"Between Logic and Reality: Modelling Inference, Action and Understanding"

Debating (Neo)logicism: Frege and the neo-Fregeans

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Abstract

The paper's aim is to determine and discuss in which sense, if any, Frege's and neo-Fregean logicism are responding to the epistemological challenge concerning our arithmetical knowledge. More precisely the paper analyses what the epistemological significance of Frege's logicist programme amounts to, namely, the objective justificatory connections obtaining between arithmetical and logical statements.
It then contrasts this result with the self-understanding of the neo-Fregeans who allegedly follow Frege's steps, but in fact take a rather different direction.

Introduction

The problem of the internal structure and the basis of our mathematical knowledge is a fundamental one in the philosophy of mathematics. It is often tied to the related issue of its epistemological source. These two issues are often presented together and labelled as "knowledge-of-sources rationale".
However we have to keep distinct, within the rationale itself, its two prongs: the normative fundationalist project in mathematics and the more factual question of grasp of mathematical knowledge. This thought is going to guide the present paper. Since it is often assumed that Frege's original route addresses both sub-issues, in this paper I will try to argue against this construal of his theory.

Firstly, an interpretation of Frege's original route will be presented and defended, which is more limited in its scope, and incompatible with the narrow epistemological reading of his theory. Secondly I shall critically concentrate on the Neo-Fregeans' programme that is supposed, in the context of epistemic significance, to be following the Fregean's one (the neo-Fregean's logicist version I find most appealing and whose work I shall try critically to address here, is Hale's and Wright's neo-logicist account[1]).

As for Frege himself, I shall abide by his distinction between the narrowly epistemological query and the task of determining the foundations for mathematics; and argue that his motivation is focused upon the latter one, even to the point of exclusivity.
My aim is not to de-philosophise his effort and portrait him as just an ingenious mathematician, but instead to locate his philosophical interest in an adequate fashion.

[1] See e.g. (Hale and Wright, 2001, 2002)

In particular, as regards Hume's principle, the main target of the present-day debate, I will argue that Frege does not come to grasp and does not invite the reader to grasp natural numbers through Hume's Principle, so that to say that Hume's Principle offers an epistemological route is to reverse the order of things. In short, Hume's Principle has only a logico-semantic priority, not a genetic, source-related, epistemic one.

Neo-Fregeans, in contrast, talk about Hume's Principle and Frege's theorem in strongly epistemic terms as offering "one clear a priori route into a recognition of the truth of... the fundamental laws of arithmetic (...)".[2]

I shall argue, albeit very tentatively, for a pessimistic conclusion to the effect that the ultimate result of all these worthy efforts might be the failure in both Frege's aims, taken at face value: proving the analyticity of arithmetic and hence determining the foundations of mathematics to be uncontentiously solid since based of logic.
In the last section I very briefly evaluate a possible escape route for the neo-Fregean logicist, namely to sustain that we could truly stipulate Hume's principle, posit certain concepts and then check their having non-empty extensions. Such a way of stipulation *tout court* would not ask for numbers to be known in advance and would be close to the Hilbert-style implicit definition.
In that case Hume's Principle would represent an epistemic path for the knowledge of arithmetic and analysis.
Such a project would unfortunately be far away from Frege's goals, given his negative attitude toward Hilbert-style definitions.
I thus limit myself to the issue of fidelity of neo-logicism to its original paradigm; I leave it open that neo-logicism might have independent high qualities that would recommend it as the best course to take.

Frege's logicism

Let me start from Frege as seen by contemporary commentators who stress the knowledge-of-sources rationale, in order to proceed to his own pronouncements a few paragraphs later.
In Frege's theory the knowledge-of-sources rationale would motivate the goal of establishing a variant of logicist view, i.e. the theses of epistemic dependence of mathematical knowledge on logical one and thus determining the epistemological source of the former.

