Acta Math. Hungar., **130 (1–2)** (2011), 35–49
DOI: 10.1007/s10474-010-0038-2
First published online November 3, 2010

CIRCULAR QUARTICS IN THE ISOTROPIC PLANE GENERATED BY PROJECTIVELY LINKED PENCILS OF CONICS

E. JURKIN

Faculty of Mining, Geology and Petroleum Engineering, University of Zagreb, Pierottijeva 6, 10000 Zagreb, Croatia
e-mail: ejurkin@rgn.hr

(Received October 19, 2009; revised March 24, 2010; accepted June 2, 2010)

Abstract. A curve in the isotropic plane is circular if it passes through the absolute point F. Its degree of circularity is defined as the number of its intersection points with the absolute line f falling into the absolute point F.

A curve of order four can be obtained as a locus of the intersections of corresponding conics of projectively linked pencils of conics. In this paper the conditions that the pencils and the projectivity have to fulfill in order to obtain a circular quartic of a certain degree of circularity have been determined analytically. The quartics of all degrees of circularity and all types (depending on their position with respect to the absolute figure) can be constructed using these results.

The results have first been stated for any projective plane and then their isotropic interpretation has been given.

1. Introduction

An *isotropic plane* $\mathcal{I}_2$ is a real projective plane where the metric is induced by a real line f and a real point F, incidental with it [6], [7]. The ordered pair (f, F) is called the *absolute figure* of the isotropic plane.

All straight lines through the absolute point F are called *isotropic lines,* and all points incidental with f are called *isotropic points.*

There are seven types of regular conics classified depending on their position with respect to the absolute figure, [1], [6]: ellipse (imaginary or real), hyperbola (of 1^{st} or 2^{nd} type), special hyperbola, parabola and circle.

A curve in the isotropic plane is *circular* if it passes through the absolute point [8]. Its *degree of circularity* is defined as the number of its intersection points with the absolute line f falling into the absolute point. If it does not share any common point with the absolute line except the absolute point, it is *entirely circular* [5].

Key words and phrases: isotropic plane, circular quartic, projectivity, pencils of conics.
2000 Mathematics Subject Classification: 51M15, 51N25.

A circular curve of order four can be 1, 2, 3 or 4-circular. The absolute line can intersect it, touch it, osculate or hyperosculate it at the absolute point. The absolute point can be simple, double or triple point of the curve. Due to that for each degree of circularity we distinguish several types of quartics.

In [2] the conditions for obtaining circular quartics by automorphic inversion have been derived. But, it has been shown that it is not possible to construct quartics of all types by using this transformation.

The aim of this paper is to construct circular quartics by using two projectively linked pencils of conics and to classify them depending on their position with respect to the absolute figure. It will be shown that it is possible to obtain quartics of all degrees of circularity and for each degree it is possible to construct quartics of all types.

1-circular quartics pass through the absolute point F and meet the absolute line f in three further points. Those three points can be different, two of them can fall together (the line touches the curve at a simple (ordinary) point or the line intersects the curve in a double point) or all three fall together (the line osculates the curve at a simple point, the line touches the curve at a double point or the line intersects the curve in a triple point). 2-circular quartics touch f at F or they have a double point at F. The other two intersection points can be different or fall together. 3-circular quartics osculate f at F, touch it at a double point or intersect it in a triple point falling into F. 4-circular quartics do not share any common point with f except F. They are called entirely circular quartics of the isotropic plane. The absolute point can be a simple point at which the line hyperosculates the curve, a double point at which the line osculates the curve, a double point at which both tangents coincide with the absolute line or a triple point at which the line touches the curve.

2. Projectively linked pencils of conics

A curve of order four can be defined as a locus of the intersections of pairs of corresponding conics in projectively linked pencils of conics [4]. Let A, B, C and D be basic conics (symmetric 3×3-matrices) of projectively linked pencils $A+\lambda B$ and $C+\lambda D$. This projectivity maps the conic A to C, B to D and $A+\lambda B$ to $C+\lambda D$. Some calculation delivers the equation

$$\mathrm{F}(\vec{x}) \equiv \vec{x}^\top B\vec{x}\cdot\vec{x}^\top C\vec{x} - \vec{x}^\top A\vec{x}\cdot\vec{x}^\top D\vec{x} = 0$$

of the quartic k. The quartic k passes through the following sixteen points: four basic points of the pencil $[A,B]$, four basic points of the pencil $[C,D]$, four intersection points of A and C and four intersection points of B and D.

Remark. We have to be aware of the fact that proportional symmetric matrices A, B, C, D and αA, βB, γC, δD represent the same four conics, but the corresponding quartics are different. For defining the projectivity beside A, C and B, D we need one more pair of conics.

We will now search for the conditions that the pencils and the projectivity have to fulfill in order to obtain a circular quartic of a certain type. The results will be first stated for a projective plane and then their isotropic interpretation will be given.

Let us start with a point $\vec{y}$ on the quartic k. We can assume that it is an intersection point of the basic conics A and C and we have $\vec{y}^\top A\vec{y} = 0$, $\vec{y}^\top C\vec{y} = 0$. The tangential behavior in this point on k is usually being studied by observing the intersections of k with arbitrary straight lines through $\vec{y}$. Such a line q will be spanned by $\vec{y}$ and a further point $\vec{z}$. It can be parametrized by

$$q \quad \dots \quad \vec{y} + t\vec{z}, \quad t \in \mathbb{R} \cup \infty.$$

We compute the intersections of k and q: They belong to the zeros of the following polynomial of degree 4 in t

