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Some recent results about Jensen's operator inequality and its converses with application to operator means

Jadranka Mićić Hot

University of Zagreb
Faculty of Mechanical Engineering
and Naval Architecture

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1.1. Overview of Jensen's inequality

Classical Jensen's inequality

J.L.W.V. Jensen, *Sur les fonctions convexes et les ingalits entre les valeurs moyennes*, Acta Mathematica **30** (1906), 175–193:

If f is a convex function on an interval $[m, M]$ for some scalars $m < M$, then

$$f(\sum_{j=1}^{k} t_j x_j) \le \sum_{j=1}^{k} t_j f(x_j). \tag{1}$$

holds for every $x_1, x_2, \cdots, x_k \in [m, M]$ and every positive real numbers $t_1, t_2, \cdots, t_k$ with $\sum_{j=1}^{k} t_j = 1$.

An operator version of (1): *Let A be a self-adjoint operator on a Hilbert space H with $\mathrm{Sp}(A) \subset [m, M]$ for some scalars $m < M$. If f is a convex function on $[m, M]$, then*

$$f((Ax, x)) \le (f(A)x, x) \tag{2}$$

for every unit vector $x \in H$.

Jessen's inequality

B.Jessen, *Bemaerkinger om konvekse Funktioner og Uligheder imellem Middelvaerdier I*, Mat.Tidsskrift **B** (1931), 17-28:

Let E be a nonempty set and $\mathfrak{L} = \{g; g : E \to \mathbb{R}\}$ satisfying:

L1: $\alpha, \beta \in \mathbb{R} \quad \wedge \quad g, h \in \mathfrak{L} \quad \Rightarrow \quad \alpha g + \beta h \in \mathfrak{L}$,

L2: $1 \in \mathfrak{L}$.

If f is a convex function on an interval $I \in \mathbb{R}$ and Φ is a unital positive linear functional, then

$$f(\Phi(g)) \leq \Phi(f(g)). \tag{3}$$

holds for every $g \in \mathfrak{L}$.

Schwarz operator inequality

C.Davis, *A Schwartz inequality for convex operator functions*, Proc. Amer. Math. Soc. **8** (1957), 42–44:

If f is an operator convex function defined on an interval I and $\Phi\colon \mathcal{A} \to B(K)$ is a unital completely positive linear map from a C^*-algebra $\mathcal{A}$ to linear operators on a Hilbert space K, then

$$f(\Phi(x)) \le \Phi(f(x)), \tag{4}$$

holds for every self-adjoint element x in $\mathcal{A}$ with spectrum in I.

Subsequently in

M. D. Choi, *A Schwarz inequality for positive linear maps on C^*-algebras*, Ill. J. Math. **18** (1974), 565–574.

noted that it is enough to assume that Φ is unital and positive. In fact, the restriction of Φ to the commutative C^*-algebra generated by self-adjoint x is automatically completely positive by Theorem 4 in

W. F.Stinespring, *Positive functions on C^*-algebras*, Proc. Amer. Math. Soc. **6** (1955), 211–216.

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F.Hansen and G.K. Pedersen, *Jensen's inequality for operators and Löwner's theorem*, Ann. Math. **258** (1982), 229–241.

proved

a Jensen type inequality:

If f is an operator convex function defined on an interval $I = [0, \alpha)$ (with $\alpha \leq \infty$ and $f(0) \leq 0$) then

$$f\left(\sum_{i=1}^{n} a_i^* x_i a_i\right) \leq \sum_{i=1}^{n} a_i^* f(x_i) a_i \tag{5}$$

holds for every n-tuple $(x_1, \ldots, x_n)$ of bounded, self-adjoint operators on an arbitrary Hilbert space H with spectra in I and for every n-tuple $(a_1, \ldots, a_n)$ operators on H with $\sum_{i=1}^{n} a_i^* a_i = \mathbf{1}$.

<<

The inequality (5) is in fact just a reformulation of (4) although this was not noticed at the time.

The restriction on the interval and the requirement $f(0) \leq 0$ in (5) was subsequently removed by

B.Mond and J.Pečarić, *On Jensen's inequality for operator convex functions*, Houston J. Math., **21** (1995), 739–753.

F.Hansen and G.K.Pederson, *Jensen's operator inequality*, Bull. London Math. Soc., **35** (2003), 553–564.

Indeed, consider an arbitrary operator convex function f defined on $[0,1)$. The function $\tilde{f}(x) = f(x) - f(0)$ satisfies the conditions of (5) and it follows

$$f\left(\sum_{i=1}^{n} a_i^* x_i a_i\right) - f(0)\mathbf{1} \leq \sum_{i=1}^{n} a_i^* \left(f(x_i) - f(0)\mathbf{1}\right) a_i = \sum_{i=1}^{n} a_i^* f(x_i) a_i - f(0)\mathbf{1}.$$

By setting $g(x) = f((\beta - \alpha)x + \alpha)$ one may reduce the statement for operator convex functions defined on an arbitrary interval $[\alpha, \beta)$ to operator convex functions defined on the interval $[0,1)$.

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By setting $g(x) = f((\beta - \alpha)x + \alpha)$ one may reduce the statement for operator convex functions defined on an arbitrary interval $[\alpha, \beta)$ to operator convex functions defined on the interval $[0,1)$.

B.Mond and J.Pečarić, *Converses of Jensen's inequality for several operators*, Rev. Anal. Numér. Théor. Approx. **23** (1994), 179–183.

proved

Jensen's operator inequality:

$$f\left(\sum_{i=1}^{n} w_i\Phi_i(x_i)\right) \leq \sum_{i=1}^{n} w_i\Phi_i(f(x_i))$$

holds for operator convex functions f defined on an interval I, where $\Phi_i : B(H) \to B(K)$ are unital positive linear maps, $x_1, \ldots, x_n$ are self-adjoint operators with spectra in I and $w_1, \ldots, w_n$ are non-negative real numbers with sum one.

In

F.Hansen and G.K.Pederson, *Jensen's operator inequality*, Bull. London Math. Soc., **35** (2003), 553–564.

a version of (5) is given **for continuous fields of operators**.

Discrete Jensen

Next, we review the basic concepts of continuous fields of (bounded linear) operators on a Hilbert space and fields of positive linear mappings, which will recur throughout the talk.

Let T be a locally compact Hausdorff space and let $\mathcal{A}$ be a C^*-algebra of operators on some Hilbert space H. We say that a field $(x_t)_{t\in T}$ of operators in $\mathcal{A}$ is continuous if the function $t \mapsto x_t$ is norm continuous on T. If in addition μ is a Radon measure on T and the function $t \mapsto \|x_t\|$ is integrable, then we can form *the Bochner integral* $\int_T x_t \, d\mu(t)$, which is the unique element in $\mathcal{A}$ such that

$$\varphi\left(\int_T x_t \, d\mu(t)\right) = \int_T \varphi(x_t) \, d\mu(t)$$

for every linear functional φ in the norm dual $\mathcal{A}^*$.

Assume furthermore that $(\Phi_t)_{t\in T}$ is a field of positive linear mappings $\Phi_t : \mathcal{A} \to \mathcal{B}$ from $\mathcal{A}$ to another C^*-algebra $\mathcal{B}$ of operators on a Hilbert space K. We say that such a field is continuous if the function $t \mapsto \Phi_t(x)$ is continuous for every $x \in \mathcal{A}$. If the C^*-algebras include the identity operators, and the field $t \mapsto \Phi_t(1)$ is integrable with integral equals 1, we say that $(\Phi_t)_{t\in T}$ is *unital*.

Finally,

F.Hansen, J.Pečarić and I.Perić, *Jensen's operator inequality and its converse*, Math. Scad., **100** (2007), 61–73.

find an inequality which contains the previous inequalities as special cases:

Theorem

Let $f : I \to \mathbb{R}$ be an operator convex functions defined on an interval I, and let $\mathcal{A}$ and $\mathcal{B}$ be a unital C^*-algebras. If $(\Phi_t)_{t\in T}$ is a unital field of positive linear mappings $\Phi_t : \mathcal{A} \to \mathcal{B}$ defined on a locally compact Hausdorff space T with a bounded Radon measure μ, then

$$f\left(\int_T \Phi_t(x_t)\, d\mu(t)\right) \leq \int_T \Phi_t(f(x_t))\, d\mu(t) \tag{6}$$

holds for every bounded continuous fields $(x_t)_{t\in T}$ of self-adjoint elements in $\mathcal{A}$ with spectra contained in I.

1.2. Overview of the Kantorovich inequality

The story of the Kantorovich inequality is a very interesting example how a mathematician makes mathematics. It provides a deep insight into how a principle raised from the Kantorovich inequality develops in the field of operator inequality on a Hilbert space, perhaps more importantly, it has given new way of thinking and methods in operator theory, noncommutative differential geometry, quantum information theory and noncommutative probability theory.

Classical Kantorovich inequality

L.V.Kantorovich, *Functional analysis and applied mathematics (in Russian)*, Uspechi Mat. Nauk., **3** (1948), 89–185.

The inequality

$$\sum_{k=1}^{\infty} \gamma_k u_k^2 \ \sum_{k=1}^{\infty} \gamma_k^{-1} u_k^2 \leq \frac{1}{4}\left[\sqrt{\frac{M}{m}}+\sqrt{\frac{m}{M}}\right]^2 \left(\sum_{k=1}^{\infty} u_k^2\right)^2 \tag{7}$$

holds, where m and M being the bounds of the numbers γ_k

$$0 < m \leq \gamma_k \leq M.$$

In the same paper he gave **an operator version of (7):**

If an operator A on H is positive such that $m\mathbf{1} \leq A \leq M\mathbf{1}$ for some scalars $0 < m < M$, then

$$\frac{(x,x)^2}{(Ax,x)(A^{-1}x,x)} \geq \frac{4}{\left[\sqrt{\frac{M}{m}}+\sqrt{\frac{m}{M}}\right]^2} \tag{8}$$

holds for every nonzero vector x in H.

The inequality

$$\sum_{k=1}^{\infty} \gamma_k u_k^2 \ \sum_{k=1}^{\infty} \gamma_k^{-1} u_k^2 \leq \frac{1}{4}\left[\sqrt{\frac{M}{m}}+\sqrt{\frac{m}{M}}\right]^2 \left(\sum_{k=1}^{\infty} u_k^2\right)^2 \tag{7}$$

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Kantorovich says in the footnote of his paper that (7) is a special case of Pólya-Szegö inequality given in the book about problems and theorems from calculus; Aufgaben 92 in:

G.Pólya and G.Szegö , "Aufgaben und Lehrsötze aus der Analysis", Springer-Verlag, 1, Berlin, 1925.

If the real number a_k *and* b_k $(k = 1, \dots, n)$ *fulfill the conditions* $0 < m_1 \le a_k \le M_1$ *and* $0 < m_2 \le b_k \le M_2$ *then*

$$1 \le \frac{\sum_{k=1}^{n} a_k^2 \ \sum_{k=1}^{n} b_k^2}{\left[\sum_{k=1}^{n} a_k b_k\right]^2} \le \frac{(M_1 M_2 + m_1 m_2)^2}{4 m_1 m_2 M_1 M_2}. \tag{9}$$

We remark that the Kantorovich constant has the form

$$\frac{1}{4}\left(\sqrt{\frac{M}{m}} + \sqrt{\frac{m}{M}}\right)^2 = \frac{(M+m)^2}{4Mm}.$$

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If the real number a_k and b_k $(k = 1, \ldots, n)$ fulfill the conditions $0 < m_1 \le a_k \le M_1$ and $0 < m_2 \le b_k \le M_2$ then

$$1 \le \frac{\sum_{k=1}^n a_k^2 \ \sum_{k=1}^n b_k^2}{\left[\sum_{k=1}^n a_k b_k\right]^2} \le \frac{(M_1 M_2 + m_1 m_2)^2}{4 m_1 m_2 M_1 M_2}. \tag{9}$$

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$$\frac{1}{4}\left(\sqrt{\frac{M}{m}} + \sqrt{\frac{m}{M}}\right)^2 = \frac{(M+m)^2}{4Mm}.$$

In the first paper which is devoted to Kantorovich inequality

W.Greub and W.Rheinboldt, *On a generalization of an inequality of L.V. Kantorovich*, Proc. Amer. Math. Soc., **10** (1959), 407–415.

it is written: *"Examining the relation between the two inequalities more closely we found that this remark is well justified and can be made even more specific in that the inequality of Pólya - Szegö in the form (9) is special case of the Kantorovich inequality (7)."*

They proved that the generalized Pólya-Szegö inequality:

$$(Ax,Ax)(Bx,Bx) \leq \frac{(M_1 M_2 + m_1 m_2)^2}{4 m_1 m_2 M_1 M_2}(Ax,Bx)^2 \quad \text{holds for all } x \in H,$$

where A and B are commuting self-adjoint operators on a Hilbert space H such that $0 < m_1 \mathbf{1} \leq A \leq M_1 \mathbf{1}$ and $0 < m_2 \mathbf{1} \leq B \leq M_2 \mathbf{1}$ is equivalent to the Kantorovich inequality:

$$(x,x)^2 \leq (Ax,x)(A^{-1}x,x) \leq \frac{(M+m)^2}{4Mm}(x,x)^2 \quad \text{holds for all } x \in H,$$

where A is a self-adjoint operator on H such that $0 < m\mathbf{1} \leq A \leq M\mathbf{1}$.