> According to Frege, mathematical objects were logical objects. Hence a knowledge of numbers calls for nothing beyond knowledge of logic and definitions.
> (Hale, Wright, *The Reason's Proper Study*, p. 1 (Intro))

So, as far as Frege's logicist programme is concerned, in the neo-Fregeans' interpretation, it allegedly shows or aims to show how mathematical knowledge is based on our capacity to grasp mathematical objects by the specifically reasoning faculties of the mind. Following a well known tradition, they take Frege's insistence

[2] Wright, On the philosophical significance of Frege's theorem, pp. 210.

on the semantic primacy of the sentential context as crucial, in conjunction with consideration of Hume's Principle:

> Where neo-Fregeanism principally differs from Frege is in its taking a more optimistic view than Frege himself came to hold of the prospects for the kind of contextual explanation of the fundamental concepts of arithmetic and analysis—the concepts of cardinal number and real number—which he considered and rejected in the central sections (§§60-68) of Grundlagen. The proposal there under consideration is that the concept of (cardinal) number may be explained, in accordance with Frege's context principle, by fixing the sense of identity-statements linking canonical singular terms for its instances— terms of the form "the number belonging to the concept F", or more briefly "the number of Fs"— and that this may be done by means of what is now widely called Hume's Principle (Hale, Wright, *The Reason Proper Study*, Intro, p.1-2)

Are we to accept such a reading of Frege's logicist project?
Let us start with few historical remarks. Following Bolzano's and his successors' steps in aiming to remove intuition and visual representation from arithmetic and analysis (if anything else, because it is misleading), Frege goes one step further.
Historically, the demand of getting a strongly rigorous proof for mathematical statements while ignoring the intuitionally based results and denying the role of our hunches goes back to the beginning of 19th century and continue through the work of primarily Bolzano and Weierstrass. In 1872 Weierstrass famously discovers the existence of functions being continuous at each point of their domain but not being differentiable in any, which has contradicted most shockingly the mathematicians' intuitions about continuous functions. And he proceeds with the elimination of self-evidence and intuition from mathematical proofs in - what Lakatos labels as - "the Weierstrassian revolution of rigour" (Lakatos, 1976, 55).
Following such demand for rigour in mathematical proofs, Frege's further step consists in aiming to determine the justification and foundation for the basic mathematical proof-steps. He nicely confronts (*Grundlagen*, §2) his demand for vigour independently from the deductive confirmation of certain mathematical results with the demand for the improvement of Euclidean rigour, which brought to the additional study of the V Euclidean axiom and the consequently results:

> …it lies deep in the nature of mathematics always to prefer proof, wherever it is possible, to inductive confirmation. Euclid proved many things that would have been granted him anyway. And it was the dissatisfaction even with the Euclidean rigour that led to the investigation of the Axiom of Parallels.[3]

Frege's further and deeper step leads him to the very foundation of mathematics.
He is not however just interested in the groundwork of mathematics in the sense of determining the **justification** of mathematical statements, but also with the **rational order** by which such justification should proceed:

[3] Wherever possible, the *Grundlagen* quotations are taken from (Beaney, 1997); alternatively they are from Austin's translation.

> After we have convinced ourselves that a boulder is immovable, by trying unsuccessfully to move it, there remains the further question, what is it that supports it so securely?
> (Frege, *Grundlagen*, §2)

It is not just that mathematicians should be rigorous in their search for subjective certainty, they should also, indeed primarily, be concerned with the **objective** foundations of mathematical knowledge.
It is not enough any more to reduce a mathematical proof into rigorous, self-evident steps in order to be sure not to have based them on intuition - as Frege nicely points out to in *Grundlagen's* paragraph 2 - it is vital to determine the objective base, the foundations of mathematics. Since even what seems to be uncontentiously true, like the case with the fifth Euclid's axiom was, could turn out to be open to discussion and further analyses, and reveal the possibility of a different outcome (in the case of Euclid's axioms, of a different geometry based on the negation of his Axiom of Parallels).
The aim is hence to determine what it is that supports the arithmetical-boulder so securely, what makes the truth of arithmetical statements *objectively* "immovable".

Event though such a demand for reliable, objective foundations is not original (Descartes having famously stated it in his *Meditations*, in his search for the true order of knowledge), the way in which Frege tries to answer it **is** novel. His main idea is to show that mathematical theorems are truths of logic, "analytic", i.e. derivable from general laws of logic and definitions.
And since logic is the arbiter of all things, in the sense that everything existing objectively has to obey the laws of logic, by proving arithmetic to be reducible to logic, we prove it to be securely grounded, objectively true.
The question to point out to is the one that concerns the existence of epistemic connotation of such a project. I will try to argue for a negative answer.
Many authors (e.g. Dummett, Kitcher, Martin-Löf) find analyticity to be for Frege an **epistemic** concept, turning on how a proposition is knowable.[4]
I think the two aspects, foundationalism and epistemology, are to be distinguished, the former being Frege's main concern, the latter not being one at all.
I will argue for it in several steps. I will distinguish the less usual notion of justification, namely the logico-semantical one from its more usual counterpart, i.e. the subject's justification, and argue that Frege is interested in the former one, relying on an analyses of the main *Grundlagen* passages; along the way I will briefly take a critical look on the work of those more recent authors who endorse the view that Frege's programme is epistemological in their, justification centred meaning of the term.

