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# Order among quasi-arithmetic operator means for convex functions


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# 1.1. Generalization of Jensen's operator inequality

**Let $T$ be a locally compact Hausdorff space and let $\mathcal{A}$ be a $C^*$-algebra of operators on some Hilbert space $H$. We say that a field $(x_t)_{t\in T}$ of operators in $\mathcal{A}$ is continuous if the function $t \mapsto x_t$ is norm continuous on $T$. If in addition $\mu$ is a Radon measure on $T$ and the function $t \mapsto \|x_t\|$ is integrable, then we can form *the Bochner integral* $\int_T x_t \, d\mu(t)$, which is the unique element in $\mathcal{A}$ such that**

$$\varphi\left(\int_T x_t \, d\mu(t)\right) = \int_T \varphi(x_t) \, d\mu(t)$$

**for every linear functional $\varphi$ in the norm dual $\mathcal{A}^*$.**

**Assume furthermore that $(\Phi_t)_{t\in T}$ is a field of positive linear mappings $\Phi_t : \mathcal{A} \to \mathcal{B}$ from $\mathcal{A}$ to another $C^*$-algebra $\mathcal{B}$ of operators on a Hilbert space $K$. We say that such a field is continuous if the function $t \mapsto \Phi_t(x)$ is continuous for every $x \in \mathcal{A}$. If the $C^*$-algebras include the identity operators, and the field $t \mapsto \Phi_t(1)$ is integrable with integral equals 1, we say that $(\Phi_t)_{t\in T}$ is *unital*.**

F.Hansen, J.Pečarić and I.Perić, *Jensen's operator inequality and its converse*, Math. Scand., **100** (2007), 61–73.

find a generalization of Jensen's operator inequality:

**Theorem**

*Let $f : I \to \mathbb{R}$ be an operator convex functions defined on an interval $I$, and let $\mathcal{A}$ and $\mathcal{B}$ be a unital $C^*$-algebras. If $(\Phi_t)_{t\in T}$ is a unital field of positive linear mappings $\Phi_t : \mathcal{A} \to \mathcal{B}$ defined on a locally compact Hausdorff space $T$ with a bounded Radon measure $\mu$, then*

$$f\left(\int_T \Phi_t(x_t)\, d\mu(t)\right) \le \int_T \Phi_t(f(x_t))\, d\mu(t) \tag{1}$$

*holds for every bounded continuous fields $(x_t)_{t\in T}$ of self-adjoint elements in $\mathcal{A}$ with spectra contained in $I$.*

# 1.2. Generalization of converses of Jensen's operator inequality

In a long research series, Mond and Pečarić established the method which gives the reverse to Jensen inequality associated with convex functions. The principle yields a rich harvest in a field of operator inequalities. We call it **the Mond-Pečarić method** for convex functions. One of the most important attributes of Mond-Pečarić method is to offer a totally new viewpoint in the field of operator inequalities.

Using the Mond-Pečarić method, F.Hansen, J.Pečarić and I.Perić generalized the previous inequality similar to what they made with Jensen's inequality.

### Theorem

*Let $(x_t)_{t\in T}$ be a bounded continuous field of self-adjoint elements in a unital $C^*$-algebra $\mathcal{A}$ with spectra in $[m,M]$ defined on a locally compact Hausdorff space $T$ equipped with a bounded Radon measure $\mu$, and let $(\Phi_t)_{t\in T}$ be a unital field of positive linear maps $\Phi_t : \mathcal{A} \to \mathcal{B}$ from $\mathcal{A}$ to another unital $C^*-$algebra $\mathcal{B}$. Let $f,g : [m,M] \to \mathbb{R}$ and $F : U \times V \to \mathbb{R}$ be functions such that $f([m,M]) \subset U$, $g([m,M]) \subset V$ and $F$ is bounded. If $F$ is operator monotone in the first variable and $f$ is convex in the interval $[m,M]$, then*

$$F\left[\int_T \Phi_t(f(x_t))\,d\mu(t), g\left(\int_T \Phi_t(x_t)d\mu(t)\right)\right] \leq \sup_{m\leq z\leq M} F\left[\alpha_f z + \beta_f, g(z)\right]\mathbf{1}.$$

*In the dual case (when $f$ is operator concave) the opposite inequality holds with* sup *instead of* inf.

