Geach's problem and a variety of consequence relation

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Pretheoretical notions on consequence relation

- There are implicit (pretheoretical, presystematic, intuitive) notions on relations of logical consequence as is exhibited in the use of adverb 'therefore'.
- The pretheoretical notions may be founded on:
  - understanding of logical terms,
  - recognition of the properties of a consequence relation, and
  - understanding of logical terms on the background of recognition of a type of consequence relation or on the background of recognition of a logical property (e.g. consistency),
  - ...
- The hypothesis on complex character notions (last case above) could explain the non-uniformity of behavior of logical terms in different contexts (e.g. within diverse sentence moods)..

An example

Example

Prior's connective

1	$\Gamma \vdash p \Rightarrow \Gamma \vdash p \texttt{ tonk } q$	$\texttt{tonk}$Intro
2	$\Gamma \vdash p \texttt{ tonk } q \Rightarrow \Gamma \vdash q$	$\texttt{tonk}$Elim
3	$\Gamma \vdash p \Rightarrow \Gamma \vdash q$	$\models$transitivit:1,2

- Connective `tonk` seems illegitimate on the background of transitive consequence relations. But its illegitimacy disappears in a non-transitive setting.

Cook's consequence: truth preservation or falsehood suppression

Roy T. Cook. What's wrong with tonk (?). *Journal of Philosophical Logic* (2005) **34**: 217–226

- Valuation $v : \mathcal{L} \to \wp\{\mathrm{t}, \mathrm{f}\}$.
- $\Gamma \models q$ iff (i) $\mathrm{t} \in v(q)$ whenever $\mathrm{t} \in v(p)$ for all $p \in \Gamma$, **or** (ii) $\mathrm{f} \notin v(q)$ whenever $\mathrm{f} \notin v(p)$ for all $p \in \Gamma$,
- Truth table for tonk (where T stands for $\{\mathrm{t}\}$, B for $\{\mathrm{t}, \mathrm{f}\}$, N for $\emptyset$, F for $\{\mathrm{f}\}$):

tonk	T	B	N	F
T	T	B	T	B
B	T	B	T	B
N	N	F	N	F
F	N	F	N	F

- The consequence relation holds for $\{p\} \models p$ tonk q in virtue of truth membership preservation and it holds for $\{p$ tonk $q\} \models q$ in virtue of falsity non-membership preservation.

Isolated treatment of connectives

Quote

Logicians who abandon transitivity, however, will need to find some other criteria by which to reject *Tonk-Logic* as illegitimate, at least if they wish to vindicate the intuition that the 'badness' of tonk traces to some violation of general requirements on legitimate logical operators, and is not specific to particular logical systems.
Cook, 223

- Cook's result gives support to the claim that operators can not be dealt with in isolation from the background notion of a consequence relation.
- I think that his results support the hypothesis that pre-theoretical notions (on logical relations and properties) are complexes of interdependent understandings (dealing with logical relations and logical terms).

A paradoxical imperative inference

1 Slip the letter into the letter-box!
2 Slip the letter into the letter-box or burn it!
3 You may: slip the letter into the letter-box or burn it.
4 You may: burn the letter.
5 Therefore, if you ought to slip the letter into the letter box, then you may burn it.

	(Purportedly) holds in virtue of	*Intuitive acceptability*
1		
2	meaning of 'or'; from 1	ambivalent
3	relations between 'must' and 'may'; from 2	affirmative
4	distributivity of "free choice permission"; from 3	mainly affirmative
5		negative

- Unexpected behavior of 'or' in 2 and 4.

Paradoxical inference again: a deontic variant

0	$p \models p \vee q$	meaning of $\vee$
1	$\mathrm{O}p \models \mathrm{O}(p \vee q)$	Scott's principle
2	$\mathrm{O}(p \vee q) \models \mathrm{P}(p \vee q)$	D axiom
3	$\mathrm{O}p \models \mathrm{P}(p \vee q)$	by $\models$ transitivity; from 1, 2
4	$\mathrm{P}(p \vee q) \models \mathrm{P}q$	by free choice permission
5	$\mathrm{O}p \models \mathrm{P}q$	by $\models$ transitivity; from 3, 4

- The consequence relation 1, which is intuitively less plausible than 4, holds in normal deontic logic while 4 does not hold.
- Scott's principle

$$\{(p_1 \wedge \ldots \wedge p_{n-1}) \to q\} \vdash (\Box p_1 \wedge \ldots \wedge \Box p_{n-1}) \to \Box q$$

($n \geq 1$) characterizes normal propositional modal logic (e.g. it may replace K axiom and neccessitation rule). It may be read as stating that "meaning relations" of propositional logic, i.e. meaning relations holding in virtue of meaning of truth-functional connectives, are preserved in the modal context.