The upshot will be that Frege's logicist project concentrates on the justificatory links between arithmetic and logic, where the notion of justification is logico-semantical (in the sense of objective grounding), not subjectively epistemological. More particularly, in connection with Hume's principle, I will endorse the view that the factual query about our mathematical knowledge does not find its answer in the grasp of the

[4] See (Dummett, 1991), (Kitcher, 1979), (Martin-Löf, 1996), (Shapiro, 2004).

Principle, since its function is limited to its logico-semantic aspect, leaving the canonically epistemic one open.

Let us start with the definition of analyticity.
In *Grundlagen* (§3), Frege says:

> When … a proposition is called *a priori or analytic* (italics mine) in my sense,… it is a judgment about the ultimate ground upon which rests the justification for holding it to be true … The problem becomes … that of finding the proof of the proposition, and of following it up right back to the primitive truths. If, carrying out this process, we come only on general laws and on definitions, then the truth is an analytic one…if, however, it is possible to give the proof without making use of truths which are not of a general nature, but belong to the sphere of some general science, then the proposition is a synthetic one.

This suggests that we should distinguish two notions of justification. The quotation above exemplifies the first one, objective logico-semantical notion of justification**.** The more usual notion of justification is less concerned with the nature of truths and more with the cognizers's thinking process, it is subject's justification that sometimes has to do with the structure of his belief-system and sometimes with the normative aspects of the very genesis of his beliefs. The two notions are independent, or so I shall argue.
Although Frege uses terms like justification, which to the contemporary reader may sound explicitly epistemic, his actual description of the object of his interest reduces it to logical semantic relations and their patterns. Contrary to what contemporary epistemologist[5] would choose to do, Frege explicitly places together the notions of aposteriori and analytic (the distinction being notably introduced in paragraph 3) and clearly distinguishes

> The question of how we arrive at the content of a judgment … from the question as to how we provide the justification for our assertion. (*Grundlagen*, §3)

The notions of analyticity and aposteriority are joined in the context of determining "the ultimate ground on which the justification for holding… (a proposition – MT) to be true rests." Frege hence does not follow the contemporary habit of locating the aposteriori into the epistemic domain and the analytic to the logico-semantical one, thus assigning them to completely disjoint fields of investigation.

For Frege the notion of justification clearly belongs to mathematics rather than to the matters concerning cognizer's mind, and is thus logico-semantical.

> In this case the question (of apriority-MT) is removed from the domain of psychology and assigned to that of mathematics, if it concerns a mathematical truth.
> (*Grundlagen*, section 3)

[5] See e.g. (Burge, 2005).

I want to take Frege’s move seriously, and following what I see as his intention, contrast the logico-semantical notion of justification, in the sense of objective grounding, with this second one, more at home on contemporary epistemology, that I shall call genetic, source-related.

One could object that there is an intermediate domain of judgement available, the one of epitemic normativity which is neither psychological in the naturalist sense, nor purely non-epistemic in the radical sense of being independent from cognizer’s justification. Ironically, for the present context, such normativity has been taken by Burge (Burge, 2005) as being central for Frege’s project.
I do not however see elements of epistemic normativity in Frege’s writings, his programme being about reducing mathematics to logic with the aim of proving it being “immovable”. The reduction of mathematics to logic is not an epistemic result, in any possible sense of the word. It may have epistemic consequences, for instance, helping mathematicians by enhancing the clarity of their grasp of fundamental mathematical notions, but it is itself not an epistemic move. After all, the epistemic path taken by the great mathematicians did very well in the history of mathematics without being logicist in its root, nor Frege criticizes it as such except for its not being sufficiently well founded, lacking the needed rigour.

We shall need the distinction between logico-semantical and genetic, source-related notions of justification a few lines below, in connection with the crucial move - the appeal to Hume’s Principle. The move will be introduced by providing some backgroud.