# 1.3. Next generalization

**Corollary**

*Let $\mathcal{A}$, $\mathcal{B}$, $(x_t)_{t\in T}$ be as above. Furthermore, let $(\phi_t)_{t\in T}$ be a field of positive linear maps $\phi_t : \mathcal{A} \to \mathcal{B}$, such that the field $t \mapsto \phi_t(\mathbf{1})$ is integrable with $\int_T \phi_t(\mathbf{1})\, d\mu(t) = k\mathbf{1}$ for some positive scalar $k$. Then the inequality*

$$f\left(\frac{1}{k}\int_T \phi_t(x_t)\, d\mu(t)\right) \leq \frac{1}{k}\int_T \phi_t(f(x_t))\, d\mu(t) \tag{2}$$

*holds for each operator convex function $f : I \to \mathbb{R}$ defined on $I$. In the dual case (when $f$ is operator concave) the opposite inequality holds in (2).*

We remark that if $\Phi(\mathbf{1}) = k\mathbf{1}$, for some positive scalar $k$ and $f$ is an operator convex function, then $f(\Phi(A)) \leq \Phi(f(A))$ is not true in general.

We remark that if $\Phi(\mathbf{1}) = k\mathbf{1}$, for some positive scalar $k$ and $f$ is an operator convex function, then $f(\Phi(A)) \leq \Phi(f(A))$ is not true in general.

## Example

A map $\Phi : \mathbf{M}_2\left(\mathbf{M}_2(\mathbb{C})\right) \to \mathbf{M}_2\left(\mathbf{M}_2(\mathbb{C})\right)$ defined by

$$\Phi\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} A+B & 0 \\ 0 & A+B \end{pmatrix} \quad \text{for } A, B \in \mathbf{M}_2(\mathbb{C})$$

is a positive linear map and $\Phi(I) = 2I$. We put $f(t) = t^2$ and

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}. \qquad \text{Then}$$

$$f\left(\Phi\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\right) - \Phi\left(f\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\right) = \begin{pmatrix} 4 & 3 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 3 & 2 \end{pmatrix} \ngeq 0.$$

# Generalization of converses of Jensen's inequality

We have a result of the Li-Mathias type:

**Theorem**

*Let $(x_t)_{t\in T}$ and $(\phi_t)_{t\in T}$ be as above, $f:[m,M]\to\mathbb{R}$, $g:[km,kM]\to\mathbb{R}$ and $F:U\times V\to\mathbb{R}$ be functions such that $(kf)([m,M])\subset U$, $g([km,kM])\subset V$ and $F$ is bounded. If $F$ is operator monotone in the first variable, then*

$$\begin{aligned}
&\inf_{km\le z\le kM} F\left[k\cdot h_1\left(\frac{1}{k}z\right), g(z)\right]\mathbf{1} \\
&\le F\left[\int_T \phi_t\left(f(x_t)\right)d\mu(t), g\left(\int_T \phi_t(x_t)d\mu(t)\right)\right] \\
&\le \sup_{km\le z\le kM} F\left[k\cdot h_2\left(\frac{1}{k}z\right), g(z)\right]\mathbf{1}
\end{aligned} \tag{3}$$

*holds for every operator convex function $h_1$ on $[m,M]$ such that $h_1\le f$ and for every operator concave function $h_2$ on $[m,M]$ such that $h_2\ge f$.*

Applying RHS of (3) for a convex function $f$ (or LHS of (3) for a concave function $f$) we obtain the following theorem:

## Theorem

*Let $(x_t)_{t\in T}$ and $(\Phi_t)_{t\in T}$ be as in § 2.1., $f : [m,M] \to \mathbb{R}$, $g : [km,kM] \to \mathbb{R}$ and $F : U \times V \to \mathbb{R}$ be functions such that $(kf)([m,M]) \subset U$, $g([km,kM]) \subset V$ and $F$ is bounded. If $F$ is operator monotone in the first variable and $f$ is convex in the interval $[m,M]$, then*

$$F\left[\int_T \Phi_t(f(x_t))\,d\mu(t), g\left(\int_T \Phi_t(x_t)d\mu(t)\right)\right] \leq \sup_{km\leq z\leq kM} F[\alpha_f z + \beta_f k, g(z)]\,\mathbf{1}. \tag{4}$$

*In the dual case (when $f$ is concave) the opposite inequality holds in (4) with* inf *instead of* sup.

