Order among quasi-arithmetic operator means for convex functions

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1.1. Generalization of Jensen’s operator inequality

Let $T$ be a locally compact Hausdorff space and let $A$ be a $C^*$-algebra of operators on some Hilbert space $H$. We say that a field $(x_t)_{t \in T}$ of operators in $A$ is continuous if the function $t \mapsto x_t$ is norm continuous on $T$. If in addition $\mu$ is a Radon measure on $T$ and the function $t \mapsto \|x_t\|$ is integrable, then we can form the Bochner integral $\int_T x_t \, d\mu(t)$, which is the unique element in $A$ such that

$$\varphi\left(\int_T x_t \, d\mu(t)\right) = \int_T \varphi(x_t) \, d\mu(t)$$

for every linear functional $\varphi$ in the norm dual $A^*$.

Assume furthermore that $(\Phi_t)_{t \in T}$ is a field of positive linear mappings $\Phi_t : A \to B$ from $A$ to another $C^*$-algebra $B$ of operators on a Hilbert space $K$. We say that such a field is continuous if the function $t \mapsto \Phi_t(x)$ is continuous for every $x \in A$. If the $C^*$-algebras include the identity operators, and the field $t \mapsto \Phi_t(1)$ is integrable with integral equals 1, we say that $(\Phi_t)_{t \in T}$ is unital.
Jensen’s inequality


Find a generalization of Jensen’s operator inequality:

**Theorem**

Let $f : I \to \mathbb{R}$ be an operator convex functions defined on an interval $I$, and let $\mathcal{A}$ and $\mathcal{B}$ be a unital $C^*$-algebras. If $(\Phi_t)_{t \in T}$ is a unital field of positive linear mappings $\Phi_t : \mathcal{A} \to \mathcal{B}$ defined on a locally compact Hausdorff space $T$ with a bounded Radon measure $\mu$, then

$$f \left( \int_T \Phi_t(x_t) \, d\mu(t) \right) \leq \int_T \Phi_t(f(x_t)) \, d\mu(t)$$

(1)

holds for every bounded continuous fields $(x_t)_{t \in T}$ of self-adjoint elements in $\mathcal{A}$ with spectra contained in $I$. 
1.2. Generalization of converses of Jensen’s operator inequality

In a long research series, Mond and Pečarić established the method which gives the reverse to Jensen inequality associated with convex functions. The principle yields a rich harvest in a field of operator inequalities. We call it the Mond-Pečarić method for convex functions. One of the most important attributes of Mond-Pečarić method is to offer a totally new viewpoint in the field of operator inequalities. Using the Mond-Pečarić method, F.Hansen, J.Pečarić and I.Perić generalized the previous inequality similar to what they made with Jensen’s inequality.
Theorem

Let \((x_t)_{t \in T}\) be a bounded continuous field of self-adjoint elements in a unital \(C^*\)-algebra \(\mathcal{A}\) with spectra in \([m, M]\) defined on a locally compact Hausdorff space \(T\) equipped with a bounded Radon measure \(\mu\), and let \((\Phi_t)_{t \in T}\) be a unital field of positive linear maps \(\Phi_t : \mathcal{A} \to \mathcal{B}\) from \(\mathcal{A}\) to another unital \(C^*\)-algebra \(\mathcal{B}\). Let \(f, g : [m, M] \to \mathbb{R}\) and \(F : U \times V \to \mathbb{R}\) be functions such that \(f([m, M]) \subset U\), \(g([m, M]) \subset V\) and \(F\) is bounded. If \(F\) is operator monotone in the first variable and \(f\) is convex in the interval \([m, M]\), then

\[
F \left[ \int_T \Phi_t(f(x_t)) \, d\mu(t), g \left( \int_T \Phi_t(x_t) \, d\mu(t) \right) \right] \leq \sup_{m \leq z \leq M} F [\alpha_f z + \beta_f, g(z)] 1.
\]

In the dual case (when \(f\) is operator concave) the opposite inequality holds with \(\sup\) instead of \(\inf\).
1.3. Next generalization

Corollary

Let \( A, B, (x_t)_{t \in T} \) be as above. Furthermore, let \((\phi_t)_{t \in T}\) be a field of positive linear maps \( \phi_t : A \to B \), such that the field \( t \mapsto \phi_t(1) \) is integrable with \( \int_T \phi_t(1) \, d\mu(t) = k \cdot 1 \) for some positive scalar \( k \). Then the inequality

\[
f \left( \frac{1}{k} \int_T \phi_t(x_t) \, d\mu(t) \right) \leq \frac{1}{k} \int_T \phi_t(f(x_t)) \, d\mu(t)
\]

