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Logics and Structures

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Abstract:
In this talk I analyse the structuralist theory of logic and the possible consequence of different logics sharing the same structure.
I point out certain aspects of such theory that might be objected to as well as the reasons as to why such a theory does not after all fulfil the (possibly unjustified) expectation of getting defined a universal logical structure.

Key words:
Structuralism, logic, universal structure

Even though structuralism is commonly associated with issues concerning the philosophy of mathematics, there is an account of it endorsed by Koslow (mainly in his book: Koslow, 1992, *A Structuralist Theory of Logic*, Cambridge University Press) which is one of the most appealing contemporary formulations of structuralism in *logic*.

But, what does structuralism in logic amount to? Is it analogous with structuralism in mathematics? And if so, in which sense: do the two versions share the same tenets or the same motivation?

I will try in this talk to answer these questions by presenting the basic ideas of Koslow's theory. I will critically analyse his views and offer reasons for holding that some aspects of his structuralist account are flawed, as it is the case with structuralism in mathematics so that matter. I will also try to show that, even though that is not one of the aims that Koslow points out to, his theory of logic fails to achieve a satisfactory answer to the question of a possible reduction of logic to structure(s).

As far as structuralism in mathematics is concerned, the statement that mathematics is about structures is twofold: mathematics is about different structures such as the vector space structure, the group structure etc., while the possibility of reducing mathematical theories to set theory or category theory, gives sense to viewing mathematics as about the (common) set-theoretic structure or the category structure. Philosophically speaking, the ontological reduction of mathematical objects to structures leads to interesting results that aim to solve some ontological, as well as epistemological, problems in the philosophy of mathematics (let me mention Resnik and Shapiro), even though it also brings to the surface some difficulties such as the problem of existing structures admitting non-trivial automorphisms.

What about logic? In which sense could logical structuralism be analogous with the mathematical one?

Could there be a (coomon) universal logical structure? As is well known, different logics are based on different principles. Examples are legion: the necessary relevant connections between the premises and the conclusion in any argument given in relevance logic are not necessarily fulfilled in classical logic.

Given the obvious disanalogies between mathematics and logic, I will try to answer the question about the possibility of existence of a common logical structure and, consequently, of a universal logic; I will do that through a discussion of the tenets of the structuralist account of logic.

The lynch-pin of Koslow's structuralist account of logic is the notion of *implication structure* and the definition of logical (and modal) operators relative to an implication structure. Let us see what these definitions amount to and what results they imply.

An implication structure is any order pair: ((S, $\Rightarrow$); where S is a non-empty set, while "$\Rightarrow$" is an implication relation.

An implication relation is (implicitly) defined as any relation that satisfies the following conditions:

(1) *Reflexivity*: $A \Rightarrow A$, for each A in S

(2) *Projection*: $A_1, A_2, \ldots, A_n \Rightarrow A_k$, for every k=1, …, n, and for each A_i in S (i = 1, …,n)

(3) *Simplification*: If $A_1, A_1, A_2, \ldots, A_n \Rightarrow B$, then $A_1, A_2, \ldots, A_n \Rightarrow B$, for all A_i (i = 1, …,n) and B in S

(4) *Permutation*: If $A_1, A_2, \ldots, A_n \Rightarrow B$, then $A_{f(1)}, A_{f(2)}, \ldots, A_{f(n)} \Rightarrow B$, for any

permutation f of $\{1, \ldots, n\}$

(5) *Dilution* (or *Thinning*): If $A_1, A_2, \ldots, A_n \Rightarrow B$, then $A_1, A_2, \ldots, A_n, C \Rightarrow B$, for

any A_i ($i = 1, \ldots, n$), B and C in S

(6) *Cut*: If $A_1, A_2, \ldots, A_n \Rightarrow B$, and $B, B_1, B_2, \ldots, B_m \Rightarrow C$, then $A_1, A_2, \ldots, A_n, B_1, B_2, \ldots, B_m \Rightarrow C$, for any A_i, B_j, B and C ($i, j = 1, \ldots, n$)

Someone might object that a different choice of constraints would be more fruitful and economical since clearly *Reflexivity* follows from *Projection*, and *Dilution* follows from *Projection* and *Cut*. Nevertheless, Koslow keeps the list of constraints for the sake of greater articulateness, based on Gentzen's theory.