# 2. Quasi-arithmetic mean

**A generalized quasi-arithmetic operator mean:**

$$M_{\varphi}(x,\Phi) := \varphi^{-1}\left(\int_T \frac{1}{k}\Phi_t\left(\varphi(x_t)\right) d\mu(t)\right), \tag{5}$$

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*under these conditions:* $(x_t)_{t\in T}$ is a bounded continuous field of operators in a $C^*$-algebra $B(H)$ with spectra in $[m,M]$ for some scalars $m<M$, $(\Phi_t)_{t\in T}$ is a field of positive linear maps $\Phi_t : B(H)\to B(K)$, such that the field $t\mapsto\Phi_t(\mathbf{1})$ is integrable with $\int_T \Phi_t(\mathbf{1})\,d\mu(t) = k\mathbf{1}$ for some positive scalar $k$ and $\varphi\in\mathcal{C}[m,M]$ is a strictly monotone function.

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# 2.1. Monotonicity

**First, we study the monotonicity of quasi-arithmetic means.**

**Theorem**

*Let* $(x_t)_{t\in T}$, $(\Phi_t)_{t\in T}$ *be as in the definition of the quasi-arithmetic mean* (5). *Let* $\psi, \varphi \in \mathcal{C}[m,M]$ *be strictly monotone functions.*

*If one of the following conditions is satisfied:*

(i) $\psi \circ \varphi^{-1}$ *is operator convex and* $\psi^{-1}$ *is operator monotone,*

(i') $\psi \circ \varphi^{-1}$ *is operator concave and* $-\psi^{-1}$ *is operator monotone,*

(ii) $\varphi^{-1}$ *is operator convex and* $\psi^{-1}$ *is operator concave,*

*then*

$$M_\varphi(x,\Phi) \le M_\psi(x,\Phi).$$

If one of the following conditions is satisfied:

(iii) $\psi \circ \varphi^{-1}$ is operator concave and $\psi^{-1}$ is operator monotone,

(iii') $\psi \circ \varphi^{-1}$ is operator convex and $-\psi^{-1}$ is operator monotone,

(iv) $\varphi^{-1}$ is operator concave and $\psi^{-1}$ is operator convex,

then

$$M_\psi(x,\Phi) \le M_\varphi(x,\Phi).$$

## Proof

*We will prove only the case (i).*
*If we put* $f = \psi \circ \varphi^{-1}$ *in the generalized Jensen's inequality and replace* $x_t$ *with* $\varphi(x_t)$, *then we obtain*

$$\begin{aligned}\psi \circ \varphi^{-1}\left(\int_T \frac{1}{k}\Phi_t(\varphi(x_t))\,d\mu(t)\right) &\le \int_T \frac{1}{k}\Phi_t\left(\psi \circ \varphi^{-1}(\varphi(x_t))\right)d\mu(t)\\ &= \int_T \frac{1}{k}\Phi_t(\psi(x_t))\,d\mu(t).\end{aligned}$$

*Since* $\psi^{-1}$ *is operator monotone, it follows*

$$\varphi^{-1}\left(\int_T \frac{1}{k}\Phi_t(\varphi(x_t))\,d\mu(t)\right) \le \psi^{-1}\left(\int_T \frac{1}{k}\Phi_t(\psi(x_t))\,d\mu(t)\right).$$

□

**Theorem**

*Let $(x_t)_{t\in T}$, $(\Phi_t)_{t\in T}$ be as in the definition of the quasi-arithmetic mean and $\psi, \varphi \in \mathcal{C}[m, M]$ be strictly monotone functions. Then*

$$M_\varphi(x, \Phi) = M_\psi(x, \Phi) \qquad \text{for all } (x_t)_{t\in T}, (\Phi_t)_{t\in T}$$

*if and only if*

$$\varphi = A\psi + B \qquad \text{for some real numbers } A \neq 0 \text{ and } B.$$