(2)

holds for each operator convex function \( f : I \to \mathbb{R} \) defined on \( I \). In the dual case (when \( f \) is operator concave) the opposite inequality holds in (2).
We remark that if $\Phi(1) = k1$, for some positive scalar $k$ and $f$ is an operator convex function, then $f(\Phi(A)) \leq \Phi(f(A))$ is not true in general.
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Example

A map $\Phi : M_2(M_2(\mathbb{C})) \to M_2(M_2(\mathbb{C}))$ defined by

$$\Phi\left( \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \right) = \begin{pmatrix} A + B & 0 \\ 0 & A + B \end{pmatrix}$$

for $A, B \in M_2(\mathbb{C})$

is a positive linear map and $\Phi(I) = 2I$. We put $f(t) = t^2$ and

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$f\left( \Phi\left( \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \right) \right) - \Phi\left( f\left( \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \right) \right) = \begin{pmatrix} 4 & 3 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 3 & 2 \end{pmatrix} \ngeq 0.$$
Generalization of converses of Jensen’s inequality

We have a result of the Li-Mathias type:

**Theorem**

Let \((x_t)_{t \in T}\) and \((\phi_t)_{t \in T}\) be as above, \(f : [m, M] \rightarrow \mathbb{R}, \ g : [km, kM] \rightarrow \mathbb{R}\)

and \(F : U \times V \rightarrow \mathbb{R}\) be functions such that \((kf)([m, M]) \subset U,\)

\(g([km, kM]) \subset V\) and \(F\) is bounded. If \(F\) is operator monotone in the first variable, then

\[
\inf_{km \leq z \leq kM} F \left[ k \cdot h_1 \left( \frac{1}{k} z \right), g(z) \right] \leq F \left[ \int_T \phi_t (f(x_t)) d\mu(t) , g \left( \int_T \phi_t (x_t) d\mu(t) \right) \right] \leq \sup_{km \leq z \leq kM} F \left[ k \cdot h_2 \left( \frac{1}{k} z \right), g(z) \right]
\]

(3)

holds for every operator convex function \(h_1\) on \([m, M]\) such that \(h_1 \leq f\)

and for every operator concave function \(h_2\) on \([m, M]\) such that \(h_2 \geq f\).
Applying RHS of (3) for a convex function $f$ (or LHS of (3) for a concave function $f$) we obtain the following theorem:

**Theorem**

Let $(x_t)_{t \in T}$ and $(\Phi_t)_{t \in T}$ be as in § 2.1., $f : [m, M] \to \mathbb{R}$, $g : [km, kM] \to \mathbb{R}$ and $F : U \times V \to \mathbb{R}$ be functions such that $(kf)([m, M]) \subset U$, $g([km, kM]) \subset V$ and $F$ is bounded. If $F$ is operator monotone in the first variable and $f$ is convex in the interval $[m, M]$, then

$$F \left[ \int_T \Phi_t(f(x_t)) \, d\mu(t), g \left( \int_T \Phi_t(x_t) \, d\mu(t) \right) \right] \leq \sup_{km \leq z \leq kM} F \left[ \alpha_f z + \beta_f k, g(z) \right].$$

(4)

In the dual case (when $f$ is concave) the opposite inequality holds in (4) with $\inf$ instead of $\sup$. 
2. Quasi-arithmetic mean

A generalized quasi-arithmetic operator mean:

\[
M_\varphi(x, \Phi) := \varphi^{-1} \left( \int_T \frac{1}{k} \Phi_t (\varphi(x_t)) \, d\mu(t) \right),
\]

(5)
2. Quasi-arithmetic mean

A generalized quasi-arithmetic operator mean:

\[ M_\varphi(x, \Phi) := \varphi^{-1} \left( \int_T \frac{1}{k} \Phi_t(\varphi(x_t)) \, d\mu(t) \right), \tag{5} \]