$$\begin{aligned}(1)\qquad \mathrm{p}(t) = \mathrm{F}(\vec{y} + t\vec{z}) &= 2t\big[\vec{y}^\top B\vec{y} \cdot \vec{y}^\top C\vec{z} - \vec{y}^\top A\vec{z} \cdot \vec{y}^\top D\vec{y}\big] \\ &+ t^2\big[\vec{y}^\top B\vec{y} \cdot \vec{z}^\top C\vec{z} + 4\vec{y}^\top B\vec{z} \cdot \vec{y}^\top C\vec{z} - 4\vec{y}^\top A\vec{z} \cdot \vec{y}^\top D\vec{z} - \vec{z}^\top A\vec{z} \cdot \vec{y}^\top D\vec{y}\big] \\ &+ 2t^3\big[\vec{y}^\top B\vec{z} \cdot \vec{z}^\top C\vec{z} + \vec{z}^\top B\vec{z} \cdot \vec{y}^\top C\vec{z} - \vec{y}^\top A\vec{z} \cdot \vec{z}^\top D\vec{z} - \vec{z}^\top A\vec{z} \cdot \vec{y}^\top D\vec{z}\big] \\ &+ t^4\big[\vec{z}^\top B\vec{z} \cdot \vec{z}^\top C\vec{z} - \vec{z}^\top A\vec{z} \cdot \vec{z}^\top D\vec{z}\big].\end{aligned}$$

The equation of the tangent of k at the regular (simple) point $\vec{y}$ is

$$(2)\qquad \vec{y}^\top B\vec{y} \cdot \vec{y}^\top C\vec{z} - \vec{y}^\top A\vec{z} \cdot \vec{y}^\top D\vec{y} = 0$$

as $t = 0$ has to be zero of multiplicity 2 of the polynomial p, and $\vec{y}$ is the intersection of k and q with multiplicity 2.

We now will discuss the following three cases:

- $\vec{y} \in B, D$,
- $\vec{y} \in B, \vec{y} \notin D$,
- $\vec{y} \notin B, D$.

If $\vec{y}$ lies on B and D, then the coefficient of t of the polynomial (1) vanishes. Therefore, $t = 0$ is zero of multiplicity 2 for every point $\vec{z}$ of the plane. In other words, if two projectively linked pencils have a common basic point, it is a singular point of the quartic k.

If $\vec{y}$ lies on B, but not on D, in general case (when A is a regular conic) the tangent of k at $\vec{y}$ contains the points $\vec{z}$ with

$$\vec{y}^\top A\vec{z} \cdot \vec{y}^\top D\vec{y} = 0.$$

Obviously the tangent of k at $\vec{y}$ is the tangent of A at $\vec{y}$.

If A is a singular conic with a singular point $\vec{y}$ ($A = a \cup \overline{a}$, $\vec{y} \in a$, $\overline{a}$), the above-mentioned equation is satisfied for every $\vec{z}$. This means that $\vec{y}$ is a singular point on k. If A is a singular conic, but $\vec{y}$ is not its singular point ($A = a \cup \overline{a}$, $\vec{y} \in a$, $\vec{y} \notin \overline{a}$), the tangent of k at $\vec{y}$ is given by the equation $\vec{y}^{\top} A\vec{z} = 0$.

If we interpret $\vec{y}$ as the absolute point F and the basic conic A as a special hyperbola, the quartic k is 1-circular (Fig. 1). By choosing a circle for the conic A, we will obtain 2-circular quartic k touching the absolute line at the absolute point (Fig. 2). It is easy to conclude that by proper choice of basic conics it is also possible to get 1-circular quartics that intersect the absolute line in three points beside the absolute point, or they intersect it in one point and touch it at another, and 2-circular quartics that touch the absolute line at the absolute point and intersect it in two points or touch it at one more.

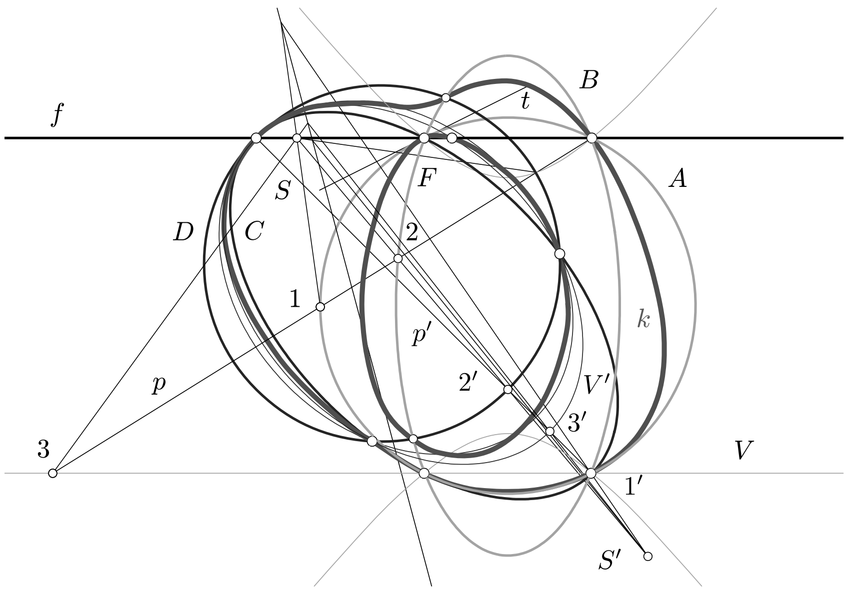

Fig. 1

It follows that if $\vec{y}$ is a basic point of one of the projectively linked pencils, the tangent of the quartic at $\vec{y}$ is identical with the tangent of the conic of the pencil linked to the conic of the second pencil which passes through $\vec{y}$.