The examples of implication relations that immediately come to mind are the notion of semantic validity or the syntactic notion of deducibility for a set of sentences of some first-order logical theory. These examples though do not even exhaust all the possibilities.

> The logical operators can act in a broad variety of settings, sentential and otherwise. In particular, the actions of the operators on structures of sets, names, and interrogatives, to cite just some nonstandard examples, are mentioned because the items in these cases fail in an obvious way to be syntactical or fail to be truth-bearers. (Koslow 1992, p.9)

An interesting example is set inclusion: given any set of subsets of a non-empty set S, the ordered pair (S, $\subseteq$) exemplifies the implication structure.

I find particularly interesting the example of erotetic logic, that will be mentioned later on.

The possibility of getting such generality of definition is certainly a great virtue of Koslow's theory and it makes structuralism in logic closer to the one in mathematics - just remember that is possible to get some rather weird group structure examples or mathematically unusual equivalence relations.

Given the definition of implication structures, the logical operators are defined *relative* to such structures, i.e. as functions defined on structures. And here again, given the possibility of non-standard implication relations, the same applies to the operators as well.

Let us take the example of the hypothetical operator $H_{\Rightarrow}$. For any elements A and B in the implication structure (S, $\Rightarrow$), $H_{\Rightarrow}(A, B)$ is the hypothetical having A as the antecedent and B as a consequent, if and only if the following conditions are fulfilled:

(H1) $A, H_{\Rightarrow}(A, B) \Rightarrow B$

(H2) $H_{\Rightarrow}(A, B)$ is the weakest element satisfying the condition (1). It means that, for any element T of the implication structure such that $A, T \Rightarrow B$, it follows that $T \Rightarrow H_{\Rightarrow}(A, B)$

Such a definition leaves open the answer to the question as to whether the hypothetical, given an implication structure, may fail to exist or not. And the following example solves the dilemma positively.

Let us take the implication structure (S, $\Rightarrow$), in which $S=\{A, B, C, D\}$ and the implication relation is given in the following way:

$$\begin{tikzcd} & A \arrow[d, Rightarrow] & \\ & B \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\ C & & D \end{tikzcd}$$

In such a structure, the hypothetical $H_{\Rightarrow}(A, B)$ does not exist (not to be confused with the fact that $A \Rightarrow B$); namely, $H_{\Rightarrow}(A, B)$ is, by definition, the weakest member T of S such that: $A, T \Rightarrow B$. Since $A \Rightarrow B$, the condition is fulfilled by any element of S, but there is no weakest element. C cannot be the weakest element since $A, D \Rightarrow B$, while $D \not\Rightarrow C$ (see the condition (H2) above). D cannot be the weakest for the same reason.

Such a definition does not put any constraints on truth conditions, syntactic features or others:

> There is no appeal to truth conditions, assertibility conditions, or any syntactical features or semantic values of the elements of the structure. (Koslow 1992, p. 78)

The fact that the elements of an implication structure are not necessarily syntactic objects having a special sign design or elements having a semantic value, is what makes the explanation/definition of the logical operators free of such constraints.

Let me present now some critical remarks concerning the structuralism theory of logic just depicted.

(1) Let us have a look at the six conditions - *Cut* - that any relation has to fulfill in order to be an implication relation. The main worry I see is about how is the left-hand side of the expression $A_1, \dots, A_n \Rightarrow B$ to be construed.

If S is a non-empty set of sets and the implication relation is set inclusion, the sequence $A_1, \dots, A_n$ is the intersection of sets. Since the intersection of sets *is* their conjunction, it turns out that in order to interpret the sequence $A_1, \dots, A_n$, i.e. in order to determine that the implication relation is set inclusion, we ought to know what the intersection, i.e. the conjunction of sets is. We therefore ought to know how a certain logical operator is defined prior to having determined an implication relation on a non-empty set. According to Koslow's theory, it certainly should be the other way round.

