

Finding the Θ -Guarded Region

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Abstract

We are given a finite set of n points (*guards*) G in the plane \mathbb{R}^2 and an angle $0 \leq \Theta \leq 2\pi$. A Θ -cone is a cone with apex angle Θ . We call a Θ -cone *empty* (with respect to G) if it does not contain any point of G . A point $p \in \mathbb{R}^2$ is called Θ -guarded if every Θ -cone with its apex located at p is non-empty. Furthermore, the set of all Θ -guarded points is called the Θ -guarded region, or the Θ -region for short.

We present several results on this topic. The main contribution of our work is to describe the Θ -region with $O(\frac{n}{\Theta})$ circular arcs, and we give an algorithm to compute it. We prove a tight $O(n)$ worst-case bound on the complexity of the Θ -region for $\Theta \geq \frac{\pi}{2}$. In case Θ is bounded from below by a positive constant, we prove an almost linear bound $O(n^{1+\varepsilon})$ for any $\varepsilon > 0$ on the complexity. Moreover, we show that there is a sequence of inputs such that the asymptotic bound on the complexity of their Θ -region is $\Omega(n^2)$.

Key words: Θ -guarded region, unoriented Θ -maxima, convex hull generalization, good Θ -illumination, α -embracing contour

1 Introduction

Illumination and guarding problems have been a popular topic of study in mathematics and computer science for several decades. One instance in this class of problems is the classical one posed by Victor Klee [13]: *How many*

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guards are necessary, and how many are sufficient to patrol the paintings and works of art in an art gallery with n walls? While this particular problem has been solved shortly after by Chvatal [6] proving a tight $\lfloor \frac{n}{3} \rfloor$ bound, many other variants in this problem class have appeared in the literature, see e.g. [16] for a general survey on the topic.

In this paper we consider a guarding problem with a fixed number of guards which have fixed positions in the plane. We concentrate on the mathematical description, the complexity, and the computation of the guarded area.

The model is as follows: We are given a finite set of points (*guards*) G in the plane \mathbb{R}^2 . A Θ -cone is a cone with apex angle Θ . We call a Θ -cone *empty* (*with respect to G*), if it does not contain any point of G in its interior. A point $p \in \mathbb{R}^2$ is called Θ -*guarded* (*with respect to G*), if every Θ -cone with apex located at p is non-empty. The set of all Θ -guarded points is called the Θ -*guarded region*, or the Θ -region for short. We consider Θ -cones as open sets, hence the Θ -region is an open set, too. The rationale behind this model is that a point is well-guarded only if it is guarded from all sides.

1.1 Previous Work

For a given set G of n points in the plane, Avis et al. [3] were the first to introduce the notion of *unoriented Θ -maxima*. They say that some point $g \in G$ is a Θ -maxima if there exists an empty Θ -cone with apex at g . Hence a point g is Θ -maxima if it is not Θ -guarded with respect to G . They present an $O(\frac{n}{\Theta} \log n)$ algorithm for computing the unoriented Θ -maximum of the set G , or to put it in other words, an algorithm to query each point in G if it is Θ -guarded or not. A slight variation of their algorithm can actually query any finite point-set P in $O(\frac{n+|P|}{\Theta} \log(n + |P|))$ time as we show in Lemma 15. They further show that the unoriented $\frac{\pi}{2}$ -maxima can be computed in $O(n)$ expected time.

Abellanas et al. [1] extend the guarding model (there it is called *good Θ -illumination*) by a range r , i.e., a guard $g \in G$ can only guard points inside the circle of radius r that is centered at g . Beneath other results they show how to check if a query point p is Θ -guarded in $O(n)$ time and output the necessary range and guards as witnesses.

Over years several generalizations of the standard convex hull of a point set have been proposed, like the α -hull [9], the k -th iterated hull [5], and the related concept of the k -hull (k -depth contour) [7].