Fig. 1 displays a 1-circular quartic k. It is generated by the projectivity linking a pencil of special hyperbolas $[A, B]$ with two basic points on absolute line f and a pencil $[C, D]$ with two basic points at F, while C is a special hyperbola. The projectivity is defined by three corresponding pairs: $A \leftrightarrow C$, $B \leftrightarrow D$ and $V \leftrightarrow V'$, where V is a singular conic formed by two lines, while the absolute line is one of them. The constructive determination of interlinking pairs of conics of pencils $[A, B]$, $[C, D]$ uses the induced projectivities on the lines p, p'.

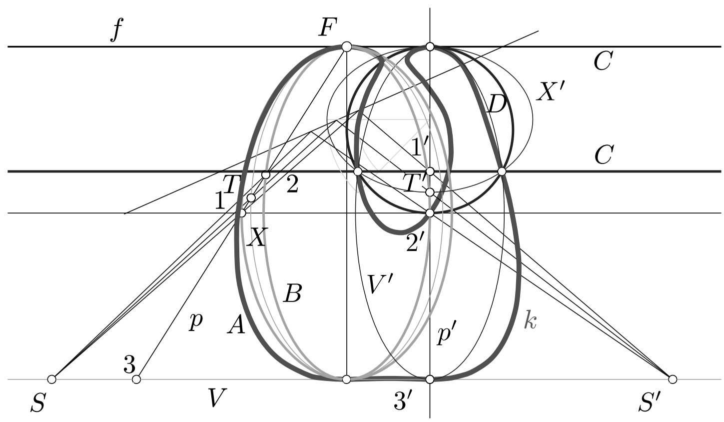

Fig. 2

The quartic k shown in Fig. 2 is 2-circular. It is a result of projectively linked pencils $[A, B]$, $[C, D]$ with the following properties: A is a circle, $[C, D]$ is a pencil of parabolas with a common isotropic point, C is a singular conic. Four intersection points of k and f form their two points of contact.

We can state:

THEOREM 1. *Let A, B, C be circular and D a non-circular conic of the isotropic plane with the absolute figure (f, F) and let $\pi : [A, B] \longmapsto [C, D]$ be a projectivity mapping a conic $A + \lambda B$ to a conic $C + \lambda D$.*

The result of π is a quartic k which is 1-circular iff A is a special hyperbola or a singular conic with a regular point F not containing f, and a 2-circular quartic touching f at F iff A is a circle or a singular conic with a regular point F containing f.

The tangent $\vec{y}^\top A\vec{z} = 0$ can also osculate the quartic k. To gain $\vec{y}$ as the intersection of q and k with multiplicity 3, the coefficient of the polynomial (1) of t^2 has to equal zero. Therefore, osculation is characterized by the condition

$$\vec{y}^\top B\vec{y} \cdot \vec{z}^\top C\vec{z} + 4\vec{y}^\top B\vec{z} \cdot \vec{y}^\top C\vec{z} - 4\vec{y}^\top A\vec{z} \cdot \vec{y}^\top D\vec{z} - \vec{z}^\top A\vec{z} \cdot \vec{y}^\top D\vec{y} = 0, \tag{3}$$

for all $\vec{z}, \vec{y}^\top A\vec{z} = 0$, while hyperosculation is characterized by (3) and the condition

$$\vec{y}^\top B\vec{z} \cdot \vec{z}^\top C\vec{z} + \vec{z}^\top B\vec{z} \cdot \vec{y}^\top C\vec{z} - \vec{y}^\top A\vec{z} \cdot \vec{z}^\top D\vec{z} - \vec{z}^\top A\vec{z} \cdot \vec{y}^\top D\vec{z} = 0, \tag{4}$$

for all $\vec{z}, \vec{y}^\top A\vec{z} = 0$. If A is a singular conic with non-singular point $\vec{y}$, $A = a \cup \overline{a}$, $\vec{y} \in a$, $\vec{y} \notin \overline{a}$, then $\vec{y}^\top A\vec{z} = 0$ is the equation of the line a. The condition (3) is fulfilled iff $\vec{y}^\top B\vec{z} \cdot \vec{y}^\top C\vec{z} = 0$ for every $\vec{z} \in a$ ($\vec{y}^\top A\vec{z} = 0$) i.e. iff one of the conics B, C touches a.

Depending on whether $[A, B]$ is a pencil of circles or parabolas (and C is passing through F) the constructed quartic k is either 3-circular or 1-circular having the absolute line as an inflexion tangent.

The tangent $\vec{y}^\top A\vec{z} = 0$ will hyperosculate the quartic k iff for every $\vec{z}$ on the tangent the equality $\vec{y}^\top B\vec{z} \cdot \vec{z}^\top C\vec{z} + \vec{z}^\top B\vec{z} \cdot \vec{y}^\top C\vec{z} = 0$ holds. This is fulfilled iff both conics B and C touch a.

By choosing for $[A, B]$ a pencil of isotropic circles and for $[C, D]$ a pencil of conics with a circle C as basic conic, while D is non-circular, an entirely circular quartic k will be obtained, Fig. 3.

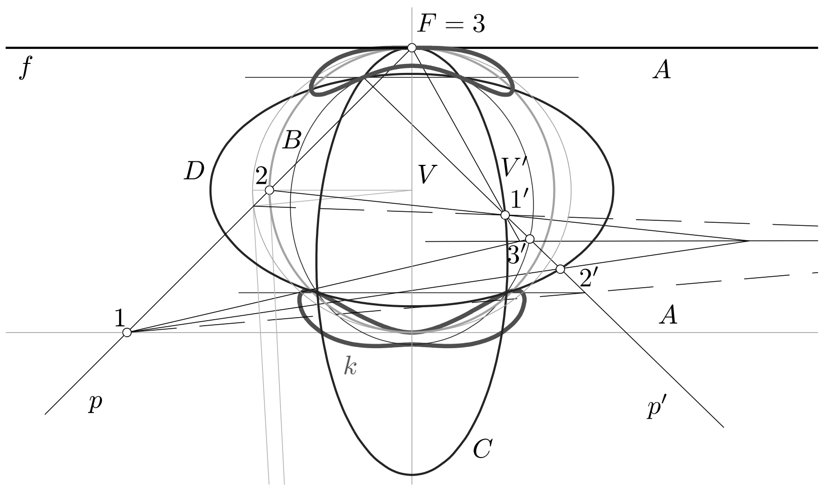

Fig. 3

Now we study the last case when $\vec{y}$ is not a basic point of either of the pencils of conics. The tangent of k at y given by the equation (2) is a linear combination of the tangents of the conics A, C at that point.