The lynch-pin of Frege’s logicism is clearly the claim that mathematics – more precisely arithmetic and analysis – is reducible to logic. Since mathematical statements are reducible to logic, we can determine their foundations via logic alone. And the task of reducing arithmetic, i.e. defining basic arithmetical terms such as numbers in logical terms only has, as such, no epistemic connotations. Shapiro, for example, even though he initially represents the programme as an epistemic one, at one point concludes by saying that:

> Perhaps this notion of foundation is as metaphysical as it is epistemic, despite the use of notions like ‘proof’ and ‘justification’. It is not a question of whether we know, for example, that 7 + 5 = 12, to take Frege’s (and Kant’s) own example. There is really no question but that we do know that. Nor is it a question of how we know that 7 + 5 = 12. We knew that proposition long before the foundational work began. Moreover, our own knowledge did not need to go, and in fact did not go, via the proposed founding definitions. We just did the sum. Frege was interested in objective grounding relations among propositions, perhaps something along the lines of Bernard Bolzano’s ground–consequence relation. This seems to drive a wedge between the state of being justified and the ultimate ground or justification of a proposition. (Shapiro, 2004, Foundations of mathematics: Metaphysics, epistemology, structure *The Philosophical Quarterly*, Vol. 54, No. 214, pp. 22-23)

I would add that such a notion of foundation in which, as Shapiro rightly points out, the main task is to determine objectively based relations among arithmetical and logical truths is clearly and exclusively logico-semantical, i.e. metaphysical.

The thesis that numbers are logical objects as well as the logico-semantical analyses of basic arithmetical truths and definitions have, as such, no epistemic connotations in the sense of determining how we perceive such relations and, more importantly, how the basic mathematical concepts and truths are epistemically accessible in the first place.

What about the epistemic aspect of analyticity?
Frege himself points out to the distinction between the epistemological query and the problem of determining the foundations for mathematics; he namely asserts:

> It frequently happens that we first discover the content of a proposition and only then provide a rigorous proof in another, more difficult way, by means of which the conditions of its validity can often also be discerned more precisely. Thus in general the question as to how we arrive at the content of a judgement has to be distinguished from the question as to how we provide the justification for our assertion.(*Grundlagen*, §3)
>
> …This would make them analytic judgments, despite the fact that they would not normally be discovered by thought alone; for we are concerned here not with the way in which they are discovered but with the kind of ground on which their proof rests; or in LEIBNIZ'S words, "the question here is not one of the history of our discoveries, which is different in different men, but of the connexion and natural order of truths, which is always the same." (*Grundlagen*, §17)

The aim of a proof is to "place the truth of a proposition beyond all doubt" (§3); in the case of mathematics it amounts to demanding "that the fundamental theorems of arithmetic, wherever possible, must be proven with the greatest rigour; since only if the utmost care is taken to avoid any gaps in the chain of inference can it be said with certainty upon what primitive truths the proof is based" (§4).

And we do that through our reason since

> for what are things independent of the reason? To answer that would be as much as to judge without judging, or to wash the fur without wetting it.
> (Frege, *Grundlagen*, §26)

It is interesting to contrast Frege's detailed articulation of logico-semantical relations with his very short, and to many interpreters puzzling remarks that do concern epistemic matters. He offers little in the way of explicit argument on how we actually grasp mathematical objects, the solution of what sustains the "mathematical boulder" so securely being the objective of his logicist project. This solution is to be found with no appeal to our intuitions or representations, but relying exclusively on reason.
However, and this might suggest the direction towards the solution of the epistemic puzzle, the laws of reason being the laws of logic. Ultimately, the rules of logic being for him those of reason, he foreshadows the epistemic route for grasping the rules of logic, encompassing arithmetic as well.

I would like to illustrate and further document this reading of Frege by applying it to one of the central topics of Frege's project, indeed the one that has been turned into *the* central topic by the neo-Fregeans.

In the route of proving in the *Grundlagen* that natural numbers are reducible to logical laws and definitions, Frege introduces (what Boolos names[6]) Hume's Principle.

> Hume's Principle (HP):
> $\forall F\ \forall G\ (n(F)=n(G) \Leftrightarrow F \approx G)$
> F, G - concepts;
> n(G) - the number of G's;
> $\approx$ equinumerosity relation

Let us apply our distinctions to this crucial move.
Hume's Principle represents for Frege a possible step in his logicist project. How we grasp Hume's Principle remains in *Grundlagen* without an answer. What is certain is that Frege's aim is to characterize the already known (mathematical) objects – natural numbers, more precisely to offer the criterion for their identity.