**under these conditions:** \((x_t)_{t \in T}\) is a bounded continuous field of operators in a \(C^*\)-algebra \(B(H)\) with spectra in \([m, M]\) for some scalars \(m < M\), \((\Phi_t)_{t \in T}\) is a field of positive linear maps \(\Phi_t : B(H) \to B(K)\), such that the field \(t \mapsto \Phi_t(1)\) is integrable with \(\int_T \Phi_t(1) \, d\mu(t) = k1\) for some positive scalar \(k\) and \(\varphi \in C[m, M]\) is a strictly monotone function.
2. Quasi-arithmetic mean

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\[ M_\varphi(x, \Phi) := \varphi^{-1}\left( \int_T \frac{1}{k} \Phi_t(\varphi(x_t)) \, d\mu(t) \right) \],

(5)

under these conditions: \((x_t)_{t \in T}\) is a bounded continuous field of operators in a \(C^*\)-algebra \(B(H)\) with spectra in \([m, M]\) for some scalars \(m < M\), \((\Phi_t)_{t \in T}\) is a field of positive linear maps \(\Phi_t : B(H) \to B(K)\), such that the field \(t \mapsto \Phi_t(1)\) is integrable with \(\int_T \Phi_t(1) \, d\mu(t) = k1\) for some positive scalar \(k\) and \(\varphi \in C[m, M]\) is a strictly monotone function.
2.1. Monotonicity

First, we study the monotonicity of quasi-arithmetic means.

Theorem

Let \((x_t)_{t \in T}, (\Phi_t)_{t \in T}\) be as in the definition of the quasi-arithmetic mean \((5)\). Let \(\psi, \phi \in C[m, M]\) be strictly monotone functions. If one of the following conditions is satisfied:

(i) \(\psi \circ \phi^{-1}\) is operator convex and \(\psi^{-1}\) is operator monotone,

(i') \(\psi \circ \phi^{-1}\) is operator concave and \(-\psi^{-1}\) is operator monotone,

(ii) \(\phi^{-1}\) is operator convex and \(\psi^{-1}\) is operator concave,

then

\[ M_\phi(x, \Phi) \leq M_\psi(x, \Phi). \]
If one of the following conditions is satisfied:

(iii) \( \psi \circ \varphi^{-1} \) is operator concave and \( \psi^{-1} \) is operator monotone,

(iii') \( \psi \circ \varphi^{-1} \) is operator convex and \( -\psi^{-1} \) is operator monotone,

(iv) \( \varphi^{-1} \) is operator concave and \( \psi^{-1} \) is operator convex,

then

\[
M_{\psi}(x, \Phi) \leq M_{\varphi}(x, \Phi).
\]
Proof

We will prove only the case (i).

If we put \( f = \psi \circ \varphi^{-1} \) in the generalized Jensen’s inequality and replace \( x_t \) with \( \varphi(x_t) \), then we obtain

\[
\psi \circ \varphi^{-1} \left( \int_T \frac{1}{k} \Phi_t(\varphi(x_t)) \, d\mu(t) \right) \leq \int_T \frac{1}{k} \Phi_t \left( \psi \circ \varphi^{-1}(\varphi(x_t)) \right) \, d\mu(t) \\
= \int_T \frac{1}{k} \Phi_t(\psi(x_t)) \, d\mu(t).
\]

Since \( \psi^{-1} \) is operator monotone, it follows

\[
\varphi^{-1} \left( \int_T \frac{1}{k} \Phi_t(\varphi(x_t)) \, d\mu(t) \right) \leq \psi^{-1} \left( \int_T \frac{1}{k} \Phi_t(\psi(x_t)) \, d\mu(t) \right).
\]
Theorem

Let \( (x_t)_{t \in T}, (\Phi_t)_{t \in T} \) be as in the definition of the quasi-arithmetic mean and \( \psi, \varphi \in C[m, M] \) be strictly monotone functions. Then

\[
M_\varphi(x, \Phi) = M_\psi(x, \Phi)
\]

for all \( (x_t)_{t \in T}, (\Phi_t)_{t \in T} \)

if and only if

\[
\varphi = A\psi + B \quad \text{for some real numbers } A \neq 0 \text{ and } B.
\]
Next, we study difference and ratio type inequalities among quasi-arithmetic means. We investigate the estimates of these inequalities, i.e. we will determine real constants $\alpha$ and $\beta$ such that

$$M_\psi(x, \Phi) - M_\phi(x, \Phi) \leq \beta \quad \text{and} \quad M_\psi(x, \Phi) \leq \alpha M_\phi(x, \Phi)$$

holds. With that in mind, we shall prove the following general result.
2.2. Difference and ratio type inequalities among quasi-arithmetic means

Next, we study difference and ratio type inequalities among quasi-arithmetic means. We investigate the estimates of these inequalities, i.e. we will determine real constants $\alpha$ and $\beta$ such that

$$M_\psi(x, \Phi) - M_\phi(x, \Phi) \leq \beta \quad \text{and} \quad M_\psi(x, \Phi) \leq \alpha M_\phi(x, \Phi)$$

holds. With that in mind, we shall prove the following general result.