Our contribution

After some general observations, we describe the structure of the boundary of the Θ -region for different values of Θ in Section 2. There we also give an easy and efficient $O(n \log n)$ time algorithm to compute the boundary in case $\Theta \geq \pi$. In the main part of the paper we concentrate on the case $\Theta < \pi$, since for these angles the problem becomes much more involved and the boundary of the Θ -region more complex to understand. In Section 3 we show that the boundary of the Θ -region is contained in an arrangement of circular arcs. In Section 4 we bound this set of arcs by $O(\frac{n}{\Theta})$. Note that in our work Θ and n are independent parameters. In particular, asymptotic bounds are stated in n and $\frac{1}{\Theta}$. For $\Theta \geq \frac{\pi}{2}$ we prove that the complexity of the Θ -region is $O(n)$. If $\Theta > \delta$ for a positive constant $\delta > 0$, we show that the complexity is $O(n^{1+\varepsilon})$, for any $\varepsilon > 0$. In Section 5 we give a generic example for a Θ -region with complexity $\Omega(n^2)$ where the angle Θ is of order $\frac{1}{n}$. In Section 6 we give an algorithm to compute the Θ -region in $O(n^{\frac{3}{2}+\xi}/\Theta + \mu \log n)$ time, for any $\xi > 0$, where μ denotes the complexity of the arrangement of the $O(\frac{n}{\Theta})$ arcs. Our algorithm is based on the Partitioning Theorem [12] and on the computation of an arrangement of circular arcs.

Remark 1 *Besides our work there is an independent and recent publication by Abellanas et al. [2]. There the complexity of the Θ -region (called the α -embracing contour) is claimed to be $O(n)$ for all constant Θ , and an algorithm that runs in $O(n^2 \log n)$ time and $O(n^2)$ space is proposed. After personal communication with the authors we agree that the claims are unfortunately not generally true for small angles.*

1.2 Remark on Plotted Pictures

The computer generated pictures are based on the value of the continuous function $f : \mathbb{R}^2 \setminus G \rightarrow (0, 2\pi]$ where $f(p) = \max\{\Theta : \exists \text{ empty } \Theta\text{-cone with apex } p\}$. The left picture of Figure 1 and the two rightmost pictures in Figure 6 are generated by plotting a grid point shaded, iff f has a value below the threshold Θ . In the right picture of Figure 1 we have mapped different intervals of function values in $[0, \pi]$ to different gray scale values to visualize isolines (along the boundary of the gray scale value) of f in this example. Although these pictures visualize only function values at grid points, one can rely on the pictures, since we deal with cones of a certain angle and not with arbitrarily thin stripes that could somehow pass between grid points.

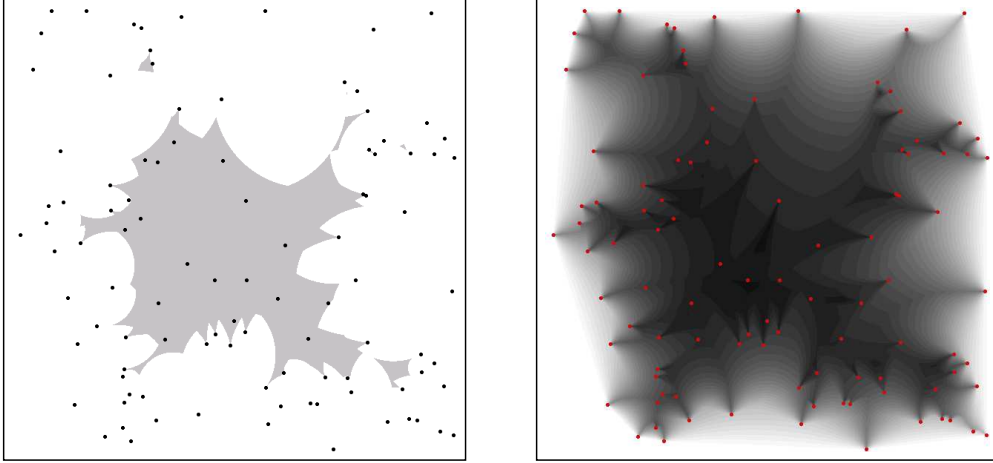


Figure 1. An example with $|G| = 50$. The Θ -region is not necessarily connected for $0 < \Theta < \pi$ (left). The isolines of function f show how components of the Θ -region disconnect for decreasing Θ in this example (right).