In the introductory paraghraps to the Hume's principle, i.e. while reading the last paraghraps of the *Grundlagen* (in particular §62), one possibile impression/reading is that the mode of presentation that Frege introduces is intrinsically an epistemic concept, as many authors have pointed out to (most famoulsy Dummett). That, if being the case, would colour Frege's logicist project with epistemic connotations.

In an alternative construal though, the mode of presentation is not related to the query of grasp of mathematical knowledge.
The fact being that for centuries mathematicians have grasped and managed to grasp properties of mathematical objects without knowing their mode of presentation endorsed by Frege. The mode of presentation in Frege's terms is objective, mathematicians can either fail to grasp it (as they did for centuries) or succeed in doing so (as in Frege's case).

We might take as an example the Earth and its mode of presentation as an object that is a planet through the propertiy of being visibile by virue of the light from a star reflected off its surface and that is (not-perfectly) sherical in its shape. Notwithstanding this mode of presentation, the Earth had been perceived as flat by astronomers now and again in the Middle Ages (even though is safe to say that mankind has known for 2000 years that the Earth is spherical, since Pythagoras' formulation)

The mode of presentation thus, construed as the way an object present itself to the world, does not reveal per se its epistemic accessibility. In this sense, it does abide by a reading of Frege's project that lacks epistemic connotations.

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[6] The principle has been called Hume's Principle by Boolos (in 'The consistency of Frege's *Foundations of Arithmetic*', in his 1998, *Logic, Logic and Logic*). Boolos gave it this title because it recalls a remark in Hume's *Treatise* (Book I, Part iii, Section 1, par. 5), and because Frege quotes Hume in *Grundlagen* (§63):

> when two numbers are so combin'd, as that the one always as unite answering to every unite of the other, we pronounce them equal…

Frege's example with Venus might look as a counter-example to my argument but, Frege characterizes the object itself merely as heavenly body; the component "star" in the "Morning star" and "Evening star" is, at least for Frege himself, a semantically inert part of the two proper names.

The aim of Frege's project is hence to get a description of arithmetic based on logic, giving mathematics the grounds for security and truth it needs.
I would like to claim that Frege does not grasp and does not invite the reader to grasp natural numbers through HP, so that to say that HP offers an epistemological route is to reverse the order of things.
Hume's Principle has a logico-semantic priority, not a genetic, source-related, epistemic one.
Let me briefly argue for this.
Firstly, Hume's Principle got formulated after twenty centuries of mathematical development. From a purely mathematical point of view, mathematicians from ancient Greeks to modern number theorists have developed the theory of numbers to its full extent.
Of course, as Frege points out, there are still philosophico-mathematical problems concerning "a concept that is fundamental to a great science" that remain open, and such an investigation of the concept of number is a task that mathematicians and philosophers should share. But, his approach is "more philosophical that many mathematicians may deem appropriate". (*Grundlagen*, Intro).
Frege is able to introduce HP due to his knowledge of mathematics in details; what he does is to encapsulate in HP the criterion of identity for mathematically well known objects.
That Frege depicts, instead of stipulating, natural numbers is also implicit both in the Caesar problem and in Frege's approach toward the so-called Hilbert-style implicit definition.
Firstly, when Frege says we know Cesar is not a number, THIS proves that in thinking of numbers he has in mind very specific (abstract) objects, because he talks about identities of the form: the number of F=x, where x is not a number. But in order to know that x is not a number, i.e. that the identity is a mixed one, we have to know what numbers are, and this is something we could have not possibly come to know just by positing Hume's Principle.
His asking whether Caesar is a number is a picturesque way of inquiring whether Hume's Principles leaves the truth value and hence the meaning of so-called "mixed" identities (like Caesar=5) undetermined. His positive answer to it suggests that positing Hume's principle is not a way to actually get the "extreme example" clarified. However, we know too well that Caesar is *not* a number. As Frege points out, "naturally, no one is going to confuse [Caesar] with the [number zero]…(§62).

So contrary to what someone might suspect, there is no vicious circle involved here, since Cesar's not being a number is a fact uncontentiously known by us prior to positing Hume's Principle.

Secondly, Frege's approach toward Hilbert-style implicit definition (usually presented as a set of axioms) is due to his view of what the aim of a (implicit) definition amounts to and is extremely critical:

> …axioms and theorems can never try to lay down the meaning of a sign or word that occurs in them, but it must already be laid down

I shall return to the issue regarding implicit definitions when discussing neo-logicism.

