Analytic Philosophy

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Debating neo-logicism

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- In this talk I will not address our official topic.

Instead I will discuss some issues having to do with another related area, namely with philosophical issues connected to logic and its application to mathematics, both in connection with Frege's and the neo-Fregeans.

- We shall have a look at the so called knowledge-of-sources rationale since it is apparently endorsed by both authorities just mentioned.

The knowledge-of-sources rationale is concerned with determining the epistemological source of our arithmetical knowledge.

- A question arises immediately: are we to distinguish, within the rationale itself, its two prongs: the normative fundationalist project in mathematics and the more factual question of grasp of mathematical knowledge?

This question is going to guide this presentation.

- I will firstly try briefly, all too briefly, to interpret Frege's original route and secondly concentrate on the Neo-Fregeans' programme.

Here is a brief preview

As for Frege himself, I shall abide by his distinction between the epistemological query and the task of determining the foundations for mathematics;

and argue that his motivation is focused upon the latter task, even to the point of exclusivity.

- My aim is not to de-philosophise his effort and portrait him as just an ingenious mathematician, but instead to locate his philosophical interest in an adequate fashion.

- In particular, as regards Hume's principle, the main target of the present-day debate, I will argue that Frege does not grasp and does not invite the reader to grasp natural numbers through Hume's Principle, so that to say that Hume's Principle offers an epistemological route is to reverse the order of things.
- In short, Hume's Principle has only a logico-semantic priority, not a genetic, source-related, epistemic one.

- Neo-Fregeans, in contrast, talk about Hume's Principle and Frege's theorem in strongly epistemic terms as offering "one clear a priori route into a recognition of the truth of... the fundamental laws of arithmetic (...)".

  (Wright, On the philosophical significance of Frege's theorem, pp. 210)

- I shall argue, albeit very tentatively, for a pessimistic conclusion to the effect that the ultimate result of all these worthy efforts might be the failure in both Frege's aims, taken at face value: proving the analyticity of arithmetic and hence determining the foundations of mathematics to be uncontentiously solid since based of logic.

- In the last section I very briefly evaluate a possible escape route for the neo-Fregean logicist, namely to sustain that we could truly stipulate Hume's principle, posit certain concepts and then check their having non-empty extensions.
- Such a way of stipulation *tout court* would not ask for numbers to be known in advance and would be close to the Hilbert-style implicit definition.

- In that case Hume's Principle would represent an epistemic path for the knowledge of arithmetic and analysis.
- Such a project would unfortunately be far away from Frege's goals, given his negative attitude toward Hilbert-style definitions.

- I limit myself to the issue of fidelity of neo-logicism to its original paradigm;

  I leave it open that neo-logicism might have independent high qualities that would recommend it as the best course to take.

- Let me start from Frege himself as seen by contemporary commentators who stress knowledge-of-sources rationale.

- In Frege's case the knowledge-of-sources rationale would consist in aiming to establish logicism in order to determine the epistemological source of our arithmetical knowledge.

> According to Frege, mathematical objects were logical objects. Hence a knowledge of numbers calls for nothing beyond knowledge of logic and definitions.

(*The Reason's Proper Study*, p. 1 (Intro))

- As far as Frege logicist programme is concerned, it allegedly shows or aims to show how mathematical knowledge is based on our capacity to grasp mathematical objects by the specifically reasoning faculties of the mind.

Let me formulate some doubts.

- Following Bolzano's steps in aiming to remove intuition and visual representation from arithmetic and analysis (if anything else, because it is misleading), Frege goes one step further.

- He is not just interested in the foundations of mathematics in the sense of determining the **justification** of mathematical statements, but also with the **rational order** by which such justification should proceed:

> After we have convinced ourselves that a boulder is immovable, by trying unsuccessfully to move it, there remains the further question, what is it that supports it so securely?
>
> (Frege, *Grundlagen*, §2)

- It is not just that mathematicians should be rigorous in their search for subjective certainty (nowadays linked to "access internalism"), they should also be concerned with the **objective** foundations of mathematical knowledge.