If A is a singular conic with the singular point $\vec{y}$, the tangent of k at $\vec{y}$ is the tangent of C. The conic C should not be a singular conic with the singular point $\vec{y}$ at the same time, as the quartic would have a singularity in $\vec{y}$ in this case.

If A is a singular conic with a regular point $\vec{y}$, $(A = a \cup \overline{a}, \vec{y} \in a, \vec{y} \notin \overline{a})$, the line a is the tangent of k at $\vec{y}$ iff C touches a.

The tangent $\vec{y}^\top A\vec{z} = 0$ of a regular conic A is the tangent of the quartic k as well, iff C touches A or C is a singular conic with the singular point $\vec{y}$.

The following theorem is an immediate consequence of our previous results:

THEOREM 2. *Let A, C be circular and B, D non-circular conics of the isotropic plane and let A and C have a point of contact at the absolute point. Let $\pi : [A,B] \longmapsto [C,D]$ be a projectivity that maps the conics $A + \lambda B$ to the conics $C + \lambda D$. The result of π is a quartic k which is 1-circular iff A and C are special hyperbolas. k is a 2-circular quartic touching f at F iff A and C are circles.*

A necessary condition to gain $\vec{y}$ as a double point is

$$\vec{y}^\top B\vec{y} \cdot \vec{y}^\top C\vec{z} - \vec{y}^\top A\vec{z} \cdot \vec{y}^\top D\vec{y} = 0, \quad \forall \vec{z} \tag{5}$$

as then $t = 0$ is a zero of multiplicity 2 of the polynomial (1) for all $\vec{z}$ (and for all lines q passing through $\vec{y}$). The equations of the tangents at $\vec{y}$ are determined by

$$\vec{y}^\top B\vec{y} \cdot \vec{z}^\top C\vec{z} + 4\vec{y}^\top B\vec{z} \cdot \vec{y}^\top C\vec{z} - 4\vec{y}^\top A\vec{z} \cdot \vec{y}^\top D\vec{z} - \vec{z}^\top A\vec{z} \cdot \vec{y}^\top D\vec{y} = 0. \tag{6}$$

The same three cases will be discussed again:

- $\vec{y} \in B, D$,
- $\vec{y} \in B, \vec{y} \notin D$,
- $\vec{y} \notin B, D$.

If both B and D pass through $\vec{y}$, the condition (5) is fulfilled. The equation (6) is reduced to

$$\vec{y}^\top B\vec{z} \cdot \vec{y}^\top C\vec{z} - \vec{y}^\top A\vec{z} \cdot \vec{y}^\top D\vec{z} = 0.$$

If A is a singular conic having $\vec{y}$ as a singular point, the tangents to k are given by

$$\vec{y}^\top B\vec{z} \cdot \vec{y}^\top C\vec{z} = 0.$$

This implies the following: If the conics of one pencil touch each other at the common basic point $\vec{y}$, one of the tangents to the quartic at that point coincides with the common tangent of conics. The other tangent to the quartic coincides with the tangent of the conic from the second pencil that is linked to a singular conic with the singular point $\vec{y}$ from the first pencil.

Defining $\vec{y}$ as the absolute point F and the conics B, C as special hyperbolas results in a 2-circular quartic k with double point in F. If one of the conics B, C is a special hyperbola and the other is a circle, k is a 3-circular quartic touching f at the double point F. If B and C are circles, the result is a 4-circular quartic having F as the double point at which both tangents coincide with the absolute line.

If A is not a singular conic with the singular point $\vec{y}$ and either B or C is a singular conic with the singular point $\vec{y}$, the tangents of k at $\vec{y}$ are given by the equation

$$\vec{y}^{\top}A\vec{z}\cdot\vec{y}^{\top}D\vec{z}=0.$$

We will obtain a 2-circular, 3-circular or 4-circular quartic depending on whether we choose A and D to be special hyperbolas, one of them to be a special hyperbola and the other a circle, or both to be circles.

If A is not a singular conic with the singular point $\vec{y}$ and neither B nor C is a singular conic with the singular point $\vec{y}$, one of the tangents to k is the tangent to A at $\vec{y}$ iff $\vec{y}^{\top}B\vec{z}=\alpha\vec{y}^{\top}A\vec{z}$ or $\vec{y}^{\top}C\vec{z}=\alpha\vec{y}^{\top}A\vec{z}$, i.e. iff B touches A or C touches A. Let us suppose this first case. Then $B-\alpha A$ is a singular conic with a singular point $\vec{y}$. Now the tangents of k at $\vec{y}$ are

$$\vec{y}^{\top}A\vec{z}\big[\alpha\vec{y}^{\top}C\vec{z}-\vec{y}^{\top}D\vec{z}\big]=0.$$

In the case of C, D touching each other at $\vec{y}$, the second tangent at that point coincides with the common tangent to the conics of the pencil $[C,D]$ (Fig. 4).

Both tangents coincide with the tangent to A iff B touches A and the conic corresponding to the singular conic $B-\alpha A$ touches A. Particularly, that holds if C, D touch A, B (Fig. 5).