Let me pass to apriority in general.
By Frege's own elucidation in §3 apriority amounts to "provide a proof from completely general laws, which themselves neither need nor admit of proof" (and from definitions not belonging to some specific area of knowledge – as Frege points out in the same paragraph). (Reader str. 93)
Such a concept is hence about the ultimate ground on which the logico-semantic justification for holding a (mathematical) proposition to be true rest upon – and as such is not epistemic.
It is not about "the psychological, physiological and physical conditions that have made it possible to form the content of the proposition in our mind". After all, prior to determining the proof we have to **know** what is the assertion whose truth we want to establish.

Frege hence (not just in *Grundlagen*) settles the question as to whether analyticity is an epistemic concept in the negative. Hume's Principle does not offer an epistemic route for grasping (natural) numbers, but rather a way for knowing/determining the ground for taking mathematical propositions to the true.

What about neo-Fregean logicism?

Contemporary neo-Fregean logicism attempts to vindicate the spirit, if not the letter, of the basic doctrines of Frege's logicism, by developing a systematic treatment of arithmetic that approaches the requirements of Frege's doctrine while avoiding the contradictoriness of Basic Law V.
The aim of neo-logicism is to develop branches of mathematics from abstraction principles and it is primarily an epistemological programme. As Shapiro points out:

> Neo-logicism is, at root, an epistemological program, attempting to determine how mathematical knowledge can be grounded. We can know facts about *the natural number* by deriving them from HP.
> (Shapiro, 2000, *Introduction* to the Abstraction and Neo-Logicism Special Issue, *Philosophia Mathematica*, Vol. 8, II, p.99)

Neo-Fregeans are themselves explicit on this one:

> The neo-Fregean thesis about arithmetic is that knowledge of its fundamental laws (essentially, the Dedekind-Peano axioms) – and hence of the existence of a range of objects which satisfy them – may be based a priori on Hume's Principle
> (Wright, 'Is Hume's Principle Analytic', in Hale and Wright, 2001, *The Reason's Proper Study*, p.321)

Neo-Fregeans[7] maintain that it is possible, following Frege himself, to define by stipulation abstract sortal concepts - that is concepts whose instances are abstract objects of a certain kind. What has to be stipulated is the truth of an abstraction principle.

The general form of an abstraction principle is the following one:

$$\forall f\, \forall g\, (\sum(f)=\sum(g) \iff f \approx g)$$

f and g - variables referring to entities of a certain kind (objects or concepts usually),
$\sum$ - a higher-order operator which forms singular terms when applied to f and g, so that $\sum(f)$ and $\sum(g)$ are singular terms referring to objects, and
$\approx$ - an equivalence relation on entities denoted by f and g

It is abstraction principles which are supposed to bear the main burden of the task of reconciling logicist or neo-Fregean logicist thesis that arithmetic and analysis are pure logic. In so far as they are stipulations they can aspire to explain in one stroke both how logic can be committed to abstract objects, and how it is possible to have knowledge of these objects.

The neo-logicists claim that

> … we can account for the necessity of at least the basic arithmetic truths and how these truths can be known a priori.
> (Shapiro, *The Measure…*, p.71)

In particular, the principle neo-logicists concentrate on is our already discussed Hume's Principle.

> If such an explanatory principle. . . can be regarded as *analytic*, then that should suffice at least to demonstrate the analyticity of arithmetic. Even if that term is found troubling,. . . it will remain that Hume's Principle like any principle serving implicitly to define a certain concept will be available without significant epistemological presupposition to one who has mastery of the concept it configures. . . So one clear a priori route into a recognition of the truth of... the fundamental laws of arithmetic. . . will have been made out.
> …
> So,... there will be an a priori route from a mastery of second-order logic to a full understanding and grasp of the truth of the fundamental laws of arithmetic. Such an epistemological route... would be an outcome still worth describing as logicism. . .
> (Wright, On the philosophical significance of Frege's theorem, in Hale and Wright, 2001, *The Reason's Proper Study*, pp. 279-280)

According to the neo-Fregeans, following Frege's priority of syntax thesis, the required condition for singular terms to refer is they occur in true statements (leaving aside for the moment some well known qualifications).

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[7]See e.g. (Hale, 1999).

The idea that abstraction principles are a legitimate way of introducing mathematical theories is problematic, since some of them, as the infamous Basic Law V, are inconsistent.
I shall concentrate in this talk on the epistemic aspect of it. It seems to me that the burning epistemic problems of neo-logicism get projected back upon Frege; in other words that Frege is being wrongly accused of something he does not actually assert. In contrast, neo-Fregeans themselves insist on the epistemic route so that they are, differently form Frege, explicit in taking their programme to be fundamentally an epistemic one.