- Event though such a demand is not original (Descartes stating it in his *Meditations*, searching for the true order of knowledge), the way in which Frege tries to solve the demand for reliable, objective foundations **is** novel. His main idea is to show that mathematical theorems are truth of logic, “analytic”, i.e. derivable from particular laws of logic and definitions.

- And since logic is the arbiter of all things, in the sense that everything existing objectively has to obey the laws of logic, by proving arithmetic to be reducible to logic, we prove it to be securely grounded, objectively true.
- Is it an epistemic project?

- Many authors (e.g. Dummett, Shapiro) find analyticity to be for Frege an **epistemic** concept, turning on how a proposition is knowable.

- I think the two aspects: foundationalism and epistemology are to be distinguished, the former being the main Frege's concern, the latter not being one at all.

- In *Grundlagen* (§3), Frege says:

When … a proposition is called *a priori or analytic* (italics mine) in my sense,… it is a judgment about the ultimate ground upon which rests the justification for holding it to be true … The problem becomes … that of finding the proof of the proposition, and of following it up right back to the primitive truths. If, carrying out this process, we come only on general laws and on definitions, then the truth is an analytic one…if, however, it is possible to give the proof without making use of truths which are not of a general nature, but belong to the sphere of some particular science, then the proposition is a synthetic one.

- It seems that we should distinguish two notions of justification. The quotation above exemplifies the first one, objective logico-semantical notion of justification.

- The more usual notion of justification is less concern with the nature of truths and more with the cognizers's thinking process, it is subject's justification that has to do sometimes with the structure of his belief-system and sometimes with the normative aspects of the very geneses of his beliefs.

- So I want to contrast the logico-semantical notion of justification with this second one that I shall call genetic, source-related.

- For Frege the notion of justification clearly belongs to mathematics rather than to the matters concerning cognizer's mind, and is thus logico-semantical.

> In this case the question (of apriority-MT) is removed from the domain of psychology and assigned to that of mathematics, if it concerns a mathematical truth.
>
> (*Grundlagen*, section 3)

- We shall need this distinction a few lines below, in connection with the crucial move - the appeal to Hume's Principle.

But, let us start from the beginning.

- The lynch-pin of Frege's logicism is clearly the claim that mathematics – more precisely arithmetic and analysis – are reducible to logic.

Since mathematical statements are reducible to logic, we can determine their foundations via logic alone, i.e. through our reason since

for what are things independent of the reason? To answer that would be as much as to judge without judging, or to wash the fur without wetting it.

(Frege, *Grundlagen*, §26)

- And the laws of reason are the laws of logic. Ultimately, the rules of logic being those of reason, he offers the epistemic route for grasping the rules of logic.

- What about the epistemic aspect of analyticity?

- Frege himself points out to the distinction between the epistemological query and the problem of determining the foundations for mathematics; he namely asserts:

It frequently happens that we first discover the content of a proposition and only then provide a rigorous proof in another, more difficult way, by means of which the conditions of its validity can often also be discerned more precisely. Thus in general the question as to how we arrive at the content of a judgement has to be distinguished from the question as to how we provide the justification for our assertion.

(Frege, *Grundlagen*, §3)

- The aim of a proof is to “place the truth of a proposition beyond all doubt” (§3); in the case of mathematics it amounts to demanding “that the fundamental theorems of arithmetic, wherever possible, must be proven with the greatest rigour; since only if the utmost care is taken to avoid any gaps in the chain of inference can it be said with certainty upon what primitive truths the proof is based” (§4).

- In the route of proving natural numbers to be reducible to logical laws and definitions, Frege introduces in the *Grundlagen* (what Boolos names) Hume's Principle (HP).