Therefore, if we consider $\vec{y}$ to be the absolute point F, the pencils of conics have to be pencils of special hyperbolas. The result of this projectivity will be a 2-circular quartic intersecting the absolute line in two further points. Those points can be different or can coincide forming either a double point or a point of contact of k with f. If we consider one of the pencils $[A,B]$, $[C,D]$ to be a pencil of special hyperbolas and the other as a pencil of circles, the absolute line f will touch the 3-circular quartic at the double point F. Further, if we consider both pencils to be pencils of circles, a 4-circular quartic k with tangents coincident with f at the double point F, will be generated.

With suitable choice of projectively linked pencils of conics 1-circular and 2-circular quartics having double points in an isotropic point (different from F) can also be obtained.

The quartic k shown in Fig. 4 is 3-circular. It is generated by a projectively linked pencil of circles $[A,B]$ and a pencil of special hyperbolas $[C,D]$ touching at the absolute point F.

If a pencil of hyperbolas had been chosen instead of a pencil of circles, and a pencil of hyperbolas having a common basic point in an isotropic point

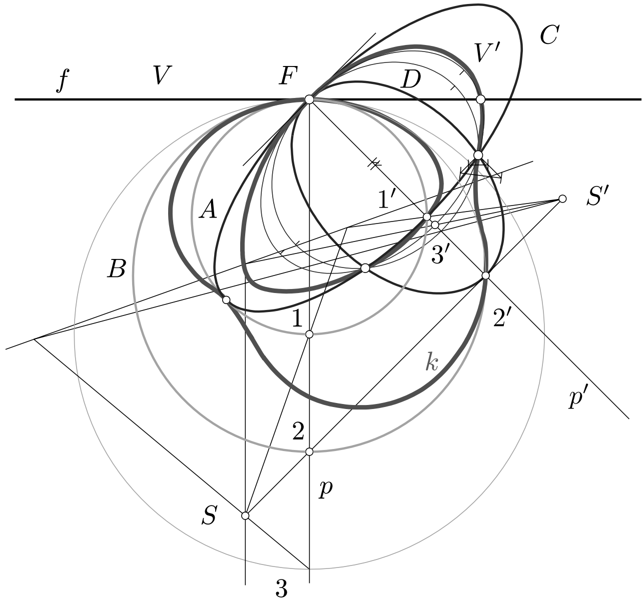

Fig. 4

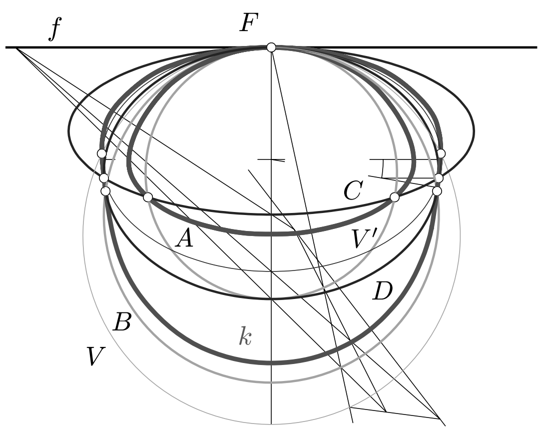

Fig. 5

instead of a pencil of special hyperbolas, and if we had defined the projectivity in the way that the singular conic V (containing f as its part) from the first pencil is mapped to a special hyperbola V' from the second, the constructed quartic would be 1-circlar.

The quartic k shown in Fig. 5 is obtained by two projectively linked pencils of circles and therefore is 4-circular.

If pencils of special hyperbolas touching at the absolute point had been chosen instead of the pencils of circles, the constructed quartic would be 2-circlar.

Previous studies lead us to:

THEOREM 3. *Let A, B, C, D be circular conics of the isotropic plane with the absolute figure (f, F). Let $\pi : [A, B] \longmapsto [C, D]$ be the projectivity which maps the conics $A + \lambda B$ to the conics $C + \lambda D$. The result of π is a quartic k having F as a double point. If A, B are circles, k touches f at F (k is 3-circular quartic). The quartic k is entirely circular iff the conic corresponding to the singular conic with the singular point F from $[A, B]$ is a circle.*

Let us observe the possibility of gaining inflexion in the double point. One of the tangents at the common basic point $\vec{y}$ of the pencils will osculate one branch of the quartic if the following equality holds for all $\vec{z}$ on the tangent

$$\vec{y}^\top B\vec{z} \cdot \vec{z}^\top C\vec{z} + \vec{z}^\top B\vec{z} \cdot \vec{y}^\top C\vec{z} - \vec{y}^\top A\vec{z} \cdot \vec{z}^\top D\vec{z} - \vec{z}^\top A\vec{z} \cdot \vec{y}^\top D\vec{z} = 0.$$

This condition is fulfilled in the case when A is a singular conic, $A = a \cup \overline{a}$, $\vec{y} \in a$, $\vec{y} \notin \overline{a}$, B touches A, and C either touches A or is a singular conic with the singular point $\vec{y}$. These assumptions provide the line a with the equation $\vec{y}^\top A\vec{z} = 0$ as an inflexion tangent at $\vec{y}$.

Let us now discuss the case of $\vec{y}$ lying on A, B, C but not on D. The condition (5) is reduced to

$$\vec{y}^\top A\vec{z} = 0, \quad \forall \vec{z}$$

which characterizes A as a singular conic (pair of lines $\{a, \overline{a}\}$ intersecting in $\vec{y}$). It follows that the conics of the pencil $[A, B]$ touch each other in the basic point $\vec{y}$. The tangents of k at $\vec{y}$ are determined by

$$4\vec{y}^\top B\vec{z} \cdot \vec{y}^\top C\vec{z} - \vec{z}^\top A\vec{z} \cdot \vec{y}^\top D\vec{y} = 0.$$

Since A is a singular conic, $\vec{z}^\top A\vec{z}$ is a product of linear factors in the coordinates of $\vec{z}$, i.e. $\vec{z}^\top A\vec{z} = (\alpha_0 z_0 + \alpha_1 z_1 + \alpha_2 z_2)(\overline{\alpha}_0 z_0 + \overline{\alpha}_1 z_1 + \overline{\alpha}_2 z_2)$.