Avoiding the paradox

- The `tonk` example shows that syntactically defined "logical" terms have different properties given the diverse types of consequence.
- (Alf) Ross' paradox and free choice permission show that logical terms may "change their behavior" in the presence of other logical terms.
  - The odd result that if anything is obligatory than everything is permitted (i.e. $\mathrm{O}p \Rightarrow \mathrm{P}q$) shows that one may have intuitions that confirm isolated consequence steps and still lack the intuition that confirms transitive closure of these steps.
- The pretheoretical understanding of logical relations may well be holistic in character: perhaps there is no unique understanding of logical terms that is *constitutive* for the understanding of consequence relations, and perhaps there is no unique understanding of admissible consequence relations that is *regulative* for the understanding of logical terms.
- In practical logic the phenomenon of unclear intuitions are noticeable. Both on the formal and on informal side the results and intuitions collide on the issues of existence of consequence relation for particular schemata and on the nature of consequence relation.

Classical consequence as a special case

In dynamic semantics the notion of consequence is generalized to the notion of processes.
Some notions "statically" reformulated:

- $p_0; \ldots; p_n \models_{test-to-test} q$ iff for all contexts σ:

$$\sigma[p_1] = \ldots = \sigma[p_n] = \sigma \rightarrow \sigma[q] = \sigma$$

- $p_0; \ldots; p_n \models_{update-to-test} q$ iff for all contexts σ:

$$\sigma[p_1]\ldots[p_n] = \sigma[p_1]\ldots[p_n][q]$$

- $p_0; \ldots; p_n \models_{ignorant-update-to-test} q$ iff for the empty context (carrying no information) 0:

$$0[p_1]\ldots[p_n] = 0[p_1]\ldots[p_n][q]$$

- We skip: update-to-update, and test-to-update variants.

Benthem, J. F. A. K. van [1996] *Exploring Logical Dynamics*, Stanford, Center for the Study of Language and Information

Information containment conception of logical consequence

Rudolf Carnap and Yehoshua Bar-Hillel. *An Outline of a Theory of Semantic Information*. Technical Report no. 247. Research Laboratory of Electronics, Massachusetts Institute of Technology, 1952.

Quote

Whenever i **L**-implies j, i asserts all that is asserted by j, and possibly more. In other words, the information carried by i includes the information carried by j as a (perhaps improper) part. Using '*In*(...)' as an abbreviation for the presystematic concept 'the information carried by . . . ', we can now state the requirement in the following way:
R3-1. $In(i)$ includes $In(j)$ iff i **L**-implies j.
By this requirement we have committed ourselves to treat information as a set or class of something. This stands in good agreement with common ways of expression,as for example, "The information supplied by this statement is more inclusive than (or is identical with, or overlaps) that supplied by the other statement."

Information containment

Jose M. Saguillo. Logical Consequence Revisited. *The Bulletin of Symbolic Logic* (1997) 3: 216-241

Quote

The information containment conception: P implies c if and only if the information of c is contained in the information of P. In this sense, if P implies c, then it would be redundant to assert c in a context where the propositions in P have already been asserted; i.e., no information would be added by asserting c.

Adding information

- Two notions "adding information" and "information as a set or class of something" show
- that sentences can do something, namely they can "add information", and
- that semantic relations occur at the level of sets, since "information [is] a set or class of something".
- Putting these two together we get that sentences act on sets.
- Two notions of "information containment" are relevant:
  1. Conclusion adds no information to **any** context that includes all information contained in premises.
  2. Conclusion adds no information to the context that includes **only** the information contained in premises.[This notion coresponds to "ignorant-update-to test" and a variant of it will be introduced later as *prima facie* consequence.]

Adding and testing

- Relative consistency testing (can an information contained in φ be added in a context σ without causing informational breakdown)
  - Acceptability testing:

$$\sigma\left[?^{consistency}\varphi\right]=\begin{cases}\sigma \text{ if } \sigma[\varphi]\neq\emptyset,\\ \emptyset \text{ otherwise.}\end{cases}$$

- Relative validity testing (will the context σ be changed by adding information contained in φ)
  - Acceptance testing:

$$\sigma\left[?^{validity}\varphi\right]=\begin{cases}\sigma \text{ if } \sigma[\varphi]=\sigma,\\ \emptyset \text{ otherwise.}\end{cases}$$

'Therefore': sentence operator or metalinguistic predicate?