Hume's Principle (HP):

$\forall F\ \forall G\ (n(F)=n(G) \Leftrightarrow F \approx G)$

F, G - concepts;

n(G) - the number of G's;

$\approx$ equinumerosity relation

- Let us apply our distinctions to this crucial move.

  Hume's Principle represents for Frege a step in his logicist project. How do we grasp Hume's Principle remains in *Grundlagen* without an answer.

- What is certain is that Frege's aim is to depict known (mathematical) objects – natural numbers, more precisely the criterion for their identity.

The aim is hence to get a description based on logic, giving mathematics the grounds for security and truth it needs.

- I would like to claim that Frege does not grasp and does not invite the reader to grasp natural numbers through Hume' Principle, so that to say that HP offers an epistemological route is to reverse the order of things.

- Hume's Principle has a logico-semantic priority, not a genetic, source-related, epistemic one.

Let me briefly argue for this.

- Firstly, Hume's Principle got formulated after twenty centuries of mathematical development. From a purely mathematical point of view, mathematicians from ancient Greeks to modern number theorists have developed the theory of numbers to its full extent.

- Of course, as Frege points out, there are still philosophico-mathematical problems concerning “a concept that is fundamental to a great science” that remain open, and such an investigation of the concept of number is a task that mathematicians and philosophers should share.

- But, his approach is “more philosophical that many mathematicians may deem appropriate”. (*Grundlagen*, Intro).

- Frege is able to introduce Hume's Principle due to his knowledge of mathematics in details; what he does is to encapsulate in HP the criterion of identity for mathematically well known objects.

- That Frege depicts, instead of stipulating, natural numbers is also implicit both in the Caesar problem and in Frege's approach toward the so-called Hilbert-style implicit definition.

- Firstly, when Frege says we know Cesar is not a number – it proves that in thinking of numbers he has in mind very specific (abstract) objects, because he talks about identities of the form: the number of F=x, where x is not a number.

  But in order to know that x is not a number, i.e. that the identity is a mixed one, we have to know what numbers are.

- How could we come to know this just by positing Hume's Principle?

His asking whether Caesar is a number is to ask whether Hume's Principles leaves the truth value and hence the meaning of so-called "mixed" identities undetermined while, "naturally, no one is going to confuse [Caesar] with the [number zero]" (§62).

- Secondly, Frege's approach toward Hilbert-style implicit definition (usually presented as a set of axioms) is due to his view of what the aim of a (implicit) definition amounts to and is extremely critical:

  > …axioms and theorems can never try to lay down the meaning of a sign or word that occurs in them, but it must already be laid down

- I shall return to the issue when discussing neo-logicism.

- Let me pass to apriority in general.

  By Frege's own elucidation in §3 apriority amounts to "provide a proof from completely general laws, which themselves neither need nor admit of proof"

- Such a concept is hence about the ultimate ground on which the logico-semantic justification for holding a (mathematical) proposition to be true rest – and as such is not epistemic.

- It is not about “the psychological, physiological and physical conditions that have made it possible to form the content of the proposition in our mind”.

After all, prior to determining the proof we have to **know** what is the assertion whose truth we want to establish.

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- Frege hence (not just in *Grundlagen*) settles the question as to whether analyticity is an epistemic concept in the negative.

Hume's Principle does not offer an epistemic route for grasping (natural) numbers, but rather a way for knowing/determining the ground for taking mathematical propositions to the true.

- What about neo-Fregean logicism?

  Contemporary neo-Fregean logicism attempts to vindicate the spirit, if not the letter, of the basic doctrines of Frege's logicism, by developing a systematic treatment of arithmetic that approaches the requirements of Frege's doctrine while avoiding the contradictoriness of Basic Law V.