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where A is a self-adjoint operator on H such that $0 < m\mathbf{1} \le A \le M\mathbf{1}$.

After the paper due to Greub and Rheinboldt was published, mathematicians concentrated their energies on the generalization of the Kantorovich inequality and the way to an even simpler proof. We will cite only some of them.

In 1960, one year after, in

W.G.Strang, *On the Kantorovich inequality*, Proc. Amer. Math. Soc., **11** (1960), p. 468.

proved a generalization:

If T is an arbitrary invertible operator on H, and $\|T\| = M, \|T^{-1}\| = m$, then

$$|(Tx,y)(x,T^{-1}y)| \leq \frac{(M+m)^2}{4Mm}(x,x)(y,y) \qquad \text{for all } x,y \in H.$$

Furthermore, the bound is best possible.

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A.H. Schopf, *On the Kantorovich inequality*, Numerische Mathematik, **2** (1960), 344-346:

Let Γ be any nonzero complex number, let $R = |\Gamma|$, and let $0 \leq r \leq R$. Let A be an operator on H such that $|A - \Gamma[A]|^2 \leq r^2[A]$, where $[A]$ is the range projection of A. Let $u \in B(K, H)$ be an operator such that $u^*[A]u$ is a projection. Then

$$(R^2 - r^2)u^*A^*Au \leq R^2(u^*A^*u)(u^*Au).$$

M.Nakamura, *A remark on a paper of Greub and Rheiboldt*, Proc. Japon. Acad., **36** (1960), 198–199.:

For $0 < m < M$, the following inequality holds true;

$$\int_m^M t d\mu(t) \cdot \int_m^M \frac{1}{t} d\mu(t) \leq \frac{(M+m)^2}{4Mm}$$

for any positive Stieltjes measure μ on $[m, M]$ with $\|\mu\| = 1$.

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B.C.Rennie, *An inequality which includes that of Kantorovich*, Amer. Math. Monthly, **70** (1963), 982.

Rennie improved a function version of the Kantorovich inequality due to Nakamura:

Let f be a measurable function on the probability space such that $0 < m \le f(x) \le M$. Then

$$\int \frac{1}{f(x)} dx \int f(x) dx \le \frac{(M+m)^2}{4mM}.$$

B.Mond, *A matrix version of Rennie's generalization of Kantorovich's inequality*, Proc. Amer. Math. Soc., **16** (1965), 1131.

Mond considered a matrix type of the Kantorovich inequality:

Let A be a positive definite Hermitian matrix with eigenvalues $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n > 0$. Then

$$(A^{-1}x, x)(Ax, x) \le \frac{(\lambda_1 + \lambda_n)^2}{4\lambda_1 \lambda_n}.$$

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$$(A^{-1}x,x)(Ax,x) \leq \frac{(\lambda_1+\lambda_n)^2}{4\lambda_1\lambda_n}.$$

Ky Fan, *Some matrix inequalities*, Abh. Math. Sem. Univ. Hamburg, **29** (1966), 185–196.

improved a generalization of the Kantorovich inequality for $f(t) = t^p$ with $p \in \mathbb{Z}$:

Let A be a positive definite Hermitian matrix of order n with all its eigenvalues contained in the close interval $[m, M]$, where $0 < m < M$. Let $x_1, \ldots, x_k$ be an finite number of vectors in the unitary n-space such that $\sum_{j=1}^{k} \|x_j\|^2 = 1$. Then for every integer $p \neq 0, 1$ (not necessarily positive), we have

$$\frac{\sum_{j=1}^{k} \left(A^p x_j, x_j\right)}{\left[\sum_{j=1}^{k} \left(A x_j, x_j\right)\right]^p} \leq \frac{(p-1)^{p-1}}{p^p} \frac{(M^p - m^p)^p}{(M-m)(mM^p - Mm^p)^{p-1}}.$$

1.3. Mond-Pečarić method

Afterwards, in the flow of a generalization by Ky Fan, and a reverse of the arithmetic-geometric mean inequality by Specht, Mond and Pečarić give definitely the meaning of "Kantorovich inequality". In 1990s, Mond and Pečarić formulate various reverses of Jensen's type inequalities. Here, it may be said that the positioning of Kantorovich inequality becomes clear for the first time in operator theory. Furthermore, they find the viewpoint of the reverse for means behind Kantorovich inequality, that is to say, Kantorovich inequality is the reverse of the arithmetic-harmonic mean inequality.

In a long research series, Mond and Pečarić established the method which gives the reverse to Jensen inequality associated with convex functions. The principle yields a rich harvest in a field of operator inequalities. We call it **the Mond-Pečarić method** for convex functions. One of the most important attributes of Mond-Pečarić method is to offer a totally new viewpoint in the field of operator inequalities.

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Here, we shall present the principle of Mond-Pečarić method:

Theorem

Let A be a self-adjoint operator on a Hilbert space H such that $m\mathbf{1} \leq A \leq M\mathbf{1}$ for some scalars $m \leq M$. If f is a convex function on $[m, M]$ such that $f > 0$ on $[m, M]$, then

$$(f(A)x, x) \leq K(m, M, f) f((Ax, x))$$

for every unit vector $x \in H$, where

$$K(m, M, f) = \max\left\{ \frac{1}{f(t)} \left(\frac{f(M) - f(m)}{M - m}(t - m) + f(m) \right) : m \leq t \leq M \right\}.$$

Proof

Since $f(t)$ is convex on $[m, M]$, we have

$$f(t) \leq \frac{f(M) - f(m)}{M - m}(t - m) + f(m) \qquad \text{for all } t \in [m, M].$$

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Proof

Since $f(t)$ is convex on $[m, M]$, we have

$$f(t) \le \frac{f(M) - f(m)}{M - m}(t - m) + f(m) \qquad \text{for all } t \in [m, M].$$

Using the operator calculus, it follows that

$$f(A) \leq \frac{f(M)-f(m)}{M-m}(A-m)+f(m)\mathbf{1}$$

and hence

$$(f(A)x,x) \leq \frac{f(M)-f(m)}{M-m}((Ax,x)-m)+f(m)$$

for every unit vector $x \in H$. Divide both sides by $f((Ax,x))$ (>0), and we get

$$\begin{aligned}\frac{(f(A)x,x)}{f((Ax,x))} &\leq \frac{\frac{f(M)-f(m)}{M-m}((Ax,x)-m)+f(m)}{f((Ax,x))} \\ &\leq \max\left\{\frac{1}{f(t)}\left(\frac{f(M)-f(m)}{M-m}(t-m)+f(m)\right) : m \leq t \leq M\right\},\end{aligned}$$

since $m \leq (Ax,x) \leq M$. $\square$

Moreover, under a general situation, we state explicitly the heart of Mond-Pečarić method:

Theorem

Let $f : [m, M] \mapsto \mathbb{R}$ be a convex continuous function, I an interval such that $I \supset f([m, M])$ and A a self-adjoint operator such that $m\mathbf{1} \le A \le M\mathbf{1}$ for some scalars $m < M$. If $F(u, v)$ is a real function defined on $I \times I$, F is bounded and non-decreasing in u, then

$$
\begin{aligned}
F\left[(f(A)x, x), f((Ax, x))\right] &\le \max_{t\in[m,M]} F\left[\frac{f(M) - f(m)}{M - m}(t - m) + f(m), f(t)\right] \\
&= \max_{\theta\in[0,1]} F\left[\theta f(m) + (1 - \theta)f(M), f(\theta m + (1 - \theta)M)\right]
\end{aligned}
$$

for every unit vector $x \in H$.

Next, we use the standard notation for a real valued continuous function $f : [m, M] \to \mathbb{R}$

$$\alpha_f := (f(M) - f(m))/(M - m) \quad \text{and} \quad \beta_f := (Mf(m) - mf(M))/(M - m).$$

Moreover, under a general situation, we state explicitly the heart of Mond-Pečarić method:

Theorem

Let $f : [m, M] \mapsto \mathbb{R}$ be a convex continuous function, I an interval such that $I \supset f([m, M])$ and A a self-adjoint operator such that $m\mathbf{1} \leq A \leq M\mathbf{1}$ for some scalars $m < M$. If $F(u, v)$ is a real function defined on $I \times I$, F is bounded and non-decreasing in u, then

$$F[(f(A)x, x), f((Ax, x))] \leq \max_{t \in [m,M]} F\left[\frac{f(M) - f(m)}{M - m}(t - m) + f(m), f(t)\right]$$

$$= \max_{\theta \in [0,1]} F[\theta f(m) + (1 - \theta)f(M), f(\theta m + (1 - \theta)M)]$$

for every unit vector $x \in H$.

Next, we use the standard notation for a real valued continuous function $f : [m, M] \to \mathbb{R}$

$$\alpha_f := (f(M) - f(m))/(M - m) \quad \text{and} \quad \beta_f := (Mf(m) - mf(M))/(M - m).$$

Using the Mond-Pečarić method, F.Hansen, J.Pečarić and I.Perić generalized the previous inequality similar to what they made with Jensen's inequality.

▸ Converses of Jensen's ineq.

Theorem

Let $(x_t)_{t\in T}$ be a bounded continuous field of self-adjoint elements in a unital C^*-algebra $\mathcal{A}$ with spectra in $[m,M]$ defined on a locally compact Hausdorff space T equipped with a bounded Radon measure μ, and let $(\Phi_t)_{t\in T}$ be a unital field of positive linear maps $\Phi_t : \mathcal{A} \to \mathcal{B}$ from $\mathcal{A}$ to another unital C^*-algebra $\mathcal{B}$. Let $f, g : [m,M] \to \mathbb{R}$ and $F : U \times V \to \mathbb{R}$ be functions such that $f([m,M]) \subset U$, $g([m,M]) \subset V$ and F is bounded. If F is operator monotone in the first variable and f is convex in the interval $[m,M]$, then

$$F\left[\int_T \Phi_t(f(x_t))\,d\mu(t), g\left(\int_T \Phi_t(x_t)d\mu(t)\right)\right] \leq \sup_{m\leq z\leq M} F[\alpha_f z+\beta_f, g(z)]\,\mathbf{1}.$$

In the dual case (when f is operator concave) the opposite inequality holds with sup *instead of* inf.

Books about the Mond-Pečarić method

- **T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo**
  Mond-Pečarić Method in Operator Inequalities
  Monographs in Inequalities **1**, Element, Zagreb, 2005.

- **M. Fujii, J. Mićić Hot, J. Pečarić and Y. Seo**
  Recent development of Mond-Pečarić Method in Operator Inequalities
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1.4. Goal of the lecture

Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras on a Hilbert spaces H and K. The goal of this lecture is to present a generalization of Jensen's operator inequality and its converses for fields of positive linear mappings $\Phi_t : \mathcal{A} \to \mathcal{B}$ such that $\int_T \phi_t(\mathbf{1})\, d\mu(t) = k\mathbf{1}$ for some positive scalar k.

At first we give general formulations of Jensen's operator inequality and it's converses. As a consequence, difference and ratio type of converses of Jensen's operator inequality are obtained.

In addition, we discuss the order among quasi-arithmetic means in a general setting. As an application we get some comparison theorems for power functions and power means.

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In addition, we discuss the order among quasi-arithmetic means in a general setting. As an application we get some comparison theorems for power functions and power means.

2.1. Generalization of Jensen's inequality

Theorem

Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras on H and K respectively. Let $(x_t)_{t\in T}$ be a bounded continuous field of self-adjoint elements in $\mathcal{A}$ with spectra in an interval I defined on a locally compact Hausdorff space T equipped with a bounded Radon measure μ. Furthermore, let $(\phi_t)_{t\in T}$ be a field of positive linear maps $\phi_t : \mathcal{A} \to \mathcal{B}$, such that the field $t \mapsto \phi_t(\mathbf{1})$ is integrable with $\int_T \phi_t(\mathbf{1})\, d\mu(t) = k\mathbf{1}$ for some positive scalar k. Then the inequality

$$f\left(\frac{1}{k}\int_T \phi_t(x_t)\, d\mu(t)\right) \le \frac{1}{k}\int_T \phi_t(f(x_t))\, d\mu(t) \tag{10}$$

holds for each operator convex function $f : I \to \mathbb{R}$ defined on I. In the dual case (when f is operator concave) the opposite inequality holds in (10).