They begin in a modest way talking in terms of explanation:

> …by stipulating, that the number of *F*s is the same as the number of *G*s just in the case the *F*s are one-one correlated with the *G*s, we can set up *number* as a sortal concept, i.e. that Hume's Principle *suffices* to explain the concept of *number* as a sortal concept.
> (Hale and Wright, *The Reason's Proper Study*, p. 15)

However, they also propose a stronger claim of an a priori route for grasping the concept of number and deriving the basic laws of arithmetic via Frege's theorem.
The problem of distinguishing the fundationalist project from the epistemological one appears more acute in the neo-Fregean's programme since they state it explicitly.
Hume' Principle allegedly gives us reasons for treating mathematical knowledge as being *a priori*, by offering an *a priori* route for acquiring mathematical knowledge, germane to the rationalist epistemology.
According to neo-Fregean logicists Frege's logicism was correct in all main respects except for two points: Frege overestimated the significance of Caesar problem and underestimated the significance of Frege' s theorem (the derivation of the axioms of arithmetic from Hume's Principle in second-order logic)
In order to avoid appealing to the disastrous Basic Law V, neo-Fregean logicists famously do not follow Frege all the way, they advocate instead adding Hume's Principle to the second-order logic as a supplementary axiom, sustaining that Frege's theorem gives reason for grounding the claim that arithmetic is analytic in Hume's Principle.
As I have said at the outset, the result is the failure in both Frege's aims, taken at face value: proving the analyticity of arithmetic and hence determining the foundations of mathematics to be uncontentiously solid since based on logic.
Neo-logicism does not prove (Frege's) analyticity of arithmetic since Hume's Principle is **not** a law of logic, as Boolos pointed out long time ago.
Instead, they say that the fact that adding Hume's Principle to second-order logic results in a consistent system that suffices for a foundation of arithmetic (all the basic laws of arithmetic are derivable within the system) and that this "constitutes a vindication of logicism, on a reasonable understanding of that thesis"[8].
But if Hume's Principle is not a law of logic, in what sense is logicism vindicated?
Next, what about the apriorist epistemology?
Neo-Fregeans sustain that

[8] Hale and Wright, Logicism in the twenty-first century, in Shapiro (ed.), 2005, *The Oxford Handbook of Philosophy of Mathematics and Logic*, p.169.

> …the case for the existence of numbers *can* be made on the basis of Hume's Principle, and it is important to the neo-Fregean that this should be so, precisely because it provides for a head-on response to the epistemological challenge posed by Benacerraf's dilemma.
>
> …provided that facts about the one-one correlation of concepts – in the basic case, sortal concepts under which only concrete objects fall – are, as we may reasonably presume, unproblematically accessible, we gain access, via Hume's principle and without any need to postulate any mysterious extrasensory faculties or so-called mathematical intuition, to corresponding truths whose formulation involves reference to numbers.
> (Hale and Wright, Logicism in the twenty-first century, in Shapiro (ed.), 2005, *The Oxford Handbook of Philosophy of Mathematics and Logic*, p.172-3)

Even though Wright insightfully emphases that in every epistemic project there are presuppositions that have to be assumed on trust, without evidential justification (in order to avoid an infinite regress), I find the question of how we grasp Hume's Principle legitimate and unpalatable for the neo-Fregean's programme. By leaving it without an answer, the epistemic project does not offer an alternative to the mysterious extrasensory faculties or so-called mathematical intuition, it just shifts the mysterious part upon the presupposed unproblematic grasp of Hume's Principle.

Hale and Wight invite us to consider an analogy. We come to know when the directions of two lines are the same through the Direction Principle
($\forall a\ \forall b$ ($d(a)=d(b) \iff a \parallel b$).
The Principle seems to be stipulated, and seems to give us the meaning the direction in virtue of pure stipulation.
Hale and Wright apply the same assumptions to the epistemology of HP. "The number of" receives its meaning from stipulation in the same way in which "direction" does.
However, the analogy does not hold in the epistemic sense since the introduction of directions of lines does not presuppose any determine objects (what do we know about properties of objects corresponding to directions of lines besides what determined by the Direction Principle?!) or any theory about directions determined intuitively or in any other way prior to the stipulation itself. We truly **stipulate** directions of lines. The analogy, upon which **Frege** rests, is that we understand what directions of lines are through the Direction Principle, in the same sense in which we can know what numbers are through Hume's Principle – but, as I have been pointing out, this is not an epistemic route *tout court*, as neo-Fregeans (and others) sustain.