2 The Shape of the Θ -Guarded Region

We start with some observations. A point $p \in \mathbb{R}^2$ does not belong to the Θ -region, if there is an empty Θ -cone with apex p . Hence, no point inside an empty cone can belong to the region, and hence, the region can not contain holes. A point p lies on the boundary of a Θ -region, if the closure¹ of each Θ -cone with apex p is non-empty, and there is at least one empty (open) Θ -cone with apex at p . The example in Figure 1 shows that the Θ -region is not necessarily connected for $0 < \Theta < \pi$. The shape of the Θ -region is invariant under translation, rotation, and scaling of G .

The shapes of all Θ -regions can be grouped according to Θ . The boundary of the π -region is just the *convex hull* $CH(G)$, because the intersection of all half-planes containing G (convex hull) is the same as removing every half-plane from \mathbb{R}^2 that does not contain any point of G (π -region). However, for $0 < \Theta < \pi$, empty (convex) Θ -cones can enter the convex hull through the edges, while for $\pi < \Theta < 2\pi$ the apexes of empty (concave) Θ -cones do not even have to touch the convex hull (see Figure 2). Therefore, the Θ -region is connected, if $\Theta \geq \pi$. Trivially, the 2π -region is the plane \mathbb{R}^2 and the 0 -region is the empty set.

Before we discuss the Θ -region for $0 < \Theta < \pi$ in Sections 3–6, we discuss the simpler case $\pi < \Theta < 2\pi$ below. Throughout the paper we use the property about *inscribed angles*: *Given a circular arc $C_{l,r}$ from l to r , then $\angle lpr = \angle lqr$ holds for all $p, q \in C_{l,r}$.* We write $C_{l,r}^\alpha$ if the inscribed angle is α . The arc end points are always given in counterclockwise order.

¹ Exceptionally we consider closed cones here.

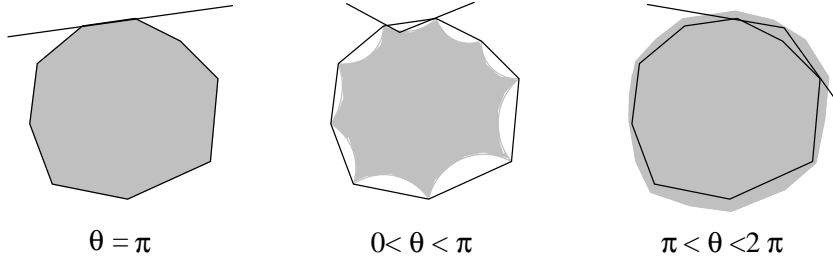


Figure 2. For $\Theta < \pi$ (resp. $\Theta > \pi$) the region lies inside (outside) the convex hull $CH(G)$ and the bounding arcs are bend inside (outside) the region.

2.1 Finding the Θ -Guarded Region for $\Theta > \pi$

As already discussed (see Figure 2), for $\Theta > \pi$ every point in the convex hull interior of G is Θ -guarded. Intuitively, the boundary of the Θ -region is drawn by the apex of an empty Θ -cone which is rotated around the convex hull $CH(G)$ such that its rays are always tangent to $CH(G)$. The following algorithm computes the boundary of the Θ -region.

We first compute the clockwise sequence of guards $G' = \{g_1, \dots, g_k\}$ defining the convex hull (see for example [15]). Formalizing the intuition given above, we construct an algorithm that outputs circular arcs defining the boundary of the Θ -region as follows. We identify all pairs $(g_i, g_j) \in G' \times G'$ with $g_i \neq g_j$, for which there exists an empty Θ -cone that is tangent to g_i and g_j , and has its apex outside the convex hull. We say that the apex of the Θ -cone can “see” the polygonal chain of $CH(G)$ from g_i to g_j . Such a pair (g_i, g_j) will always have the property, that the lines supporting the convex hull edges $(g_j, g_{\text{succ}(j)})$ and $(g_{\text{pred}(i)}, g_i)$ have an angle of intersection not greater than Θ , and that the lines supporting $(g_i, g_{\text{succ}(i)})$ and $(g_{\text{pred}(j)}, g_j)$ have an angle of intersection greater than Θ . The sequence of all these pairs (g_i, g_j) and the corresponding circular arcs $C_{g_j, g_i}^{2\pi - \Theta}$, that are defined by g_i, g_j , and the apex of the Θ -cone that is tangent to g_i and g_j , can be computed by a cyclic scan over the sequence G' . The arc end points of the Θ -region boundary can be computed as the intersection points of each circular arc $C_{g_j, g_i}^{2\pi - \Theta}$ with the supporting lines through $(g_j, g_{\text{succ}(j)})$ and $(g_{\text{pred}(i)}, g_i)$. Consequently the Θ -region has the same complexity than the convex hull. The running time of the algorithm is dominated by the convex hull construction in $O(n \log n)$ time. We summarize the above in the following Lemma.