Proof

The function $t \mapsto \frac{1}{k}\Phi_t(x_t) \in \mathcal{B}$ is continuous and bounded, hence integrable with respect to the bounded Radon measure μ. Furthermore, the integral is an element in the multiplier algebra $M(\mathcal{B})$ acting on K and we may organize the set $CB(T,\mathcal{A})$ of bounded continuous functions on T with values in $\mathcal{A}$ as a normed involutive algebra by applying the point-wise operations with

$$\|(y_t)_{t\in T}\| = \sup_{t\in T}\|y_t\| \qquad (y_t)_{t\in T} \in CB(T,\mathcal{A}),$$

The norm is already complete and satisfy the C^*-identity. It follows that $f((x_t)_{t\in T}) = (f(x_t))_{t\in T}$. Then the mapping $\pi : CB(T,\mathcal{A}) \to M(\mathcal{B}) \subseteq B(K)$ defined by

$$\pi((x_t)_{t\in T}) = \int_T \frac{1}{k}\Phi_t(x_t)\, d\mu(t),$$

is a unital positive linear map. Using a Schwarz inequality we obtain the statement of the theorem. □

We remark that if $\Phi(\mathbf{1}) = k\mathbf{1}$, for some positive scalar k and f is an operator convex function, then $f(\Phi(A)) \le \Phi(f(A))$ is not true in general.

Example

A map $\Phi : \mathbf{M}_2(\mathbf{M}_2(\mathbb{C})) \to \mathbf{M}_2(\mathbf{M}_2(\mathbb{C}))$ defined by

$$\Phi\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} A+B & 0 \\ 0 & A+B \end{pmatrix} \quad \text{for } A, B \in \mathbf{M}_2(\mathbb{C})$$

is a positive linear map and $\Phi(I) = 2I$. We put $f(t) = t^2$ and

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}. \qquad \text{Then}$$

$$f\left(\Phi\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\right) - \Phi\left(f\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\right) = \begin{pmatrix} 4 & 3 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 3 & 2 \end{pmatrix} \not\geq 0.$$

We remark that if $\Phi(\mathbf{1}) = k\mathbf{1}$, for some positive scalar k and f is an operator convex function, then $f(\Phi(A)) \leq \Phi(f(A))$ is not true in general.

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2.2. Generalization of converses of Jensen's inequality

We obtain a result of the Li-Mathias type:

Theorem

Let $(x_t)_{t\in T}$ and $(\phi_t)_{t\in T}$ be as in § 2.1., $f:[m,M]\to\mathbb{R}$, $g:[km,kM]\to\mathbb{R}$ and $F:U\times V\to\mathbb{R}$ be functions such that $(kf)([m,M])\subset U$, $g([km,kM])\subset V$ and F is bounded. If F is operator monotone in the first variable, then

$$\begin{aligned}
&\inf_{km\le z\le kM} F\left[k\cdot h_1\left(\frac{1}{k}z\right),g(z)\right]\mathbf{1}\\
\le\ & F\left[\int_T \phi_t\left(f(x_t)\right)d\mu(t),\, g\left(\int_T \phi_t(x_t)d\mu(t)\right)\right] \qquad (11)\\
\le\ & \sup_{km\le z\le kM} F\left[k\cdot h_2\left(\frac{1}{k}z\right),g(z)\right]\mathbf{1}
\end{aligned}$$

holds for every operator convex function h_1 on $[m,M]$ such that $h_1\le f$ and for every operator concave function h_2 on $[m,M]$ such that $h_2\ge f$.

Proof

We only prove RHS. Let h_2 be operator concave function and $\leq h_2$ on $[m, M]$. Thus, $f(x_t) \leq h_2(x_t)$ for every $t \in T$. Then

$$\int_T \Phi_t(f(x_t))\, d\mu(t) \leq \int_T \Phi_t(h_2(x_t))\, d\mu(t).$$

Furthermore, the generalized Jensen's inequality gives $\int_T \Phi_t(f(x_t))\, d\mu(t) \leq k \cdot h_2\left(\frac{1}{k}\int_T \Phi_t(x_t)\, d\mu(t)\right)$. *Using operator monotonicity of $F(\cdot, v)$, we obtain*

$$\begin{aligned} &F\left[\int_T \Phi_t(f(x_t))\, d\mu(t), g\left(\int_T \Phi_t(x_t) d\mu(t)\right)\right] \\ &\leq F\left[k \cdot h_2\left(\frac{1}{k}\int_T \Phi_t(x_t)\, d\mu(t)\right), g\left(\int_T \Phi_t(x_t) d\mu(t)\right)\right] \\ &\leq \sup_{km \leq z \leq kM} F\left[k \cdot h_2\left(\frac{1}{k} z\right), g(z)\right] \mathbf{1}. \qquad \square \end{aligned}$$

Applying RHS of (11) for a convex function f (or LHS of (11) for a concave function f) we obtain the following theorem:

Theorem

Let $(x_t)_{t\in T}$ and $(\Phi_t)_{t\in T}$ be as in § 2.1., $f:[m,M]\to\mathbb{R}$, $g:[km,kM]\to\mathbb{R}$ and $F:U\times V\to\mathbb{R}$ be functions such that $(kf)([m,M])\subset U$, $g([km,kM])\subset V$ and F is bounded. If F is operator monotone in the first variable and f is convex in the interval $[m,M]$, then

$$F\left[\int_T \Phi_t(f(x_t))\,d\mu(t), g\left(\int_T \Phi_t(x_t)d\mu(t)\right)\right] \leq \sup_{km\leq z\leq kM} F\left[\alpha_f z+\beta_f k, g(z)\right]\mathbf{1}. \tag{12}$$

In the dual case (when f is concave) the opposite inequality holds in (12) with inf *instead of* sup.

Proof

If f is convex the inequality $f(z)\leq\alpha_f z+\beta_f$ holds for every $z\in[m,M]$ and we put $h_2(z)=\alpha_f z+\beta_f$ in RHS of (11). □

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Theorem

Let $(x_t)_{t\in T}$ and $(\Phi_t)_{t\in T}$ be as in § 2.1., $f:[m,M]\to\mathbb{R}$, $g:[km,kM]\to\mathbb{R}$ and $F:U\times V\to\mathbb{R}$ be functions such that $(kf)([m,M])\subset U$, $g([km,kM])\subset V$ and F is bounded. If F is operator monotone in the first variable and f is convex in the interval $[m,M]$, then

$$F\left[\int_T \Phi_t(f(x_t))\,d\mu(t), g\left(\int_T \Phi_t(x_t)d\mu(t)\right)\right] \le \sup_{km\le z\le kM} F\left[\alpha_f z+\beta_f k, g(z)\right]\mathbf{1}. \tag{12}$$

In the dual case (when f is concave) the opposite inequality holds in (12) with inf *instead of* sup.

Proof

If f is convex the inequality $f(z)\le\alpha_f z+\beta_f$ holds for every $z\in[m,M]$ and we put $h_2(z)=\alpha_f z+\beta_f$ in RHS of (11). □

Applying this theorem for the function $F(u,v) = u - \lambda v$, we obtain:

Corollary

Let $(x_t)_{t\in T}$ and $(\Phi_t)_{t\in T}$ be as in § 2.1. and f be convex.
(i) If λg is convex differentiable, then

$$\int_T \Phi_t(f(x_t))\,d\mu(t) \le \lambda\, g\left(\int_T \Phi_t(x_t)d\mu(t)\right) + C\mathbf{1}, \tag{13}$$

where $C = \alpha_f z_0 + \beta_f k - \lambda g(z_0)$, $z_0 = g'^{-1}(\alpha_f/\lambda)$ for $\lambda g'(km) \le \alpha_f \le \lambda g'(kM)$; $z_0 = km$ for $\lambda g'(km) \ge \alpha_f$, and $z_0 = kM$ for $\lambda g'(kM) \le \alpha_f$.
(ii) If λg is concave differentiable, then the constant C in (13) can be written more precisely as

$$C = \begin{cases} \alpha_f kM + \beta_f k - \lambda g(kM) & \text{for } \ \alpha_f - \lambda\alpha_{g,k} \ge 0, \\ \alpha_f km + \beta_f k - \lambda g(km) & \text{for } \ \alpha_f - \lambda\alpha_{g,k} \le 0, \end{cases}$$

where $\alpha_{g,k} = \frac{g(kM)-g(km)}{kM-km}$.

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Setting $\Phi_t(A_t) = \langle A_t \xi_t, \xi_t \rangle$ for $\xi_t \in H$ and $t \in T$ in this corollary, then:

Corollary

Let $(A_t)_{t \in T}$ be a continuous field of positive operators on a Hilbert space H defined on a locally compact Hausdorff space T equipped with a bounded Radon measure μ. We assume the spectra are in $[m, M]$ for some $0 < m < M$. Let furthermore $(\xi_t)_{t \in T}$ be a continuous field of vectors in H such that $\int_T \|\xi_t\|^2 d\mu(t) = k$ for some scalar $k > 0$. Then for any real λ, q, p

$$\int_T \langle A_t^p \xi_t, \xi_t \rangle d\mu(t) - \lambda \left(\int_T \langle A_t \xi_t, \xi_t \rangle d\mu(t) \right)^q \leq C, \tag{14}$$

where the constant $C \equiv C(\lambda, m, M, p, q, k)$ is

$$C = \begin{cases} (q-1)\lambda \left(\frac{\alpha_p}{\lambda q} \right)^{\frac{q}{q-1}} + \beta_p k & \text{for } \ \lambda q m^{q-1} \leq \frac{\alpha_p}{k^{q-1}} \leq \lambda q M^{q-1}, \\ kM^p - \lambda (kM)^q & \text{for } \ \frac{\alpha_p}{k^{q-1}} \geq \lambda q M^{q-1}, \\ km^p - \lambda (km)^q & \text{for } \ \frac{\alpha_p}{k^{q-1}} \leq \lambda q m^{q-1}, \end{cases} \tag{15}$$

in the case $\lambda q(q-1)>0$ and $p\in\mathbb{R}\setminus(0,1)$
or

$$C=\begin{cases} kM^p-\lambda(kM)^q & \text{for } \alpha_p-\lambda k^{q-1}\alpha_q\geq 0,\\ km^p-\lambda(km)^q & \text{for } \alpha_p-\lambda k^{q-1}\alpha_q\leq 0,\end{cases} \tag{16}$$

in the case $\lambda q(q-1)<0$ and $p\in\mathbb{R}\setminus(0,1)$.

In the dual case: $\lambda q(q-1)<0$ and $p\in(0,1)$ the opposite inequality holds in (14) with the opposite condition while determining the constant C in (15). But in the dual case: $\lambda q(q-1)>0$ and $p\in(0,1)$ the opposite inequality holds in (14) with the opposite condition while determining the constant C in (16).
Constants α_p and β_p in terms above are the constants α_f and β_f associated with the function $f(z)=z^p$, i.e.

$$\alpha_p=\frac{M^p-m^p}{M-m},\quad \beta_p=\frac{M\,m^p-m\,M^p}{M-m}.$$

Setting $\Phi_t(A_t) = \langle \exp(rA_t)\xi_t, \xi_t\rangle$ for $\xi_t \in H$ and $t \in T$ we have:

Corollary

Let $(A_t)_{t\in T}$ and $(\xi_t)_{t\in T}$ be as in previous corollary. Then for any real number $r \neq 0$ we have

$$\int_T \langle \exp(rA_t)\xi_t, \xi_t\rangle d\mu(t) - \exp\left(r\int_T \langle A_t\xi_t, \xi_t\rangle d\mu(t)\right) \leq C_1, \tag{17}$$

$$\int_T \langle \exp(rA_t)\xi_t, \xi_t\rangle d\mu(t) \leq C_2 \exp\left(r\int_T \langle A_t\xi_t, \xi_t\rangle d\mu(t)\right), \tag{18}$$

where constants $C_1 \equiv C_1(r,m,M,k)$ and $C_2 \equiv C_2(r,m,M,k)$ are

$$C_1 = \begin{cases} \frac{\alpha}{r}\ln\left(\frac{\alpha}{re}\right) + k\beta & \textit{for} \;\; re^{rkm} \leq \alpha \leq re^{rkM}, \\ kM\alpha + k\beta - e^{rkM} & \textit{for} \;\; re^{rkM} \leq \alpha, \\ km\alpha + k\beta - e^{rkm} & \textit{for} \;\; re^{rkm} \geq \alpha \end{cases}$$

$$C_2 = \begin{cases} \frac{\alpha}{re} e^{kr\beta/\alpha} & \text{for} \quad kre^{rm} \le \alpha \le kre^{rM}, \\ k\mathrm{e}^{(1-k)rm} & \text{for} \quad kre^{rm} \ge \alpha, \\ k\mathrm{e}^{(1-k)rM} & \text{for} \quad kre^{rM} \le \alpha. \end{cases}$$

Constants α and β in terms above are the constants α_f and β_f associated with the function $f(z) = \mathrm{e}^{rz}$, i.e.

$$\alpha = \frac{\mathrm{e}^{rM} - \mathrm{e}^{rm}}{M-m}, \quad \beta_p = \frac{M\mathrm{e}^{rm} - m\mathrm{e}^{rM}}{M-m}.$$

Applying the inequality $f(x) \le \frac{M-x}{M-m} f(m) + \frac{x-m}{M-m} f(M)$ (for a convex function f on $[m,M]$) to positive operators $(A_t)_{t\in T}$ and using $0 < A_t \le \|A_t\| \mathbf{1}$, we obtain the following theorem, which is a generalization of results from

R. Drnovšek, T. Kosem, *Inequalities between* $f(\|A\|)$ *and* $\|f(|A|)\|$, Math. Inequal. Appl. **8** (2005) 1–6.

$$C_2 = \begin{cases} \frac{\alpha}{re} e^{kr\beta/\alpha} & \text{for} \quad kre^{rm} \le \alpha \le kre^{rM}, \\ k\mathrm{e}^{(1-k)rm} & \text{for} \quad kre^{rm} \ge \alpha, \\ k\mathrm{e}^{(1-k)rM} & \text{for} \quad kre^{rM} \le \alpha. \end{cases}$$

Constants α and β in terms above are the constants α_f and β_f associated with the function $f(z) = \mathrm{e}^{rz}$, i.e.