Here is my final worry: the only way out for the neo-Fregean logicist might be to sustain that we could truly stipulate Hume's principle, posit certain concepts and then check their having non-empty extensions. Such a way of stipulation *tout court* would not ask for numbers to be known in advance and would be close to the Hilbert-style implicit definition. As Shapiro explains ("The Good, the Bad and the Ugly") there are important differences between Hilbert-style and neo-Fregeans logicists' stipulations the crucial one in this context being that

> With the exception of logical terminology (connectives, etc.), *no* term in a Hilbert-style implicit definition comes with the previously established meaning or extension.
> …
> For Hilbert, the satisfiability (or relative consistency) of the set of axioms is sufficient for their truth, whereas for the neo-logicist, a crucial issue is the uniqueness of the objects referred to by the relevant terms involved.

Could Hilbert-style reading of Hume's Principle help?

The acceptance of a Hilbert-style implicit definition would raise a new version of Julius Cesar worry: by using Hume's Principle as a **Hilbert-style** implicit definition, it would not be possible to depict certain, unique objects; remember Hilbert's quip that "table, chairs and beer mugs" could be taken as satisfiers of axioms normally taken to refer to points and lines.
No Hilbert-style implicit definition can uniquely determine the objects it refers to and it's not its aim either. As Ebert and Shapiro rightly notice: "the connection to intuition or observation is broken for good."[9]
On the other hand, the history of mathematics shows examples that did work this way. Let us remember, e.g. Cardano's stipulation of "imaginary numbers"[10] – at first they here stipulated as numbers whose square was a negative number and it took almost 300 years before Gauss determined their geometrical interpretation and hence explained "the true metaphysics of the imaginary numbers"[11].
Maybe a similar, truly stipulative, positing route might be open for Hume's principle and natural numbers as well. In that case Hume's Principle would represent an epistemic path for the knowledge of arithmetic and analysis.

Such a project would unfortunately be far away from Frege's goals, given his approach toward Hilbert-style definitions, and its aspect according to which singular terms do not have to uniquely determine the objects they refer to.

With the adapted concept of analyticity and Hilbert-style implicit definition, However, the essential core of Frege's logicist programme – contrary to the neo-Fregeans' aim - would **not** to be preserved.

Conclusion

I have started from the contrast between two concepts of justification: genetic and foundationalist. In contrast to the lot of mainstream contemporary work I have argued that Frege's interest is far from the genetic-epistemological and is limited to the philosophical-mathematical task of determining the foundations for mathematics.

I offer an interpretation of Frege's original route, which is more limited in its scope, and incompatible with the narrow epistemological reading of his theory.

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[9] Ebert and Shapiro, The Good, the Bad and the Ugly, p. 6.
[10] 1545, *Ars Magna*.
[11] 1831.

I try to argue for such an interpretation through the analyses of *Grundlagen* and some mathematico-logico-historical remarks. I emphasise the logico-semantical both aspect and priority of the notion of analyticity and the implicit definition of numbers in Hume's Principle, and show that Frege's aim is to determine the objective justificatory connections between arithmetic and logic, which is normative foundationalist in its root.
Going one step further, I argue that in Frege's project these notions, or any other for that matter, have no source-related, epistemic priority.

Secondly, I critically concentrate on the Neo-Fregeans' programme, more precisely on the neo-logicists' version that I find most appealing, which is Hale's and Wright's neo-logicist account. Such a project is supposed, in the context of epistemic significance, to be following the Fregean's one, but I argue to the contrary. I hence endorse the view that Neo-Fregeans do not manage to fulfil either of Frege's aims, taken at face value: proving the analyticity of arithmetic and hence determining the foundations of mathematics to be uncontentiously solid since based of logic. I do not however analyse those features, which are legion, that makes neo-Fregean logicism a worthwhile project.

Furthermore, I defend the view that accepting the neo-logicist aim, we cannot say that Hume's Principle proves the apriority of mathematics since, in order to stipulate Hume's Principle, we have to grasp objects somehow, it's hence question-begging to say that grasping arithmetic is viable through Hume's Principle.
Frege himself does not pretend to grasp natural numbers through Hume's Principle.
Hume's Principle has only a logico-semantic priority, not a genetic, source-related, epistemic one.

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