Recurrences and characters of Feigin-Stoyanovsky's type subspaces

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Abstract. We present some results on the recurrence relations and character formulas for Feigin-Stoyanovsky's type subspaces of standard $\widetilde{\mathfrak{sl}(\ell+1,\mathbb{C})}$-modules.

1. Introduction

This expository note should serve as short introduction to an interesting and fruitful area of research concerning Feigin-Stoyanovsky's type subspaces of standard modules of affine Lie algebras. Special emphasis is given to the construction of recurrence relations for characters of these subspaces for affine Lie algebra $\widetilde{\mathfrak{sl}(\ell+1,\mathbb{C})}$, as well as to an effort to obtain the character formulas in some cases. We present also the analogous results for principal subspaces, a similarly defined class of subspaces of standard modules for affine Lie algebras.

For historical background of the subject please refer to introductory remarks of e.g. [**21, 24, 3, 14**].

Denote by $\mathfrak{g}$ simple Lie algebra $\mathfrak{sl}(\ell+1,\mathbb{C})$, $\mathfrak{h}$ its Cartan subalgebra and R the corresponding root system with fixed simple roots $\alpha_1,\dots,\alpha_\ell$. We have the known triangular decomposition $\mathfrak{g}=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+$, with fixed root vectors x_α, $\alpha\in R$. Identify $\mathfrak{h}$ and $\mathfrak{h}^*$ via Killing form $\langle\cdot,\cdot\rangle$ normalized so that for the maximal root θ the relation $\langle\theta,\theta\rangle=2$ holds. Also, denote by $Q=Q(R)$ and $P=P(R)$ the root and weight lattices respectively, with fundamental weights denoted by $\omega_1,\dots,\omega_\ell$. For later use define $\omega_0=0$.

We proceed to the affine Lie algebra $\tilde{\mathfrak{g}}$ associated to $\mathfrak{g}$:

$$\tilde{\mathfrak{g}}=\mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}c\oplus\mathbb{C}d,$$

c denoting the canonical central element and d the degree operator, with Lie product given in the usual way (cf. [**16**]). Let us write $x(n)=x\otimes t^n$ for $x\in\mathfrak{g}$, $n\in\mathbb{Z}$, and denote $x(z)=\sum_{n\in\mathbb{Z}}x(n)z^{-n-1}$. Denote by $\Lambda_0,\dots,\Lambda_\ell$ the corresponding fundamental weights of $\tilde{\mathfrak{g}}$.

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For given integral dominant weight $\Lambda = k_0\Lambda_0 + k_1\Lambda_1 + \cdots + k_\ell\Lambda_\ell$, denote by $L(\Lambda)$ the standard $\tilde{\mathfrak{g}}$-module with highest weight Λ, and by v_Λ a fixed highest weight vector of $L(\Lambda)$. Let $k = \Lambda(c) = k_0 + k_1 + \cdots + k_\ell$ be the level of $L(\Lambda)$.

2. Definition of Feigin-Stoyanovsky's type subspaces

We are now ready to define object of research presented in this note. First, for fixed minuscule weight $\omega = \omega_\ell$ define

$$\Gamma = \{\alpha \in R \mid \langle \alpha, \omega \rangle = 1\} = \{\gamma_1, \gamma_2, \ldots, \gamma_\ell \mid \gamma_i = \alpha_i + \cdots + \alpha_\ell\}.$$

This gives us a $\mathbb{Z}$-grading of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1, \tag{2.1}$$

with $\mathfrak{g}_0 = \mathfrak{h} + \sum_{\langle \alpha, \omega \rangle = 0} \mathfrak{g}_\alpha$, $\mathfrak{g}_{\pm 1} = \sum_{\alpha \in \pm\Gamma} \mathfrak{g}_\alpha$, and correspondingly the $\mathbb{Z}$-grading

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_{-1} + \tilde{\mathfrak{g}}_0 + \tilde{\mathfrak{g}}_1,$$

having denoted $\tilde{\mathfrak{g}}_0 = \mathfrak{g}_0 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$, $\tilde{\mathfrak{g}}_{\pm 1} = \mathfrak{g}_{\pm 1} \otimes \mathbb{C}[t, t^{-1}]$. Note that

$$\tilde{\mathfrak{g}}_1 = \text{span}\{x_\gamma(n) \mid \gamma \in \Gamma, n \in \mathbb{Z}\}$$

is a commutative subalgebra and a $\tilde{\mathfrak{g}}_0$-module.

Definition 2.1. For a standard $\tilde{\mathfrak{g}}$-module $L(\Lambda)$, *Feigin-Stoyanovsky's type subspace of* $L(\Lambda)$ is

$$W(\Lambda) = U(\tilde{\mathfrak{g}}_1) \cdot v_\Lambda, \tag{2.2}$$

where $U(\tilde{\mathfrak{g}}_1)$ is the universal enveloping algebra of $\tilde{\mathfrak{g}}_1$.

Feigin-Stoyanovsky's type subspaces are constructed analogously to principal subspaces (sometimes called also Feigin-Stoyanovsky's principal subspaces), first appeared in [**23**]: for a $\tilde{\mathfrak{g}}$-module $L(\Lambda)$ define *principal subspace of* $L(\Lambda)$ as

$$W(\Lambda) = U(\hat{\mathfrak{n}}_+) \cdot v_\Lambda, \tag{2.3}$$

with $\hat{\mathfrak{n}}_+ = \mathfrak{n}_+ \otimes \mathbb{C}[t, t^{-1}]$.