If one of the conics B, C is a singular conic with the singular point $\vec{y}$, the tangents to k coincide with a, $\overline{a}$.

We conclude: If one of the pencils of conics is a pencil of pairs of lines with the vertex in a point different from the basic points of the second pencil, that vertex will be a double point of the quartic. There is only one conic from the second pencil passing through the vertex of the first. The pair of lines corresponding to that conic is the pair of tangents to the quartic at the double point.

We can also state: If the conics from first pencil touch each other at a point $\vec{y}$ not being a basic point of the second pencil and the singular conic with the singular point $\vec{y}$ from the first pencil is mapped to the singular conic with the singular point $\vec{y}$ from the second pencil, then k has $\vec{y}$ as a double point at which tangents coincide with the singular conic with the singular point $\vec{y}$ from the first pencil. Particularly, if $a = \overline{a}$, a cusp will occur in the double point.

If none of B, C is a singular conic with the singular point $\vec{y}$, one of the tangents to k will be one of the lines a, $\overline{a}$ iff one of the conics B, C touches that line.

If B touches a, $\vec{y}^\top B\vec{z} = \beta a$, the pencil $[A, B]$ contains conics osculating each other at $\vec{y}$. The tangents are:

$$a\big[4\beta \cdot \vec{y}^\top C\vec{z} - \vec{y}^\top D\vec{y} \cdot \overline{a}\big] = 0.$$

The tangent a osculates the quartic iff for all $\vec{z} \in a$ the equality

$$\vec{y}^\top B\vec{z} \cdot \vec{z}^\top C\vec{z} + \vec{z}^\top B\vec{z} \cdot \vec{y}^\top C\vec{z} - \vec{y}^\top A\vec{z} \cdot \vec{z}^\top D\vec{z} - \vec{z}^\top A\vec{z} \cdot \vec{y}^\top D\vec{z} = 0$$

holds as in that case $t = 0$ is the zero of the polynomial (1) with multiplicity 2. This condition is obviously fulfilled iff C touches a, too.

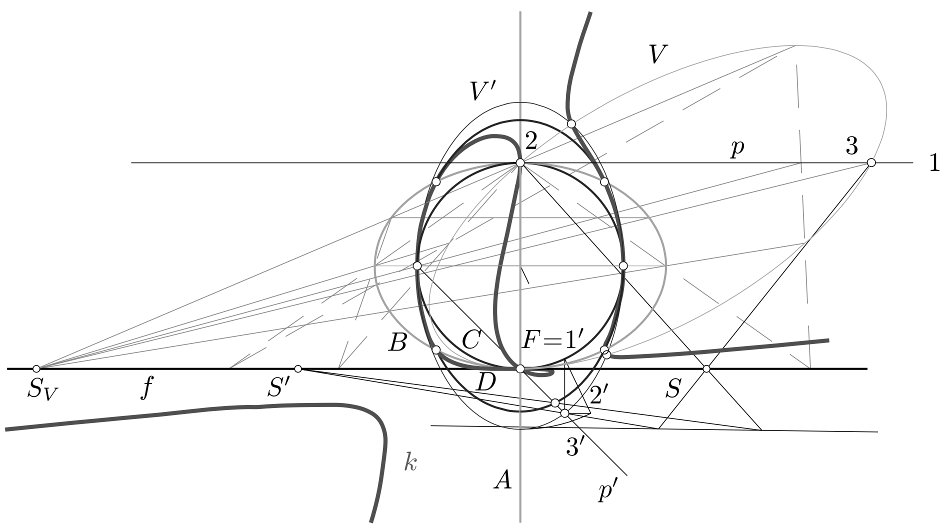

Fig. 6

By identifying a with the absolute line f and $\vec{y}$ with the absolute point F an entirely circular quartic will be obtained (Fig. 6).

If the conic A is a double line touched by the conics B, C, both tangents to k coincide with that line.

We conclude:

THEOREM 4. *Let A, B, C be circular and D non-circular conic of the isotropic plane with the absolute figure (f,F) and let $\pi : [A,B] \longmapsto [C,D]$ be the projectivity mapping the conics $A+\lambda B$ to the conics $C+\lambda D$. The result of π is a quartic k having a double point in F iff A is a pair of isotropic lines. If $[A,B]$ is a pencil of circles osculating at F, k touches f at F (i.e. k is 3-circular).*

Let us now consider the case of $\vec{y}$ on A, C but not on B, D.

If A is a singular conic with the singular point $\vec{y}$, the condition (5) is fulfilled iff C is a singular conic with the singular point $\vec{y}$. The tangents at the double point are determined by

$$\vec{y}^\top B\vec{y}\cdot\vec{z}^\top C\vec{z}-\vec{z}^\top A\vec{z}\cdot\vec{y}^\top D\vec{y}=0.$$

Obviously, the lines of which A consists cannot be the tangents to k.

If A is not a singular conic with the singular point $\vec{y}$, the condition (5) is fulfilled iff $\vec{y}$ is a singular point of the conic $\vec{y}^\top B\vec{y}\cdot C-\vec{y}^\top D\vec{y}\cdot A$. It follows that C necessarily touches A, however that condition is not sufficient.