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Applying the inequality $f(x) \le \frac{M-x}{M-m} f(m) + \frac{x-m}{M-m} f(M)$ (for a convex function f on $[m, M]$) to positive operators $(A_t)_{t \in T}$ and using $0 < A_t \le \|A_t\| \mathbf{1}$, we obtain the following theorem, which is a generalization of results from

R. Drnovšek, T. Kosem, *Inequalities between* $f(\|A\|)$ *and* $\|f(|A|)\|$, Math. Inequal. Appl. **8** (2005) 1–6.

Theorem

Let f be a convex function on $[0,\infty)$ and let $\|\cdot\|$ be a normalized unitarily invariant norm on $B(H)$ for some finite dimensional Hilbert space H. Let $(\Phi_t)_{t\in T}$ be a field of positive linear maps $\Phi_t : B(H) \to B(K)$, where K is a Hilbert space, defined on a locally compact Hausdorff space T equipped with a bounded Radon measure μ. If the field $t \mapsto \Phi_t(\mathbf{1})$ is integrable with $\int_T \Phi_t(\mathbf{1})\, d\mu(t) = k\mathbf{1}$ for some positive scalar k, then for every continuous field of positive operators $(A_t)_{t\in T}$ we have

$$\int_T \Phi_t(f(A_t))\, d\mu(t) \le kf(0)\mathbf{1} + \int_T \frac{f(\|A_t\|) - f(0)}{\|A_t\|} \phi_t(A_t)\, d\mu(t).$$

Especially, for $f(0) \le 0$, the inequality

$$\int_T \Phi_t(f(A_t))\, d\mu(t) \le \int_T \frac{f(\|A_t\|)}{\|A_t\|} \phi_t(A_t)\, d\mu(t).$$

is valid.

In the present context and by using **subdifferentials** we can give an application of the first theorem in this section.

Theorem

Let $(x_t)_{t\in T}$, $(\Phi_t)_{t\in T}$ be as in § 2.1., $f:[m,M]\to\mathbb{R}$, $g:[km,kM]\to\mathbb{R}$ and $F:U\times V\to\mathbb{R}$ be functions such that $(kf)([m,M])\subset U$, $g([km,kM])\subset V$, F is bounded and $f(y)+l(y)(t-y)\in U$ for every $y,t\in[m,M]$ where l is the subdifferential of f. If F is operator monotone in the first variable and f is convex on $[m,M]$, then

$$F\left[\int_T \phi_t(f(x_t))\,d\mu(t), g\left(\int_T \phi_t(x_t)\,d\mu(t)\right)\right] \geq \inf_{km\leq z\leq kM} F\left[f(y)k+l(y)(z-yk), g(z)\right]\mathbf{1} \tag{19}$$

holds for every $y\in[m,M]$. In the dual case (when f is concave) the opposite inequality holds in (19) with sup *instead of* inf.

In the present context and by using **subdifferentials** we can give an application of the first theorem in this section.

Theorem

Let $(x_t)_{t\in T}$, $(\Phi_t)_{t\in T}$ *be as in § 2.1.,* $f:[m,M]\to\mathbb{R}$, $g:[km,kM]\to\mathbb{R}$ *and* $F:U\times V\to\mathbb{R}$ *be functions such that* $(kf)([m,M])\subset U$, $g([km,kM])\subset V$, F *is bounded and* $f(y)+l(y)(t-y)\in U$ *for every* $y,t\in[m,M]$ *where* l *is the subdifferential of* f. *If* F *is operator monotone in the first variable and* f *is convex on* $[m,M]$, *then*

$$\begin{aligned} & F\left[\int_T \phi_t(f(x_t))\,d\mu(t), g\left(\int_T \phi_t(x_t)\,d\mu(t)\right)\right] \\ & \geq \inf_{km\leq z\leq kM} F\left[f(y)k+l(y)(z-yk), g(z)\right]\mathbf{1} \end{aligned} \tag{19}$$

holds for every $y\in[m,M]$. *In the dual case (when* f *is concave) the opposite inequality holds in (19) with* $\sup$ *instead of* $\inf$.

Though $f(z) = \ln z$ is operator concave, the Schwarz inequality $\phi(f(x)) \leq f(\phi(x))$ does not hold in the case of non-unital ϕ. Therefore we have the following application of results above.

Corollary

Let $(x_t)_{t\in T}$, $(\Phi_t)_{t\in T}$ be as in § 2.1. and $0 < m < M$. Then

$$C_1 \mathbf{1} \leq \int_T \Phi_t(\ln(x_t))\, d\mu(t) - \ln\left(\int_T \Phi_t(x_t) d\mu(t)\right) \leq C_2 \mathbf{1},$$

$$C_1 = \begin{cases} k\beta + \ln(\mathrm{e}/L(m,M)) & \text{for } km \leq L(m,M) \leq kM, \\ \ln\left(M^{k-1}/k\right) & \text{for } kM \leq L(m,M), \\ \ln\left(m^{k-1}/k\right) & \text{for } km \geq L(m,M), \end{cases}$$

$$C_2 = \begin{cases} \ln\left(\frac{L(m,M)^k k^{k-1}}{\mathrm{e}^k m}\right) + \frac{m}{L(m,M)} & \text{for } m \leq kL(m,M) \leq M \\ \ln\left(M^{k-1}/k\right) & \text{for } kL(m,M) \geq M, \\ \ln\left(m^{k-1}/k\right) & \text{for } kL(m,M) \leq m, \end{cases}$$

where $L(m,M)$ is the logarithmic mean, $\beta = \frac{M\ln m - m\ln M}{M-m}$.

By using subdifferentials, we also give an application of the first theorem in this section.

Theorem

Let $(x_t)_{t\in T}$, $(\Phi_t)_{t\in T}$ be as in § 2.1. and $f:[m,M]\to\mathbb{R}$ is a convex function then

$$
\begin{aligned}
& f(y)k\mathbf{1}+l(y)\left(\int_T \Phi_t(x_t)d\mu(t)-yk\mathbf{1}\right)\le \int_T \Phi_t(f(x_t))\,d\mu(t) \\
\le\quad & f(x)k\mathbf{1}-x\int_T \Phi_t(l(x_t))\,d\mu(t)+\int_T \Phi_t(l(x_t)x_t)\,d\mu(t) \qquad (20)
\end{aligned}
$$

for every $x,y\in[m,M]$, where l is the subdifferential of f. In the dual case (f is concave) the opposite inequality holds.

<<

Proof Since f is convex we have $f(x)\ge f(y)+l(y)(x-y)$ for every $x,y\in[m,M]$. Then $f(x_t)\ge f(y)\mathbf{1}+l(y)(x_t-y\mathbf{1})$ for $t\in T$. Applying the positive linear maps Φ_t and integrating, LHS of (20) follows. The RHS of (20) follows similarly by using the variable y. □

3. Quasi-arithmetic mean

A generalized quasi-arithmetic operator mean:

$$M_\varphi(x,\Phi) := \varphi^{-1}\left(\int_T \frac{1}{k}\Phi_t(\varphi(x_t))\,d\mu(t)\right), \tag{21}$$

under these conditions: $(x_t)_{t\in T}$ is a bounded continuous field of positive operators in a C^*-algebra $B(H)$ with spectra in $[m,M]$ for some scalars $0<m<M$, $(\Phi_t)_{t\in T}$ is a field of positive linear maps $\Phi_t : B(H)\to B(K)$, such that the field $t\mapsto \Phi_t(\mathbf{1})$ is integrable with $\int_T \Phi_t(\mathbf{1})\,d\mu(t) = k\mathbf{1}$ for some positive scalar k and $\varphi\in\mathcal{C}[m,M]$ is a strictly monotone function.

This mean is well-defined, since $m\mathbf{1}\le x_t\le M\mathbf{1}$ for every $t\in T$, then

- $\varphi(m)\,\mathbf{1}\le \int_T \frac{1}{k}\Phi_t(\varphi(x_t))\,d\mu(t)\le\varphi(M)\,\mathbf{1}$ if φ is increasing,
- $\varphi(M)\,\mathbf{1}\le \int_T \frac{1}{k}\Phi_t(\varphi(x_t))\,d\mu(t)\le\varphi(m)\,\mathbf{1}$ if φ is decreasing.

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- $\varphi(M)\,\mathbf{1}\le \int_T \frac{1}{k}\Phi_t(\varphi(x_t))\,d\mu(t)\le \varphi(m)\,\mathbf{1}$ if φ is decreasing.

3.1. Monotonicity

First, we study the monotonicity of quasi-arithmetic means.

Theorem

Let $(x_t)_{t\in T}$, $(\Phi_t)_{t\in T}$ be as in the definition of the quasi-arithmetic mean (21). *Let $\psi,\varphi \in \mathcal{C}[m,M]$ be strictly monotone functions.*
If one of the following conditions is satisfied:

- (i) $\psi\circ\varphi^{-1}$ *is operator convex and* ψ^{-1} *is operator monotone,*
- (i') $\psi\circ\varphi^{-1}$ *is operator concave and* $-\psi^{-1}$ *is operator monotone,*
- (ii) φ^{-1} *is operator convex and* ψ^{-1} *is operator concave,*

then

$$M_\varphi(x,\Phi) \le M_\psi(x,\Phi).$$

If one of the following conditions is satisfied:

(iii) $\psi \circ \varphi^{-1}$ is operator concave and ψ^{-1} is operator monotone,

(iii') $\psi \circ \varphi^{-1}$ is operator convex and $-\psi^{-1}$ is operator monotone,

(iv) φ^{-1} is operator concave and ψ^{-1} is operator convex,

then

$$\psi^{-1}(\Phi(\psi(A))) \leq \varphi^{-1}(\Phi(\varphi(A))).$$

Proof

We will prove only the case (i) and (ii).
(i) If we put $f = \psi \circ \varphi^{-1}$ in the generalized Jensen's inequality Generalized Jensen's ineq. *and replace x_t with $\varphi(x_t)$, then we obtain*

$$\begin{aligned} \psi \circ \varphi^{-1} \left(\int_T \frac{1}{k} \Phi_t(\varphi(x_t))\, d\mu(t) \right) &\leq \int_T \frac{1}{k} \Phi_t \left(\psi \circ \varphi^{-1}(\varphi(x_t)) \right) d\mu(t) \\ &= \int_T \frac{1}{k} \Phi_t(\psi(x_t))\, d\mu(t). \end{aligned}$$

(continued)

Since ψ^{-1} is operator monotone, it follows

$$\varphi^{-1}\left(\int_T \frac{1}{k}\Phi_t(\varphi(x_t))\,d\mu(t)\right) \le \psi^{-1}\left(\int_T \frac{1}{k}\Phi_t(\psi(x_t))\,d\mu(t)\right).$$

(ii) Since φ^{-1} is operator convex, it follows that

$$\varphi^{-1}(\Phi(\varphi(A))) \le \Phi(\varphi^{-1}\circ\varphi(A)) = \Phi(A).$$

Similarly, since ψ^{-1} is operator concave, we have

$$\Phi(A) \le \psi^{-1}(\Phi(\psi(A))).$$

Using two inequalities above, we have the means order in this case. $\square$

Theorem

Let $(x_t)_{t\in T}$, $(\Phi_t)_{t\in T}$ be as in the definition of the quasi-arithmetic mean and $\psi,\varphi\in\mathcal{C}[m,M]$ be strictly monotone functions. Then

$$M_\varphi(x,\Phi)=M_\psi(x,\Phi) \qquad \text{for all } (x_t)_{t\in T},\ (\Phi_t)_{t\in T}$$

if and only if

$$\varphi=A\psi+B \qquad \text{for some real numbers } A\neq 0 \text{ and } B.$$

There are many references about operator monotone or operator convex functions, see e.g.