Lemma 2 *The boundary of the Θ -region for $\Theta > \pi$ can be computed in $O(n \log n)$ time and its complexity is $|CH(G)|$.*

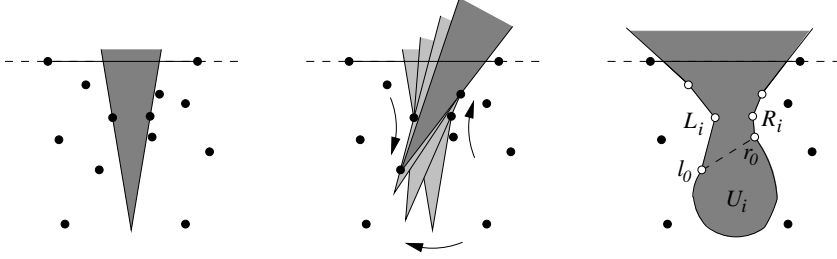


Figure 3. The construction of a tunnel.

3 The Boundary of the Θ -Region

From now on we assume that the angle is $0 < \Theta < \pi$. Here we give a mathematical description of the Θ -region. First we come back to the inscribed angles and explain its meaning for our setting. Let $e = (l, r) \in G \times G$ be any pair of guards. Then the set of points where we can place the apex of an empty Θ -cone passing through the line segment (l, r) in the same direction is bounded by the circular arc, incident to l and r having inscribed angles Θ , and its chord lr . We denote this closed circular segment with $D_{l,r}^\Theta$ (or D_e for short) and its bounding circular arc, as above, with $C_{l,r}^\Theta$ (or C_e for short). Because of the orientation, the circular segment is described uniquely.

The construction of the Θ -region is motivated by the idea of locally removing sets T_i of unguarded points from the convex hull $CH(G)$ such that the remaining part matches the Θ -region (see Figure 2, middle), i.e. we aim for

$$\Theta\text{-region} = CH(G) \setminus \left(\bigcup_{i \in I} T_i \right) \quad (1)$$

for specific sets T_i . Next we give the construction for the sets T_i . Consider any empty Θ -cone c that has at least a guard on each ray (see Figure 3, left). First we turn the cone clockwise while pushing the cone towards the point set, such that it always stays empty but touches a guard on each boundary (see Figure 3, middle). We end this motion when the apex of the cone reaches the position of a guard, say l_0 . Afterwards we start again with cone c , i.e. in the original position, and rotate the cone in a similar way counterclockwise until the apex reaches the position of another guard, say r_0 . We extend our notions. With L_i (resp. R_i) we denote the set of guards that are incident to the left (resp. right) ray of a cone during the construction (the white points in Figure 3, right). We call the closure of the union of all cones, which are used during the construction, the *tunnel* T_i with respect to L_i and R_i , or tunnel for short (shaded region in Figure 3, right). Note that the index set I in Formula (1) enumerates over all tunnels.

Note that Formula (1) describes the Θ -region of G , because each empty Θ -

cone that intersects $CH(G)$ lies in at least one tunnel T_i : Let c be such a cone. We can identify a tunnel by moving c in the direction of its medial axis until one of its rays is tangent to a guard. Then we let the empty cone slide along that point without rotation until the second ray is also tangent to a guard. According to our construction there is a tunnel that contains this cone and hence the cone c in its original position.