It is clear from (2.2) and (2.3) that Feigin-Stoyanovsky's type subspaces and principal subspaces of the same standard module coincide in the case of $\ell = 1$. Otherwise, in order to avoid notational confusion, we will state which of the two here defined subspaces of the corresponding standard $\tilde{\mathfrak{g}}$-module we have in mind when using the notation $W(\Lambda)$ (or it will be clear from the context).

At the end of this section note that, concerning the main line of exposition in this note, our definition of Feigin-Stoyanovsky's type subspaces is restricted to $\mathfrak{g} = \mathfrak{sl}(\ell + 1, \mathbb{C})$ and $\omega = \omega_\ell$. But, these subspaces can easily be defined in the more general setting of any simple Lie algebra $\mathfrak{g}$, provided a $\mathbb{Z}$-grading $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ such that $\mathfrak{h} \subset \mathfrak{g}_0$ is given. Similar observation holds about generalization of definition (2.3) to the case of any simple Lie algebra.

3. Combinatorial bases

As in previous chapter, let $\mathfrak{g} = \mathfrak{sl}(\ell+1, \mathbb{C})$ and $\omega = \omega_\ell$. From Poincaré-Birkhoff-Witt theorem it follows that a Feigin-Stoyanovsky's type subspace $W(\Lambda)$ is spanned by set of monomial vectors

$$\{x(\pi)v_\Lambda | x(\pi) = \ldots x_{\gamma_1}(-2)^{a_\ell} x_{\gamma_\ell}(-1)^{a_{\ell-1}} \cdots x_{\gamma_1}(-1)^{a_0}, a_i \in \mathbb{Z}_+, i \in \mathbb{Z}_+\}. \tag{3.1}$$

It is an important and interesting problem to reduce the spanning set (3.1) to monomial basis of $W(\Lambda)$, i.e. basis consisting of monomial vectors. In [**19**] this reduction was obtained by Primc for Feigin-Stoyanovsky's type subspaces of arbitrary level standard modules. Later it turned out that in this case basis elements are parametrized by so-called $(k, \ell+1)$-admissible configurations (first described in [**9, 10**]). More precisely, for level k standard $\tilde{\mathfrak{g}}$-module $L(\Lambda)$ with highest weight $\Lambda = k_0\Lambda_0 + k_1\Lambda_1 + \cdots + k_\ell\Lambda_\ell$, we say that a monomial vector $x(\pi)v_\Lambda = \ldots x_{\gamma_1}(-2)^{a_\ell} x_{\gamma_\ell}(-1)^{a_{\ell-1}} \cdots x_{\gamma_1}(-1)^{a_0} v_\Lambda \in W(\Lambda)$ is $(k, \ell+1)$-*admissible for* Λ if it satisfies *difference conditions*

$$a_i + \cdots + a_{i+\ell} \leq k, \quad i \in \mathbb{Z}_+ \tag{3.2}$$

and *initial conditions*

$$\begin{aligned} a_0 &\leq k_0 \\ a_0 + a_1 &\leq k_0 + k_1 \\ &\ldots \\ a_0 + a_1 + \cdots + a_{\ell-1} &\leq k_0 + \cdots + k_{\ell-1}. \end{aligned} \tag{3.3}$$

We have the following theorem:

THEOREM 3.1. *The set of* $(k, \ell+1)$*-admissible monomial vectors for* Λ *is a basis of* $W(\Lambda)$.

Furthermore, in [**20**] Primc constructed monomial bases with suitable combinatorial description in the case of arbitrary classical simple Lie algebra and for all possible choices of (2.1), but only for Feigin-Stoyanovsky's type subspace of basic module.

It seems that construction of combinatorial bases is also a hard problem when principal subspaces are concerned. So far, there has been progress in the case of principal subspaces for $\mathfrak{sl}(\ell+1,\mathbb{C})\tilde{}$: Georgiev in [**13**] constructed combinatorial bases, so-called quasi-particle bases, for principal subspaces of all level 1 standard $\mathfrak{sl}(\ell+1,\mathbb{C})\tilde{}$-modules, as well as for some classes of principal subspaces of higher level standard modules. Linear independence of these bases was proven using Dong-Lepowsky's intertwining operators (cf. [**8**]).

The use of intertwining operators proved interesting in further exploration of both Feigin-Stoyanovsky's type subspaces and principal subspaces. Namely, Capparelli, Lepowsky and Milas in [**6, 7**] use intertwining operators to calculate recursions for formal characters and, consequently, to obtain character formulas for principal subspaces of standard $\mathfrak{sl}(3,\mathbb{C})\tilde{}$-modules (we give more detailed information on this line of research in the next section).

Although Capparelli, Lepowsky and Milas have not explicitly worked on combinatorial bases of principal subspaces, their use of intertwining operators inspired Primc in [**21**] to obtain a simpler proof of Theorem 3.1.

Working also on $\mathfrak{g} = \mathfrak{sl}(\ell+1,\mathbb{C})$, but in more general setting of an arbitrary choice for ω (allowing it to be *any* of the fundamental weights $\omega_1, \ldots, \omega_\ell$) - therefore covering all possible $\mathbb{Z}$-gradings (2.1), Trupčević in [**24, 25**] also uses intertwining operators to prove linear independence of combinatorial bases for Feigin-Stoyanovsky's type subspaces of all standard $\tilde{\mathfrak{g}}$-modules at arbitrary integer level.

