# Two generalizations of column-convex polygons


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**Abstract**

Column-convex polygons were first counted by area several decades ago, and the result was found to be a simple, rational, generating function. The aim of our recent work (summarized in this paper) is to generalize that result. Let a $p$-column polyomino be a polyomino whose columns can have $1, 2, \ldots, p$ connected components. Then column-convex polygons are equivalent to 1-convex polyominoes. The area generating function of even the simplest generalisation, namely to 2-column polyominoes, is unlikely to be solvable. We therefore define two classes of polyominoes which interpolate between column-convex polygons and 2-column polyominoes. We write down the area generating functions of those two classes and then give an asymptotic analysis. The growth constants of the both classes are greater than the growth constant of column-convex polyominoes.


## 1 Introduction

The enumeration of polyominoes is a topic of great interest to chemists, physicists and combinatorialists alike [16]. In chemical terms, any polyomino (with hexagonal cells) is a possible benzenoid hydrocarbon. In physics, determining the number of $n$-celled polyominoes is related to the study of two-dimensional percolation phenomena. In combinatorics, polyominoes are of interest in their


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own right because several polyomino models have mathematically appealing exact solutions.

One very popular polyomino model is that of column-convex polygons [1]. We will consider two versions of column-convex polygons: the first composed of square cells and the second of hexagonal cells. Both versions have a rational area generating function. For the version with square cells, the area generating function was found independently by Pólya [15] in 1938 *or* 1969 and by Temperley [17] in 1956. That was perhaps the earliest major result in polyomino enumeration. For the version with hexagonal cells, the area generating function was found by Klarner in 1967 [14]. The growth constant of square-celled column-convex polygons is $\mu = 3.205569\ldots$, while the growth constant of hexagonal-celled column-convex polygons is $\mu = 3.863130\ldots$. (By the growth constant we mean the limit $\lim_{n\to\infty} \sqrt[n]{a_n}$, where $a_n$ denotes the number of $n$-celled elements in a given set of polyominoes.) In both cases the area generating function is a simple pole, so that $a_n \sim const. \times \lambda^n$.

There exist some models which are supersets of column-convex polygons and are still solvable. Those models are called *m-convex polygons* [13], *prudent polygons* [11], *cheesy polyominoes* [6], *polyominoes with cheesy blocks* [7], *column-subconvex polyominoes* [9], and *simplex-duplex polyominoes* [8]. The former two models can be enumerated by perimeter and area, whereas the latter four models have been enumerated only by area.

In this paper, we present two models: simple-$p$-column polyominoes and column-subconvex polyominoes. In a simple-$p$-column polyomino, a column may have $1,\ 2, \ldots,\ p$ connected components. However, columns with more than one component must not be adjacent to one another. In this paper we discuss the simplest version, simple-2-column polyominoes. In a column-subconvex polyomino, a column (again) may have one or two connected components. Two-component columns are allowed to be adjacent to one another, but the gap within a two-component column must not be greater than $m$ cells in size, where $m$ is a positive integer which we fix in advance. (If there were no other requirements besides "a column may have one or two connected components", the model would still be too hard, *i.e.,* not amenable to exact enumeration.)

Column-subconvex polyominoes are somewhat easier to deal with when cells are hexagons than when cells are squares. Thus, we computed the area generating function for simple-2-column polyominoes with square cells and for $m = 1$ column-subconvex polyominoes with hexagonal cells. Both of those generating functions are complicated $q$-series. In our computations, we made use of Bousquet-Mélou's [3] and Svrtan's [10] upgraded version of the Temperley methodology [17].

The computations are rather long and intricate. Therefore in this paper the detailed computations are omitted, and will be published elsewhere subsequently [8, 9]. Preliminary versions can be found at [18].

[1] We distinguish between polygons and polyominoes in that the former cannot have internal holes. As a consequence, the *perimeter* generating function for polygons has a non-zero radius of convergence, whereas for polyominoes the radius of convergence is zero.

In Section 2, we define the models. In Section 3, we immediately state the formula for $G(q, w)$, a generating function for simple-2-column polyominoes. In Section 4, we discuss the asymptotic behaviour of $G(q, w)$. In Section 5, we give the formula for $C_1(q, x)$, a generating function for $m = 1$ column-subconvex polyominoes on the hexagonal lattice. In Section 6, we discuss the asymptotic behaviour of $C_1(q, x)$. In Section 7 we conclude, outlining further work prompted by our results.

## 2 Definitions of the models

There are three regular tilings of the Euclidean plane, namely the triangular tiling, the square tiling, and the hexagonal tiling. We adopt the convention that every square or hexagonal tile has two horizontal edges. In a regular tiling, a tile is often referred to as a *cell*. A plane figure $P$ is a *polyomino* if $P$ is a union of finitely many cells and the interior of $P$ is connected. Observe that, if a union of *hexagonal* cells is connected, then it possesses a connected interior as well, as a connected union of hexagonal tiles must be connected through shared edges. Topologically, a connected union of square cells may be connected only at a shared vertex. Such unions are forbidden by the definition of polyominoes however.

Let $P$ and $Q$ be two polyominoes. We consider $P$ and $Q$ to be equal if and only if there exists a translation $f$ such that $f(P) = Q$.

Given a polyomino $P$, it is useful to partition the cells of $P$ according to their horizontal projection. Each block of that partition is a *column* of $P$. Note that a column of a polyomino is not necessarily a connected set. On the other hand, it may happen that every column of a polyomino $P$ is a connected set. In this case, the polyomino $P$ is a *column-convex polygon*. See Figure 1.

By a *2-column polyomino*, we mean a polyomino in which columns with three or more connected components are not allowed. Thus, each column of a 2-column polyomino has either one or two connected components.

A *simple-2-column polyomino* is such a 2-column polyomino in which consecutive two-component columns are not allowed. If $c$ is a column of a simple-2-column polyomino, and $c$ is a (left or right) neighbour of a two-component column, then $c$ must be a one-component column. See Figures 2 and 4.