- If one thinks about the semantics as something to do with the actions performed on "sets of something", then one is not obliged to treat natural language expressions 'therefore' and 'might' as a metalinguistic predicate.

Example

Denote by $\mathcal{L}_0$ the language in which some logical constants occur. Then we need a meta language $\mathcal{L}_1$ to state that a sentence $p \in \mathcal{L}_0$ is a consequence of a set of sentences $\Gamma \subseteq \mathcal{L}_0$ since operator 'therefore' does not belong to the language $\mathcal{L}_0$.

- It may seem odd that by saying 'p therefore q'
  - either (i) the speaker mentions sentences p and q (using their names) but does not use them,
  - or (ii) the speaker simultaneously uses and mentions p and q since she is asserting p and q (using the sentences) as well as (mentioning them while) asserting the existence of consequence relation between 'p' and 'q'.

Lowering

- One may choose whether to treat `might` and `therefore` as logical operators or as metalogical predicates.

metalogical predicate	*logical operator*
$\mathtt{therefore}(\Gamma, p)$ i.e. $\Gamma \models p$	$\mathtt{therefore}\ p \in \mathcal{L}_0$
$\mathtt{might}(\Gamma, p)$ i.e. $\Gamma \cup \{p\} \not\models \bot$	$\mathtt{might}\ p \in \mathcal{L}_0$

- The "logical operator option" is taken when we interpret some natural language sentences as "test functions" both for stating relative consistency and for stating relative validity (context validity), i.e

$$sentence\_function(context) = \begin{cases} context \text{ if the condition is met,} \\ failure \text{ otherwise.} \end{cases}$$

  - Advantages:
    - Adverb 'therefore' is treated unambiguously (instead of signifying different relations in the contexts with different logics).
    - Gain in sensitivity to different phenomena of "information containment".
    - The drawback is that correctness of the use of 'therefore' need not imply existence of a consequence relation.
  - The advantage of might-operator is that the statements on consistency become part of the object language.

Geach's description

Peter Geach. Dr. Kenny on practical inference. *Analysis* (1966) **26**: 76–79

Quote

Some years ago I read a letter in a political weekly to some such effect as this. 'I do not dispute Col. Bogey's premises, nor the logic of his inference. But even if a conclusion is validly drawn from acceptable premises, we are not obliged to accept it if those premises are incomplete; and unfortunately there is a vital premise missing from the Colonel's argument-the existence of Communist China.' I do not know what Col. Bogey's original argument had been; whether this criticism of it could be apt depends on whether it was a piece of indicative or of practical reasoning. Indicative reasoning from a set of premises, if valid, could of course not be invalidated because there is a premise "missing" from the set. But a piece of practical reasoning from a set of premises can be invalidated thus: your opponent produces a fiat you have to accept, and the addition of this to the fiats you have already accepted yields a combination with which your conclusion is inconsistent.

Defeasibility of conclusion and completeness of premises

The consequence relation described by Geach has two notable properties:

- ("locality") conclusion holds in virtue of premises but it can be defeated by additional premises;
- (existence of the limit) if the premises are complete the conclusion cannot be defeated (where 'conclusion is defeated' means 'premises are acceptable and conclusion is not acceptable').
- By 'Geach's problem' I mean a problem of devising modeltheoretic notion of consequence relation that captures the pretheoretical notions of conclusion defeasibility and of "completeness of premises".

Tarskian consequence relation

Properties of Tarskian consequence relation

(Reflexivity) $$\frac{}{\Gamma \models p, \text{ for all } p \in \Gamma}$$

(Monotony) $$\frac{\Gamma \models p}{\Gamma, \Delta \models p}$$

(Transitivity) $$\frac{\Gamma \models p, \text{ for all } p \in \Delta \qquad \Delta \models q}{\Gamma, \Delta \models q}$$

- Pretheoretical notion given in Geach's quote[1] is a notion of nonmonotonic consequence relation.
- The example shows that prethoretical notions concern the properties of a variety of consequence relation.