No point in T_i is Θ -guarded, but only its boundary can contribute to the boundary of the Θ -region. First we consider its straight-line boundaries. Since $\Theta < \pi$, each point of a straight-line boundary can be crossed infinitesimally by an empty Θ -cone. That means, there are open neighborhoods of unguarded points around each point of a straight-line boundary, and hence they can not contribute to the Θ -region boundary. Points beyond the straight-line boundaries belong to different tunnels and will be processed independent from T_i .

Therefore we only have to consider the curved boundary of T_i . Observe that during the construction, the apex of the rotating cone is drawing a *sequence of circular arcs* between l_0 and r_0 which we will formalize next. We define the set

$$U_i := \bigcap_{(l,r) \in L_i \times R_i} D_{l,r}^\Theta \quad (2)$$

as the intersection of all circular segments for guard pairs in $L_i \times R_i$. In the following Lemma we state that we can derive the curved boundary of T_i from these circular segments. Let h_i be the closed half plane which is bounded by the line through l_0 and r_0 and contains the sequence of arcs.

Lemma 3 *Let T_i , U_i , and h_i be as defined above. Then $T_i \cap h_i = U_i$.*

Proof. (**Superset.**) Let $p \in U_i$. Assume there is no empty cone with apex p through tunnel T_i . This means that there is at least a pair $(l, r) \in L_i \times R_i$ with the property that $\angle lpr < \Theta$. Hence $p \notin D_{l,r}^\Theta$ which is a contradiction. (**Subset.**) Let p be the apex of an empty Θ -cone through tunnel T_i . This means that $\angle lpr \geq \Theta$ for all $(l, r) \in L_i \times R_i$, and hence p lies in all corresponding circular segments $D_{l,r}^\Theta$. \square

It follows that the Θ -region boundary is contained in the curved boundary of the union of the sets U_i , i.e.

$$\partial\Theta\text{-region} \subseteq \partial \bigcup_{i \in I} U_i = \partial \bigcup_{i \in I} \left(\bigcap_{(l,r) \in L_i \times R_i} D_{l,r}^\Theta \right), \quad (3)$$

where i enumerates over all tunnels. We observe that the intersections of

$|L_i| \cdot |R_i|$ circular segments $D_{l,r}$ in Formulae (2) and (3) are too pessimistic. During the construction of a tunnel we collect all guard pairs $(l, r) \subset L_i \times R_i$, that are incident to the rotating cone simultaneously, in E_i . Since the touching point of L_i (resp. R_i) can only change in one direction to its neighbor in the sequence of L_i (resp. R_i), that leads to a set E_i of size $|L_i| + |R_i| - 1$. Therefore we may reduce the intersection of the circular segments in Formula (2) to

$$U_i := \bigcap_{(l,r) \in E_i} D_{l,r} \cap h_i$$

for which Lemma 3 is still valid. With \mathcal{C} we denote the set of all circular arcs that appear in the boundary of a set U_i .

4 Upper Bounds on the Worst-Case Complexity

Now we discuss the worst-case complexity of the Θ -region and state the asymptotic bounds in the number n of guards and the reciprocal value of the angle, i.e. $\frac{1}{\Theta}$. During the analysis of the complexity, we distinguish cases according to the value of Θ . We already know that the 0-region is the empty set. Since G is a discrete set the Θ -region is also the empty set for values close to 0.

Lemma 4 *The Θ -region for $\Theta \leq \frac{2\pi}{n}$ is the empty set.*

Proof. Consider the n rays emanating from a point $p \in \mathbb{R}^2 \setminus G$ through the guards in G . Then the rays form at least one empty cone with angle of at least $\frac{2\pi}{n}$ which contains an empty Θ -cone. Hence p is unguarded. We can argue similarly for the guards $p \in G$. \square

According to the right term in Formula (3) the complexity of the Θ -region is hidden in an arrangement of circular arcs. Since there are at most $O(n^2)$ different circular arcs, two for each guard pair, the complexity of the Θ -region is trivially $O(n^4)$. Now we show that the set \mathcal{C} of circular arcs is of $O(\frac{n}{\Theta})$ size. Hence the complexity of the Θ -region is $O(\frac{n^2}{\Theta^2})$.