A polyomino $P$ is a *level $m$ column-subconvex polyomino* if the following holds:

- $P$ is a 2-column polyomino,
- if a column of $P$ has two connected components, then the gap between the components consists of at most $m$ cells.

Observe that in a column-subconvex polyomino, two-component columns may be adjacent to one another. See Figures 3 and 4.

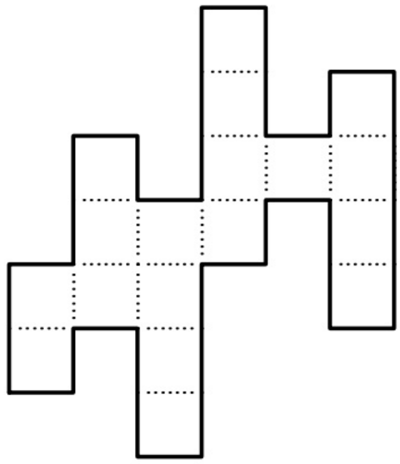

Figure 1: A column-convex polygon.

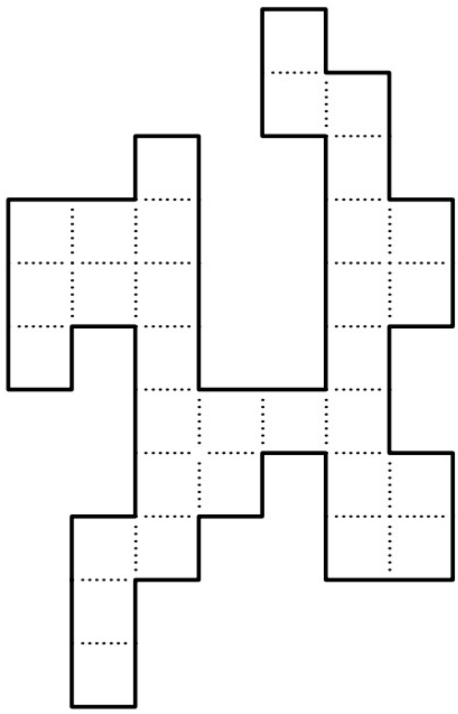

Figure 2: A simple-2-column polyomino.

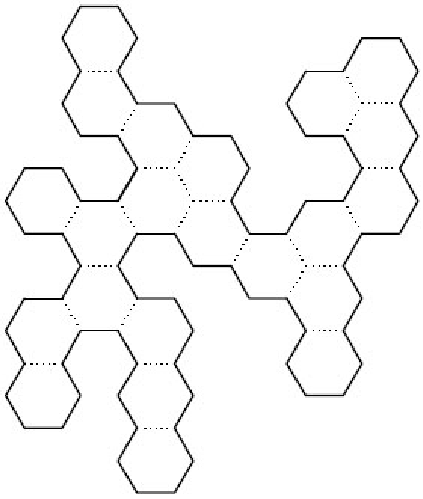

Figure 3: A level one column-subconvex polyomino.

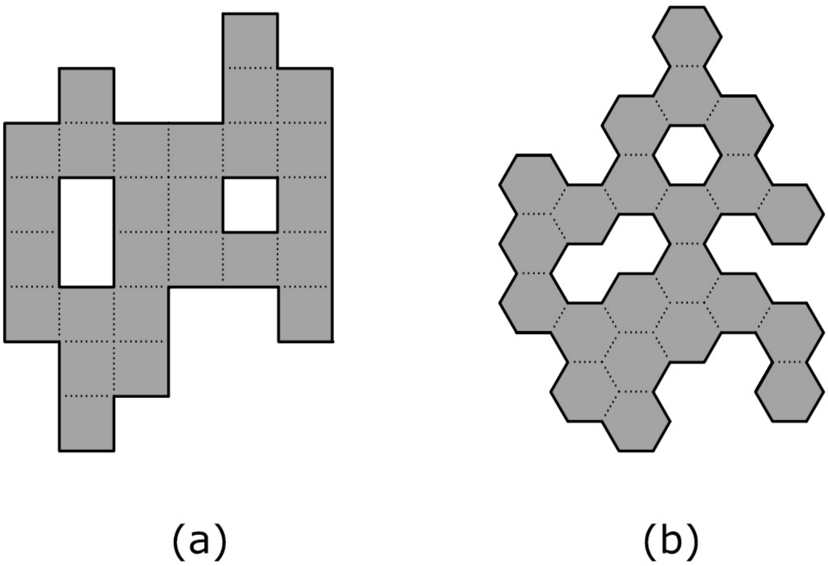


Figure 4: (a) Simple-2-convex polyominoes can have internal holes. (b) The same holds for level one column-subconvex polyominoes.

# 3 The area generating function for simple-2-column polyominoes with square cells

If a polyomino $P$ is made up of $n$ cells, we say that the *area* of $P$ is $n$. Let $\mathcal{P}$ denote the set of all simple-2-column polyominoes with square cells. Let $a$ denote the area of a polyomino $P$, and $m$ the number of two-component columns of $P$.