**Theorem**

Let $(x_t)_{t \in T}$, $(\Phi_t)_{t \in T}$ be as in the definition of the quasi-arithmetic mean. Let $\psi, \phi \in C[m, M]$ be strictly monotone functions and $F : [m, M] \times [m, M] \to \mathbb{R}$ be a bounded and operator monotone function in its first variable.
If one of the following conditions is satisfied:

(i) $\psi \circ \varphi^{-1}$ is convex and $\psi^{-1}$ is operator monotone,

(i') $\psi \circ \varphi^{-1}$ is concave and $-\psi^{-1}$ is operator monotone,

then

$$F \left[ M_\psi(x, \Phi), M_\varphi(x, \Phi) \right] \leq \sup_{0 \leq \theta \leq 1} F \left[ \psi^{-1} \left( \theta \psi(m) + (1 - \theta)\psi(M) \right), \varphi^{-1} \left( \theta \varphi(m) + (1 - \theta)\varphi(M) \right) \right].$$

(6)
(continued)

If one of the following conditions is satisfied:

(i) \( \psi \circ \varphi^{-1} \) is convex and \( \psi^{-1} \) is operator monotone,

(ii) \( \psi \circ \varphi^{-1} \) is concave and \( \psi^{-1} \) is operator monotone,

then

\[
F \left[ M_\psi(x, \Phi), M_\varphi(x, \Phi) \right] \leq \sup_{0 \leq \theta \leq 1} F \left[ \psi^{-1} \left( \theta \psi(m) + (1 - \theta)\psi(M) \right), \varphi^{-1} \left( \theta \varphi(m) + (1 - \theta)\varphi(M) \right) \right] 1.
\]

If one of the following conditions is satisfied:

(ii) \( \psi \circ \varphi^{-1} \) is concave and \( \psi^{-1} \) is operator monotone,

(ii') \( \psi \circ \varphi^{-1} \) is convex and \( -\psi^{-1} \) is operator monotone,

then the opposite inequality is valid in (6) with \( \inf \) instead of \( \sup \).
Remark

It is particularly interesting to observe inequalities when the function $F$ in Theorem for $F(u, v)$ has the form $F(u, v) = u - v$ and $F(u, v) = v^{-1/2}uv^{-1/2}$ ($v > 0$).

E.g. if (i) or (i') of this theorem is satisfied, then

$$M_\psi(x, \Phi) \leq M_\varphi(x, \Phi)$$

$$+ \sup_{0 \leq \theta \leq 1} \left\{ \psi^{-1} (\theta \psi(m) + (1 - \theta)\psi(M)) - \varphi^{-1} (\theta \varphi(m) + (1 - \theta)\varphi(M)) \right\} \mathbf{1},$$

If in addition $\varphi > 0$, then

$$M_\psi(x, \Phi) \leq \sup_{0 \leq \theta \leq 1} \left\{ \frac{\psi^{-1} (\theta \psi(m) + (1 - \theta)\psi(M))}{\varphi^{-1} (\theta \varphi(m) + (1 - \theta)\varphi(M))} \right\} M_\varphi(x, \Phi).$$
We will investigate the above inequalities, with different assumptions. For this purpose, we introduce some notations for real valued continuous functions \( \psi, \varphi \in C[m, M] \)

\[
\begin{align*}
a_{\psi, \varphi} &= \frac{\psi(M) - \psi(m)}{\varphi(M) - \varphi(m)}, \\
b_{\psi, \varphi} &= \frac{M\psi(m) - M\psi(M)}{\varphi(M) - \varphi(m)}.
\end{align*}
\]
We will investigate the above inequalities, with different assumptions. For this purpose, we introduce some notations for real valued continuous functions $\psi, \varphi \in C[m, M]$

\[
\begin{align*}
\alpha_{\psi, \varphi} &= \frac{\psi(M) - \psi(m)}{\varphi(M) - \varphi(m)}, \\
\beta_{\psi, \varphi} &= \frac{M \psi(m) - M \psi(M)}{\varphi(M) - \varphi(m)}.
\end{align*}
\]

**Theorem**

Let $(x_t)_{t \in T}$, $(\Phi_t)_{t \in T}$ be as in the definition of the quasi-arithmetic mean and $\psi, \varphi \in C[m, M]$ be strictly monotone functions. Let $\psi \circ \varphi^{-1}$ be convex (resp. concave).