## 2.2. Difference and ratio type inequalities among quasi-arithmetic means

**Next, we study difference and ratio type inequalities among quasi-arithmetic means. We investigate the estimates of these inequalities, i.e. we will determine real constants $\alpha$ and $\beta$ such that**

$$M_\psi(x,\Phi) - M_\varphi(x,\Phi) \le \beta\,\mathbf{1} \qquad \text{and} \qquad M_\psi(x,\Phi) \le \alpha M_\varphi(x,\Phi)$$

**holds. With that in mind, we shall prove the following general result.**

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**holds. With that in mind, we shall prove the following general result.**

**Theorem**

*Let $(x_t)_{t\in T}$, $(\Phi_t)_{t\in T}$ be as in the definition of the quasi-arithmetic mean. Let $\psi,\varphi \in \mathcal{C}[m,M]$ be strictly monotone functions and $F:[m,M]\times[m,M]\to\mathbb{R}$ be a bounded and operator monotone function in its first variable.*

(continued)

If one of the following conditions is satisfied:

(i) $\psi \circ \varphi^{-1}$ is convex and $\psi^{-1}$ is operator monotone,

(i') $\psi \circ \varphi^{-1}$ is concave and $-\psi^{-1}$ is operator monotone,

then

$$F\left[M_\psi(x,\Phi), M_\varphi(x,\Phi)\right] \tag{6}$$
$$\leq \sup_{0\leq\theta\leq 1} F\left[\psi^{-1}\left(\theta\psi(m)+(1-\theta)\psi(M), \varphi^{-1}\left(\theta\varphi(m)+(1-\theta)\varphi(M)\right)\right)\right] \mathbf{1}.$$

(continued)

If one of the following conditions is satisfied:

(i) $\psi\circ\varphi^{-1}$ is convex and $\psi^{-1}$ is operator monotone,

(i') $\psi\circ\varphi^{-1}$ is concave and $-\psi^{-1}$ is operator monotone,

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If one of the following conditions is satisfied:

(ii) $\psi\circ\varphi^{-1}$ is concave and $\psi^{-1}$ is operator monotone,

(ii') $\psi\circ\varphi^{-1}$ is convex and $-\psi^{-1}$ is operator monotone,

then the opposite inequality is valid in (6) with inf instead of sup.

## Remark

*It is particularly interesting to observe inequalities when the function F in Theorem for $F(u,v)$ has the form $F(u,v) = u - v$ and $F(u,v) = v^{-1/2}uv^{-1/2}$ $(v > 0)$.*
*E.g. if (i) or (i') of this theorem is satisfied, then*

$$M_\psi(x,\Phi) \le M_\varphi(x,\Phi)$$
$$+ \sup_{0\le\theta\le1} \left\{ \psi^{-1}\left(\theta\psi(m) + (1-\theta)\psi(M)\right) - \varphi^{-1}\left(\theta\varphi(m) + (1-\theta)\varphi(M)\right) \right\} \mathbf{1},$$

*If in addition $\varphi > 0$,then*

$$M_\psi(x,\Phi) \le \sup_{0\le\theta\le1} \left\{ \frac{\psi^{-1}\left(\theta\psi(m) + (1-\theta)\psi(M)\right)}{\varphi^{-1}\left(\theta\varphi(m) + (1-\theta)\varphi(M)\right)} \right\} M_\varphi(x,\Phi).$$

We will investigate the above inequalities, with different assumptions. For this purpose, we introduce some notations for real valued continuous functions $\psi, \varphi \in \mathcal{C}[m, M]$

$$a_{\psi,\varphi} = \frac{\psi(M)-\psi(m)}{\varphi(M)-\varphi(m)}, \quad b_{\psi,\varphi} = \frac{M\psi(m)-M\psi(M)}{\varphi(M)-\varphi(m)}.$$

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$$a_{\psi,\varphi} = \frac{\psi(M)-\psi(m)}{\varphi(M)-\varphi(m)}, \quad b_{\psi,\varphi} = \frac{M\psi(m)-M\psi(M)}{\varphi(M)-\varphi(m)}.$$

**Theorem**

*Let $(x_t)_{t\in T}$, $(\Phi_t)_{t\in T}$ be as in the definition of the quasi-arithmetic mean and $\psi, \varphi \in \mathcal{C}[m, M]$ be strictly monotone functions. Let $\psi \circ \varphi^{-1}$ be convex (resp. concave).*

