Kochen-Specker Experiments that Are Contained in a Single 24-24 One vs. Those that Are Not

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Abstract

We show that all possible 388 4-dim Kochen-Specker (KS) setups with 18 through 23 vectors and 844 setups with 24 vectors all with component values from \{-1,0,1\} can be obtained by stripping vectors off a single system provided—in effect—by Asher Peres 20 years ago. In addition to them, we have found a number of other KS setups with 22 through 24 vectors.

Key words: Kochen-Specker sets, MMP diagrams, lattice theory

The Kochen-Specker (KS) theorem has recently been given renewed attention due to developments of both experimental and computational techniques as well as new theoretical results.

The experiments were carried out for spin-$\frac{1}{2} \otimes \frac{1}{2}$ particles (correlated photons or spatial and spin neutron degrees of freedom), and therefore in this paper we provide results only for them. The first experiments and their designs \cite{1, 2, 3, 4, 5} were not literal KS setups. They made use of state-dependent vector orientations that were additionally “translated” into new measurable observables according to ingenious keys found by their authors, because they could not be directly implemented by reading off the orientations of the vectors from the original setup. The most recent designs and experiments \cite{6, 7, 8} dispense with state-dependent vectors, though.

The new theoretical results concern the conditions under which such experiments are feasible at all \cite{9, 10, 11, 12} and a possibility to formulate the KS theorem for single qubits \cite{13, 14}. Such results and experiments enable applications in quantum computation (e.g., restrictions imposed on complex configurations of quantum gates, implementations of KS configurations of quantum gates that rule out classical solutions, etc.).
All that prompted the need to find out how many possible KS setups there are. Several authors discovered a number of them containing from 18 to 24 vectors for systems with four degrees of freedom and with vector component values from the set \{-1,0,1\}. [15, 16, 17, 18] Later, new algorithms for exhaustive generation of all possible KS setups for any degree of freedom based on the lattice theory and interval analysis were designed. [19, 20, 21, 22]

Based on these algorithms, computer programs were written which enabled us to prove that Cabello’s 18 vector setup with 9 orthogonal tetrads is the smallest possible one and to find all KS sets of vectors with components from \{-1,0,1\} for up to and including 22 vectors. [20, 22] The only KS set with components that was not from \{-1,0,1\} was the one with 24 vectors which we presented in Fig. 3 (b) of [20].

Further computation on our clusters and new algorithms and programs then provided us with definitive results on all possible KS vector sets with up to and including 23 vectors with respect to several chosen sets containing 24 vectors. In [21] we have shown that Peres’ original 24-vector set can be extended to the one with 24 orthogonal tetrads—we call it the 24-24 set. In this paper, we show that, surprisingly, the 24-24 set properly contains all of the smaller aforementioned sets and also that there are no KS sets with component values from the set \{-1,0,1\} that are not contained in it. We also found 37 new KS sets with 22 through 24 vectors with component values from other sets (not from \{-1,0,1\}). In the end we obtained all 18 through 23 vector sets and additional 844 24 vector sets by simply stripping vectors off the 24-24-set and filtering it to find those that do allow 0-1 values (one 1 and three 0s in each tetrad).

To obtain our results we used the algorithms that are described in detail in [20] and some others that will be described here.

We start by describing vectors as vertices (points) and orthogonalities between them as edges (lines connecting vertices), thus obtaining MMP diagrams [19, 21, 23] which are defined as follows:

1. Every vertex belongs to at least one edge;
2. Every edge contains at least 3 vertices;
3. Edges that intersect each other in \(n - 2\) vertices contain at least \(n\) vertices;

We denote vertices of MMP diagrams by \(1, 2, \ldots, A, B, \ldots a, b, \ldots\). There is no
upper limit for the number of vertices and/or edges in our algorithms and/or programs.

Isomorphism-free generation of MMP diagrams follows the general principles established by [24], which we now recount briefly. Deleting an edge from an MMP diagram, together with any vertices that lie only on that edge, yields another MMP diagram (perhaps the vacuous one with no vertices). Consequently, every MMP diagram can be constructed by starting with the vacuous diagram and adding one edge at a time, at each stage having an MMP diagram. We can represent this process as a rooted tree whose vertices correspond to MMP diagrams whose vertices and edges have unique labels. The vacuous diagram is at the root of the tree, and for any other diagram its parent node is the diagram formed by deleting the edge with the highest label. The isomorph rejection problem is to prune this tree until it contains just one representative of each isomorphism class of diagram.