In Theorem 1 below, we state a formula for the generating function

$$G(q,w) = \sum_{P \in \mathcal{P}} q^a \cdot w^m.$$

**Theorem 1** *The generating function $G(q,w)$ is given by*

$$G(q,w) = \frac{NUM}{DEN}, \tag{1}$$

*where*

$$\begin{aligned}
NUM \;=\;& (1-q)^4(\tilde\alpha + \tilde\gamma + 2\tilde\alpha\tilde\eta - 2\tilde\gamma\tilde\epsilon) + q^2w(1-q)^2(\tilde\iota + \tilde\lambda - \tilde\alpha\tilde\kappa - \tilde\alpha\tilde\mu \\
& + \tilde\beta\tilde\iota + \tilde\beta\tilde\lambda - \tilde\gamma\tilde\kappa - \tilde\gamma\tilde\mu + \tilde\delta\tilde\iota + \tilde\delta\tilde\lambda - 2\tilde\epsilon\tilde\lambda + 2\tilde\eta\tilde\iota + 2\tilde\alpha\tilde\zeta\tilde\lambda - 2\tilde\alpha\tilde\eta\tilde\kappa \\
& - 2\tilde\alpha\tilde\eta\tilde\mu + 2\tilde\alpha\tilde\theta\tilde\lambda - 2\tilde\beta\tilde\epsilon\tilde\lambda + 2\tilde\beta\tilde\eta\tilde\iota + 2\tilde\gamma\tilde\epsilon\tilde\kappa + 2\tilde\gamma\tilde\epsilon\tilde\mu - 2\tilde\gamma\tilde\zeta\tilde\iota - 2\tilde\gamma\tilde\theta\tilde\iota \\
& - 2\tilde\delta\tilde\epsilon\tilde\lambda + 2\tilde\delta\tilde\eta\tilde\iota) + 2q^2w(1-q^2)(\tilde\alpha\tilde\lambda - \tilde\gamma\tilde\iota), \\
DEN \;=\;& (1-q)^4(1 - \tilde\beta + \tilde\delta - \tilde\epsilon + \tilde\eta - \tilde\alpha\tilde\zeta + \tilde\alpha\tilde\theta + \tilde\beta\tilde\epsilon - \tilde\beta\tilde\eta + \tilde\gamma\tilde\zeta - \tilde\gamma\tilde\theta \\
& - \tilde\delta\tilde\epsilon + \tilde\delta\tilde\eta) - 2(1-q)^3(\tilde\gamma + \tilde\alpha\tilde\eta - \tilde\gamma\tilde\epsilon) \\
& - 2q^2w(1-q)^2(\tilde\kappa - \tilde\beta\tilde\mu + \tilde\delta\tilde\kappa - \tilde\epsilon\tilde\kappa + \tilde\zeta\tilde\iota - \tilde\zeta\tilde\lambda + \tilde\eta\tilde\kappa - \tilde\alpha\tilde\zeta\tilde\mu + \tilde\alpha\tilde\theta\tilde\kappa \\
& + \tilde\beta\tilde\epsilon\tilde\mu - \tilde\beta\tilde\eta\tilde\mu - \tilde\beta\tilde\theta\tilde\iota + \tilde\beta\tilde\theta\tilde\lambda + \tilde\gamma\tilde\zeta\tilde\mu - \tilde\gamma\tilde\theta\tilde\kappa - \tilde\delta\tilde\epsilon\tilde\kappa + \tilde\delta\tilde\zeta\tilde\iota - \tilde\delta\tilde\zeta\tilde\lambda + \tilde\delta\tilde\eta\tilde\kappa) \\
& - 4q^2w(1-q)(\tilde\beta\tilde\lambda - \tilde\gamma\tilde\kappa + \tilde\alpha\tilde\zeta\tilde\lambda - \tilde\alpha\tilde\eta\tilde\kappa - \tilde\beta\tilde\epsilon\tilde\lambda + \tilde\beta\tilde\eta\tilde\iota + \tilde\gamma\tilde\epsilon\tilde\kappa - \tilde\gamma\tilde\zeta\tilde\iota) \\
& - 2q^3w(1-q)(\tilde\iota + \tilde\alpha\tilde\kappa - \tilde\alpha\tilde\mu - \tilde\beta\tilde\iota + \tilde\delta\tilde\iota - \tilde\epsilon\tilde\lambda + \tilde\eta\tilde\iota - \tilde\alpha\tilde\zeta\tilde\lambda + \tilde\alpha\tilde\eta\tilde\kappa \\
& - \tilde\alpha\tilde\eta\tilde\mu + \tilde\alpha\tilde\theta\tilde\lambda + \tilde\beta\tilde\epsilon\tilde\lambda - \tilde\beta\tilde\eta\tilde\iota - \tilde\gamma\tilde\epsilon\tilde\kappa + \tilde\gamma\tilde\epsilon\tilde\mu + \tilde\gamma\tilde\zeta\tilde\iota - \tilde\gamma\tilde\theta\tilde\iota - \tilde\delta\tilde\epsilon\tilde\lambda + \tilde\delta\tilde\eta\tilde\iota) \\
& - 4q^3w(\tilde\alpha\tilde\lambda - \tilde\gamma\tilde\iota),
\end{aligned}$$

$$\begin{aligned}
\tilde\beta &= \sum_{i=1}^{\infty} \frac{(-3)^{i-1} q^{i^2+2i-2} w^{i-1}}{(1-q)^{2i-2} \cdot \left[\prod_{k=1}^{i-1}(1-q^k)\right]^4 \cdot (1-q^i)^2}, \\
\tilde\gamma &= \sum_{i=1}^{\infty} \frac{(-3)^{i-1} q^{i^2+4i} w^{i}}{(1-q)^{2i} \cdot \left[\prod_{k=1}^{i-1}(1-q^k)\right]^4 \cdot (1-q^i)^3},
\end{aligned}$$