Baranović in [**3**] gives a combinatorial description (in terms of difference and initial conditions) of bases for Feigin-Stoyanovsky's type subspaces for level 1 standard modules for affine Lie algebra of type $D_\ell^{(1)}$, and for a specific choice of (2.1). She then extends her method to obtain combinatorial bases in the case of level 2 standard modules of affine Lie algebra $D_4^{(1)}$.

Finding combinatorial bases for Feigin-Stoyanovsky's type subspaces in other cases remains an open problem.

4. Exact sequences, recurrences and characters

In the main section of this note we present some results concerning recurrence relations for characters (i.e. generating functions for dimensions of the homogeneous components) of both Feigin-Stoyanovsky's type subspaces and principal subspaces. For precise definitions of characters please consult papers mentioned below, because definitions may differ.

Let us first present an overview of results obtained so far for principal subspaces, having in mind they apply to Feigin-Stoyanovsky's type subspaces in the case of $\tilde{\mathfrak{g}} = \mathfrak{sl}(2,\mathbb{C})\tilde{}$.

By describing the dual spaces of principal subspaces in terms of symmetric polynomial forms vanishing on certain hyperplanes, Feigin and Stoyanovsky in [**23**] obtained character formulas for principal subspaces of integer level standard $\mathfrak{sl}(2,\mathbb{C})\tilde{}$-modules, as well as for vacuum standard $\mathfrak{sl}(3,\mathbb{C})\tilde{}$-modules.

As already mentioned in previous section, Georgiev in [**13**] used intertwining operators between standard modules to obtain quasi-particle bases. Consequently, he was able to calculate character formulas for principal subspaces of fundamental $\mathfrak{sl}(\ell+1,\mathbb{C})\tilde{}$-modules, while for higher integer levels he calculated characters for $W(k_0\Lambda_0 + k_j\Lambda_j)$, $k_0, k_j \in \mathbb{Z}_+$, and $j = 1,\ldots,\ell$.

Capparelli, Lepowsky and Milas extended Georgiev's method to obtain the following exact sequences of principal subspaces of level k standard $\mathfrak{sl}(2,\mathbb{C})\tilde{}$-modules:

$$0 \longrightarrow W((k-i)\Lambda_0 + i\Lambda_1) \xrightarrow{e_{(k)}^{\alpha/2}} W(i\Lambda_0 + (k-i)\Lambda_1) \xrightarrow{o\mathcal{Y}(v_{(k-1)\Lambda_0+\Lambda_1})} \longrightarrow W((i-1)\Lambda_0 + (k-i+1)\Lambda_1) \longrightarrow 0$$

$$0 \longrightarrow W(k\Lambda_0) \xrightarrow{e_{(k)}^{\alpha/2}} W(k\Lambda_1) \longrightarrow 0,$$

for all $i = 1,\ldots,k$. Here $e_{(k)}^{\alpha/2}$ represents certain linear map between standard modules, and $o\mathcal{Y}(v_{(k-1)\Lambda_0+\Lambda_1})$ coefficients of suitably chosen intertwining operators associated to standard modules (cf. [**6, 7**] for details). As a direct consequence of this result they obtained recursions for characters of principal subspaces appearing above (cf. (4.9) and (4.10) in [**7**]). It turned out that these recursions precisely equal the previously known Rogers-Selberg recursions, whose solution has already been given by Andrews (cf. [**1, 2**]). Thus they directly recovered formulas for characters of principal subspaces of level k standard $\mathfrak{sl}(2,\mathbb{C})\tilde{}$-modules, confirming results of [**23, 13**]:

$$\chi(W(i\Lambda_0 + (k-i)\Lambda_1))(z;q) = \sum_{n\geq 0} \sum_{\substack{N_1+\cdots+N_k=n \\ N_1\geq\cdots\geq N_k\geq 0}} \frac{q^{N_1^2+\cdots+N_k^2+N_{i+1}+\cdots+N_k}}{(q)_{N_1-N_2}\cdots(q)_{N_{k-1}-N_k}(q)_{N_k}} z^n,$$

for every $i = 0, \ldots, k$, with $(q)_n = (1-q)(1-q^2)\cdots(1-q^n)$ for $n \geq 0$, z and q being formal variables.

As a continuation of the above mentioned approach, Calinescu in [**4**] obtained a family of exact sequences for principal subspaces of basic $\tilde{\mathfrak{sl}}(\ell+1,\mathbb{C})$-modules:

$$\begin{aligned}
&0 \longrightarrow W(\Lambda_1) \xrightarrow{e^{\lambda^1}} W(\Lambda_0) \xrightarrow{\mathcal{Y}_c(e^{\lambda_1},z)} W(\Lambda_1) \longrightarrow 0,\\
&0 \longrightarrow W(\Lambda_2) \xrightarrow{e^{\lambda^2}} W(\Lambda_0) \xrightarrow{\mathcal{Y}_c(e^{\lambda_2},z)} W(\Lambda_2) \longrightarrow 0,\\
&\vdots\\
&0 \longrightarrow W(\Lambda_\ell) \xrightarrow{e^{\lambda^\ell}} W(\Lambda_0) \xrightarrow{\mathcal{Y}_c(e^{\lambda_\ell},z)} W(\Lambda_\ell) \longrightarrow 0,
\end{aligned}$$