To find diagrams that cannot be ascribed $0$-$1$ values we apply an algorithm which we call states$_{01}$ and which is based on the lattice theory of Hilbert space states. The algorithm is an exhaustive search of MMP diagrams with backtracking. The criterion for assigning $0$-$1$ (dispersion-free) states is that each edge must contain exactly one vertex assigned to 1, with the others assigned to 0. As soon as a vertex on an edge is assigned a 1, all other vertices on that edge become constrained to 0, and so on.

To find KS vectors we follow the idea put forward in [19, 21] and proceed so as to require that their number, i.e. the number of vertices within edges, corresponds to the dimension of the experimental space $\mathbb{R}^n$ and that edges correspond to $n(n-1)/2$ equations resulting from inner products of vectors being equal to zero which means orthogonality. So, e.g., an edge of length 4, BCDE, represents the following 6 equations:

\[
\begin{align*}
\mathbf{a}_B \cdot \mathbf{a}_C &= a_B a_1 a_C + a_B a_2 a_C + a_B a_3 a_C + a_B a_4 a_C = 0, \\
\mathbf{a}_B \cdot \mathbf{a}_D &= a_B a_1 a_D + a_B a_2 a_D + a_B a_3 a_D + a_B a_4 a_D = 0, \\
\mathbf{a}_B \cdot \mathbf{a}_E &= a_B a_1 a_E + a_B a_2 a_E + a_B a_3 a_E + a_B a_4 a_E = 0, \\
\mathbf{a}_C \cdot \mathbf{a}_D &= a_C a_1 a_D + a_C a_2 a_D + a_C a_3 a_D + a_C a_4 a_D = 0, \\
\mathbf{a}_C \cdot \mathbf{a}_E &= a_C a_1 a_E + a_C a_2 a_E + a_C a_3 a_E + a_C a_4 a_E = 0, \\
\mathbf{a}_D \cdot \mathbf{a}_E &= a_D a_1 a_E + a_D a_2 a_E + a_D a_3 a_E + a_D a_4 a_E = 0.
\end{align*}
\]

(1)

Each possible combination of edges for a chosen number of vertices corresponds to a system of such nonlinear equations. A solution to systems which
correspond to MMP diagrams without 0-1 states is a set of components of KS vectors we want to find. Thus the main method for finding all KS vectors is to exhaustively generate all MMP diagrams, then pick out all those diagrams that cannot have 0-1 states, then establish the correspondence between the latter diagrams and the equations for the vectors as shown in Eq. (1), and finally solve the systems of the so obtained equations.

To find solutions in the set \{-1,0,1\} we use the program \textit{vectorfind}, and to find solutions in the set of real numbers we use the interval analysis as described in detail in [20, 22]. There is no other upper limit for the number of vertices and edges of the generated MMP diagrams and solved equations apart from the computational power of today’s supercomputers.

In Table 1 we give the numbers of all 18- through 24-vector sets with component values from \{-1,0,1\} we generated and solved with the help of the aforementioned algorithms and programs. Vector sets with vector component values from other sets then \{-1,0,1\} are given at the end of the paper.

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Table 1: KS setups for systems with 4 degrees of freedom with up to 24 vectors with component values from \{-1,0,1\}.

We reported on the properties of the KS sets with 18 through (including) 22 vectors in [20].\footnote{Notice that here (as opposed to [20]) the sets with loops of size 2 and 3 are put together.} It took two weeks on our cluster with 500 3.4 GHz processors (recalculated) in 2004. For the present results, we ran a parallel computation for 23-vector sets and obtained the 275 sets given in the 6th row of Table 1. This took about two months on our cluster. We then analyzed the data and conjectured that all sets with solutions from \{-1,0,1\} might be
subsets of the aforementioned 24-24 set. Our program subgraph confirmed the conjecture.

That meant that we can actually get all 18- through 24-vector sets by stripping vectors and tetrads—vertices and edges in MMP notation—of the 24-24 set and filtering it with our state01 program described in [20]. We wrote the program subset to generate all subsets (i.e. MMP diagrams with edges removed) of the 24-24 set. From these, we determined the ones with 18 through 23 vectors that are isomorphic with the ones we previously obtained on our cluster. It is interesting that all such stripped sets filtered by state01 have solutions. In addition, we determined (again filtering the output of subset) 844 24-vector sets with 12 through 23 tetrads (MMP diagrams with 24 vertices with 12 through 23 edges). They are given in the seventh row of Table 1. All that, i.e., obtaining all 1232 sets shown in Table 1 with their vector component values from the 24-24-set, took a few minutes on a single PC.

In that way we can even get new sets with up to 41 vectors (upper limit for the solutions from \{-1,0,1\} [20]) simply by adding new vectors and tetrads to the sets from the 7th row of Table 1.