$$
\begin{aligned}
\tilde{\zeta} &= \sum_{i=1}^{\infty} \frac{(-3)^{i-1} q^{i^2+2i-2} w^{i-1}}{(1-q)^{2i-2} \cdot \left[\prod_{k=1}^{i-1}(1-q^k)\right]^4 \cdot (1-q^i)^{\overline{2}}} \\
&\quad \cdot \left(2i-2+4\cdot\sum_{k=1}^{i-1}\frac{q^k}{1-q^k}+\frac{\overline{2}q^i}{1-q^i}\right), \\
\tilde{\eta} &= \sum_{i=1}^{\infty} \frac{(-3)^{i-1} q^{i^2+4i} w^{i}}{(1-q)^{2i} \cdot \left[\prod_{k=1}^{i-1}(1-q^k)\right]^4 \cdot (1-q^i)^{\overline{3}}} \\
&\quad \cdot \left(2i+4\cdot\sum_{k=1}^{i-1}\frac{q^k}{1-q^k}+\frac{\overline{3}q^i}{1-q^i}\right), \\
\tilde{\kappa} &= \frac{1}{2}\cdot\sum_{i=1}^{\infty} \frac{(-3)^{i-1} q^{i^2+2i-2} w^{i-1}}{(1-q)^{2i-2} \cdot \left[\prod_{k=1}^{i-1}(1-q^k)\right]^4 \cdot (1-q^i)^{\overline{2}}} \\
&\quad \cdot \left[\left(2i-2+4\cdot\sum_{k=1}^{i-1}\frac{q^k}{1-q^k}+\frac{\overline{2}q^i}{1-q^i}\right)^2\right. \\
&\quad \left. -2i+2+4\cdot\sum_{k=1}^{i-1}\frac{q^{2k}}{(1-q^k)^2}+\frac{\overline{2}q^{2i}}{(1-q^i)^2}\right], \\
\tilde{\lambda} &= \frac{1}{2}\cdot\sum_{i=1}^{\infty} \frac{(-3)^{i-1} q^{i^2+4i} w^{i}}{(1-q)^{2i} \cdot \left[\prod_{k=1}^{i-1}(1-q^k)\right]^4 \cdot (1-q^i)^{\overline{3}}} \\
&\quad \cdot \left[\left(2i+4\cdot\sum_{k=1}^{i-1}\frac{q^k}{1-q^k}+\frac{\overline{3}q^i}{1-q^i}\right)^2\right. \\
&\quad \left. -2i+4\cdot\sum_{k=1}^{i-1}\frac{q^{2k}}{(1-q^k)^2}+\frac{\overline{3}q^{2i}}{(1-q^i)^2}\right].
\end{aligned}
$$

In the above formulae, it will be noticed that (a) some of the numbers have an overline, and (b) no result is given for $\tilde{\alpha}$, $\tilde{\delta}$, $\tilde{\epsilon}$, $\tilde{\theta}$, $\tilde{\iota}$, and $\tilde{\mu}$.

This is both to save space, and to highlight the close similarity between certain quantities. For all the quantities defined above, the overlines may be ignored. To obtain the formula for $\tilde{\alpha}$ from the formula for $\tilde{\beta}$, replace $\overline{2}$ by 1. To obtain the formula for $\tilde{\delta}$ from the formula for $\tilde{\gamma}$, replace the $\overline{3}$ by 4 and change $(-3)^{i-1}$ to $(-3)^i$. To obtain the formula for $\tilde{\epsilon}$ from the formula for $\tilde{\zeta}$, and also to obtain the formula for $\tilde{\iota}$ from the formula for $\tilde{\kappa}$, replace each of the $\overline{2}$'s by 1. To obtain the formula for $\tilde{\theta}$ from the formula for $\tilde{\eta}$, and also to obtain the formula for $\tilde{\mu}$ from the formula for $\tilde{\lambda}$, replace each of the $\overline{3}$'s by 4 and change $(-3)^{i-1}$ into $(-3)^i$.

By setting $w = 0$, from Theorem 1 we obtain the well known result, discovered independently by Temperley [17] and Pólya [15]:

**Corollary 1** *The area generating function for column-convex polygons with square cells is given by*

$$G(q, 0) = \frac{q(1-q)^3}{1 - 5q + 7q^2 - 4q^3} .$$

# 4 The asymptotic analysis of $G(q, w)$

We write $[q^n]f$ to denote the coefficient of $q^n$ in a power series $f = f(q)$. Note that $G(q, 0)$ is dominated by a simple pole at the smallest zero of $1 - 5q + 7q^2 - 4q^3$, which is at $q = q_c = 0.311957055278\ldots$. From the solution above for $G(q, w)$, it is a straightforward matter to generate many hundreds of terms of the series $G(q, 1)$, corresponding to the area generating function of simple-2-column polyominoes. We have $G(q, 1) = q + 2q^2 + 6q^3 + 19q^4 + 63q^5 + 216q^6 + 758q^7 + 2693q^8 + 9608q^9 + 34269q^{10} + 121946q^{11} + 432701q^{12} + 1531246q^{13} + \ldots$. The solution is too complicated to permit an analytic analysis of the asymptotics, so we resort to numerical methods. Fortunately, in this instance our methods are able to achieve almost any required accuracy.

One of the simplest things to try is to look at the ratio of successive terms. In the presence of an algebraic singularity, of the form $F(x) = \sum a_n x^n \sim A(1 - \mu x)^{-\gamma}$, one has

$$r_n = a_n / a_{n-1} = \mu(1 + (\gamma - 1)/n + \mathrm{o}(1/n)).$$

Depending on the nature of the singularity, the correction term $\mathrm{o}(1/n)$ can usually be made considerably sharper.

Writing $G(q, 1) = \sum b_n q^n$, we find $b_{50}/b_{49} = 3.522019842$,
$b_{100}/b_{99} = 3.5220198128815885$, $b_{150}/b_{149} = 3.52201981288158483006767$,
$b_{200}/b_{199} = 3.52201981288158483006752097715664$,
and $b_{250}/b_{249} = 3.5220198128815848300675209771568684365 3$. It can be seen that each additional 50 terms adds approximately 8 significant digits to the estimate of $\mu$, the limiting value of the ratios. Thus the ratios are approaching $\mu$ when extrapolated against $1/n$, with zero slope, corresponding to a simple pole singularity, as might have been expected by analogy with the behaviour of $G(q, 0)$.

We can now write the asymptotics much more precisely. We have that

$$[q^n]G(q, 1) = \lambda \mu^n + \mathrm{o}(\rho^{-n})$$

for any $1 < \rho < \rho_c$, where $\mu = 3.52201981288158483006752097715686843653\ldots$. We will shortly provide an estimate of $\rho_c$. To calculate the amplitude $\lambda$, we simply compute the sequence $[q^n]G(q, 1)/\mu^n$, which is also rapidly convergent, so that we may write $\lambda = 0.11944287040486708431326423705270432958 6\ldots$ where we are confident that our estimates of $\mu$ and $\lambda$ are correct to all quoted digits.