[1] But even if a conclusion is validly drawn from acceptable premises, we are not obliged to accept it if those premises are incomplete

Imperatives

Geach's description of practical argument can be illustrated using a modified variant of Von Wright-Lemmon's syntax and semantics for change expressions. Imperatives are commanded changes and can be analyzed as two part sentences combining two kinds of direction of fit:

$$!(\overset{BEFORE}{\underbrace{initial\_situation}_{word-to-world\ \ fit}} / \overset{AFTER}{\underbrace{resulting\_situation}_{world-to-word\ \ fit}})$$

Basic semantics of imperatives:

- Imperatives are commanded actions.
  - Produce A: $!(\neg A/A)$; Suppress A: $!(\neg A/\neg A)$; Maintain A: $!(A/A)$; Destroy A: $!(A/\neg A)$; See to it that A: $!(\top/A)$
- Imperative $!(p/q)$ is true iff (i) in the initial situation p is the case, (ii) q is the case in the imperative future, (iii) q is possible in the future, (iv) q is avoidable in the future. [2]

[2] The problem of practical reasoning is to find out which one is the actual and which one is the ideal world (on the basis of available facts and commands).

A simple system

Definition

Syntax

Atom is a finite set of propositional letters. Language $\mathcal{L}_{PL}$

$$\begin{array}{rcl} a & \in & Atom \\ \mathcal{L}_{PL} & : & := a \mid \top \mid \varphi \mid \neg\varphi \mid \varphi \wedge \psi \end{array}$$

Language $\mathcal{L}_{!}$

$$\begin{array}{rcl} p, q & \in & \mathcal{L}_{PL} \\ \mathcal{L}_{!} & : & := \cdot\,(p/\top) \mid !\,(p/q) \mid \boxdot\,(\top/q) \mid \neg\varphi \mid \varphi \wedge \psi \end{array}$$

Language $\mathcal{L}_{!might}$

$$\begin{array}{rcl} p & \in & \mathcal{L}_{!} \\ \mathcal{L}_{!might} & : & := p \mid might\ p \mid \end{array}$$

Semantics

Definition

$\mathbf{W}_0 = \wp Atom$
Structures
$\Sigma = \{\langle W, R_!, R_F\rangle \mid W \subseteq \mathbf{W}_0, R_! \subseteq W \times W, R_F \subseteq W \times W\}$

Definition

Ignorant structure: $0 = \langle \mathbf{W}_0, \mathbf{W}_0 \times \mathbf{W}_0, \mathbf{W}_0 \times \mathbf{W}_0\rangle$

Semantics

Definition

Valuation for $p, q \in \mathcal{L}_{PL}$
$w \models p$ iff $p \in w_i$ for propositional letters p
$w \models \neg p$ iff $w \not\models p$
$w \models p \wedge q$ iff $w \models p$ and $w \models q$

Definition

Truth at w in σ
$\sigma, w \models \cdot(p/\top)$ iff $w \models p$ and $R_!(w, v)$ or $R_F(w, v)$ for some v
$\sigma, w \models !(p/q)$ iff (i) $w \models p$ and (ii) $v \models q$ for all v such that $R_!(w, v)$, and (iii) $u \models q$ for some u such that $R_F(w, u)$, and (iv) $z \not\models q$ for some z such that $R_F(w, z)$
$\sigma, w \models \boxdot(\top/p)$ iff $v \models p$ for all v such that $R_!(w, v)$ or $R_F(w, v)$
$\sigma, w \models \neg\varphi$ iff $\sigma, w \not\models \varphi$
$\sigma, w \models \varphi \wedge \psi$ iff $\sigma, w \models \varphi$ and $\sigma, w \models \psi$
$\sigma, w \models \mathtt{might}\ \varphi$ iff $\sigma, v \models \varphi$ for some v

Prima facie consequence

Definition

Validity in $\sigma = \langle W \times R_! \times R_F \rangle$
$\sigma \models p$ iff $\sigma, w \models p$ for all $w \in W$ where $\sigma = \langle W, R_!, R_F \rangle$

Definition (Substructure)

$\sigma \leq \sigma'$ iff $W \subseteq W'$ and $R_! \subseteq R'_!$ and $R_F \subseteq R'_F$ (where $\sigma = \langle W, R_!, R_F \rangle$ and $\sigma' = \langle W', R'_!, R'_F \rangle$).

Definition (Minimal structure)

$(0 \mid p) = \sigma$ iff $\sigma \models p$ and if $\sigma' \models p$, then $\sigma' \leq \sigma$.

Prima facie consequence; completeness of premises

Definition

$(0 \mid \Gamma) = \bigcap_{p \in \Gamma} (0 \mid p)$

Definition

$\Gamma \models_{prima\ facie} p$ iff $(0 \mid \Gamma) \models p$

Definition

Let $(0 \mid \Gamma) = \langle W, R_!, R_F \rangle$. Γ is a complete set iff $|mem_1(R_!)| = 1$ and $|mem_2(R_!)| = 1$.