- The aim of neo-logicism is to develop branches of mathematics from abstraction principles and it is primarily an epistemological programme. As Shapiro points out:

> Neo-logicism is, at root, an epistemological program, attempting to determine how mathematical knowledge can be grounded. We can know facts about *the natural number* by deriving them from HP.
>
> (Shapiro, 2000, *Introduction* to the Abstraction and Neo-Logicism Special Issue, *Philosophia Mathematica*, Vol. 8, II, p.99)

- Neo-Fregeans are themselves explicit on this one:

> The neo-Fregean thesis about arithmetic is that knowledge of its fundamental laws (essentially, the Dedekind-Peano axioms) – and hence of the existence of a range of objects which satisfy them – may be based a priori on Hume's Principle

(Wright, 'Is Hume's Principle Analytic', in Hale and Wright, 2001, *The Reason's Proper Study*, p.321)

- Neo-Fregeans maintain that it is possible, following Frege himself, to define by stipulation abstract sortal concepts - that is concepts whose instances are abstract objects of a certain kind. What has to be stipulated is the truth of an abstraction principle.

- The general form of an abstraction principle is the following one:

$$\forall f \, \forall g \, (\sum(f) = \sum(g) \iff f \approx g)$$

- f and g - variables referring to entities of a certain kind (objects or concepts usually),
- $\sum$ - a higher-order operator which forms singular terms when applied to f and g, so that $\sum(f)$ and $\sum(g)$ are singular terms referring to objects, and
- $\approx$ - an equivalence relation on entities denoted by f and g

- It is abstraction principles which are supposed to bear the main burden of the task of reconciling logicist or neo-Fregean logicist thesis that arithmetic and analysis are pure logic.

- In so far as they are stipulations they can aspire to explain in one stroke both how logic can be committed to abstract objects, and how it is possible to have knowledge of these objects.

- The neo-logicists claim that

> we can account for the necessity of at least the basic arithmetic truths and how these truths can be known a priori.
>
> (Shapiro, *The Measure…*, p.71)

If such an explanatory principle. . . can be regarded as *analytic*, then that should suffice at least to demonstrate the analyticity of arithmetic. Even if that term is found troubling,. . . it will remain that Hume's Principle like any principle serving implicitly to define a certain concept will be available without significant epistemological presupposition to one who has mastery of the concept it configures. . .

So one clear a priori route into a recognition of the truth of... the fundamental laws of arithmetic. . . will have been made out.

(Wright, *The Reason's Proper Study*, pp. 279)

So,... there will be an a priori route from a mastery of second-order logic to a full understanding and grasp of the truth of the fundamental laws of arithmetic. Such an epistemological route... would be an outcome still worth describing as logicism. . .

(Wright, On the philosophical significance of Frege's theorem, in Hale and Wright, 2001, *The Reason's Proper Study*, pp. 279-280)

In the remaining part I shall concentrate on the epistemic aspect of neo-Fregean logicism.

- It seems to me that the burning epistemic problems of neo-logicism get projected back upon Frege; in other words that Frege is being wrongly burdened with something he does not actually assert.

- In contrast, neo-Fregeans themselves insist on the epistemic route so that they are, differently from Frege, explicit in taking their programme to be fundamentally an epistemic one.

- They begin in a modest way talking in terms of explanation:

> …Hume's Principle *suffices* to explain the concept of *number* as a sortal concept.

(Hale and Wright, *The Reason's Proper Study*, p. 15)

- However, they also propose a stronger claim of an apriori route for grasping the concept of number and deriving the basic laws of arithmetic via Frege's theorem.

- The problem of distinguishing the fundationalist project from the epistemological one appears more acute in the neo-Fregean's programme since they state it explicitly.

- Hume' Principle allegedly offers an *apriori* route for acquiring mathematical knowledge, germane to the rationalist epistemology.

- In order to avoid appealing to the disastrous Basic Law V, neo-Fregean logicists famously do not follow Frege all the way, they advocate instead adding Hume's Principle to the second-order logic as a supplementary axiom, sustaining that Frege's theorem gives reason for grounding the claim that arithmetic is analytic in Hume's Principle.