(2) Let me present now a critical remark concerning the structuralist account in erotetic logic, i.e. the logic of interrogatives.

How are operators for interrogatives to be defined?

Q - a set of interrogatives

S - a set of sentences inclusive of the sentential direct answers to the questions in Q

$S' = S \cup Q$.

Interestingly enough, according to Koslow a direct answer need not be a true one:

> We shall use the term "interrogative" to include any question that has a *direct answer*. The most important feature of the direct answers to a question is that they are statements that, whether they are true or false, tell the questioner exactly what he wants to know – neither more nor less. (Koslow 1992, p.220)

The implication relation on S' ('$\Rightarrow_q$') is defined as follows:

$M_1?, M_2?, \dots, M_n?$ and $R?$ - questions in Q,

$F_1, F_2, \dots, F_m$ and G - statements of S (the set of M's or the set of F's may be empty but not both), and

A_i - a direct answer to the question $M_i?$ (i=1,…,n).

We then define:

(1.) $F_1, F_2, \dots, F_m, M_1?, M_2?, \dots, M_n? \Rightarrow_q R?$ if and only if there is some direct answer B to the question $R?$ such that

$F_1, F_2, \dots, F_m, A_1, A_2, \dots, A_n \Rightarrow B$

(2.) $F_1, F_2, \dots, F_m, M_1?, M_2?, \dots, M_n? \Rightarrow_q G$ if and only if

$F_1, F_2, \ldots, F_m, A_1, A_2, \ldots, A_n \Rightarrow G$

I find such a definition problematic. Namely, let the set of *F*'s be empty (for the sake of simplicity), and let us examine the case in which a question implies a statement (the second condition in the definition). Let the statement *G* be any false statement, e.g. a false answer to the question *R?*. In this case, whether $M_1? \Rightarrow_q G$ or not depends on whether $A_1 \Rightarrow G$, and the latter depends on what answer A_1 (to the question $M_1?$) we choose. If the answer we choose is a false one, then $M_1? \Rightarrow_q G$, otherwise $M_1? \not\Rightarrow_q G$. More generally, the same problem appears whenever the statement *G* is false. In this case, given a collection of interrogatives $M_i?$ (i=1,…n), their respective direct answers A_i, and a set of true statements F_i (i=1,…n), there is nothing in Koslow's definition that allows us to uniquely determine whether $F_1, F_2, \ldots, F_m, M_1?, M_2?, \ldots, M_n? \Rightarrow_q G$ or not.

(3) Let me finally present a more general difficulty with the structuralist account of logic. Koslow shows that, even though we might expect that certain results that hold given the operators classically defined hold in non-standard cases too, it is not the case.

The cases of implication structures in which $(((A\rightarrow B)\rightarrow A)\rightarrow A)$ is not a theorem show us that much. Since these are features of the system, not of the structure**,** so it is not odd that such results are not necessarily present in non-standard implication structures. Nevertheless, to have a true hypothetical whenever the antecedent is false is not an unimportant result given the operators classically defined.

So, two problems seem to arise at this point: how do we get from the semantic-and-synthatic-features-free definitions to the syntactic rules for formula formation or the (semantic) truth tables? From prevent us from doing it is the fact that they do not follow from the **structurally** defined operators.

And how can a system have so many basic features that are not, in some form or another, already present in the structure? In we think of the mathematical case, all the basic properties of, e.g. vectors (whatever system that we take – geometrical vectors, real numbers, etc.) are already present in the structure. And that is basically what makes the study of structures so fruitful in mathematics.

In the presented theory, on the other hand, the characterization of the operators relative to the implication structure obtain neither the semantic nor the synthactic results expected in the defined implication structures.

That is why I think that the depicted structuralist account of logic fails, notwithstanding its being original in many philosophical and logical aspects, to successfully solve the task of characterizing the logical operators relative to implication structures. For all the presented critical notes make us skeptical as to whether a general logical structure might be determined.