By analogy with some other solved polygon models [11] we hoped to identify $\mu$ as an algebraic number, but have been unable to find a convincing representation in terms of the solution of any polynomial of degree less than 20. We also consider it likely that $\lambda$ is a rational function of $\mu$, but have not been able to identify it.

With the singularity being a simple pole, subdominant terms are exponentially small. We can estimate the location of the first such singularity by the method of differential approximants [16] and find a conjugate pair of singularities at $q = q^* = 0.400e^{\pm i\pi/8.88}$. Thus $\rho_c$ defined above is given, approximately, by $0.400 \times 3.522 \approx 1.41$. Evidence of the phase factor can be seen by calculating a "correction series", with coefficients given by $[q^n]G(q,1) - \lambda\mu^n$. These coefficients have a periodicity in their sign pattern of about 9 terms, corresponding to a phase factor close to $e^{\pm i\pi/9}$, exactly as found.

We can also write $G(q,w)$ as $\sum_n G_n(q)w^n$, where

$$G_0(q) = G(q,0) = \frac{q(1-q)^3}{\Lambda},$$

with $\Lambda = 1 - 5q + 7q^2 - 4q^3$,

$$G_1(q) = \frac{q^5(2 + 3q^3 - 4q^4 + q^5 + 2q^6 - 7q^7 + 4q^8 + q^9)}{(1-q^2)^3\Lambda^2},$$

$$G_2(q) = \frac{q^7(1 + 2q + \cdots + 19q^{25} + q^{26})}{(1-q^2)^6(1-q^3)^3\Lambda^3},$$

$$G_3(q) = \frac{q^{12}(9 + 36q + \cdots + 32q^{41} + q^{42})}{(1-q^2)^6(1-q)^2(1-q^3)^4(1-q^4)^3\Lambda^4}.$$

From this structure, we can make several remarks. Firstly, note that each term in the expansion has a pole at the zero of $\Lambda$, whereas the sum of the terms has a pole closer to the origin at $q = 1/\mu$, as shown above. Secondly, note that $\Lambda^{n+1}G_n(q)$ is a rational function with denominators given by powers of cyclotomic polynomials of steadily increasing degree. If, as seems likely, this pattern persists, the zeros on the unit circle in the complex $q$ plane will become dense. Such a function cannot be differentiably finite in $w$ [4]. While this does not, in principle, exclude the possibility that $G(q,1)$ could be D-finite, it would have to be a pathological function indeed that behaved in this way. Of course, pathological functions exist, so our argument is just that– a plausibility argument, and not a proof.

# 5 The area generating function for level one column-subconvex polyominoes with hexagonal cells

Let $\mathcal{S}$ denote the set of all level one column-subconvex polyominoes with hexagonal cells. Let $a$ denote the area of a polygon $P$, and $m$ the number of columns of $P$.

In Theorem 2 below, we state a formula for the generating function

$$A_1(q,x) = \sum_{P \in \mathcal{S}} q^a \cdot x^m.$$

**Theorem 2** *The generating function $C_1(q,x)$ is given by*

$$C_1(q,x) = \frac{\sum_{n=1}^{3} num_n}{\sum_{n=1}^{6} den_n}, \tag{2}$$

*where*

$$\begin{aligned}
num_1 &= (q - 8q^2 + 28q^3 - 56q^4 + 70q^5 - 56q^6 + 28q^7 - 8q^8 + q^9)x \\
&\quad + (-4q^3 + 24q^4 - 58q^5 + 72q^6 - 48q^7 + 16q^8 - 2q^9)x^2 \\
&\quad + (5q^5 - 12q^6 + 12q^7 - 6q^8 + q^9)x^3,
\end{aligned}$$

$$\begin{aligned}
num_2 &= [(-q + 5q^2 - 9q^3 + 5q^4 + 5q^5 - 9q^6 + 5q^7 - q^8)x \\
&\quad + (-4q^3 + 18q^4 - 32q^5 + 28q^6 - 12q^7 + 2q^8)x^2 \\
&\quad + (5q^5 - 7q^6 + 5q^7 - q^8)x^3]\beta,
\end{aligned}$$

$$num_3 = (-2q^4 + 8q^5 - 12q^6 + 8q^7 - 2q^8)x^2\delta,$$

$$\begin{aligned}
den_1 &= 1 - 9q + 36q^2 - 84q^3 + 126q^4 - 126q^5 + 84q^6 - 36q^7 + 9q^8 - q^9 \\
&\quad + (-2q + 10q^2 - 16q^3 - 2q^4 + 40q^5 - 58q^6 + 40q^7 - 14q^8 + 2q^9)x \\
&\quad + (7q^3 - 36q^4 + 69q^5 - 60q^6 + 21q^7 - q^9)x^2 \\
&\quad + (-10q^5 + 10q^6 - 6q^7 + 2q^8)x^3,
\end{aligned}$$

$$\begin{aligned}
den_2 &= [(2q^2 - 12q^3 + 30q^4 - 40q^5 + 30q^6 - 12q^7 + 2q^8)x \\
&\quad + (4q^3 - 22q^4 + 46q^5 - 46q^6 + 22q^7 - 4q^8)x^2 \\
&\quad + (-10q^5 + 10q^6 - 6q^7 + 2q^8)x^3]\alpha,
\end{aligned}$$

$$\begin{aligned}
den_3 &= [-1 + 8q - 28q^2 + 56q^3 - 70q^4 + 56q^5 - 28q^6 + 8q^7 - q^8 \\
&\quad + (2q - 6q^2 - 2q^3 + 30q^4 - 50q^5 + 38q^6 - 14q^7 + 2q^8)x \\
&\quad + (13q^3 - 41q^4 + 48q^5 - 26q^6 + 7q^7 - q^8)x^2]\beta,
\end{aligned}$$

