarios etc.

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