(i) *If $\psi^{-1}$ is operator monotone and subadditive (resp. superadditive) on $\mathbb{R}$, then*

## (continued)

$$M_\psi(x,\Phi) \leq M_\varphi(x,\Phi) + \psi^{-1}(\beta)\mathbf{1} \tag{7}$$
$$(\text{resp. } M_\psi(x,\Phi) \geq M_\varphi(x,\Phi) + \psi^{-1}(\beta)\mathbf{1}\ ),$$

(i') if $-\psi^{-1}$ is operator monotone and subadditive (resp. superadditive) on $\mathbb{R}$, then the opposite inequality is valid in (7),

(ii) if $\psi^{-1}$ is operator monotone and superadditive (resp. subadditive) on $\mathbb{R}$, then

$$M_\psi(x,\Phi) \leq M_\varphi(x,\Phi) - \varphi^{-1}(-\beta)\mathbf{1} \tag{8}$$
$$(\text{resp. } M_\psi(x,\Phi) \geq M_\varphi(x,\Phi) - \varphi^{-1}(-\beta)\mathbf{1}\ ),$$

(ii') if $-\psi^{-1}$ is operator monotone and superadditive (resp. subadditive) on $\mathbb{R}$, then the opposite inequality is valid in (8), Furthermore, if $\psi \circ \varphi^{-1}$ is strictly convex (resp. strictly concave) differentiable, then the constant $\beta \equiv \beta(m, M, \varphi, \psi)$ can be written more precisely as $\beta = a_{\psi,\varphi} z_0 + b_{\psi,\varphi} - \psi \circ \varphi^{-1}(z_0)$, where $z_0$ is the unique solution of the equation $(\psi \circ \varphi^{-1})'(z) = a_{\psi,\varphi}$, $(\varphi_m < z_0 < \varphi_M)$.

## Proof

*We will prove only the case (i). Putting in Theorem* $F(u,v) = u - \lambda v$, $\lambda = 1$, $f = g = \psi \circ \varphi^{-1}$ *and replacing* $\Phi_t$ *by* $\frac{1}{k}\Phi_t$, *we have*

$$\begin{aligned} \int_T \frac{1}{k}\Phi_t(\psi(x_t))\, d\mu(t) &= \int_T \frac{1}{k}\Phi_t\left(\psi \circ \varphi^{-1}(\varphi(x_t))\right) d\mu(t) \\ &\leq \psi \circ \varphi^{-1}\left(\int_T \frac{1}{k}\Phi_t(\varphi(x_t))\, d\mu(t)\right) + \beta \mathbf{1}, \end{aligned}$$

*where* $\beta$ *as in the theorem statement. Since* $\psi^{-1}$ *is operator monotone and subadditive on* $\mathbb{R}$, *then we obtain*
$M_\psi(x,\Phi) \leq \psi^{-1}\left(\psi \circ \varphi^{-1}\left(\int_T \Phi_t(\varphi(x_t))\, d\mu(t)\right) + \beta\mathbf{1}\right) \leq M_\varphi(x,\Phi) + \psi^{-1}(\beta)\mathbf{1}$.

$\square$

## Theorem

*Let $(x_t)_{t\in T}$, $(\Phi_t)_{t\in T}$ be as in the definition of the quasi-arithmetic mean and $\psi, \varphi \in \mathcal{C}[m,M]$ be strictly monotone functions. Let $\psi \circ \varphi^{-1}$ be convex and $\psi > 0$ on $[m,M]$.*

(i) *If $\psi^{-1}$ is operator monotone and submultiplicative on $\mathbb{R}$, then*

$$M_{\psi}(x,\Phi) \leq \psi^{-1}(\alpha)\, M_{\varphi}(x,\Phi), \tag{9}$$

(i') *if $-\psi^{-1}$ is operator monotone and submultiplicative on $\mathbb{R}$, then the opposite inequality is valid in (9),*

## Theorem

*Let $(x_t)_{t\in T}$, $(\Phi_t)_{t\in T}$ be as in the definition of the quasi-arithmetic mean and $\psi, \varphi \in \mathcal{C}[m,M]$ be strictly monotone functions. Let $\psi\circ\varphi^{-1}$ be convex and $\psi > 0$ on $[m,M]$.*