It is left to us to observe the case of a triple point on k. The necessary condition to gain a triple point at $\vec{y}$ is that the coefficients of the polynomial (1) next to t and t^2 vanish:

$$\vec{y}^\top B\vec{y}\cdot\vec{y}^\top C\vec{z}-\vec{y}^\top A\vec{z}\cdot\vec{y}^\top D\vec{y}=0, \tag{7}$$

$$\vec{y}^\top B\vec{y}\cdot\vec{z}^\top C\vec{z}+4\vec{y}^\top B\vec{z}\cdot\vec{y}^\top C\vec{z}-4\vec{y}^\top A\vec{z}\cdot\vec{y}^\top D\vec{z}-\vec{z}^\top A\vec{z}\cdot\vec{y}^\top D\vec{y}=0,\quad \forall\vec{z}.$$

The tangents are given by

$$\vec{y}^\top B\vec{z}\cdot\vec{z}^\top C\vec{z}+\vec{z}^\top B\vec{z}\cdot\vec{y}^\top C\vec{z}-\vec{y}^\top A\vec{z}\cdot\vec{z}^\top D\vec{z}-\vec{z}^\top A\vec{z}\cdot\vec{y}^\top D\vec{z}=0. \tag{8}$$

If $\vec{y}$ is a basic point of the pencils $[A,B]$, $[C,D]$, the condition (7) is reduced to

$$\vec{y}^\top B\vec{z}\cdot\vec{y}^\top C\vec{z}-\vec{y}^\top A\vec{z}\cdot\vec{y}^\top D\vec{z}=0,\quad \forall\vec{z}. \tag{9}$$

Now we will discuss the following two cases: Either A is or is not a singular conic with a singular point $\vec{y}$.

In the first case the condition above is fulfilled iff at least one of the conics B, C is a singular conic with a singular point $\vec{y}$. We arrive at three subcases:

(1) B, C are singular conics with a singular point $\vec{y}$. Tangents to the quartic are described by $\vec{z}^\top A\vec{z} \cdot \vec{y}^\top D\vec{z} = 0$, or written in the other way

$$a \cdot \overline{a} \cdot \vec{y}^\top D\vec{z} = 0,$$

where a, $\overline{a}$ are lines forming A or more precisely linear factors representing them. A cusp will occur iff either D touches a (or $\overline{a}$) or $a = \overline{a}$.

Therefore, $[A, B]$ is a pencil of degenerated conics (pairs of lines) and $[C, D]$ is a pencil of conics touching at the vertex $\vec{y}$. One of the tangents at the triple point $\vec{y}$ is the common tangent of the conics from $[C, D]$. The other two tangents coincide with the lines from the first pencil that are linked to the singular conic with the singular point $\vec{y}$ from the second pencil.

(2) The conic $B = b \cup \overline{b}$, $\vec{y} \in b, \overline{b}$. Tangents are now

$$b \cdot \overline{b} \cdot \vec{y}^\top C\vec{z} - a \cdot \overline{a} \cdot \vec{y}^\top D\vec{z} = 0.$$

One of the tangents coincide with a iff C touches a.

(3) The conic $C = c \cup \overline{c}$, $\vec{y} \in c$, $\overline{c}$. Tangents are determined by

$$\vec{y}^\top B\vec{z} \cdot c \cdot \overline{c} - a \cdot \overline{a} \cdot \vec{y}^\top D\vec{z} = 0.$$

One of the tangents coincides with a iff B touches a. This leads to the statement: If the conics of one pencil have a common tangent at the point $\vec{y}$ and if the conics of the second pencil have a common tangent at the point $\vec{y}$ (not necessarily the same line) and if a singular conic with a singular point $\vec{y}$ from the first pencil is linked to a singular conic with a singular point $\vec{y}$ from the second pencil, then the common basic point is a triple point of a quartic. Furthermore, if $\vec{y}$ is the intersection of the conics from one pencil with multiplicity 3, their common tangent will be tangent to the quartic.

Fig. 7 presents an entirely circular quartic generated by the projectively linked pencil of pairs of isotropic line and a pencil of circles. By exchanging $[C, D]$ with a pencil of special hyperbolas, we will obtain a 3-circular quartic. It is also possible to get a 1-circular quartic if we interpret $[C, D]$ as a pencil of special hyperbolas with one further basic point on the absolute line and at the same time $[A, B]$ is a pencil of pairs of lines with the vertex in the isotropic point.

In the case when A is not a singular conic with a singular point $\vec{y}$, it necessarily follows from (9) that either $\vec{y}^\top B\vec{z} = \omega\vec{y}^\top A\vec{z}$ or $\vec{y}^\top C\vec{z} = \omega\vec{y}^\top A\vec{z}$ holds. Let us suppose $\vec{y}^\top B\vec{z} = \omega\vec{y}^\top A\vec{z}$ which implies that the conics from $[A, B]$ intersect in $\vec{y}$ with multiplicity 2 and $\vec{y}$ is a singular point on the singular conic $B - \omega A$ from that pencil. The condition (7) becomes

$$\vec{y}^\top A\vec{z} \cdot \left[\omega\vec{y}^\top C\vec{z} - \vec{y}^\top D\vec{z}\right] = 0, \quad \forall\vec{z}$$