where e^{λ^j} are certain linear maps, and $\mathcal{Y}_c(e^{\lambda_j},z)$ correspond to suitable intertwining operators (cf. [**4**] for details). From these sequences Calinescu gets a complete set of recursions for characters of principal subspace $W(\Lambda_0)$ (with $A = (a_{ij})_{1\leq i,j\leq \ell}$ being Cartan matrix of $\mathfrak{sl}(\ell+1,\mathbb{C})$):

$$\begin{aligned}
\chi(W(\Lambda_0))(z_1,\ldots,z_\ell;q) &= \chi(W(\Lambda_0))(z_1,\ldots,(z_jq)^{\frac{a_{jj}}{2}},\ldots,z_\ell;q)+\\
&+(z_jq)^{\frac{a_{jj}}{2}}\chi(W(\Lambda_0))(z_1q^{a_{j1}},z_2q^{a_{j2}},z_3q^{a_{j3}},\ldots,z_\ell q^{a_{j\ell}};q),
\end{aligned}$$

for $j = 1, \ldots, \ell$. By directly solving this system she obtains following character formulas for principal subspaces of basic $\tilde{\mathfrak{sl}}(\ell+1,\mathbb{C})$-modules:

$$\chi(W(\Lambda_i))(z_1,\ldots,z_\ell;q) = \sum_{n_1,\ldots,n_\ell\geq 0} \frac{q^{n_1^2+\cdots+n_\ell^2+n_i-n_2n_1-\cdots-n_\ell n_{\ell-1}}}{(q)_{n_1}\cdots(q)_{n_\ell}} z_1^{n_1}\cdots z_\ell^{n_\ell},$$

for $i = 0, \ldots, \ell$ (and, for $j = 0$, n_0 appearing in the numerator set to be zero), thus confirming Georgiev's result (cf. formula (4.20) in [**13**]).

In [**5**] Calinescu applies this approach to obtain two families of exact sequences for principal subspaces of arbitrary level k standard $\tilde{\mathfrak{sl}}(3,\mathbb{C})$-modules:

$$\begin{aligned}
&0 \longrightarrow W(i\Lambda_1+(k-i)\Lambda_2) \xrightarrow{e^{\lambda}_{(k)}} W(i\Lambda_0+(k-i)\Lambda_1) \xrightarrow{\mathcal{Y}_c(v_{(k-1)\Lambda_0+\Lambda_1},z)}\\
&\qquad \longrightarrow W((i-1)\Lambda_0+(k-i+1)\Lambda_1) \longrightarrow 0\\
&0 \longrightarrow W((k-i)\Lambda_1+i\Lambda_2) \xrightarrow{e^{\beta}_{(k)}} W(i\Lambda_0+(k-i)\Lambda_2) \xrightarrow{\mathcal{Y}_c(v_{(k-1)\Lambda_0+\Lambda_2},z)}\\
&\qquad \longrightarrow W((i-1)\Lambda_0+(k-i+1)\Lambda_2) \longrightarrow 0,
\end{aligned}$$

for any i with $1 \leq i \leq k$. Given these sequences Calinescu derived a system of recurrences for $W(i\Lambda_0+(k-i)\Lambda_j)$, $1 \leq i \leq k$ and $j = 1, 2$ (cf. Theorem 4.2 in [**5**]), and was able to show that it is satisfied by the following formulas previously obtained in [**13**]:

$$\begin{aligned}
&\chi(W(i\Lambda_0+(k-i)\Lambda_j))(z_1,z_2;q) =\\
&= \sum_{n_1,n_2\geq 0} \sum_{\substack{\sum_{t=1}^k N_{1,t}=n_1\\ N_{1,1}\geq\cdots\geq N_{1,k}\geq 0\\ \sum_{t=1}^k N_{2,t}=n_2\\ N_{2,1}\geq\cdots\geq N_{2,k}\geq 0}} \frac{q^{\sum_{t=1}^k(N_{1,t}^2+N_{2,t}^2-N_{1,t}N_{2,t})+\sum_{t=1}^k(N_{1,t}\delta_{1,j_t}+N_{2,t}\delta_{2,j_t})}}{(q)_{N_{1,1}-N_{1,2}}\cdots(q)_{N_{1,k}}(q)_{N_{2,1}-N_{2,2}}\cdots(q)_{N_{2,k}}} z_1^{n_1}z_2^{n_2},
\end{aligned}$$

where $j_t = 0$ for $0 \le t \le i$ and $j_t = j$ for $i < t \le k$. Going a step further, Calinescu calculates previously unknown character formulas for $W(i\Lambda_1 + (k-i)\Lambda_2)$ for $1 \le i \le k-1$ (cf. Corollary 4.1 in [**5**]).

Let us now turn to Feigin-Stoyanovsky's type subspaces. Since these subspaces do not differ from principal subspaces in the case of $\mathfrak{sl}(2, \mathbb{C})$, we will start presenting the results obtained for $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$. Feigin, Jimbo, Miwa, Mukhin and Takeyama in [**10**] embed the dual space of principal subspace $W(\Lambda)$ for level k standard modules into the space of symmetric polynomials, where they introduce the so-called Gordon filtration. By explicitly calculating components of the associated graded space (using vertex operators), they obtained principally specialized character formulas for $W(\Lambda)$ in the case of $\Lambda = k_0\Lambda_0 + k_1\Lambda_1$, with $k_0, k_1 \in \mathbb{Z}_+$ such that $k_0 + k_1 = k$:

$$\tag{4.1} \chi(W(k_0\Lambda_0 + k_1\Lambda_1))(z; q) = $$
$$\sum_{n\ge 0} \sum_{\substack{l_1+l_2=n \\ l_1,l_2\ge 0}} \sum_{\substack{\sum_j j m_{i,j}=l_i \\ i=1,2}} \frac{q^{{}^t mAm - (\mathrm{diag} A)\cdot m + 2c_{k_0}^{(3)}\cdot m}}{(q^2)_{m_{1,1}} \cdots (q^2)_{m_{1,k}} (q^2)_{m_{2,1}} \cdots (q^2)_{m_{2,k}}} q^{l_2} z^n,$$

where

$$A = \left(\begin{array}{c|c} A^{(2)} & B^{(3)} \\ \hline B^{(3)} & A^{(2)} \end{array} \right)$$
$$A^{(2)} = (A_{ab}^{(2)})_{1\le a,b\le k}, \quad A_{ab}^{(2)} = 2\min(a,b)$$
$$B^{(3)} = (B_{ab}^{(3)})_{1\le a,b\le k}, \quad B_{ab}^{(3)} = \max(0, a+b-k)$$
$$c_{k_0}^{(3)} = (\underbrace{0,\ldots,0,1,2,\ldots,k-k_0}_{k}, \underbrace{0,\ldots,0}_{k})$$
$$m = {}^t(m_{1,1},\ldots,m_{1,k},m_{2,1},\ldots,m_{2,k}).$$

We now provide an exposition of new results obtained for $\mathfrak{g} = \mathfrak{sl}(\ell+1, \mathbb{C})$ in [**14, 15**]. In these papers we use both intertwining operators between standard $\tilde{\mathfrak{g}}$-modules and the fact that the combinatorial bases are known (cf. Theorem 3.1) to obtain the exact sequences of Feigin-Stoyanovsky's type subspaces for standard modules at arbitrary integer level, as well as the accompanying systems of recurrence relations for their formal characters. Furthermore, we were able to obtain the character formulas in some cases.

We review shortly the vertex operator construction of fundamental $\tilde{\mathfrak{g}}$-modules, as well as a definition of Dong-Lepowsky's intertwining operators (cf. [**11, 22, 12, 8**] for details).

Denote by $M(1)$ the Fock space for the homogeneous Heisenberg subalgebra and by $\mathbb{C}[P]$ the group algebra of the weight lattice (with a basis e^λ, $\lambda \in P$). It is a well-known fact that $M(1) \otimes \mathbb{C}[P]$ obtains the structure of $\tilde{\mathfrak{g}}$-module by extending the action of Heisenberg subalgebra via the vertex operator formula

$$\tag{4.2} x_\alpha(z) = E^-(-\alpha, z)E^+(-\alpha, z)e_\alpha z^\alpha,$$

where x_α are properly chosen root vectors, and $z^\alpha = 1 \otimes z^\alpha$, $z^\alpha e^\lambda = z^{\langle \alpha, \lambda \rangle}$, $E^\pm(\alpha, z) = E^\pm(\alpha, z) \otimes 1 = \exp\left(\sum_{n>0} \alpha(\pm n) z^{\mp n}/(\pm n) \right) \otimes 1$. Then

$$M(1) \otimes \mathbb{C}[P] = L(\Lambda_0) + L(\Lambda_1) + \cdots + L(\Lambda_\ell)$$

and $L(\Lambda_i) = M(1) \otimes e^{\omega_i}\mathbb{C}[Q]$ with highest weight vectors $v_{\Lambda_i} = 1 \otimes e^{\omega_i}$, $i = 0, \ldots, \ell$.

We use certain coefficients of the following intertwining operators

$$\mathcal{Y}(1\otimes e^{\lambda},z)=E^-(-\lambda,z)E^+(-\lambda,z)e_\lambda z^\lambda e^{i\pi\lambda}c(\cdot,\lambda), \tag{4.3}$$

with $\lambda\in P$ and $e_\lambda=1\otimes e_\lambda=1\otimes e^{\lambda}\epsilon(\lambda,\cdot)$ (cf. [**8**]).

Namely, for $\lambda_i:=\omega_i-\omega_{i-1}$, $i=1,\dots,\ell$, define

$$[i]:=\mathrm{Res}z^{-1-\langle\lambda_i,\omega_{i-1}\rangle}c_i\mathcal{Y}(1\otimes e^{\lambda_i},z),\quad i=1,\dots,\ell.$$

By using (4.2) and (4.3) one can prove that for suitably chosen constants c_i the following relations for $[i]$ hold:

$$L(\Lambda_0)\xrightarrow{[1]}L(\Lambda_1)\xrightarrow{[2]}L(\Lambda_2)\xrightarrow{[3]}\dots\xrightarrow{[\ell-1]}L(\Lambda_{\ell-1})\xrightarrow{[\ell]}L(\Lambda_\ell)$$

$$v_{\Lambda_0}\xrightarrow{[1]}v_{\Lambda_1}\xrightarrow{[2]}v_{\Lambda_2}\xrightarrow{[3]}\dots\xrightarrow{[\ell-1]}v_{\Lambda_{\ell-1}}\xrightarrow{[\ell]}v_{\Lambda_\ell}. \tag{4.4}$$