$$den_4 = [(2q^4 - 8q^5 + 12q^6 - 8q^7 + 2q^8)x^2 + (-4q^6 + 8q^7 - 4q^8)x^3]\gamma,$$

$$den_5 = [(6q^4 - 22q^5 + 30q^6 - 18q^7 + 4q^8)x^2 + (4q^6 - 4q^7)x^3]\delta,$$

$$den_6 = [(2q^4 - 6q^5 + 6q^6 - 2q^7)x^2 + (4q^6 - 4q^7)x^3](\alpha\delta - \beta\gamma),$$

$$\alpha = \sum_{i=1}^{\infty} \frac{x^i q^{\frac{i(i+5)}{2}}}{(1-q)^i \left[\prod_{k=1}^{i}(1-q^{k+1})\right]^2},$$

$$\beta = \sum_{i=1}^{\infty} \frac{x^i q^{\frac{i(i+5)}{2}}}{(1-q)^i \left[\prod_{k=1}^{i-1}(1-q^{k+1})\right]^2 (1-q^{i+1})},$$

$$\gamma = \sum_{i=1}^{\infty} \frac{x^i q^{\frac{i(i+5)}{2}} \left(\frac{i}{q} + 2\sum_{j=1}^{i} \frac{q^j}{1-q^{j+1}}\right)}{(1-q)^i \left[\prod_{k=1}^{i}(1-q^{k+1})\right]^2},$$

$$\delta = \sum_{i=1}^{\infty} \frac{x^i q^{\frac{i(i+5)}{2}} \left(\frac{i}{q} + 2\sum_{j=1}^{i-1} \frac{q^j}{1-q^{j+1}} + \frac{q^i}{1-q^{i+1}}\right)}{(1-q)^i \left[\prod_{k=1}^{i-1}(1-q^{k+1})\right]^2 (1-q^{i+1})}.$$

**Proof.** Let $S$ denote the set of all level one column-subconvex polyominoes.

When we build a column-subconvex polyomino from left to right, adding one column at a time, the intermediate figures need not all be polyominoes, and therefore need not all be elements of $S$. We say that a figure $P$ is an *incomplete level one column-subconvex polyomino* if $P$ itself is not an element of $S$, but $P$ is a "left factor" of an element of $S$. Notice that, if $P$ is an incomplete level one column-subconvex polyomino, then the last (*i.e.*, the rightmost) column of $P$ necessarily has a hole.

Let $T$ denote the set of all incomplete level one column-subconvex polyominoes.

Let $P$ be an element of $S \cup T$ and let $P$ have at least two columns. Then we define:

- the *body* of $P$ to be all of $P$, except the rightmost column of $P$,
- the *lower pivot cell* of $P$ to be the lower right neighbour of the lowest cell of the second last column of $P$,
- the *upper pivot cell* of $P$ to be the upper right neighbour of the highest cell of the second last column of $P$.

We shall deal with the following generating functions:

$$A(q,t) = \sum_{P \in S} q^{area\ of\ P} \cdot t^{the\ height\ of\ the\ last\ column\ of\ P},$$

$$A_1 = A(q,1), \qquad B_1 = \frac{\partial A}{\partial t}(q,1),$$

$$
\begin{aligned}
C(q,u,v) &= \sum_{P\in T} q^{\mathit{area\ of\ P}} \cdot u^{\substack{\mathit{the\ height\ of\ the\ upper}\\ \mathit{component\ of\ the\ last\ column\ of\ P}}} \cdot \\
&\quad \cdot v^{\substack{\mathit{the\ height\ of\ the\ lower}\\ \mathit{component\ of\ the\ last\ column\ of\ P}}},
\end{aligned}
$$

$$
D(u) = C(q,u,1), \qquad E(v) = C(q,1,v), \qquad C_1 = C(q,1,1).
$$

In the above definitions, by the height of a holed column we mean the height of the upper component plus the height of the lower component plus one. (One is the height of the hole).

Now we are going to partition the set $S$ into six subsets: $S_\alpha$, $S_\beta$, $S_\gamma$, $S_\delta$, $S_\epsilon$ and $S_\zeta$. The parts of the series $A$ that come from the sets $S_\alpha, \dots,\ S_\zeta$ will be denoted $A_\alpha, \dots,\ A_\zeta$, respectively.

By $S_\alpha$ we denote the set of level one column-subconvex polyominoes which have only one column. We have $A_\alpha = \frac{qt}{1-qt}$.

By $S_\beta$ we denote the set of all $P \in S \backslash S_\alpha$ which have the following properties: the body of $P$ lies in $S$, the last column of $P$ has no hole, and the lower pivot cell of $P$ is contained in $P$. We have $A_\beta = \frac{qt}{(1-qt)^2} \cdot A_1$.

By $S_\gamma$ we denote the set of all $P \in S \backslash S_\alpha$ which have the following properties: the body of $P$ lies in $S$, the last column of $P$ has no hole, and the lower pivot cell of $P$ is not contained in $P$. We have $A_\gamma = \frac{qt}{1-qt} \cdot B_1$.

By $S_\delta$ we denote the set of all $P \in S \backslash S_\alpha$ which have the following properties: the body of $P$ lies in $S$ and the last column of $P$ has a hole. We have $A_\delta = \frac{q^2t^3}{(1-qt)^2} \cdot (B_1 - A_1)$.

By $S_\epsilon$ we denote the set of all $P \in S \backslash S_\alpha$ which have the following properties: the body of $P$ lies in $T$ and the last column of $P$ has no hole. We have $A_\epsilon = \frac{q^2t^2}{(1-qt)^2} \cdot C_1$.

By $S_\zeta$ we denote the set of all $P \in S \backslash S_\alpha$ which have the following properties: the body of $P$ lies in $T$ and the last column of $P$ has a hole. We have $A_\zeta = \frac{2q^3t^4}{(1-qt)^3} \cdot C_1 - \frac{2q^2t^3}{(1-qt)^3} \cdot D(qt)$.