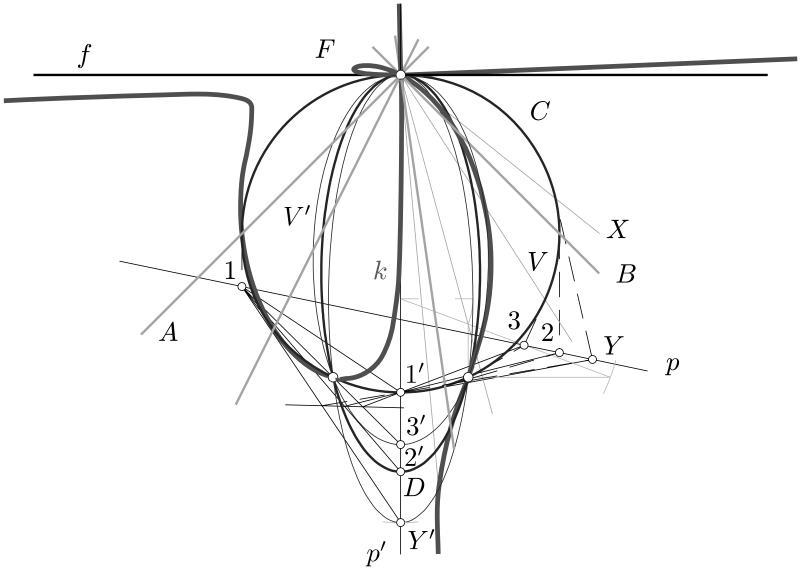

Fig. 7

and is equivalent to the fact that $\vec{y}$ is the singular point on the singular conic $D-\omega C$ from the pencil $[C,D]$. We are now in a position to conclude: $\vec{y}$ is a triple point of the quartic k iff the singular conic with the singular point $\vec{y}$ from $[A,B]$ is mapped to the singular conic with the singular point $\vec{y}$ from $[C,D]$.

The tangents to k at $\vec{y}$ are given by

$$\vec{y}^{\top} A\vec{z}\cdot\left[\vec{z}^{\top}(D-\omega C)\vec{z}\right]-\vec{y}^{\top} C\vec{z}\cdot\left[\vec{z}^{\top}(B-\omega A)\vec{z}\right]=0.$$

Since the equations of the singular conics can be written as the products of linear factors in the coordinates of $\vec{z}$, $\vec{z}^{\top}(B-\omega A)\vec{z}=h\cdot\overline{h}$, $\vec{z}^{\top}(D-\omega C)\vec{z}=g\cdot\overline{g}$, the equation above is transformed into

$$\vec{y}^{\top} A\vec{z}\cdot g\cdot\overline{g}-\vec{y}^{\top} C\vec{z}\cdot h\cdot\overline{h}=0.$$

One of the tangents to the quartic is identical to the common tangent to the conics from the first pencil iff one of the lines $\vec{y}^{\top} C\vec{z}=0$, $h=0$, $\overline{h}=0$ is identical to the line $\vec{y}^{\top} A\vec{z}=0$. Similarly, a tangent to the quartic is identical with the common tangent to the conics from the second pencil iff one of the lines $\vec{y}^{\top} A\vec{z}=0$, $g=0$, $\overline{g}=0$ is the line $\vec{y}^{\top} C\vec{z}=0$.

Therefore:

– In the case when the conics from one pencil osculate each other (and the conics from the second pencil touch each other), one tangent to the quartic at the triple point coincides with the common tangent to the conics from that pencil.

– If both pencils have the same common tangent, the tangent will be the tangent to the quartic at the triple point.

Giving isotropic interpretation to these results we can state:

THEOREM 5. *Let* A, B *and* C, D *be conics of the isotropic plane with* F *as an intersection point with multiplicity* 2. *Let* $\pi : [A, B] \longmapsto [C, D]$ *be the projectivity which maps the conic* $A + \lambda B$ *to the conic* $C + \lambda D$. *The result of* π *is a quartic* k *having a triple point in* F *iff the singular conic with the singular point* F *from* $[A, B]$ *is mapped to the singular conic with a singular point* F *from* $[C, D]$. *The quartic* k *is entirely circular iff one of the pencils contains isotropic circles osculating at* F *or both pencils are pencils of circles.*

3. Conclusion

We did not observe all the possible cases of projectively linked pencils of conics, but those that we did observe allow us to construct the quartics of all degrees of circularity and all types.

REMARKS. 1. In this paper we have observed the projectivity $\pi : [A, B] \longmapsto [C, D]$. Fixing $B = C$ and studying the projectivity $\pi : [A, B] \longmapsto [B, D]$ in the same manner would allow us to construct almost all types of circular quartics. The problem in this less general case appears when we try to find the conditions for obtaining a quartic osculating the absolute line. It is caused by the fact that if $\vec{y}$ is an intersection of A and B with multiplicity n, then it is an intersection of A and k with multiplicity $2n$.

2. Similar observations have been made in [3] for the entirely circular quartics in the hyperbolic plane.

References

[1] J. Beban-Brkić, Isometric invariants of conics in the isotropic plane – Classification of conics, *J. Geom. Graph.*, **6** (2002), 17–26.

[2] E. Jurkin, Automorphic inversion and circular quartics in isotropic plane, *KoG*, **12** (2008), 19–26.

[3] E. Jurkin and O. Röschel, Envelopes of absolute quadratic systems of conics of the hyperbolic plane I, *Math. Pannonica*, **19** (2008), 1–13.

[4] E. F. Kummer, Über einige besondere Arten von Flächen vierten Grades, *Berl. Monatsber.* (1872), 474–489.

[5] D. Palman, Vollständig zirkuläre Kurven 4. Ordnung der isotropen Ebene, *Rad Hrvat. Akad. Znan. Umjet.*, **10** (1991), 15–32.

[6] H. Sachs, *Ebene Isotrope Geometrie*, Wieweg (Braunschweig–Wiesbaden, 1987).

[7] H. Sachs, Vollständig zirkuläre Kurven n-ter Ordnung der isotropen Ebene, *Stud. Sci. Math. Hung.*, **24** (1989), 377–383.

[8] V. Szirovicza, *Circular Cubics inA Isotropic and Hyperbolic Plane*, PhD thesis (Zagreb, 2001) (in Croatian).