By using commutator formula for intertwining operators one shows the important fact that $[i]$ commute with the action of $x(\pi)$, with $x(\pi)$ defined as in (3.1):

$$x(\pi)[i]=[i]x(\pi),\quad i=1,\dots,\ell. \tag{4.5}$$

Furthermore, we will use the so-called simple current operator, a linear bijection $[\omega]=e^{\omega}\epsilon(\cdot,\omega)$ on $M(1)\otimes\mathbb{C}[P]$ such that

$$L(\Lambda_0)\xrightarrow{[\omega]}L(\Lambda_\ell)\xrightarrow{[\omega]}L(\Lambda_{\ell-1})\xrightarrow{[\omega]}\dots\xrightarrow{[\omega]}L(\Lambda_1)\xrightarrow{[\omega]}L(\Lambda_0)$$

$$[\omega]v_{\Lambda_0}=v_{\Lambda_\ell},\quad [\omega]v_{\Lambda_i}=x_{\gamma_i}(-1)v_{\Lambda_{i-1}},\quad i=1,\dots,\ell, \tag{4.6}$$

together with important property

$$x(\pi)[\omega]=[\omega]x(\pi^+), \tag{4.7}$$

with $x(\pi^+)$ standing for monomial obtained from $x(\pi)$ by raising degrees of all its factors by one.

Considering higher level k standard $\tilde{\mathfrak{g}}$-modules, we will use the fact that for $\Lambda=k_0\Lambda_0+\dots+k_\ell\Lambda_\ell$ such that $k_0+\dots+k_\ell=k$, module $L(\Lambda)$ is embedded in the appropriate k-fold tensor product of fundamental modules with the highest weight vector

$$v_\Lambda=v_{\Lambda_\ell}^{\otimes k_\ell}\otimes\dots\otimes v_{\Lambda_1}^{\otimes k_1}\otimes v_{\Lambda_0}^{\otimes k_0}.$$

For $i=1,\dots,\ell$ and $j=0,\dots,\ell$ denote by $[i]_j=1^{\otimes(k-j)}\otimes[i]\otimes 1^{\otimes(j-1)}$ linear maps between level k standard $\tilde{\mathfrak{g}}$-modules, keeping properties (4.4) and (4.5) of $[i]$. On k-fold tensor products of standard $\tilde{\mathfrak{g}}$-modules we use also $[\omega]^{\otimes k}$, with formulas analogous to (4.6) and (4.7).

We are now ready to state the result on exactness. Fix $K=(k_0,\dots,k_\ell)$ such that $k_0+\dots+k_\ell=k$, $k_i\in\mathbb{Z}_+$, $i=0,\dots,\ell$. Denote $W=W_{k_0,k_1,\dots,k_\ell}=W(\Lambda)$ for $\Lambda=k_0\Lambda_0+\dots+k_\ell\Lambda_\ell$, and by v highest weight vector of $L(\Lambda)$. Define also $m=\sharp\{i=0,\dots,\ell-1\mid k_i\neq 0\}$ and for $t=0,\dots,m-1$ set

$$I_t=\{\{i_0,\dots,i_{t-1}\}|0\le i_0\le\dots\le i_{t-1}\le\ell-1,k_{i_j}\neq 0,j=0,\dots,t-1\}. \tag{4.8}$$

Now, denote

$$W_{I_t}=W_{k_0,\dots,k_{i_0}-1,k_{i_0+1}+1,\dots,k_{i_{t-1}}-1,k_{i_{t-1}+1}+1,\dots,k_\ell},$$

and by v_{I_t} the corresponding highest weight vector.

Define $U(\tilde{\mathfrak{g}}_1)-$homogeneous mappings $\varphi_t:\sum_{I_t}W_{I_t}\to\sum_{I_{t+1}}W_{I_{t+1}}$ by

$$\varphi_t|_{W_{I_t}}=\sum_{\substack{i,k_i\neq 0\\ i\notin I_t}}(-1)^{\sharp\{j\in I_t|j<i\}}[i]_{k_0+\dots+k_{i-1}},$$

meaning its action on corresponding highest weight vectors is given by

$$\varphi_t(v_{I_t}) = \sum_{\substack{i,k_i\neq 0\\ i\notin I_t}} (-1)^{\sharp\{j\in I_t | j<i\}} v_{I_t\cup\{i\}}.$$

THEOREM 4.1. *The following sequence is exact:*

$$0 \to W_{k_\ell,k_0,k_1,\dots,k_{\ell-1}} \xrightarrow{[\omega]^{\otimes k}} W \xrightarrow{\varphi_0} \sum_{I_1} W_{I_1} \xrightarrow{\varphi_1} \dots \xrightarrow{\varphi_{m-1}} W_{I_m} \to 0.$$

EXAMPLE 4.2. For Feigin-Stoyanovsky's type subspaces of level 2 standard $\widetilde{\mathfrak{sl}(3,\mathbb{C})}$-modules we have the following family of exact sequences:

$$\begin{aligned}
&0 \to W_{0,2,0} \to W_{2,0,0} \to W_{1,1,0} \to 0\\
&0 \to W_{0,1,1} \to W_{1,1,0} \to W_{0,2,0} \oplus W_{1,0,1} \to W_{0,1,1} \to 0\\
&0 \to W_{1,1,0} \to W_{1,0,1} \to W_{0,1,1} \to 0\\
&0 \to W_{0,0,2} \to W_{0,2,0} \to W_{0,1,1} \to 0\\
&0 \to W_{1,0,1} \to W_{0,1,1} \to W_{0,0,2} \to 0\\
&0 \to W_{2,0,0} \to W_{0,0,2} \to 0
\end{aligned}$$

The proof of Theorem 4.1 relies on the interplay between initial conditions (3.3) for various dominant integral weights at the fixed level k (note that difference conditions (3.2) are the same for all Feigin-Stoyanovsky's type subspaces at the same integer level), and on the use of properties (4.4) - (4.7), cf. [**14**] for details.