- R.Bhatia, *Matrix Anaysis*, Springer, New York, 1997.
- M. Uchiyama, *A new majorization between functions, polynomials, and operator inequalities*, J. Funct. Anal. **231** (2006), 231-244.

Using this we have the following two corollaries.

Theorem

Let $(x_t)_{t\in T}$, $(\Phi_t)_{t\in T}$ be as in the definition of the quasi-arithmetic mean and $\psi,\varphi \in \mathcal{C}[m,M]$ be strictly monotone functions. Then

$$M_\varphi(x,\Phi) = M_\psi(x,\Phi) \qquad \text{for all } (x_t)_{t\in T},\ (\Phi_t)_{t\in T}$$

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Using this we have the following two corollaries.

Corollary

Let $(x_t)_{t\in T}$ be a bounded continuous field of positive operators in a C^*-algebra $B(H)$ with spectra in $[m,M]\subset(0,\infty)$ and $(\Phi_t)_{t\in T}$ is a field of positive linear maps $\Phi_t: B(H)\to B(K)$, such that $\int_T \Phi_t(\mathbf{1})\,d\mu(t)=k\mathbf{1}$ for some positive scalar k. Let φ and ψ be continuous strictly monotone functions from $[0,\infty)$ into itself.
If one of the following conditions is satisfied:

- (i) $\psi\circ\varphi^{-1}$ *and* ψ^{-1} *are operator monotone,*
- (ii) $\varphi\circ\psi^{-1}$ *is operator convex,* $\varphi\circ\psi^{-1}(0)=0$ *and* ψ^{-1} *is operator monotone,*

then $M_\varphi(x,\Phi)\geq M_\psi(x,\Phi)$ *holds.*

Proof

In the case (i) we use the assertion that a real valued continuous function f on an interval $I=[\alpha,\infty)$ and bounded below is operator monotone on I if and only if f is operator concave on I.

Corollary

Let $(x_t)_{t\in T}$ be a bounded continuous field of positive operators in a C^*-algebra $B(H)$ with spectra in $[m,M]\subset(0,\infty)$ and $(\Phi_t)_{t\in T}$ is a field of positive linear maps $\Phi_t : B(H)\to B(K)$, such that $\int_T \Phi_t(\mathbf{1})\,d\mu(t)=k\mathbf{1}$ for some positive scalar k. Let φ and ψ be continuous strictly monotone functions from $[0,\infty)$ into itself.
If one of the following conditions is satisfied:

- (i) $\psi\circ\varphi^{-1}$ *and* ψ^{-1} *are operator monotone,*
- (ii) $\varphi\circ\psi^{-1}$ *is operator convex,* $\varphi\circ\psi^{-1}(0)=0$ *and* ψ^{-1} *is operator monotone,*

then $M_\varphi(x,\Phi)\geq M_\psi(x,\Phi)$ *holds.*

Proof

In the case (i) we use the assertion that a real valued continuous function f on an interval $I=[\alpha,\infty)$ and bounded below is operator monotone on I if and only if f is operator concave on I.

(continued)

In the case (ii) we use the assertion that if f is a real valued function from $[0,\infty)$ into itself such that $f(0)=0$ and f is operator convex, then f^{-1} is operator monotone. It follows that $\psi\circ\varphi^{-1}$ is operator concave. □

Corollary

Let $(x_t)_{t\in T}$ and $(\Phi_t)_{t\in T}$ be as in previous corollary. Let $\varphi(u)=\alpha u+\beta$, $\alpha\neq 0$, and ψ be an strictly monotone function from $[0,\infty)$ into itself. If one of the following conditions is satisfied:

(i) *ψ is operator convex and ψ^{-1} is operator concave,*

(ii) *ψ is operator convex and $\psi(0)=0$,*

(iii) *ψ^{-1} is operator convex and ψ is operator concave,*

(iv) *ψ^{-1} is operator convex and $\psi^{-1}(0)=0$,*

then $M_\varphi(x,\Phi)=M_1(x,\Phi)\leq M_\psi(x,\Phi)$ holds.

(continued)

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(i) *ψ is operator convex and ψ^{-1} is operator concave,*

(ii) *ψ is operator convex and $\psi(0)=0$,*

(iii) *ψ^{-1} is operator convex and ψ is operator concave,*

(iv) *ψ^{-1} is operator convex and $\psi^{-1}(0)=0$,*

then $M_\varphi(x,\Phi)=M_1(x,\Phi)\leq M_\psi(x,\Phi)$ holds.

3.2. Difference and ratio type inequalities

Next, we study difference and ratio type inequalities among quasi-arithmetic means. We investigate the estimates of these inequalities, i.e. we will determine real constants α and β such that

$$M_\psi(x,\Phi) - M_\varphi(x,\Phi) \le \beta\,\mathbf{1} \qquad \text{and} \qquad M_\psi(x,\Phi) \le \alpha M_\varphi(x,\Phi)$$

holds. With that in mind, we shall prove the following general result.

Theorem

Let $(x_t)_{t\in T}$, $(\Phi_t)_{t\in T}$ be as in the definition of the quasi-arithmetic mean. Let $\psi,\varphi \in \mathcal{C}[m,M]$ be strictly monotone functions and $F:[m,M]\times[m,M] \to \mathbb{R}$ be a bounded and operator monotone function in its first variable.

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Theorem

Let $(x_t)_{t\in T}$, $(\Phi_t)_{t\in T}$ be as in the definition of the quasi-arithmetic mean. Let $\psi,\varphi \in \mathcal{C}[m,M]$ be strictly monotone functions and $F : [m,M]\times[m,M] \to \mathbb{R}$ be a bounded and operator monotone function in its first variable.

(continued)

If one of the following conditions is satisfied:

(i) $\psi \circ \varphi^{-1}$ is convex and ψ^{-1} is operator monotone,

(i') $\psi \circ \varphi^{-1}$ is concave and $-\psi^{-1}$ is operator monotone,

then

$$F\left[M_\psi(x,\Phi), M_\varphi(x,\Phi)\right] \tag{22}$$
$$\leq \sup_{0\leq\theta\leq 1} F\left[\psi^{-1}\left(\theta\psi(m)+(1-\theta)\psi(M), \varphi^{-1}\left(\theta\varphi(m)+(1-\theta)\varphi(M)\right)\right)\right] \mathbf{1}.$$

If one of the following conditions is satisfied:

(ii) $\psi \circ \varphi^{-1}$ is concave and ψ^{-1} is operator monotone,

(ii') $\psi \circ \varphi^{-1}$ is convex and $-\psi^{-1}$ is operator monotone,

then the opposite inequality is valid in (22) with inf instead of sup.

Proof

We will prove only the case (i). By using the Mond-Pečarić method and the functional calculus, we obtain that

$$\psi(x_t) \le \frac{\varphi_M \mathbf{1} - \varphi(x_t)}{\varphi_M - \varphi_m} \psi \circ \varphi^{-1}(\varphi_m) + \frac{\varphi(x_t) - \varphi_m \mathbf{1}}{\varphi_M - \varphi_m} \psi \circ \varphi^{-1}(\varphi_M)$$

holds for every $t \in T$, where $\varphi_m := \min\{\varphi(m), \varphi(M)\}$, $\varphi_M := \max\{\varphi(m), \varphi(M)\}$. It follows

$$\int_T \frac{1}{k} \Phi_t(\psi(x_t))\, d\mu(t) \le B\psi(m) + (\mathbf{1} - B)\psi(M),$$

where

$$B = \frac{\varphi(M)\mathbf{1} - \int_T \frac{1}{k} \Phi_t(\varphi(x_t))\, d\mu(t)}{\varphi(M) - \varphi(m)}, \qquad 0 \le B \le \mathbf{1}.$$

We have $M_\psi(x, \Phi) \le \psi^{-1}(B\psi(m) + (\mathbf{1} - B)\psi(M))$ and $M_\varphi(x, \Phi) = \varphi^{-1}(B\varphi(m) + (\mathbf{1} - B)\varphi(M))$. Finally, using operator monotonicity of $F(\cdot, v)$, we obtain (22). $\square$

Corollary

Let $(x_t)_{t\in T}$, $(\Phi_t)_{t\in T}$ *be as in the definition of the quasi-arithmetic mean. Let* $\psi,\varphi\in\mathcal{C}[m,M]$ *be strictly monotone functions and* $F:[m,M]\times[m,M]\to\mathbb{R}$ *be a bounded and operator monotone function in its first variable, such that* $F(z,z)=C$ *for all* $z\in[m,M]$.

If one of the following conditions is satisfied:

(i) $\psi\circ\varphi^{-1}$ *is operator convex and* ψ^{-1} *is operator monotone,*

(i') $\psi\circ\varphi^{-1}$ *is operator concave and* $-\psi^{-1}$ *is operator monotone,*

then

$$F\left[M_\psi(x,\Phi),M_\varphi(x,\Phi)\right]\geq C\mathbf{1} \tag{23}$$

holds.

If one of the following conditions is satisfied:

(ii) $\psi\circ\varphi^{-1}$ *is operator concave and* ψ^{-1} *is operator monotone,*

(ii') $\psi\circ\varphi^{-1}$ *is operator convex and* $-\psi^{-1}$ *is operator monotone,*

then the reverse inequality is valid in (23). □

Corollary

Let $(x_t)_{t\in T}$, $(\Phi_t)_{t\in T}$ be as in the definition of the quasi-arithmetic mean. Let $\psi,\varphi \in \mathcal{C}[m,M]$ be strictly monotone functions and $F:[m,M]\times[m,M]\to\mathbb{R}$ be a bounded and operator monotone function in its first variable, such that $F(z,z)=C$ for all $z\in[m,M]$.
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(i) $\psi\circ\varphi^{-1}$ *is operator convex and* ψ^{-1} *is operator monotone,*

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(ii') $\psi\circ\varphi^{-1}$ *is operator convex and* $-\psi^{-1}$ *is operator monotone,*

then the reverse inequality is valid in (23). □

Proof

Suppose (i) or (i'). Applying the monotonicity of quasi-arithmetic means, we have $M_\varphi(x,\Phi) \le M_\psi(x,\Phi)$. *Using assumptions about function* F, *it follows*
$F\left[M_\psi(x,\Phi), M_\varphi(x,\Phi)\right] \ge F\left[M_\varphi(x,\Phi), M_\varphi(x,\Phi)\right] = C\mathbf{1}$. □

Remark

It is particularly interesting to observe inequalities when the function F *in Theorem* ▸ F(u,v) *has the form* $F(u,v) = u - v$ *and*
$F(u,v) = v^{-1/2}uv^{-1/2}$ $(v > 0)$.
E.g. if (i) or (i') of this theorem is satisfied, then

$$M_\psi(x,\Phi) \le M_\varphi(x,\Phi)$$
$$+ \sup_{0\le\theta\le 1}\left\{\psi^{-1}\left(\theta\psi(m) + (1-\theta)\psi(M)\right) - \varphi^{-1}\left(\theta\varphi(m) + (1-\theta)\varphi(M)\right)\right\}\mathbf{1},$$

(continued)

If in addition $\varphi > 0$,then

$$M_{\psi}(x,\Phi) \leq \sup_{0\leq\theta\leq 1}\left\{\frac{\psi^{-1}\left(\theta\psi(m)+(1-\theta)\psi(M)\right)}{\varphi^{-1}\left(\theta\varphi(m)+(1-\theta)\varphi(M)\right)}\right\} M_{\varphi}(x,\Phi).$$

We will investigate the above inequalities, with different assumptions. For this purpose, we introduce some notations for real valued continuous functions $\psi,\varphi \in \mathcal{C}[m,M]$

$$a_{\psi,\varphi} = \frac{\psi(M)-\psi(m)}{\varphi(M)-\varphi(m)}, \quad b_{\psi,\varphi} = \frac{M\psi(m)-M\psi(M)}{\varphi(M)-\varphi(m)}.$$

Theorem

Let $(x_t)_{t\in T}$, $(\Phi_t)_{t\in T}$ be as in the definition of the quasi-arithmetic mean and $\psi,\varphi \in \mathcal{C}[m,M]$ be strictly monotone functions. Let $\psi\circ\varphi^{-1}$ be convex (resp. concave).

(i) *If ψ^{-1} is operator monotone and subadditive (resp. superadditive) on $\mathbb{R}^+$, then*

(continued)

If in addition $\varphi > 0$,then

$$M_\psi(x,\Phi) \leq \sup_{0\leq\theta\leq 1} \left\{ \frac{\psi^{-1}\left(\theta\psi(m)+(1-\theta)\psi(M)\right)}{\varphi^{-1}\left(\theta\varphi(m)+(1-\theta)\varphi(M)\right)} \right\} M_\varphi(x,\Phi).$$

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$$a_{\psi,\varphi} = \frac{\psi(M)-\psi(m)}{\varphi(M)-\varphi(m)}, \qquad b_{\psi,\varphi} = \frac{M\psi(m)-M\psi(M)}{\varphi(M)-\varphi(m)}.$$

Theorem

Let $(x_t)_{t\in T}$, $(\Phi_t)_{t\in T}$ be as in the definition of the quasi-arithmetic mean and $\psi,\varphi \in \mathcal{C}[m,M]$ be strictly monotone functions. Let $\psi\circ\varphi^{-1}$ be convex (resp. concave).