(i) *If $\psi^{-1}$ is operator monotone and submultiplicative on $\mathbb{R}$, then*

$$M_\psi(x,\Phi) \le \psi^{-1}(\alpha)\, M_\varphi(x,\Phi), \tag{9}$$

(i') *if $-\psi^{-1}$ is operator monotone and submultiplicative on $\mathbb{R}$, then the opposite inequality is valid in (9),*

(ii) *if $\psi^{-1}$ is operator monotone and supermultiplicative on $\mathbb{R}$, then*

$$M_\psi(x,\Phi) \le \left[\psi^{-1}(\alpha^{-1})\right]^{-1} M_\varphi(x,\Phi), \tag{10}$$

(ii') *if $-\psi^{-1}$ is operator monotone and supermultiplicative on $\mathbb{R}$, then the opposite inequality is valid in (10),*

*where*

## (continued)

$$\alpha = \max_{\varphi_m \le z \le \varphi_M} \left\{ \frac{a_{\psi,\varphi} z + b_{\psi,\varphi}}{\psi \circ \varphi^{-1}(z)} \right\} \quad \left( \text{resp. } \alpha = \min_{\varphi_m \le z \le \varphi_M} \left\{ \frac{a_{\psi,\varphi} z + b_{\psi,\varphi}}{\psi \circ \varphi^{-1}(z)} \right\} \right).$$

Furthermore, if $\psi \circ \varphi^{-1}$ is strictly convex differentiable, then the constant $\alpha \equiv \alpha(m, M, \varphi, \psi)$ can be written more precisely as

$$\alpha = \frac{a_{\psi,\varphi} z_0 + b_{\psi,\varphi}}{\psi \circ \varphi^{-1}(z_0)},$$

where $z_0$ is the unique solution of the equation $(\psi \circ \varphi^{-1})'(a_{\psi,\varphi} z + a_{\psi,\varphi}) = a_{\psi,\varphi} \cdot \psi \circ \varphi^{-1}(z)$, $(\varphi_m < z_0 < \varphi_M)$.

(continued)

$$\alpha = \max_{\varphi_m \le z \le \varphi_M} \left\{ \frac{a_{\psi,\varphi} z + b_{\psi,\varphi}}{\psi \circ \varphi^{-1}(z)} \right\} \quad \left( \text{resp. } \alpha = \min_{\varphi_m \le z \le \varphi_M} \left\{ \frac{a_{\psi,\varphi} z + b_{\psi,\varphi}}{\psi \circ \varphi^{-1}(z)} \right\} \right).$$

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where $z_0$ is the unique solution of the equation
$(\psi \circ \varphi^{-1})'(a_{\psi,\varphi} z + a_{\psi,\varphi}) = a_{\psi,\varphi} \cdot \psi \circ \varphi^{-1}(z)$, $(\varphi_m < z_0 < \varphi_M)$.

**Remark**

**We can obtain order among quasi-arithmetic means using the function order of positive operator.**

# 2.3. Power means

If we put $\varphi(t) = t^r$ and $\psi(t) = t^s$ in Theorem about monotonicity of quasi-arithmetic means, then we obtain the order among power means:

Let $(A_t)_{t\in T}$, $(\Phi_t)_{t\in T}$be as above. Let $A_t$ be a positive operator and $\int_T \Phi_t(\mathbf{1})\, d\mu(t) = k\mathbf{1}$ for some positive scalar $k$. Then

$$\left(\frac{1}{k}\int_T \Phi_t\left(A_t^r\right) d\mu(t)\right)^{1/r} \leq \left(\frac{1}{k}\int_T \Phi_t\left(A_t^s\right) d\mu(t)\right)^{1/s}$$

holds for either $r \leq s$, $r \notin (-1,1)$, $s \notin (-1,1)$ or $1/2 \leq r \leq 1 \leq s$ or $r \leq -1 \leq s \leq -1/2$.

In the remaining cases we need to use the function order of positive operator.

# 3. Future research
## 3.1 Jensen's inequality for operators without operator convexity

Jensen's opeator inequality would be false if we replaced an operator convex function by general convex.

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E.g.