We proceed by defining formal character of $W = W(\Lambda)$.

DEFINITION 4.3. For $x(\pi) = \dots x_{\gamma_1}(-2)^{a_\ell} x_{\gamma_\ell}(-1)^{a_{\ell-1}} \cdots x_{\gamma_1}(-1)^{a_0}$ define *degree* $d(x(\pi)) = \sum_{j=0}^{\infty}\sum_{i=1}^{\ell}(j+1)a_{i+j\ell-1}$ and *weight* $w(x(\pi)) = \sum_{j=0}^{\infty}\sum_{i=1}^{\ell}\gamma_i a_{i+j\ell-1}$. *Formal character of* $W = W(\Lambda)$ is given by

$$\chi(W)(z_1,\dots,z_\ell;q) = \sum \dim W^{m,n_1,\dots,n_\ell} q^m z_1^{n_1}\cdots z_\ell^{n_\ell},$$

with $W^{m,n_1,\dots,n_\ell}$ denoting the component of W spanned by basis monomial vectors $x(\pi)v$ of degree m and weight $n_1\gamma_1 + \dots + n_\ell\gamma_\ell$.

As a direct consequence of Theorem 4.1 we were able to obtain systems of relations connecting characters of *all* Feigin-Stoyanovsky's type subspaces of arbitrary integer level k standard $\widetilde{\mathfrak{sl}(\ell+1,\mathbb{C})}$-modules:

$$\begin{aligned}
\sum_{I\in D(K)} (-1)^{|I|}\chi(W_I)(z_1,\dots,z_\ell;q) &=\\
= (z_1 q)^{k_0}\dots(z_\ell q)^{k_{\ell-1}}\chi(W_{k_\ell,k_0,\dots,k_{\ell-1}})&(z_1 q,\dots,z_\ell q;q),
\end{aligned}\tag{4.9}$$

where $D(K)$ denotes the set of all I_{t+1} as defined in (4.8).

EXAMPLE 4.4. For Feigin-Stoyanovsky's type subspaces of level 2 standard $\tilde{\mathfrak{sl}}(3,\mathbb{C})$-modules we have the following system of relations:

$$\begin{aligned}
\chi(W_{2,0,0})(z_1,z_2;q) &= \chi(W_{1,1,0})(z_1,z_2;q) + (z_1q)^2\chi(W_{0,2,0})(z_1q,z_2q;q)\\
\chi(W_{1,1,0})(z_1,z_2;q) &= \chi(W_{0,2,0})(z_1,z_2;q) + \chi(W_{1,0,1})(z_1,z_2;q) -\\
&\qquad - \chi(W_{0,1,1})(z_1,z_2;q) + (z_1q)(z_2q)\chi(W_{0,1,1})(z_1q,z_2q;q)\\
\chi(W_{1,0,1})(z_1,z_2;q) &= \chi(W_{0,1,1})(z_1,z_2;q) + z_1q\chi(W_{1,1,0})(z_1q,z_2q;q)\\
\chi(W_{0,2,0})(z_1,z_2;q) &= \chi(W_{0,1,1})(z_1,z_2;q) + (z_2q)^2\chi(W_{0,0,2})(z_1q,z_2q;q)\\
\chi(W_{0,1,1})(z_1,z_2;q) &= \chi(W_{0,0,2})(z_1,z_2;q) + z_2q\chi(W_{1,0,1})(z_1q,z_2q;q)\\
\chi(W_{0,0,2})(z_1,z_2;q) &= \chi(W_{2,0,0})(z_1q,z_2q;q).
\end{aligned}$$

If we now write

$$\chi(W_{k_0,\dots,k_\ell})(z_1,\dots,z_\ell;q) = \sum_{n_1,\dots,n_\ell\geq 0} A^{n_1,\dots,n_\ell}_{k_0,\dots,k_\ell}(q) z_1^{n_1}\cdots z_\ell^{n_\ell}, \tag{4.10}$$

where $A^{n_1,\dots,n_\ell}_{k_0,\dots,k_\ell}(q)$ are formal series in formal variable q, system (4.9) reads

$$\sum_{I\in D(K)} (-1)^{|I|} A_I^{n_1,\dots,n_\ell}(q) = q^{n_1+\cdots+n_\ell} A^{n_1-k_0,\dots,n_\ell-k_{\ell-1}}_{k_\ell,k_0,\dots,k_{\ell-1}}(q). \tag{4.11}$$

It is not hard to prove that the system (4.11) consists of relations that are *recursive* and that it has a *unique* solution (not obvious by itself), cf. Propositions 6.2 and 6.3 in [**14**].