(i) If $\psi^{-1}$ is operator monotone and subadditive (resp. superadditive) on $\mathbb{R}$, then
(continued)

\[ M_\psi(x, \Phi) \leq M_\phi(x, \Phi) + \psi^{-1}(\beta)1 \quad (7) \]
\[ (\text{resp. } M_\psi(x, \Phi) \geq M_\phi(x, \Phi) + \psi^{-1}(\beta)1) \]

(i') if \(-\psi^{-1}\) is operator monotone and subadditive (resp. superadditive) on \(\mathbb{R}\), then the opposite inequality is valid in (7),

(ii) if \(\psi^{-1}\) is operator monotone and superadditive (resp. subadditive) on \(\mathbb{R}\), then

\[ M_\psi(x, \Phi) \leq M_\phi(x, \Phi) - \phi^{-1}(-\beta)1 \quad (8) \]
\[ (\text{resp. } M_\psi(x, \Phi) \geq M_\phi(x, \Phi) - \phi^{-1}(-\beta)1) \]

(ii') if \(-\psi^{-1}\) is operator monotone and superadditive (resp. subadditive) on \(\mathbb{R}\), then the opposite inequality is valid in (8),
Furthermore, if $\psi \circ \varphi^{-1}$ is strictly convex (resp. strictly concave) differentiable, then the constant $\beta \equiv \beta(m, M, \varphi, \psi)$ can be written more precisely as

$$\beta = a_{\psi, \varphi} z_0 + b_{\psi, \varphi} - \psi \circ \varphi^{-1}(z_0),$$

where $z_0$ is the unique solution of the equation

$$(\psi \circ \varphi^{-1})'(z) = a_{\psi, \varphi}, \quad (\varphi_m < z_0 < \varphi_M).$$

**Proof**

*We will prove only the case (i).* Putting in Theorem $F(u, v) = u - \lambda v$, $\lambda = 1$, $f = g = \psi \circ \varphi^{-1}$ and replacing $\Phi_t$ by $\frac{1}{k} \Phi_t$, we have

$$\int_T \frac{1}{k} \Phi_t(\psi(x_t)) \, d\mu(t) = \int_T \frac{1}{k} \Phi_t \left( \psi \circ \varphi^{-1}(\varphi(x_t)) \right) \, d\mu(t)$$

$$\leq \psi \circ \varphi^{-1} \left( \int_T \frac{1}{k} \Phi_t(\varphi(x_t)) \, d\mu(t) \right) + \beta 1,$$

where $\beta$ as in the theorem statement. Since $\psi^{-1}$ is operator monotone and subadditive on $\mathbb{R}$, then we obtain

$$M_\psi(x, \Phi) \leq \psi^{-1} \left( \psi \circ \varphi^{-1} \left( \int_T \Phi_t(\varphi(x_t)) \, d\mu(t) \right) + \beta 1 \right) \leq M_\varphi(x, \Phi) + \psi^{-1}(\beta) 1.$$  

$\square$
Theorem

Let \((x_t)_{t \in T}, (\Phi_t)_{t \in T}\) be as in the definition of the quasi-arithmetic mean and \(\psi, \varphi \in C[m, M]\) be strictly monotone functions. Let \(\psi \circ \varphi^{-1}\) be convex and \(\psi > 0\) on \([m, M]\).

(i) If \(\psi^{-1}\) is operator monotone and submultiplicative on \(\mathbb{R}\), then

\[
M_{\psi}(x, \Phi) \leq \psi^{-1}(\alpha) M_{\varphi}(x, \Phi),
\]  

(9)

(ii) if \(\psi^{-1}\) is operator monotone and supermultiplicative on \(\mathbb{R}\), then the opposite inequality is valid in (9),

\[
M_{\psi}(x, \Phi) \leq \frac{1}{\psi^{-1}(\alpha - 1)} M_{\varphi}(x, \Phi),
\]  

(10)
Theorem

Let \((x_t)_{t \in T}, (\Phi_t)_{t \in T}\) be as in the definition of the quasi-arithmetic mean and \(\psi, \varphi \in C[m, M]\) be strictly monotone functions. Let \(\psi \circ \varphi^{-1}\) be convex and \(\psi > 0\) on \([m, M]\).