For a higher number of vertices we might find KS sets that do not contain any of the sets from Table 1 as their subsets. If their vectors have their component values from the set \{-1,0,1\}, they should have loops of order higher than six because they should not have any of the above 1,231 sets as their subsets. With today’s computer power, such a search is not feasible, though.

We analyzed the obtained vector sets and obtained the properties we present below. All the vector sets contain a hexagon MMP loop 1234,4567, 789A,ABCD,DEFG, GHI1 which is always given in our figures [except in Fig. 1 (b)] and for which we assume it is present whenever we give a new KS set. For instance, for 20-10 from Fig. 1 (a) we just write: H68F, IJK5, 1J9B, 4KEC.

The set 20-10 contains the smallest system 18-9. To determine the orientations of its vectors, we use the program vectorfind. It gives: \{\{1,0,0,1\} \{0,1,0,0\} \{0,0,1,0\} \{1,0,0,-1\} \{1,0,1,0\} \{-1,-1,1,1\} \{-1,1,0,0\} \{-1,-1,0,-1\} \{0,1,0,1\} \{-1,1,1,1\} \{-1,-1,1,1\} \{0,0,1,1\} \{1,1,1,1\} \{-1,1,1,1\} \{0,1,-1,0\}\}, where the first curly bracket corresponds to vector 1, the second to 2, etc.

Previously, we found two smallest (20-11) KS sets that do not contain the smallest 18-9 set (Fig. 4 (a) and (b) of [20]) and two smallest (22-13) sets that contain neither of the previous sets (Fig. 4 (c) and (d) of [20]).
Figure 1: KS sets: (a) 20-10; (b) the same 20-10 but redrawn so as to match the visual appearance of 18-9 in Fig. 1 from [25] and Fig. 3 of [20]; (c) 23-14 which contains neither 18-19 nor (a), (b), (c) from Fig. 3 of [20].

Our new results show that there are three 23-14s that contain neither the above 18-9, nor the two 20-11s, nor the first of the above 22-13s. One of them, 12JI, 1JLA, 35CE, 678K, 9ABL, CDEM, FGHN, GNK7, is given in Fig. 1 (c). It contains (d) from Fig. 3 of [20].

There are also two 23-14s that contain neither the above 18-9, nor the two 20-11s, nor the second of the above 22-13s. One of them, 12JI, 1J9B, 345K, 4KEC, 6LMB, 9ABM, FGHN, GNL7, is given in Fig. 2 (a). It contains (c) from Fig. 3 of [20].

Figure 2: (a) 23-14 which contains neither 18-19 nor (a), (b), (d) from Fig. 3 of [20]; (b) 24-15 set (the only one that exists) that does not contain any of the previous sets; (c) 24-20 that contains all previous sets;

The vectors components for the two KS sets are
\[
\begin{align*}
\{0,0,0,1\} & \{0,0,1,0\} \\
\{1,1,0,0\} & \{0,1,1,1\} \\
\{1,-1,1,1\} & \{1,1,1,1\} \\
\{0,1,0,-1\} & \{0,0,1,0\} \\
\{1,0,0,1\} & \{1,0,0,1\} \\
\{1,0,0,1\} & \{0,1,-1,-1\} \\
\{1,-1,1,1\} & \{1,1,-1,1\} \\
\{1,-1,-1,1\} & \{1,1,-1,1\} \\
\{0,1,0,-1\} & \{0,0,1,0\}
\end{align*}
\]
\[ \{1,0,0,0\}\{1,1,-1,-1\}\{0,1,1,0\}\{0,1,0,1\} \quad \text{and} \quad \{0,0,0,1\}\{1,0,0,0\}\{0,1,1,0\} \]
\[ \{0,1,-1,0\}\{1,0,0,-1\}\{1,1,1,1\}\{1,-1,1,1\}\{1,1,0,0\}\{0,0,1,1\}\{1,-1,0,0\}\{1,1,1,-1\} \]
\[ \{1,1,-1,1\}\{1,1,-1,-1\}\{0,1,0,-1\}\{1,0,1,0\}\{1,0,-1,0\}\{0,1,0,0\}\{0,0,1,0\}\{1,0,0,1\} \]
\[ \{1,1,-1,-1\}\{0,0,1,-1\}\{0,1,1,0\}\{0,1,-1,0\}\{1,1,1,1\}\] respectively.

In Fig. 2 (b) we give the only set (24-15) that does not contain any of the previous sets. The set (c) is the one which contains all the previous sets. Their MMP notations can easily be read off their figures.