- As I've said at the outset, the result is the failure in both Frege's aims, taken at face value: proving the analyticity of arithmetic and hence determining the foundations of mathematics to be uncontentiously solid since based on logic.

- Neo-logicism does not prove (Frege's) analyticity of arithmetic since Hume's Principle is **not** a law of logic, as Boolos pointed out long time ago.

- Instead, they say that the fact that adding Hume's Principle to second-order logic results in a consistent system that suffices for a foundation of arithmetic (all the basic laws of arithmetic are derivable within the system) and that this "constitutes a vindication of logicism, on a reasonable understating of that thesis".

(Hale and Wright, Logicism in the twenty-first century, in Shapiro (ed.), 2005, *The Oxford Handbook of Philosophy of Mathematics and Logic*, p.169)

- What about the apriorist epistemology?

  Neo-Fregeans sustain that

…the case for the existence of numbers *can* be made on the basis of Hume's Principle, and … it provides for a head-on response to the epistemological challenge posed by Benacerraf's dilemma.

…provided that facts about the one-one correlation of concepts – in the basic case, sortal concepts under which only concrete objects fall – are, as we may reasonably presume, unproblematically accessible, we gain access, via Hume's principle and without any need to postulate any mysterious extrasensory faculties or so-called mathematical intuition, to corresponding truths whose formulation involves reference to numbers.

(Hale and Wright, Logicism in the twenty-first century, in Shapiro (ed.), 2005, *The Oxford Handbook of Philosophy of Mathematics and Logic*, p.172-3)

- Even though Wright correctly emphases that in every epistemic project there are presuppositions that have to be assumed on trust, without evidential justification (in order to avoid an infinite regress), I find the question of how we grasp Hume's Principle legitimate and unpalatable for the neo-Fregean's programme.

- By leaving it without an answer, the epistemic project does not offer an alternative to the mysterious extrasensory faculties or so-called mathematical intuition, it just shifts the mysterious part upon the presupposed unproblematic grasp of Hume's Principle.

- Here is my final worry: the only way out for the neo-Fregean logicist might be to sustain that we could truly stipulate Hume's principle, posit certain concepts and then check their having non-empty extensions.

- Such a way of stipulation *tout court* would not ask for numbers to be known in advance and would be close to the Hilbert-style implicit definition.

- As Shapiro explains ("The Good, the Bad and the Ugly") there are important differences between Hilbert-style and neo-Fregeans logicists' stipulations the crucial one in this context being that:

With the exception of logical terminology (connectives, etc.), *no* term in a Hilbert-style implicit definition comes with the previously established meaning or extension.

…

For Hilbert, the satisfiability (or relative consistency) of the set of axioms is sufficient for their truth, whereas for the neo-logicist, a crucial issue is the uniqueness of the objects referred to by the relevant terms involved.

- Could Hilbert-style reading of Hume's Principle help?

- The acceptance of a Hilbert-style implicit definition would raise a new version of Julius Cesar worry: by using Hume's Principle as a **Hilbert-style** implicit definition instead, it would not be possible to depict certain, unique objects; remember Hilbert's quip that "table, chairs and beer mugs" could be taken as satisfiers of axioms normally taken to refer to points and lines.

- No Hilbert-style implicit definition can uniquely determine the objects its definition refers to and it's not its aim either. As Ebert and Shapiro rightly notice: "the connection to intuition or observation is broken for good" ('The Good, the Bad and the Ugly', p. 6.)

- On the other hand, the history of mathematics shows examples that did work this way. Let us remember, e.g. Cardano's stipulation of "imaginary numbers" – at first they here stipulated as numbers whose square was a negative number and it took almost 300 years before Gauss determined their geometrical interpretation and hence explained "the true metaphysics of the imaginary numbers"

- Maybe a similar, truly stipulative, positing route might be open for Hume's principle and natural numbers as well. In that case Hume's Principle would represent an epistemic path for the knowledge of arithmetic and analysis.