Inserting the expressions for $A_\alpha, \dots,\ A_\zeta$ into the equation $A = A_\alpha + \dots + A_\zeta$, we obtain

$$
\begin{aligned}
A &= \frac{qt}{1-qt} + \frac{qt}{(1-qt)^2} \cdot A_1 + \frac{qt}{1-qt} \cdot B_1 + \frac{q^2t^3}{(1-qt)^2} \cdot (B_1 - A_1) \\
&\quad + \frac{q^2t^2}{(1-qt)^2} \cdot C_1 + \frac{2q^3t^4}{(1-qt)^3} \cdot C_1 - \frac{2q^2t^3}{(1-qt)^3} \cdot D(qt). \qquad (3)
\end{aligned}
$$

Similarly, we partition the set $T$ into five subsets: $T_\alpha$, $T_\beta$, $T_\gamma$, $T_\delta$ and $T_\epsilon$. The parts of the series $C$ that come from the sets $T_\alpha, \dots,\ T_\epsilon$ are denoted $C_\alpha, \dots,\ C_\epsilon$, respectively.

By $T_\alpha$ we denote the set of incomplete level one column-subconvex polyominoes which have only one column. We have $C_\alpha = \frac{q^2uv}{(1-qu)(1-qv)}$.

By $T_\beta$ we denote the set of all $P \in T\backslash T_\alpha$ which have the following properties: the body of $P$ lies in $S$, and the hole of the last column of $P$ coincides either with the lower pivot cell of $P$ or with the upper pivot cell of $P$. We have $C_\beta = \frac{2q^2uv}{(1-qu)(1-qv)} \cdot A_1$.

By $T_\gamma$ we denote the set of all $P \in T\backslash T_\alpha$ which have the following properties: the body of $P$ lies in $S$, and the hole of the last column of $P$ lies either below the lower pivot cell of $P$ or above the upper pivot cell of $P$. We have $C_\gamma = \frac{q^2uv}{(1-qu)^2(1-qv)} \cdot A_1 + \frac{q^2uv}{(1-qu)(1-qv)^2} \cdot A_1$.

By $T_\delta$ we denote the set of all $P \in T\backslash T_\alpha$ which have the following properties: the body of $P$ lies in $T$, and the hole of the last column of $P$ touches the hole of the second last column of $P$. We have $C_\delta = \frac{2q^2uv}{(1-qu)(1-qv)} \cdot C_1$.

By $T_\epsilon$ we denote the set of all $P \in T\backslash T_\alpha$ which have the following properties: the body of $P$ lies in $T$, and the hole of the last column of $P$ does not touch the hole of the second last column of $P$. We have $C_\epsilon = \frac{q^2uv}{(1-qu)(1-qv)^2} \cdot D(qv) + \frac{q^2uv}{(1-qu)^2(1-qv)} \cdot D(qu)$.

Inserting the expressions for $C_\alpha, \ldots, C_\epsilon$ into the equation $C = C_\alpha + \ldots + C_\epsilon$, we obtain a functional equation for $C$. For our purposes, it will be enough to state the case $v = 1$ of that functional equation. With the notation

$$F = 1 + \frac{3-2q}{1-q} \cdot A_1 + 2C_1 + \frac{1}{1-q} \cdot D(q), \tag{4}$$

the case $v = 1$ of the functional equation for $C$ reads

$$D(u) = \frac{q^2u}{(1-q)(1-qu)^2} \cdot A_1 + \frac{q^2u}{(1-q)(1-qu)} \cdot F + \frac{q^2u}{(1-q)(1-qu)^2} \cdot D(qu). \tag{5}$$

The iteration of (5) produces

$$\begin{aligned} D(u) \quad = \quad & \left\{ \sum_{i=1}^{\infty} \frac{q^{\frac{i(i+3)}{2}} u^i}{(1-q)^i \cdot \left[\prod_{k=1}^{i} (1-q^k u)\right]^2} \right\} \cdot A_1 \\ & + \left\{ \sum_{i=1}^{\infty} \frac{q^{\frac{i(i+3)}{2}} u^i}{(1-q)^i \cdot \left[\prod_{k=1}^{i-1} (1-q^k u)\right]^2 \cdot (1-q^i u)} \right\} \cdot F. \end{aligned} \tag{6}$$

Then we set up a system of six linear equations in six unknowns: $A_1$, $B_1$, $C_1$, $D(q)$, $D'(q)$ and $F$. One of the six equations is (4), and the other five are obtained as follows:

- by setting $t = 1$ in (3),

- by differentiating (3) with respect to $t$ and then setting $t = 1$,
- by setting $u = 1$ in (5),
- by setting $u = q$ in (6),
- by differentiating (6) with respect to $u$ and then setting $u = q$.

Once the linear system is solved, the proof of the theorem is complete. (To solve the linear system, we made use of the computer algebra package *Maple*.)

$\square$

# 6 The asymptotic analysis of $C_1(q, x)$

Our analysis of the $C_1(q, x)$ series parallels that given in Section 4, as the singularities here are also simple poles. From the solution above for $C_1(q, x)$, it is again a straightforward matter to generate many hundreds of terms of the series $C_1(q, 1)$, corresponding to the area generating function of level one column-subconvex polyominoes. We have $C_1(q, 1) = q + 3q^2 + 11q^3 + 44q^4 + 184q^5 + 786q^6 + 3391q^7 + 14683q^8 + 63619q^9 + 275506q^{10} + 1192134q^{11} + 5154794q^{12} + 22278047q^{13} + 96250859q^{14} + \ldots$. The solution is too complicated to permit an analytic analysis of the asymptotics, so we again resort to numerical methods. As above, the ratios of successive terms are rapidly convergent, enabling us to estimate that the dominant singularity is at
$\mu = 4.3191392437297882262941251868138189849416008 1$. The asymptotics are given by

$$[q^n]C_1(q, 1) = \lambda\mu^n + \mathrm{o}(\rho^{-n})$$

for any $1 < \rho < \rho_c$, where $\mu$ is given above and
$\lambda = 0.12242810045612224320502391150597363330617138 3\ldots$ where we are confident that our estimates of $\mu$ and $\lambda$ are correct to all quoted digits. Again, we have been unable to find a convincing representation of $\mu$ in terms of the solution of any polynomial of degree less than 20. We also consider it likely that $\lambda$ is a rational function of $\mu$, but have not been able to identify it.