(i) If \(\psi^{-1}\) is operator monotone and submultiplicative on \(\mathbb{R}\), then

\[
M_\psi(x, \Phi) \leq \psi^{-1}(\alpha) M_\varphi(x, \Phi),
\]

(ii) If \(\psi^{-1}\) is operator monotone and supermultiplicative on \(\mathbb{R}\), then

\[
M_\psi(x, \Phi) \leq \left[\psi^{-1}(\alpha^{-1})\right]^{-1} M_\varphi(x, \Phi),
\]

where
(continued)

\[ \alpha = \max_{\varphi_m \leq z \leq \varphi_M} \left\{ \frac{a_{\psi,\varphi}z + b_{\psi,\varphi}}{\psi \circ \varphi^{-1}(z)} \right\} \quad \text{(resp.} \quad \alpha = \min_{\varphi_m \leq z \leq \varphi_M} \left\{ \frac{a_{\psi,\varphi}z + b_{\psi,\varphi}}{\psi \circ \varphi^{-1}(z)} \right\} \text{)}.\]

Furthermore, if \( \psi \circ \varphi^{-1} \) is strictly convex differentiable, then the constant \( \alpha \equiv \alpha(m, M, \varphi, \psi) \) can be written more precisely as

\[ \alpha = \frac{a_{\psi,\varphi}z_0 + b_{\psi,\varphi}}{\psi \circ \varphi^{-1}(z_0)}, \]

where \( z_0 \) is the unique solution of the equation

\[ (\psi \circ \varphi^{-1})'(a_{\psi,\varphi}z + a_{\psi,\varphi}) = a_{\psi,\varphi} \cdot \psi \circ \varphi^{-1}(z), \quad (\varphi_m < z_0 < \varphi_M). \]
\[
\alpha = \max_{\varphi_m \leq z \leq \varphi_M} \left\{ \frac{a_{\psi,\varphi} z + b_{\psi,\varphi}}{\psi \circ \varphi^{-1}(z)} \right\} \quad \text{(resp.} \quad \alpha = \min_{\varphi_m \leq z \leq \varphi_M} \left\{ \frac{a_{\psi,\varphi} z + b_{\psi,\varphi}}{\psi \circ \varphi^{-1}(z)} \right\} \).
\]

Furthermore, if \( \psi \circ \varphi^{-1} \) is strictly convex differentiable, then the constant \( \alpha \equiv \alpha(m, M, \varphi, \psi) \) can be written more precisely as

\[
\alpha = \frac{a_{\psi,\varphi} z_0 + b_{\psi,\varphi}}{\psi \circ \varphi^{-1}(z_0)},
\]

where \( z_0 \) is the unique solution of the equation

\[
(\psi \circ \varphi^{-1})'(a_{\psi,\varphi} z + a_{\psi,\varphi}) = a_{\psi,\varphi} \cdot \psi \circ \varphi^{-1}(z), \quad (\varphi_m < z_0 < \varphi_M).
\]

**Remark**

We can obtain order among quasi-arithmetic means using the function order of positive operator.
2.3. Power means

If we put $\varphi(t) = t^r$ and $\psi(t) = t^s$ in Theorem about monotonicity of quasi-arithmetic means, then we obtain the order among power means:

Let $(A_t)_{t \in T}, (\Phi_t)_{t \in T}$ be as above. Let $A_t$ be a positive operator and $\int_T \Phi_t(1) \, d\mu(t) = k1$ for some positive scalar $k$. Then

$$\left( \frac{1}{k} \int_T \Phi_t(A_t^r) \, d\mu(t) \right)^{1/r} \leq \left( \frac{1}{k} \int_T \Phi_t(A_t^s) \, d\mu(t) \right)^{1/s}$$

holds for either $r \leq s$, $r \not\in (-1, 1)$, $s \not\in (-1, 1)$ or $1/2 \leq r \leq 1 \leq s$ or $r \leq -1 \leq s \leq -1/2$.