The vector components for the two KS sets are \( \{0,0,0,1\}\{1,0,0,0\}\{0,1,1,0\} \)
\( \{0,1,-1,0\}\{1,1,1,1\}\{1,0,0,-1\}\{1,-1,1,1\}\{0,1,0,-1\}\{1,0,1,0\}\{0,0,1,0\}\{1,0,0,1\} \) and \( \{0,0,0,1\}\{0,1,1,0\} \)
\( \{1,0,0,0\}\{0,1,-1,0\}\{1,1,1,1\}\{1,0,0,-1\}\{1,-1,1,1\}\{1,0,1,0\}\{1,1,1,-1\}\{0,1,0,0\} \) respectively.

KS sets with vectors with component values from sets other than \(-1,0,1\) are less numerous than the ones with values from \{-1,0,1\}. They are not our primary target in this paper and we shall present only several examples below while the exhaustive generation of these sets is underway. [26]

All 37 KS sets with 22 through 24 vectors with component values from sets other than \(-1,0,1\) would have component values from \{-1,0,1\} if we discarded vectors that share only one tetrad. But we clearly cannot do so because we have to have all vectors in every tetrad to be able to assign 1 to one and 0 to three of them. This confirms the results obtained in [20, 27].

All of these sets contain the 18-9 set. The smallest one is 22-11: 25BE, 1AJK, JFLM, 68FH, 39IC. shown in Fig. 3 (a) It contains 20-10 from Fig. 1 (a).

Figure 3: KS sets with vectors whose components are not from \(-1,0,1\): (a) 22-11; (b) 23-12; (c) 24-14.
The vector components for this KS set are: \{{0,1,0,0}\{0,0,1,0\}{1,0,0,0}\{0,0,1}\{1/\sqrt{2},1/\sqrt{2},0,0\}\{1/2,-(1/2),-(1/\sqrt{2}),0\}\{1/2,-(1/2),1/\sqrt{2},0\}\{1/2,-(1/2),-(1/\sqrt{2}),0\}\{1/\sqrt{2},1/\sqrt{2},0,0\}\{1/2,1/2,1/\sqrt{2},0\}\{1/2,-(1/2),0,0\}\{1/\sqrt{2},-(1/2),-(1/\sqrt{2}),0\}\{1/2,-(1/2),-(1/\sqrt{2}),0\}\{1/2,-(1/\sqrt{2}),-(1/2),1/2\}\{1/\sqrt{2},1/\sqrt{2},0,0\}\{1/2,1/2,1/\sqrt{2},0\}\{1/2,-(1/2),0,0\}\{1/\sqrt{2},-(1/2),-(1/\sqrt{2}),0\}\{1/2,-(1/2),-(1/\sqrt{2}),0\}\{0,0,1,0\}\{1/\sqrt{2},0,1,2,1/2\}\{1/\sqrt{2},1/2,1/\sqrt{2},0\}\{1/2,-(1/2),1/\sqrt{2},0\}\{1/2,0,-(1/\sqrt{2}),0\}\{1/\sqrt{2},0,-(1/\sqrt{2}),0\}\{0,0,0,1\}\{1/\sqrt{2},0,1,2,1/2\}\{1/\sqrt{2},1/2,1/\sqrt{2},0\}\{1/2,-(1/2),0,0\}\{0,0,-(1/\sqrt{2}),1/2\}\{-(1/\sqrt{2}),0,-(1/\sqrt{2}),1/2\}\{\sqrt{3}/2,-(1/2,\sqrt{2}),-(1/2),1/4\}\{1/2,\sqrt{3}/2,\sqrt{3}/4,-(\sqrt{3}/4)\}}.

23-12 KS set shown in Fig. 3 (b) contains the 18-9, the 20-10, and the 22-11. And 24-14 set shown in Fig. 3 (c) contains the 18-9, the 20-10, a 21-11, the 22-11, a 22-12, a 22-13, and a 23-12. To avoid further visual clutter we do not give their vector components here. (Some further such components the reader can find in [26].)

We sum up our results as follows. All possible 388 KS setups for systems with 4 degrees of freedom with 18 through 23 vectors and 844 KS setups with 24 vectors with component values from \{-1,0,1\} can be obtained by “peeling” vectors off a single system provided—in effect—by Asher Peres 20 years ago. But we would not know that the setups with 18 through 23 vectors obtained by such peeling exhaust all possible KS setups up to 23 vectors without extensive computation we carried out. And the computation would not have been feasible without putting together the theory of hypergraphs, lattice theory, and interval analysis, and many algorithms and programs we devised for the purpose.

There exist sets with 22 and more vectors with component values that are not from \{-1,0,1\} and that are not isomorphic to any of the 1,233 sets mentioned above. Unlike the “\{-1,0,1\} sets,” they can be obtained only by extensive generation of MMP diagrams and computation of their properties, which we are currently carrying out. [26]

References