Back to Ross' paradox

The letter is not burned.	(i)	$\cdot\,(\neg B / \top)$
It is not possible that the letter is in the letter box and that it is burned.	(ii)	$\boxdot\,(\top / \neg L \vee \neg B)$
Put the letter into the letter box!	(iii)	$!\,(\neg L / L)$
Put the letter into the letter box or burn it!	(iv)	$!\,(\neg L \wedge \neg B / L \vee B)$
It might be good to burn the letter!	(v)	$\texttt{might}\,!\,(\neg B / B)$

Creating the largest structure by eliminating relations

- For $\cdot(p/\top)$ remove all arrows starting at $\neg p$-worlds. For $!(p/q)$ test wheter there is an R_F arrow pointing to a q world and an R_F arrow pointing to a $\neg q$ world; if so, remove all $R_!$ arrows starting in a $\neg p$ world or ending in a $\neg q$ world; otherwise, remove all arrows. For $\boxdot(\top/p)$ remove all arrows ending in $\neg p$-worlds.
- Disjunction introduction partially vindicated

$$\left\{ \underset{(i)}{\cdot(\neg B/\top)}, \underset{(ii)}{\boxdot(\top/\neg L \vee \neg B)}, \underset{(iii)}{!(\neg L/L)} \right\} \models_{prima\ facie} \underset{(iv)}{!(\neg B \wedge \neg L/B \vee L)} \quad (1)$$

Initial situation			Imperative future			Possible future		
w_1	$\{B, L\}$	× by (i)	w_1	$\{B, L\}$	× by (ii)	w_1	$\{B, L\}$	× by (ii)
w_2	$\{B\}$	× by (i)	w_2	$\{B\}$	× by (iii)	w_2	$\{B\}$	
w_3	$\{L\}$	× by (iii)	w_3	$\{L\}$		w_3	$\{L\}$	
w_4	$\emptyset$		w_4	$\emptyset$	× by (iii)	w_4	$\emptyset$	

- Free choice permission partially vindicated (here modified to choice offering imperative and suggestion)

$$\left\{ !\underbrace{(\neg B \wedge \neg L / B \vee L)}_{(iv)} \right\} \models_{prima\ \ facie} \texttt{might}\ !\underbrace{(\neg B / B)}_{(v)} \qquad (2)$$

Initial situation			Imperative future			Possible future	
w_1	$\{B, L\}$	$\times$ by (iv)	w_1	$\{B, L\}$		w_1	$\{B, L\}$
w_2	$\{B\}$	$\times$ by (iv)	w_2	$\{B\}$		w_2	$\{B\}$
w_3	$\{L\}$	$\times$ by (iv)	w_3	$\{L\}$		w_3	$\{L\}$
w_4	$\emptyset$		w_4	$\emptyset$	$\times$ by (iv)	w_4	$\emptyset$

- Avoiding the paradox ($\mathrm{O}p \Rightarrow \mathrm{P}q$)

The relation $\models_{prima\ facie}$ is not transitive and in this case the unwanted conclusion does not follow;

$$\left\{ \underset{(i)}{\cdot(\neg B/\top)}, \underset{(ii)}{\boxdot(\top/\neg L \vee \neg B)}, \underset{(iii)}{!(\neg L/L)} \right\} \not\models_{prima\ facie} \mathtt{might}\ \underset{(v)}{!(\neg B/B)} \quad (3)$$

since $\{\cdot(\neg B/\top), \boxdot(\top/\neg L \vee \neg B), !(\neg L/L), !(\neg B/B)\}$ is not satisfiable.

Initial situation			Imperative future			Possible future		
w_1	$\{B, L\}$	$\times$ by (i)	w_1	$\{B, L\}$	$\times$ by (ii)	w_1	$\{B, L\}$	$\times$ by (ii)
w_2	$\{B\}$	$\times$ by (i)	w_2	$\{B\}$	$\times$ by (iii)	w_2	$\{B\}$	$\times$ by (v)
w_3	$\{L\}$	$\times$ by (iii)	w_3	$\{L\}$	$\times$ by (v)	w_3	$\{L\}$	$\times$ by (v)
w_4	$\emptyset$	$\times$ by (v)	w_4	$\emptyset$	$\times$ by (iii)	w_4	$\emptyset$	$\times$ by (v)

Conjectures

- The language practices do not support the hypothesis that understanding of meanings of logical terms is constitutive for the understanding of consequence relations.
- The language practices do not support the hypothesis that understanding of consequence relations is regulative for the understanding of meaning of logical terms.
- My conjecture is that understandings of logical terms and logical relations come to us bundled together as a collection of open notions.