In the remaining cases we need to use the function order of positive operator.
3. Future research

3.1 Jensen’s inequality for operators without operator convexity

Jensen’s operator inequality would be false if we replaced an operator convex function by general convex.

E.g.

We put mappings \( \Phi_1, \Phi_2 : M_3(\mathbb{C}) \to M_2(\mathbb{C}) \) as follows:

\[
\Phi_1((a_{ij})_{1 \leq i,j \leq 3}) = (a_{ij})_{1 \leq i,j \leq 2}, \quad \Phi_2 = \Phi_1.
\]

Then

\[
\Phi_1(I_3) + \Phi_2(I_3) = I_2.
\]

If

\[
A_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

then we have

\[
(\Phi_1(A_1) + \Phi_2(A_2)) \not\leq (80 \ 40 \\ 40 \ 24) = \Phi_1(A_1) + \Phi_2(A_2).
\]

Namely we have no relation between

\[
(\Phi_1(A_1) + \Phi_2(A_2))
\]

and

\[
\Phi_1(A_1) + \Phi_2(A_2)
\]

under the operator order.
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E.g.

We put mappings $\Phi_1, \Phi_2 : M_3(\mathbb{C}) \to M_2(\mathbb{C})$ as follows:

$\Phi_1((a_{ij})_{1 \leq i,j \leq 3}) = \frac{1}{2}(a_{ij})_{1 \leq i,j \leq 2}$, $\Phi_2 = \Phi_1$. Then $\Phi_1(I_3) + \Phi_2(I_3) = I_2$. If $A_1 = 2 \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ and $A_2 = 2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, then we have

$(\Phi_1(A_1) + \Phi_2(A_2))^4 = \begin{pmatrix} 16 & 0 \\ 0 & 0 \end{pmatrix} \not\leq \begin{pmatrix} 80 & 40 \\ 40 & 24 \end{pmatrix} = \Phi_1(A_1^4) + \Phi_2(A_2^4)$.

Namely we have no relation between $(\Phi_1(A_1) + \Phi_2(A_2))^4$ and $\Phi_1(A_1^4) + \Phi_2(A_2^4)$ under the operator order.
We observe that in above example we have the following bounds of matrices $A_1, A_2, A$: $A = \Phi_1(A_1) + \Phi_2(A_2) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ and $[m, M] = [0, 2]$, $[m_1, M_1] \subset [-1.60388, 4.49396]$, $[m_2, M_2] = [0, 2]$, i.e. 

$$(m, M) \subset [m_1, M_1] \cup [m_2, M_2]$$

similarly as in Figure 1.a).

![Figure 1. Spectral conditions for a convex function $f$](image)
Future research

Jensen’s inequality

But, if $A_1 = \begin{pmatrix} -14 & 0 & 1 \\ 0 & -2 & -1 \\ 1 & -1 & -1 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 15 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 15 \end{pmatrix}$, then we have $\left( \phi_1(A_1) + \phi_2(A_2) \right)^4 = \begin{pmatrix} \frac{1}{16} & 0 \\ 0 & 0 \end{pmatrix} \preceq \begin{pmatrix} 89660 & -247 \\ -247 & 51 \end{pmatrix} = \phi_1(A_1^4) + \phi_2(A_2^4)$.

So we have that an equality of Jensen’ type now is valid.

In this case we have $A = \Phi_1(A_1) + \Phi_2(A_2) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}$ and

$[m, M] = [0, 0.5]$, $[m_1, M_1] \subset [-14.077, -0.328566]$, $[m_2, M_2] = [2, 15]$, i.e.

$$(m, M) \cap [m_1, M_1] = \emptyset \quad \text{and} \quad (m, M) \cap [m_2, M_2] = \emptyset.$$ 

similarly as in Figure 1.b).