(i) *If ψ^{-1} is operator monotone and subadditive (resp. superadditive) on $\mathbb{R}^+$, then*

(continued)

$$M_{\psi}(x,\Phi) \leq M_{\varphi}(x,\Phi) + \psi^{-1}(\beta)\mathbf{1} \tag{24}$$
$$(\text{resp. } M_{\psi}(x,\Phi) \geq M_{\varphi}(x,\Phi) + \psi^{-1}(\beta)\mathbf{1}\ ),$$

(i') if $-\psi^{-1}$ is operator monotone and subadditive (resp. superadditive) on $\mathbb{R}^{+}$, then the opposite inequality is valid in (24),

(ii) if ψ^{-1} is operator monotone and superadditive (resp. subadditive) on $\mathbb{R}$, then

$$M_{\psi}(x,\Phi) \leq M_{\varphi}(x,\Phi) - \varphi^{-1}(-\beta)\mathbf{1} \tag{25}$$
$$(\text{resp. } M_{\psi}(x,\Phi) \geq M_{\varphi}(x,\Phi) - \varphi^{-1}(-\beta)\mathbf{1}\ ),$$

(ii') if $-\psi^{-1}$ is operator monotone and superadditive (resp. subadditive) on $\mathbb{R}$, then the opposite inequality is valid in (25), where

$$\beta = \max_{\varphi_m \leq z \leq \varphi_M} \left\{ a_{\psi,\varphi} z + b_{\psi,\varphi} - \psi \circ \varphi^{-1}(z) \right\} \ \left(\text{resp. } \beta = \min_{\varphi_m \leq z \leq \varphi_M} \{\cdots\} \right).$$

Furthermore, if $\psi \circ \varphi^{-1}$ is strictly convex (resp. strictly concave) differentiable, then the constant $\beta \equiv \beta(m, M, \varphi, \psi)$ can be written more precisely as $\beta = a_{\psi,\varphi} z_0 + b_{\psi,\varphi} - \psi \circ \varphi^{-1}(z_0)$, where z_0 is the unique solution of the equation $(\psi \circ \varphi^{-1})'(z) = a_{\psi,\varphi}$, $(\varphi_m < z_0 < \varphi_M)$.

Proof

We will prove only the case (i). Putting in Corollary $F(u,v) = u - \lambda v$ $\lambda = 1$, $f = g = \psi \circ \varphi^{-1}$ *and replacing* Φ_t *by* $\frac{1}{k}\Phi_t$, *we have*

$$\begin{aligned} \int_T \frac{1}{k}\Phi_t(\psi(x_t))\,d\mu(t) &= \int_T \frac{1}{k}\Phi_t\left(\psi \circ \varphi^{-1}(\varphi(x_t))\right)d\mu(t) \\ &\le \psi \circ \varphi^{-1}\left(\int_T \frac{1}{k}\Phi_t(\varphi(x_t))\,d\mu(t)\right) + \beta \mathbf{1}, \end{aligned}$$

where β *as in the theorem statement. Since* ψ^{-1} *is operator monotone and subadditive on* $\mathbb{R}^+$, *then we obtain*
$M_\psi(x,\Phi) \le \psi^{-1}\left(\psi \circ \varphi^{-1}\left(\int_T \Phi_t(\varphi(x_t))\,d\mu(t)\right) + \beta\mathbf{1}\right) \le M_\varphi(x,\Phi) + \psi^{-1}(\beta)\mathbf{1}$.

□

Theorem

Let $(x_t)_{t\in T}$, $(\Phi_t)_{t\in T}$ *be as in the definition of the quasi-arithmetic mean and* $\psi,\varphi \in \mathcal{C}[m,M]$ *be strictly monotone functions. Let* $\psi\circ\varphi^{-1}$ *be convex and* $\psi > 0$ *on* $[m,M]$.

(i) *If* ψ^{-1} *is operator monotone and submultiplicative on* $\mathbb{R}^+$, *then*

$$M_\psi(x,\Phi) \le \psi^{-1}(\alpha)\, M_\varphi(x,\Phi), \tag{26}$$

(i') *if* $-\psi^{-1}$ *is operator monotone and submultiplicative on* $\mathbb{R}^+$, *then the opposite inequality is valid in* (26),

(ii) *if* ψ^{-1} *is operator monotone and supermultiplicative on* $\mathbb{R}$, *then*

$$M_\psi(x,\Phi) \le \left[\psi^{-1}(\alpha^{-1})\right]^{-1} M_\varphi(x,\Phi), \tag{27}$$

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where

(continued)

$$\alpha = \max_{\varphi_m \le z \le \varphi_M} \left\{ \frac{a_{\psi,\varphi} z + b_{\psi,\varphi}}{\psi \circ \varphi^{-1}(z)} \right\} \quad \left(\text{resp. } \alpha = \min_{\varphi_m \le z \le \varphi_M} \left\{ \frac{a_{\psi,\varphi} z + b_{\psi,\varphi}}{\psi \circ \varphi^{-1}(z)} \right\} \right).$$

Furthermore, if $\psi \circ \varphi^{-1}$ is strictly convex differentiable, then the constant $\alpha \equiv \alpha(m, M, \varphi, \psi)$ can be written more precisely as

$$\alpha = \frac{a_{\psi,\varphi} z_0 + b_{\psi,\varphi}}{\psi \circ \varphi^{-1}(z_0)},$$

where z_0 is the unique solution of the equation
$(\psi \circ \varphi^{-1})'(a_{\psi,\varphi} z + a_{\psi,\varphi}) = a_{\psi,\varphi} \cdot \psi \circ \varphi^{-1}(z)$, $(\varphi_m < z_0 < \varphi_M)$.

Remark

We can obtain order among quasi-arithmetic means using the function order of positive operator in the same way as we will demonstrate for power functions in the next section.

(continued)

$$\alpha = \max_{\varphi_m \le z \le \varphi_M} \left\{ \frac{a_{\psi,\varphi} z + b_{\psi,\varphi}}{\psi \circ \varphi^{-1}(z)} \right\} \quad \left(\text{resp. } \alpha = \min_{\varphi_m \le z \le \varphi_M} \left\{ \frac{a_{\psi,\varphi} z + b_{\psi,\varphi}}{\psi \circ \varphi^{-1}(z)} \right\} \right).$$

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Remark

We can obtain order among quasi-arithmetic means using the function order of positive operator in the same way as we will demonstrate for power functions in the next section.

If we put $\varphi(t) = t^r$ and $\psi(t) = t^s$ in Theorem about monotonicity of quasi-arithmetic means, then we obtain the order among power means:

Let $(A_t)_{t\in T}$ is a bounded continuous field of positive operators in a C^*-algebra $B(H)$ with spectra in $[m, M]$ for some scalars $0 < m < M$, defined on a locally compact Hausdorff space T equipped with a bounded Radon measure μ and $(\Phi_t)_{t\in T}$ is a field of positive linear maps $\Phi_t : B(H) \to B(K)$, such that $\int_T \Phi_t(\mathbf{1})\, d\mu(t) = \mathbf{1}$ for some positive scalar k. Then

$$\left(\int_T \Phi_t\left(A_t^r\right) d\mu(t)\right)^{1/r} \leq \left(\int_T \Phi_t\left(A_t^s\right) d\mu(t)\right)^{1/s}$$

holds for either $r \leq s$, $r \notin (-1, 1)$, $s \notin (-1, 1)$ or $1/2 \leq r \leq 1 \leq s$ or $r \leq -1 \leq s \leq -1/2$.

In the remaining cases we need to use the function order of positive operator.

4. Power functions

We consider the order among the following power functions:

$$I_r(x,\Phi) := \left(\int_T \Phi_t\left(x_t^r\right) d\mu(t) \right)^{1/r} \qquad \text{if} \quad r \in \mathbb{R}\backslash\{0\}, \tag{28}$$

with these conditions: $(x_t)_{t\in T}$ is a bounded continuous field of positive elements in a unital C^*-algebra $\mathcal{A}$ with spectra in $[m,M]$ for some scalars $0<m<M$, defined on a locally compact Hausdorff space T equipped with a bounded Radon measure μ and $(\Phi_t)_{t\in T}$ is a field of positive linear maps $\Phi_t : \mathcal{A} \to \mathcal{B}$ from $\mathcal{A}$ to another unital C^*-algebra $\mathcal{B}$, such that the field $t \mapsto \Phi_t(\mathbf{1})$ is integrable with $\int_T \Phi_t(\mathbf{1})\, d\mu(t) = k\mathbf{1}$ for some positive scalar k.

J.Mićić, J.Pečarić and Y.Seo, *Converses of Jensen's operator inequality*, Oper. and Matr. (2010), accepted

4.1. Some previous results for the ratio

We wish to observe the ratio type order among power functions. In order this to get we need some previous results given in the following lemmas.

Lemma

Let $(x_t)_{t\in T}$ and $(\Phi_t)_{t\in T}$ be as in the definition of the power function (28).
If $0 < p \leq 1$, then

$$\int_T \Phi_t\left(x_t^p\right) d\mu(t) \leq k^{1-p} \left(\int_T \Phi_t(x_t) d\mu(t) \right)^p . \tag{29}$$

If $-1 \leq p < 0$ or $1 \leq p \leq 2$, then the opposite inequality holds in (29).

Proof

We obtain this lemma by applying generalized Jensen's inequality. □

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We obtain this lemma by applying generalized Jensen's inequality. □

Lemma

Let $(x_t)_{t\in T}$ and $(\Phi_t)_{t\in T}$ be as in the definition of the power function. If $0<p\le 1$, then

$$k^{1-p}K(m,M,p)\left(\int_T \Phi_t(x_t)d\mu(t)\right)^p \le \int_T \Phi_t\left(x_t^p\right)d\mu(t) \le k^{1-p}\left(\int_T \Phi_t(x_t)d\mu(t)\right)^p,$$

if $-1\le p<0$ or $1\le p\le 2$, then

$$k^{1-p}\left(\int_T \Phi_t(x_t)d\mu(t)\right)^p \le \int_T \Phi_t\left(x_t^p\right)d\mu(t) \le k^{1-p}K(m,M,p)\left(\int_T \Phi_t(x_t)d\mu(t)\right)^p,$$

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\begin{aligned}
k^{1-p}K(m,M,p)\left(\int_T \Phi_t(x_t)d\mu(t)\right)^p &\leq \int_T \Phi_t\left(x_t^p\right)d\mu(t) \\
&\leq k^{1-p}\left(\int_T \Phi_t(x_t)d\mu(t)\right)^p,
\end{aligned}
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\end{aligned}
$$

(continued)

if $p < -1$ or $p > 2$, then

$$k^{1-p}K(m,M,p)^{-1}\left(\int_T \Phi_t(x_t)d\mu(t)\right)^p \le \int_T \Phi_t\left(x_t^p\right)d\mu(t) \le k^{1-p}K(m,M,p)\left(\int_T \Phi_t(x_t)d\mu(t)\right)^p,$$

where $K(m,M,p) \equiv K(h,p)$, $h = \frac{M}{m} \ge 1$, is **the generalized Kantorovich constant** defined by

$$K(h,p) = \frac{h^p - h}{(p-1)(h-1)}\left(\frac{p-1}{p}\frac{h^p-1}{h^p-h}\right)^p, \qquad \text{for all } p \in \mathbb{R}.$$

Proof

We obtain this lemma by applying Corollary ▸ $F(u,v) = u - \lambda v$. □

A generalization of the Kantorovich inequality is firstly initiated by Ky Fan in 1966.

We shall need some properties of the generalized Specht ratio.

In 1960 Specht estimated the upper boundary of the arithmetic mean by the geometric one for positive numbers: For $x_1, \cdots, x_n \in [m, M]$ with $M \geq m > 0$,

$$\frac{(h-1)h^{\frac{1}{h-1}}}{e \log h} \sqrt[n]{x_1 \cdots x_n} \geq \frac{x_1 + \cdots x_n}{n},$$

where $h = \frac{M}{m} (\geq 1)$. **The Specht ratio** is defined by

$$S(h) = \frac{(h-1)h^{\frac{1}{h-1}}}{e \ln h} \ (h \neq 1) \quad \text{and} \quad S(1) = 1.$$

We have the representation of the Specht ratio by the limit of Kantorovich constant $\lim_{r \to 0} K(h^r, \frac{p}{r}) = S(h^p)$.