We solved (4.11) (and consequently (4.9)) in case of general ℓ and $k=1$. In other words, we obtained character formulas in form of (4.10) for Feigin-Stoyanovsky's type subspaces $W(\Lambda_i)$ of fundamental $\tilde{\mathfrak{sl}}(\ell+1,\mathbb{C})$-modules, $i=0,\dots,\ell$:

$$\chi(W(\Lambda_i))(z_1,\dots,z_\ell;q) = \sum_{n_1,\dots,n_\ell\geq 0} \frac{q^{n_1^2+\cdots+n_\ell^2+n_1n_2+\cdots+n_{\ell-1}n_\ell+n_1+\cdots+n_i}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_\ell}} z_1^{n_1}\cdots z_\ell^{n_\ell}.$$

We were also able to prove the following result:

THEOREM 4.5. *Let $\Lambda = k_0\Lambda_0 + k_1\Lambda_1 + k_2\Lambda_2$ be the highest weight of the level k standard $\tilde{\mathfrak{sl}}(3,\mathbb{C})$-module $L(\Lambda)$. For the character of Feigin-Stoyanovsky's type subspace $W(k_0\Lambda_0 + k_1\Lambda_1 + k_2\Lambda_2)$ the following formula holds:*

$$\chi(W(k_0\Lambda_0 + k_1\Lambda_1 + k_2\Lambda_2))(z_1,z_2;q) =$$

$$= \sum_{n_1,n_2\geq 0} \sum_{\substack{\sum_{i=1}^k N_{1,i}=n_1\\ N_{1,1}\geq\cdots\geq N_{1,k}\geq 0\\ \sum_{i=1}^k N_{2,i}=n_2\\ N_{2,k}\geq\cdots\geq N_{2,1}\geq 0}} \frac{q^{\sum_{i=1}^k (N_{1,i}^2+N_{2,i}^2+N_{1,i}N_{2,i})}\cdot L^{N_{1,1},\dots,N_{1,k},N_{2,1},\dots,N_{2,k}}_{k_0,k_1,k_2}(q)}{(q)_{N_{1,1}-N_{1,2}}\cdots(q)_{N_{1,k}}(q)_{N_{2,k}-N_{2,k-1}}\cdots(q)_{N_{2,1}}} z_1^{n_1}z_2^{n_2},$$

where $L^{N_{1,1},\dots,N_{1,k},N_{2,1},\dots,N_{2,k}}_{k_0,k_1,k_2}(q)$ represents the "linear" term of the nominator:

$$L^{N_{1,1},\dots,N_{1,k},N_{2,1},\dots,N_{2,k}}_{k_0,k_1,k_2}(q) =$$

$$= \sum_{p\in P_{k_1+k_2}} q^{p_1N_{1,1}+\cdots+p_kN_{1,k}+p_1'N_{2,1}+\cdots+p_k'N_{2,k}} \prod_{i=1}^k (1-\delta_{p_i-p_{i+1},-1}q^{N_{1,i}-N_{1,i+1}}),$$

with $P_{k_1+k_2} = \{(p_1,\ldots,p_k) \in \{0,1\}^k \,|\, \sum_{i=1}^k p_i = k_1+k_2\}$ *and* $N_{1,k+1}, p_{k+1}$ *set to be zero. Also,* p' *denotes a* k*-tuple obtained from* p *by changing all except the first* k_2 1*'s to zeros.*

One should mention that formulas in Theorem 4.5 represent a complete set of full characters for Feigin-Stoyanovsky's type subspaces of all standard $\tilde{\mathfrak{sl}}(3,\mathbb{C})$-modules, which specially reinstalls the result (4.1). Theorem is proven by directly checking that formulas for $\chi(W(k_0\Lambda_0 + k_1\Lambda_1 + k_2\Lambda_2))(z_1,z_2;q)$ satisfy the corresponding system (4.9); more precisely, by checking that matching $A^{n_1,n_2}_{k_0,k_1,k_2}(q)$ satisfy (4.11). This is done in a somewhat similar fashion as in Andrews' method for solving Rogers-Selberg recursions (cf. [**1, 15**]).

Example 4.6. Let us present "linear" terms in the nominators of character formulas for Feigin-Stoyanovsky's type subspaces of level 2 standard $\tilde{\mathfrak{sl}}(3,\mathbb{C})$-modules:

$$L^{N_{1,1},N_{1,2},N_{2,1},N_{2,2}}_{2,0,0}(q) = 1$$
$$L^{N_{1,1},N_{1,2},N_{2,1},N_{2,2}}_{1,1,0}(q) = q^{N_{1,2}}$$
$$L^{N_{1,1},N_{1,2},N_{2,1},N_{2,2}}_{1,0,1}(q) = q^{N_{1,1}+N_{2,1}} + q^{N_{1,2}+N_{2,2}}(1-q^{N_{1,1}-N_{1,2}})$$
$$L^{N_{1,1},N_{1,2},N_{2,1},N_{2,2}}_{0,2,0}(q) = q^{N_{1,1}+N_{1,2}}$$
$$L^{N_{1,1},N_{1,2},N_{2,1},N_{2,2}}_{0,1,1}(q) = q^{N_{1,1}+N_{1,2}+N_{2,1}}$$
$$L^{N_{1,1},N_{1,2},N_{2,1},N_{2,2}}_{0,0,2}(q) = q^{N_{1,1}+N_{1,2}+N_{2,1}+N_{2,2}}.$$

5. Further directions

There are further possible directions for the programme that uses intertwining operators and combinatorial basis of Feigin-Stoyanovsky's type subspaces for computing exact sequences of those subspaces and solving systems of recurrences for their characters. Namely, one could solve system (4.11) for some higher choices of ℓ and/or k. Furthermore, since all the "ingredients" are already here, it would be interesting to try to pursue these ideas in the context of [**24, 25**] or [**3**], although this seems to be substantially more challenging task then in [**14, 15**].

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