With the singularity being a simple pole, subdominant terms are exponentially small. We can estimate the location of the first such singularity by the method of differential approximants [16] and find a conjugate pair at $q = q^* = 0.399878e^{\pm i\pi/9.4864}$. Thus $\rho_c$ defined above is given, approximately, by $0.399878 \times 4.3191 \approx 1.727$. Evidence of the phase factor can, as in the previous case, be seen by calculating a "correction series", with coefficients given by $[q^n]C_1(q, 1) - \lambda\mu^n$. These coefficients again have a periodicity in their sign pattern of about 9 terms, corresponding to a phase factor close to $e^{\pm i\pi/9}$, exactly as found.

We can also write $C_1(q, x)$ as $\sum_{n>0} C_1^{(n)}(q)x^n$, where

$$C_1^{(1)}(q) = \frac{q}{1 - q},$$

$$C_1^{(2)}(q) = \frac{2q^2(1-q+q^3)}{(1-q)^5(1+q)},$$
$$C_1^{(3)}(q) = \frac{q^3(4+q+\cdots-2q^7-q^8)}{(1-q)^8(1+q)^2(1+q+q^2)},$$
$$C_1^{(4)}(q) = \frac{2q^4(4+6q+\cdots-4q^{12}-2q^{13})}{(1-q)^{11}(1+q)^3(1+q+q^2)^2(1+q^2)}.$$

From this structure, we note that $C_1^{(n)}(q)$ is a rational function with denominators given by powers of cyclotomic polynomials of steadily increasing degree. (Indeed, the increases are very systematic, so that one could readily conjecture the pattern). If, as seems likely, this pattern persists, the zeros on the unit circle in the complex $q$ plane will become dense. As noted above, such a function cannot be differentiably finite in $x$ [4]. As for the previous case, this is a plausibility argument, rather than a proof, that $C_1(q, 1)$ is not D-finite.

# 7 Further work

Our next goal will be to find the area generating function for simple-2-column polyominoes with hexagonal cells. That should not be difficult because we already have a method which works with square cells. It is usually possible to make such a method work when cells are hexagons as well.

Unlike the simple, rational expression given above for the area generating function of column-convex polygons, the area generating function of convex polygons is a sum of rational functions of $q$-series [2]—not unlike our solution for the area generating function of simple-2-column polyominoes, though not as complicated. For convex polygons, it is also possible to find the generating function by perimeter. This was first given in [5] and was later obtained independently in [12]. However, if one asks for the perimeter generating function of simple-2-column polyominoes, it turns out that this has zero radius of convergence. We show this by a very simple argument.

Consider a square of side $2n+1$ sites. This clearly has perimeter $8n+4$. Then construct a simple-2-column polyomino by placing a single square (of perimeter 4) in any of the square cells of the second column, except the top and bottom. This can be done in $2n-1$ ways. Now repeat this for the fourth column, the sixth column, up to the $2n^{th}$ column. We have therefore placed $n$ squares inside the large square, so the total perimeter of our object, which is a simple-2-column polyomino, is now $12n+4$. The squares can be placed in $(2n-1)^n$ ways. Thus if $p_{2n}$ denotes the number of simple-2-column polyominoes of perimeter $2n$, we have $p_{12n+4} \geq (2n-1)^n$. The large $n$ limit of $\frac{1}{2n}\log p_{2n}$ diverges, hence the radius of convergence is zero. While this does not mean that the perimeter generating function is uninteresting, it would be a whole new research project to study the nature of the singularity, and its significance, and will not be discussed further in this article.

In terms of possible extensions of this work, it is probably possible to compute the area generating function of *simple-2-column*$_2$ polyominoes. Here, by a simple-2-column$_2$ polyomino we mean a 2-column polyomino in which runs of two consecutive two-component columns are allowed, but it is forbidden for three consecutive columns to each have two connected components. One reason for doing this is that the growth factor $\mu$ is expected to be greater than that for simple-2-column polyominoes, and may set the benchmark in this regard. At present the situation is that for column-convex polygons the growth constant is $\mu = 3.205569\ldots$, while for simple-2-column polyominoes the growth constant is $\mu = 3.5220198\ldots$. For polyominoes the best lower bound [1] is $\mu \geq 3.980137$, which is quite close to the best estimate [16] $\mu \approx 4.0625696$. The polyomino model with a growth constant closest to the actual value for polyominoes is a directed model called *multi-directed polyominoes* [4] with a growth constant of $\mu \approx 3.58$. It would be interesting to compute the area of simple-2-column$_2$ polyominoes to see if they had a growth constant closer still to that for polyominoes.

As regards column-subconvex polyominoes, the above argument may be repeated *mutatis mutandis* to show that the perimeter generating function will also have zero radius of convergence. The growth constant for this model, when enumerated by area, is $\mu = 4.319139\ldots$, which may be compared to the best estimate for hexagonal polyominoes [16] of $\mu \approx 5.1831453$. The enumeration by area of the level two model is possible, but takes a lot of effort. We did perform that enumeration, which will be published subsequently, and the formula for the area generating function of level two column-subconvex polyominoes is available at [19]. The growth constant is found to increase to $\mu = 4.50948\ldots$ in that case.

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(This file is a *Maple 9.5* worksheet. The file can also be obtained from Svjetlan Feretić via e-mail.)