The generalized Specht ratio is defined for $h > 0$ and $r, s \in \mathbb{R}$ by:

$$\Delta(h, r, s) = \begin{cases} K(h^r, \frac{s}{r})^{\frac{1}{s}} & \text{if} \quad rs \neq 0, \\ \Delta(h, 0, s) = S(h^s)^{\frac{1}{s}} & \text{if} \quad r = 0, \\ \Delta(h, r, 0) = S(h^r)^{-\frac{1}{r}} & \text{if} \quad s = 0. \end{cases}$$

Obviously, the generalized Specht ratio for $rs \neq 0$:

$$\Delta(h,r,s) = \left\{ \frac{r(h^s - h^r)}{(s-r)(h^r-1)} \right\}^{1/s} \left\{ \frac{s(h^r - h^s)}{(r-s)(h^s-1)} \right\}^{-1/r}, \quad h = \frac{M}{m}.$$

Lemma

Let $M > m > 0$, $r \in \mathbb{R}$ and

$$\Delta(h,r,1) = \frac{r(h-h^r)}{(1-r)(h^r-1)} \left(\frac{h^r-h}{(r-1)(h-1)} \right)^{-1/r}, \qquad h = \frac{M}{m}.$$

1. *A function $\Delta(r) \equiv \Delta(h,r,1)$ is strictly decreasing for all $r \in \mathbb{R}$,*
2. $\lim_{r\to 1} \Delta(h,r,1) = 1$ *and* $\lim_{r\to 0} \Delta(h,r,1) = S(h)$, *where $S(h)$ is the Specht ratio,*
3. $\lim_{r\to\infty} \Delta(h,r,1) = 1/h$ *and* $\lim_{r\to-\infty} \Delta(h,r,1) = h$.

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Lemma

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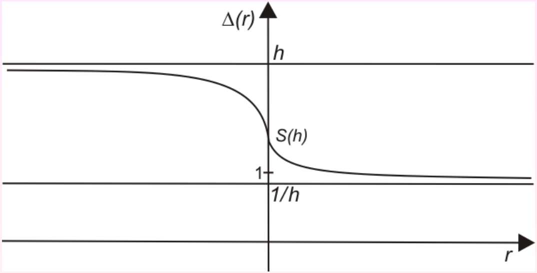


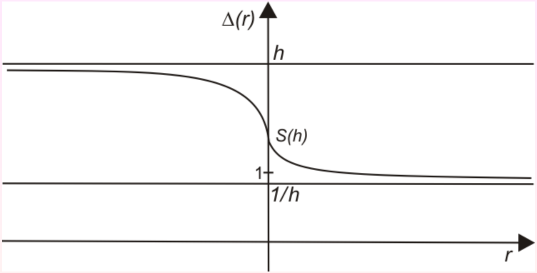
Figure 1. *Function* $\Delta(r) \equiv \Delta(h, r, 1)$

Proof

We use differential calculus. Refer to

J. Mićić and J. Pečarić, Order among power means of positive operators, II, Sci. Math. Japon. Online (2009), 677–693.

$\square$



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Also, we need the following function order of positive operators:

Theorem

If $A, B \in B_+(H)$, $A \geq B > 0$ *such that* $\mathrm{Sp}(A) \subseteq [n, N]$ *and* $\mathrm{Sp}(B) \subseteq [m, M]$ *for some scalars* $0 < n < N$ *and* $0 < m < M$*, then*

$$\begin{array}{llll} K(n,N,p) & A^p \geq B^p > 0 & \text{for all} & p > 1, \\ K(m,M,p) & A^p \geq B^p > 0 & \text{for all} & p > 1, \\ K(n,N,p) & B^p \geq A^p > 0 & \text{for all} & p < -1, \\ K(m,M,p) & B^p \geq A^p > 0 & \text{for all} & p < -1. \end{array}$$

Refer to

<<

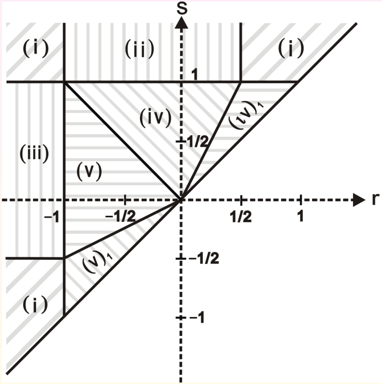
J.Mićić, J.Pečarić and Y.Seo, *Function order of positive operators based on the Mond-Pečarić method*, Linear Algebra and Appl., **360** (2003), 15–34.

J.Pečarić and J.Mićić, *Some functions reversing the order of positive operators*, Linear Algebra and Appl., **396** (2005), 175–187.

4.2. Ratio type inequalities with power functions

We give the ratio type order among power functions.

We observe regions (i) – $(\mathrm{v})_1$ as in Figure 2.



(i)	$r \leq s$, $s \notin (-1,1)$, $r \notin (-1,1)$ **or** $1/2 \leq r \leq 1 \leq s$ **or** $r \leq -1 \leq s \leq -1/2$,
(ii)	$s \geq 1$, $-1 < r < 1/2$, $r \neq 0$,
(iii)	$r \leq -1$, $-1/2 < s < 1$, $s \neq 0$,
(iv)	$-s \leq r < s/2$, $r \neq 0$, $0 < s \leq 1$,
$(\mathrm{iv})_1$	$r \leq s \leq 2r$, $0 < s \leq 1$,
(v)	$r/2 < s \leq -r$, $s \neq 0$, $-1 \leq r < 0$,
$(\mathrm{v})_1$	$2s \leq r \leq s$, $-1 \leq r < 0$.

Figure 2. *Regions in the (r, s)-plain*

Theorem

Let regions (i) – (v)$_1$ *be as in Figure 2.*
If (r,s) *in* (i), *then*

$$k^{\frac{s-r}{rs}}\,\Delta(h,r,s)^{-1}\,I_s(x,\Phi)\le I_r(x,\Phi)\le k^{\frac{s-r}{rs}}\,I_s(x,\Phi),$$

if (r,s) *in* (ii) *or* (iii), *then*

$$k^{\frac{s-r}{rs}}\,\Delta(h,r,s)^{-1}\,I_s(x,\Phi)\le I_r(x,\Phi)\le k^{\frac{s-r}{rs}}\,\Delta(h,r,s)\,I_s(x,\Phi),$$

if (r,s) *in* (iv), *then*

$$k^{\frac{s-r}{rs}}\,\Delta(h,s,1)^{-1}\Delta(h,r,s)^{-1}\,I_s(x,\Phi)\le I_r(x,\Phi)$$
$$\le k^{\frac{s-r}{rs}}\,\min\{\Delta(h,r,1),\Delta(h,s,1)\Delta(h,r,s)\}\,I_s(x,\Phi),$$

if (r,s) *in* (v) *or* (iv)$_1$ *or* (v)$_1$, *then*

$$k^{\frac{s-r}{rs}}\,\Delta(h,s,1)^{-1}\Delta(h,r,s)^{-1}\,I_s(x,\Phi)\le I_r(x,\Phi)\le k^{\frac{s-r}{rs}}\,\Delta(h,s,1)\,I_s(x,\Phi).$$

Proof

This theorem follows from second lemma by putting $p = s/r$ or $p = r/s$ and then using the Löwner-Heinz theorem, function order of positive operators and we choose better bounds by using third lemma. □

As an application, we can obtain the ratio type order among of the weighted power means of operators:

$$M_r(x,\Phi) := \left(\int_T \frac{1}{k} \Phi_t \left(x_t^r \right) d\mu(t) \right)^{1/r} \qquad \text{if} \quad r \in \mathbb{R} \backslash \{0\}$$

at the same conditions as above.

Since a field $(\frac{1}{k}\Phi_t)_{t \in T}$ in this case is unital, this result will be given in §5.1.

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4.3. Some previous results for the difference

We wish to observe the difference type order among power functions. We need some previous results.

Lemma

Let $(x_t)_{t\in T}$ and $(\Phi_t)_{t\in T}$ be as in the definition of the power function. If $0 < p \le 1$, then

$$\alpha_p \int_T \Phi_t(x_t)d\mu(t) + k\beta_p \mathbf{1} \le \int_T \Phi_t(x_t^p)d\mu(t) \le k^{1-p}\left(\int_T \Phi_t(x_t)d\mu(t)\right)^p, \tag{30}$$

if $-1 \le p < 0$ or $1 \le p \le 2$, then

$$k^{1-p}\left(\int_T \Phi_t(x_t)d\mu(t)\right)^p \le \int_T \Phi_t(x_t^p)d\mu(t) \le \alpha_p \int_T \Phi_t(x_t)d\mu(t) + k\beta_p \mathbf{1}, \tag{31}$$

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(continued)

if $p < -1$ or $p > 2$, then

$$py^{p-1}\int_T \Phi_t(x_t)d\mu(t) + k(1-p)y^p\mathbf{1} \le \int_T \Phi_t(x_t^p)d\mu(t) \le \alpha_p \int_T \Phi_t(x_t)d\mu(t) + k\beta_p\mathbf{1} \tag{32}$$

for every $y \in [m, M]$. Constants α_p and β_p are the constants α_f and β_f associated with the function $f(z) = z^p$.

Proof

RHS of (30) *and LHS of* (31) *follows from the generalized Jensen's inequality. LHS of* (30) *and RHS of* (31) *and (32) follow from Corollary* ▸ $F(u,v) = u - \lambda v$ *for* $f(z) = z^p$, $g(z) = z$ *and* $\lambda = \alpha_p$. *LHS of (32) follows from Theorem* ▸ Subdifferentials *by putting* $f(y) = y^p$ *and* $l(y) = py^{p-1}$.

Remark

Setting $y = (\alpha_p/p)^{1/(p-1)} \in [m, M]$ *in the last inequality we obtain*

$$\alpha_p \int_T \Phi_t(x_t)d\mu(t) + k(1-p)(\alpha_p/p)^{p/(p-1)}\mathbf{1} \leq \int_T \Phi_t(x_t^p)d\mu(t)$$
$$\leq \alpha_p \int_T \Phi_t(x_t)d\mu(t) + k\beta_p\mathbf{1} \qquad \text{for } p < -1 \text{ or } p > 2. \tag{33}$$

Furthermore, setting $y = m$ *or* $y = M$ *gives*

$$pm^{p-1} \int_T \Phi_t(x_t)d\mu(t) + k(1-p)m^p\mathbf{1} \leq \int_T \Phi_t(x_t^p)d\mu(t)$$
$$\leq \alpha_p \int_T \Phi_t(x_t)d\mu(t) + k\beta_p\mathbf{1} \tag{34}$$

or

$$pM^{p-1} \int_T \Phi_t(x_t)d\mu(t) + k(1-p)M^p\mathbf{1} \leq \int_T \Phi_t(x_t^p)d\mu(t)$$
$$\leq \alpha_p \int_T \Phi_t(x_t)d\mu(t) + k\beta_p\mathbf{1}. \tag{35}$$

(continued)

We remark that the operator in LHS of (34) is positive for $p > 2$, since

$$\begin{aligned} 0 < km^p\mathbf{1} &\leq pm^{p-1}\int_T \Phi_t(x_t)d\mu(t) + k(1-p)m^p\mathbf{1} \\ &\leq k(pm^{p-1}M + (1-p)m^p)\mathbf{1} < kM^p\mathbf{1} \end{aligned} \tag{36}$$

and the operator in LHS of (35) is positive for $p < -1$, since

$$\begin{aligned} 0 < kM^p\mathbf{1} &\leq pM^{p-1}\int_T \Phi_t(x_t)d\mu(t) + k(1-p)M^p\mathbf{1} \\ &\leq k(pM^{p-1}m + (1-p)M^p)\mathbf{1} < km^p\mathbf{1}. \end{aligned} \tag{37}$$

(We have the inequality $pm^{p-1}M + (1-p)m^p < M^p$ in RHS of (36) and $pM^{p-1}m + (1-p)M^p < m^p$ in RHS of (37) by using Bernoulli's inequality.)

We shall need some properties of a constant $C(m,M,p)$ (this type of a generalized Kantorovich constant for difference) defined by

$$C(m,M,p) = (p-1)\left(\frac{1}{p}\frac{M^p - m^p}{M-m}\right)^{p/(p-1)} + \frac{Mm^p - mM^p}{M-m} \quad \text{for all } p \in \mathbb{R}.$$

If we put $f(t) = t^p$ in a difference type reverse of Jensen's inequality (obtain by using the Mond-Pečarić method), then we have a difference type reverse of Hölder-McCarthy inequality:

Let A be a self-adjoint operator such that $m\mathbf{1} \le A \le M\mathbf{1}$ for some scalars $m \le M$. Then

$$0 \le (A^p x, x) - (Ax, x)^p \le C(m,M,p) \qquad \text{for all } p \notin [0,1]$$

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$$C(m,M,p) \le (A^p x, x) - (Ax, x)^p \le 0 \qquad \text{for all } p \in [0,1]$$

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for every unit vector $x \in H$.

We collect basic properties of $C(m,M,p)$ in Lemma 2.59 in our the first book.