It is no coincidence that Jensen’ operator inequality is valid in this example. In the following theorem we prove a general result when Jensen’s operator inequality holds for convex functions.
Theorem

Let \((A_1, \ldots, A_n)\) be \(n\)-tuple of self-adjoint operators \(A_i \in B(H)\) with bounds \(m_i\) and \(M_i\), \(m_i \leq M_i\), \(i = 1, \ldots, n\). Let \((\Phi_1, \ldots, \Phi_n)\) be \(n\)-tuple of positive linear mappings \(\Phi_i : B(H) \rightarrow B(K)\), \(i = 1, \ldots, n\), such that \(\sum_{i=1}^{n} \Phi_i(1_H) = 1_K\). If

\[
(m_A, M_A) \cap [m_i, M_i] = \emptyset \quad \text{for} \ i = 1, \ldots, n,
\]

where \(m_A\) and \(M_A\), \(m_A \leq M_A\), are bounds of a self-adjoint operator \(A = \sum_{i=1}^{n} \Phi_i(A_i)\), then

\[
f \left( \sum_{i=1}^{n} \Phi_i(A_i) \right) \leq \sum_{i=1}^{n} \Phi_i(f(A_i)) \quad \text{(11)}
\]

holds for every continuous convex function \(f : I \rightarrow \mathbb{R}\) provided that the interval \(I\) contains all \(m_i, M_i\).

If \(f : I \rightarrow \mathbb{R}\) is concave, then the reverse inequality is valid in \((11)\).
3.2 Quasi-arithmetic means without operator convexity

Now, we can observe the monotonicity of the quasi-arithmetic mean without operator convexity. E.g.
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**Theorem**

Let \((A_1, \ldots, A_n)\) and \((\Phi_1, \ldots, \Phi_n)\) be as above. Let \(m_i\) and \(M_i\), \(m_i \leq M_i\) be bounds of \(A_i\), \(i = 1, \ldots, n\). Let \(\varphi, \psi : I \to \mathbb{R}\) be continuous strictly monotone functions on an interval \(I\) which contains all \(m_i, M_i\). Let \(m_\varphi\) and \(M_\varphi\), \(m_\varphi \leq M_\varphi\), be bounds of the mean \(M_\varphi(A, \Phi, n)\), such that

\[
(m_\varphi, M_\varphi) \cap [m_i, M_i] = \emptyset \quad i = 1, \ldots, n.
\]

If one of the following conditions is satisfied:

(i) \(\psi \circ \varphi^{-1}\) is convex and \(\psi^{-1}\) is operator monotone,

(ii) \(\psi \circ \varphi^{-1}\) is concave and \(-\psi^{-1}\) is operator monotone,

then

\[
M_\varphi(A, \Phi, n) \leq M_\psi(A, \Phi, n).
\]
3.3 Converse inequalities with condition on spectra

Finally, we can observe converse to all above inequalities with condition on spectra. E.g.
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**Theorem**

Let $A_1, \ldots, A_n \in \mathcal{B}(H)$ be self-adjoint operators with bounds $m_1 \leq M_1, \ldots, m_n \leq M_n$ and $[m, M]$ be an interval with endpoints $m = \min\{m_1, \ldots, m_n\}$ and $M = \max\{M_1, \ldots, M_n\}$ where $m < M$. Let $\Phi_1, \ldots, \Phi_n : \mathcal{B}(H) \to \mathcal{B}(K)$ be positive linear mappings such that $\sum_{i=1}^n \Phi_i(1_H) = 1_K$. Let $A = \sum_{i=1}^n \Phi_i(A_i) \in \mathcal{B}(K)$ be the sum of operators $\Phi_i(A_i)$ with bounds $m_A \leq M_A$. Let the spectral conditions

$$(m_A, M_A) \cap [m_i, M_i] = \emptyset \quad \text{for } i = 1, \ldots, n$$

are valid. Let $f : [m, M] \to \mathbb{R}$ be continuous function.
(continued)

If \( f \) is convex, then

\[
\sum_{i=1}^{n} \Phi_i(f(A_i)) \leq \max_{m_A \leq t \leq M_A} \left\{ f_{[m,M]}^{cho}(t) - \alpha f(t) \right\} 1_K + \alpha f\left( \sum_{i=1}^{n} \Phi_i(A_i) \right)
\]

holds for every strictly positive \( \alpha \in \mathbb{R} \).

If \( f \) is convex, and strictly positive on \([m_A, M_A]\), then

\[
\sum_{i=1}^{n} \Phi_i(f(A_i)) \leq \max_{m_A \leq t \leq M_A} \left\{ \frac{f_{[m,M]}^{cho}(t)}{f(t)} \right\} f\left( \sum_{i=1}^{n} \Phi_i(A_i) \right).
\]

Also, we can observe the order among quasi-arithmetic means and power means under the same condition on spectra.
References: These papers are submitted in some journals or they are manuscripts.