We put mappings $\Phi_1, \Phi_2 : M_3(\mathbb{C}) \to M_2(\mathbb{C})$ as follows:
$\Phi_1((a_{ij})_{1\le i,j\le 3}) = \frac{1}{2}(a_{ij})_{1\le i,j\le 2}$, $\Phi_2 = \Phi_1$. Then $\Phi_1(I_3) + \Phi_2(I_3) = I_2$. If

$$A_1 = 2\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \textit{and} \quad A_2 = 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$ then we have

$$(\Phi_1(A_1) + \Phi_2(A_2))^4 = \begin{pmatrix} 16 & 0 \\ 0 & 0 \end{pmatrix} \not\leq \begin{pmatrix} 80 & 40 \\ 40 & 24 \end{pmatrix} = \Phi_1\left(A_1^4\right) + \Phi_2\left(A_2^4\right).$$

Namely we have no relation between $(\Phi_1(A_1) + \Phi_2(A_2))^4$ and $\Phi_1\left(A_1^4\right) + \Phi_2\left(A_2^4\right)$ under the operator order.

We observe that in above example we have the following bounds of matrices $A_1, A_2, A$: $A = \Phi_1(A_1) + \Phi_2(A_2) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ and $[m, M] = [0, 2]$, $[m_1, M_1] \subset [-1.60388, 4.49396]$, $[m_2, M_2] = [0, 2]$, i.e.

$$(m, M) \subset [m_1, M_1] \cup [m_2, M_2]$$

similarly as in Figure 1.a).

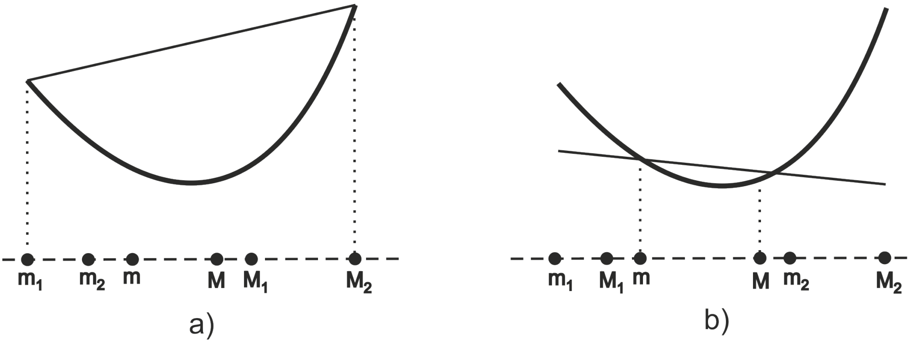


**Figure 1.** Spectral conditions for a convex function $f$

But, if $A_1 = \begin{pmatrix} -14 & 0 & 1 \\ 0 & -2 & -1 \\ 1 & -1 & -1 \end{pmatrix}$ *and* $A_2 = \begin{pmatrix} 15 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 15 \end{pmatrix}$, then we have $(\Phi_1(A_1) + \Phi_2(A_2))^4 = \begin{pmatrix} \frac{1}{16} & 0 \\ 0 & 0 \end{pmatrix} \leq \begin{pmatrix} 89660 & -247 \\ -247 & 51 \end{pmatrix} = \Phi_1\left(A_1^4\right) + \Phi_2\left(A_2^4\right)$. So we have that an equality of Jensen' type now is valid.

In this case we have $A = \Phi_1(A_1) + \Phi_2(A_2) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}$ and $[m, M] = [0, 0.5]$, $[m_1, M_1] \subset [-14.077, -0.328566]$, $[m_2, M_2] = [2, 15]$, i.e.

$$(m, M) \cap [m_1, M_1] = \emptyset \quad \text{and} \quad (m, M) \cap [m_2, M_2] = \emptyset.$$

similarly as in Figure 1.b).

It is no coincidence that Jensen' operator inequality is valid in this example. In the following theorem we prove a general result when Jensen's operator inequality holds for convex functions.