Lemma

Let $M > m > 0$, $r \in \mathbb{R}$ and

$$C(m^r, M^r, 1/r) := \frac{1-r}{r}\left(r\frac{M-m}{M^r-m^r}\right)^{1/(1-r)} + \frac{M^r m - m^r M}{M^r - m^r}.$$

1. *A function $C(r) \equiv C(m^r, M^r, 1/r)$ is strictly decreasing for all $r \in \mathbb{R}$,*
2. $\lim_{r\to 1} C(m^r, M^r, 1/r) = 0$ *and*
   $\lim_{r\to 0} C(m^r, M^r, 1/r) = L(m,M)\ln S(M/m)$,
   where $L(m,M)$ is the logarithmic mean and $S(h)$ is the Specht ratio.
3. $\lim_{r\to\infty} C(m^r, M^r, 1/r) = m - M$ *and* $\lim_{r\to-\infty} C(m^r, M^r, 1/r) = M - m$.

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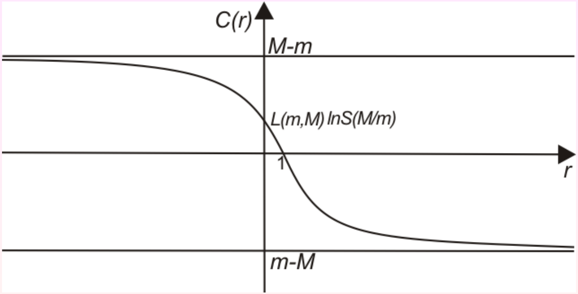
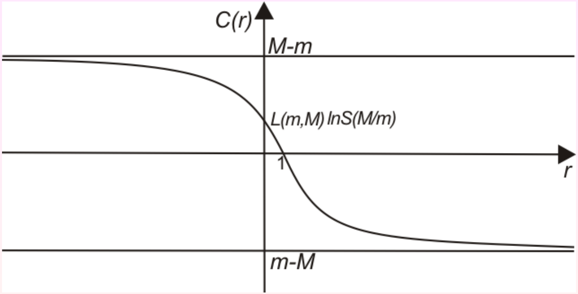


Figure 3. *Function* $C(r) \equiv C(m^r, M^r, 1/r)$

Proof

We use differential calculus. Refer to

- *J. Mićić and J. Pečarić, Order among power means of positive operators, II, Sci. Math. Japon. Online (2009), 677–693.*



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$\square$

Also, we need the following function order of positive operators:

Theorem

Let A, B be positive operators in $B(H)$.
If $A \geq B > 0$ and the spectrum $\mathrm{Sp}(B) \subseteq [m, M]$ for some scalars $0 < m < M$, then

$$A^p + C(m, M, p)\mathbf{1} \geq B^p \qquad \text{for all} \quad p \geq 1.$$

But, if $A \geq B > 0$ and the spectrum $\mathrm{Sp}(A) \subseteq [m, M]$, $0 < m < M$, then

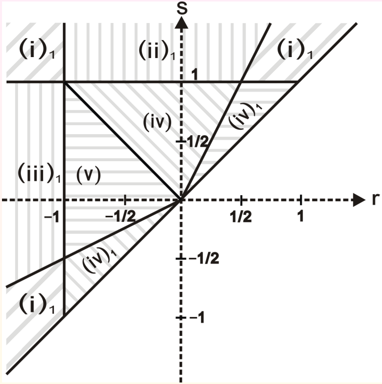
$$B^p + C(m, M, p)\mathbf{1} \geq A^p \qquad \text{for all} \quad p \leq -1,$$

Refer to Order.

4.4. Difference type inequalities with power functions

We give the difference type order among power functions.

We observe regions $(\mathrm{i})_1 - (\mathrm{v})_1$ as in Figure 4.



$(\mathrm{i})_1$	$r \leq -1,\ s \geq 1$ **or** $s/2 \leq r \leq s,\ s \geq 1$ **or** $r \leq s \leq r/2,\ r \leq -1,$
$(\mathrm{ii})_1$	$s \geq 1,\ -1 < r < s/2,\ r \neq 0,$
$(\mathrm{iii})_1$	$r \leq -1,\ r/2 < s < 1,\ s \neq 0,$
(iv)	$-s \leq r < s/2,\ r \neq 0,\ 0 < s \leq 1,$
$(\mathrm{iv})_1$	$r \leq s \leq 2r,\ 0 < s \leq 1,$
(v)	$r/2 < s \leq -r,\ s \neq 0,\ -1 \leq r < 0,$
$(\mathrm{v})_1$	$2s \leq r \leq s,\ -1 \leq r < 0.$

Figure 4. *Regions in the (r,s)-plain*

Theorem

Let regions $(\mathrm{i})_1 - (\mathrm{v})_1$ *be as in Figure 4.*
Then

$$C_2\mathbf{1} \le I_s(x,\Phi) - I_r(x,\Phi) \le C_1\mathbf{1}, \tag{38}$$

where constants $C_1 \equiv C_1(m,M,s,r,k)$ *and* $C_2 \equiv C_2(m,M,s,r,k)$ *are*

$$C_1 = \begin{cases} \widetilde{\Delta}_k, & \text{for } (r,s) \text{ in } (\mathrm{i})_1 \text{ or } (\mathrm{ii})_1 \text{ or } (\mathrm{iii})_1; \\ \widetilde{\Delta}_k + \min\{C_k(s), C_k(r)\}, & \text{for } (r,s) \text{ in } (\mathrm{iv}) \text{ or } (\mathrm{v}) \text{ or } (\mathrm{iv})_1 \text{ or } (\mathrm{v})_1; \end{cases}$$

$$C_2 = \begin{cases} (k^{1/s} - k^{1/r})\, m, & \text{for } (r,s) \text{ in } (\mathrm{i})_1; \\ \widetilde{D}_k, & \text{for } (r,s) \text{ in } (\mathrm{ii})_1; \\ \overline{D}_k, & \text{for } (r,s) \text{ in } (\mathrm{iii})_1; \\ \max\left\{\widetilde{D}_k - C_k(s), \left(k^{1/s} - k^{1/r}\right) m - C_k(r)\right\}, & \text{for } (r,s) \text{ in } (\mathrm{iv}); \\ \max\left\{\overline{D}_k - C_k(r), \left(k^{1/s} - k^{1/r}\right) m - C_k(s)\right\}, & \text{for } (r,s) \text{ in } (\mathrm{v}); \\ (k^{1/s} - k^{1/r})\, m - \min\{C_k(r), C_k(s)\}, & \text{for } (r,s) \text{ in } (\mathrm{iv})_1 \text{ or } (\mathrm{v})_1. \end{cases}$$

(continued)

A constant $\widetilde{\Delta}_k \equiv \widetilde{\Delta}_k(m, M, r, s)$ is

$$\widetilde{\Delta}_k = \max_{\theta\in[0,1]} \left\{ k^{1/s}[\theta M^s + (1-\theta)m^s]^{1/s} - k^{1/r}[\theta M^r + (1-\theta)m^r]^{1/r} \right\},$$

$\widetilde{D}_k \equiv \widetilde{D}_k(m, M, r, s)$ is

$$\widetilde{D}_k = \min\left\{ \left(k^{\frac{1}{s}} - k^{\frac{1}{r}}\right) m, k^{\frac{1}{s}} m \left(s\frac{M^r - m^r}{rm^r} + 1 \right)^{\frac{1}{s}} - k^{\frac{1}{r}} M \right\},$$

$\overline{D}_k \equiv \overline{D}_k(m, M, r, s) = -\widetilde{D}_k(M, m, s, r)$
and $C_k(p) \equiv C_k(m, M, p)$ is

$$C_k(p) = k^{1/p} \cdot C(m^p, M^p, 1/p) \qquad \text{for } p \neq 0.$$

5. Weighted power means

As an application §4, we can obtain the order among of the weighted power means of operators:

$$M_r(x,\Phi) := \left(\int_T \Phi_t\left(x_t^r\right) d\mu(t) \right)^{1/r} \qquad \text{if} \quad r \in \mathbb{R}\backslash\{0\}$$

with these conditions: $(x_t)_{t\in T}$ is a bounded continuous field of positive operator in a C^*-algebra $B(H)$ with spectra in $[m,M]$ for some scalars $0<m<M$, defined on a locally compact Hausdorff space T equipped with a bounded Radon measure μ and $(\Phi_t)_{t\in T}$ is a field of positive linear maps $\Phi_t : B(H) \to B(K)$, such that $\int_T \Phi_t(\mathbf{1})\,d\mu(t) = \mathbf{1}$.

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5.1. Ratio type inequalities

Corollary

Let regions (i) – (v)$_1$ *be as in Figure 2.*
If (r,s) *in* (i), *then*

$$\Delta(h,r,s)^{-1}\, M_s(x,\Phi) \le M_r(x,\Phi) \le M_s(x,\Phi),$$

if (r,s) *in* (ii) *or* (iii), *then*

$$\Delta(h,r,s)^{-1}\, M_s(x,\Phi) \le M_r(x,\Phi) \le \Delta(h,r,s)\, M_s(x,\Phi),$$

if (r,s) *in* (iv), *then*

$$\begin{aligned}\Delta(h,s,1)^{-1}\Delta(h,r,s)^{-1}\, M_s(x,\Phi) &\le M_r(x,\Phi)\\ &\le \min\{\Delta(h,r,1),\Delta(h,s,1)\Delta(h,r,s)\}\, M_s(x,\Phi),\end{aligned}$$

if (r,s) *in* (v) *or* (iv)$_1$ *or* (v)$_1$, *then*

$$\Delta(h,s,1)^{-1}\Delta(h,r,s)^{-1}\, M_s(x,\Phi) \le M_r(x,\Phi) \le \Delta(h,s,1)\, M_s(x,\Phi).$$

5.2. Difference type inequalities

Corollary

Let regions (i) – (v)$_1$ *be as in Figure 2.*
If (r,s) *in* (i), *then*

$$0 \leq M_s(x,\Phi) - M_r(x,\Phi) \leq \tilde{\Delta}\mathbf{1},$$

if (r,s) *in* (ii), *then*

$$\left(m \left(\frac{s}{r}\frac{M^r}{m^r} + 1 - \frac{s}{r} \right)^{1/s} - M \right) \mathbf{1} \leq M_s(x,\Phi) - M_r(x,\Phi) \leq \tilde{\Delta}\mathbf{1},$$

if (r,s) *in* (iii), *then*

$$\left(m - M \left(\frac{r}{s}\frac{m^s}{M^s} + 1 - \frac{r}{s} \right)^{1/r} \right) \mathbf{1} \leq M_s(x,\Phi) - M_r(x,\Phi) \leq \tilde{\Delta}\mathbf{1},$$

(continued)

if (r,s) in (iv), then

$$\max\{m\left(\frac{s}{r}\frac{M^r}{m^r}+\frac{r-s}{r}\right)^{1/s}-M-\ C(m^s,M^s,1/s),-C\left(m^r,M^r,1/r\right)\}\mathbf{1}$$
$$M_s(x,\Phi)-M_r(x,\Phi)\le\left(\tilde{\Delta}+C(m^s,M^s,1/s)\right)\mathbf{1},$$

if (r,s) in (v) or $(\mathrm{iv})_1$ or $(\mathrm{v})_1$, then

$$-C(m^s,M^s,1/s)\mathbf{1}\le M_s(x,\Phi)-M_r(x,\Phi)\le\left(\tilde{\Delta}+C(m^s,M^s,1/s)\right)\mathbf{1},$$

where $\tilde{\Delta}=\max_{\theta\in[0,1]}\left\{[\theta M^s+(1-\theta)m^s]^{1/s}-[\theta M^r+(1-\theta)m^r]^{1/r}\right\}$.

6. Chaotic order version of these results

We can obtain chaotic order among quasi-arithmetic means in a general setting and chaotic order among power means.

E.g.

If $\psi \circ \varphi^{-1}$ is operator convex and ψ is operator monotone then

$$M_{\varphi}(x, \Phi) \ll M_{\psi}(x, \Phi).$$

We are working on this.

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If $\psi \circ \varphi^{-1}$ is operator convex and ψ is operator monotone then

$$M_{\varphi}(x,\Phi) \ll M_{\psi}(x,\Phi).$$

We are working on this.

For $r \in \mathbb{R}$ we define the r-th power operator mean as

$$M_r(x,\Phi) := \begin{cases} \left(\int_T \frac{1}{k}\Phi_t(x_t^r)\, d\mu(t)\right)^{1/r}, & r \neq 0; \\ \exp\left(\int_T \frac{1}{k}\Phi_t(\ln(x_t^r))\, d\mu(t)\right)^{1/r}, & r = 0. \end{cases}$$

The limit $\mathbf{s} - \lim_{r\to 0} M_r(x,\Phi) = M_0(x,\Phi)$ exists.

Theorem

If $r, s \in \mathbb{R}$, $r < s$, then

$$\Delta(h,r,s)^{-1} M_s(x,\Phi) \ll M_r(x,\Phi) \ll M_s(x,\Phi).$$

where $\Delta(h,r,s)$ is the generalized Specht ratio. Gen. Specht ratio

Thank you very much for your attention