## Theorem

*Let $(A_1,\dots,A_n)$ be $n-$tuple of self-adjoint operators $A_i \in B(H)$ with bounds $m_i$ and $M_i$, $m_i \le M_i$, $i=1,\dots,n$. Let $(\Phi_1,\dots,\Phi_n)$ be $n-$tuple of positive linear mappings $\Phi_i : B(H) \to B(K)$, $i=1,\dots,n$, such that $\sum_{i=1}^n \Phi_i(1_H) = 1_K$. If*

$$(m_A, M_A) \cap [m_i, M_i] = \emptyset \quad \textit{for } i=1,\dots,n,$$

*where $m_A$ and $M_A$, $m_A \le M_A$, are bounds of a self-adjoint operator $A = \sum_{i=1}^n \Phi_i(A_i)$, then*

$$f\left(\sum_{i=1}^n \Phi_i(A_i)\right) \le \sum_{i=1}^n \Phi_i(f(A_i)) \tag{11}$$

*holds for every continuous convex function $f : I \to \mathbb{R}$ provided that the interval $I$ contains all $m_i, M_i$.*
*If $f : I \to \mathbb{R}$ is concave, then the reverse inequality is valid in (11).*

# 3.2 Quasi-arithmetic means without operator convexity

Now, we can observe the monotonicity of the quasi-arithmetic mean without operator convexity. E.g.

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**Theorem**

*Let $(A_1,\dots,A_n)$ and $(\Phi_1,\dots,\Phi_n)$ be as above. Let $m_i$ and $M_i$, $m_i \le M_i$ be bounds of $A_i$, $i=1,\dots,n$. Let $\varphi,\psi : I \to \mathbb{R}$ be continuous strictly monotone functions on an interval $I$ which contains all $m_i, M_i$. Let $m_\varphi$ and $M_\varphi$, $m_\varphi \le M_\varphi$, be bounds of the mean $M_\varphi(\mathbf{A},\Phi,n)$, such that*

$$(m_\varphi, M_\varphi) \cap [m_i, M_i] = \emptyset \qquad i=1,\dots,n.$$

*If one of the following conditions is satisfied:*

(i) $\psi\circ\varphi^{-1}$ *is convex and* $\psi^{-1}$ *is operator monotone,*

(i') $\psi\circ\varphi^{-1}$ *is concave and* $-\psi^{-1}$ *is operator monotone,*

*then*

$$M_\varphi(\mathbf{A},\Phi,n) \le M_\psi(\mathbf{A},\Phi,n).$$

# 3.3 Converse inequalities with condition on spectra

Finally, we can observe converse to all above inequalities with condition on spectra. E.g.

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**Theorem**

*Let $A_1,\ldots,A_n \in \mathcal{B}(H)$ be self-adjoint operators with bounds $m_1 \leq M_1,\ldots,m_n \leq M_n$ and $[m,M]$ be an interval with endpoints $m = \min\{m_1,\ldots,m_n\}$ and $M = \max\{M_1,\ldots,M_n\}$ where $m < M$. Let $\Phi_1,\ldots,\Phi_n : \mathcal{B}(H) \to \mathcal{B}(K)$ be positive linear mappings such that $\sum_{i=1}^n \Phi_i(1_H) = 1_K$. Let $A = \sum_{i=1}^n \Phi_i(A_i) \in \mathcal{B}(K)$ be the sum of operators $\Phi_i(A_i)$ with bounds $m_A \leq M_A$. Let the spectral conditions*

$$(m_A, M_A) \cap [m_i, M_i] = \emptyset \quad \text{for } i = 1,\ldots,n$$

*are valid. Let $f : [m,M] \to \mathbb{R}$ be continuous function.*

(continued)

If $f$ is convex, then

$$\sum_{i=1}^{n} \Phi_i(f(A_i)) \le \max_{m_A \le t \le M_A} \left\{ f_{[m,M]}^{cho}(t) - \alpha f(t) \right\} 1_K + \alpha f\left( \sum_{i=1}^{n} \Phi_i(A_i) \right)$$

holds for every strictly positive $\alpha \in \mathbb{R}$.
If $f$ is convex, and strictly positive on $[m_A, M_A]$, then

$$\sum_{i=1}^{n} \Phi_i(f(A_i)) \le \max_{m_A \le t \le M_A} \left\{ \frac{f_{[m,M]}^{cho}(t)}{f(t)} \right\} f\left( \sum_{i=1}^{n} \Phi_i(A_i) \right).$$

Also, we can observe the order among quasi-arithmetic means and power means under the same condition on spectra.

**References:** These papers are submitted in some journals or they are manuscripts.