Theorem 5 *The set \mathcal{C} of circular arcs, which defines the boundary of the Θ -region, is of $O(\frac{n}{\Theta})$ size.*

Proof. Instead of counting the arcs directly we count their end points. Let p be an arc end point of a tunnel as shown in Figure 4, left. In this position a ray of the rotating cone is incident to two guards at once. We assume without loss of generality that two guards lie on the left ray. We focus on the guard l_k that is closer to the apex and count how often a guard can be in this

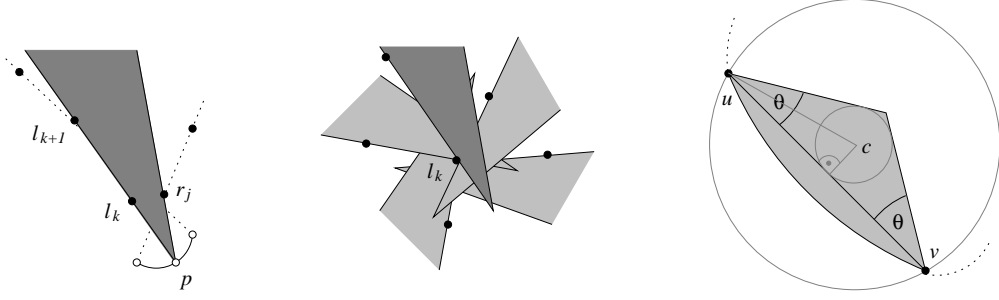


Figure 4. An end point p of a circular arc in the boundary of a tunnel (left). A situation that is described in the proof of Theorem 5 (middle). The auxiliary construction that is described in the proof of Theorem 6 (right).

situation. Clearly the number is bounded by $n - 1$ because there are no more other guards. On the other hand we observe that the empty Θ -cones in this situation can not intersect each other beyond the second guard on the left ray (see Figure 4, middle). Hence there can be at most $\lfloor \frac{2\pi}{\Theta} \rfloor$ different such cones. With the same argumentation for the right ray we bound the number of arc end points per guard by $2\lfloor \frac{2\pi}{\Theta} \rfloor$ and hence the total number of arc end points by $O(\frac{n}{\Theta})$. \square

From the last Theorem we can derive, that if the angle $\Theta > \delta$ is bounded by a constant $\delta > 0$, then the number of arcs in \mathcal{C} is $O(n)$ and the complexity of the Θ -region is $O(n^2)$. With an auxiliary construction we can even further improve this result.

Theorem 6 *If the angle $\Theta > \delta$ is bounded by a constant $\delta > 0$, then the complexity of the Θ -region is $O(n^{1+\epsilon})$, for any $\epsilon > 0$.*

Proof. We make use of the following construction. Let $a \in \mathcal{C}$ be an arc in the boundary of tunnel T_i , and let u and v be the end points of a . The line segment (u, v) and a are the boundary of a circular segment, say d_a . Now we glue a triangle t_a at the edge (u, v) of d_a , which has an angle of $\min\{\Theta, \frac{\pi}{4}\}$ at u and v , and denote this new object with $F_a := d_a \cup t_a$ (see Figure 4, right). We state that the triangle t_a is a subset of T_i : Assume there is a guard $g \in t_a$. Then the angles $\angle guv$ and $\angle gv u$ are smaller than Θ . Hence two empty Θ -cones with apexes u and v would belong to different tunnels what is a contradiction to $a \subset \partial T_i$. Furthermore, because of the angle at u and v the set of empty Θ -cones with apexes at points in a have to cover t_a , what completes the proof of the statement.

We repeat the above construction for each arc $a \in \mathcal{C}$ and collect the new objects F_a in the set \mathcal{F} . We repeat the definition of α -fatness from Efrat et al. [10]: *An object F is α -fat for some fixed $\alpha > 1$, if there exist two concentric disks $\mathcal{D} \subseteq F \subseteq \mathcal{D}'$ such that the ratio $\frac{\rho'}{\rho}$ between the radii of \mathcal{D}' and \mathcal{D} is at most α .* We state that there is an $\alpha > 1$ such that the objects $F_a \in \mathcal{F}$ are α -fat:

The worst-case scenario occurs when the arc a is almost a straight-line. Hence we concentrate on the proof that the triangle t_a is α -fat (see again Figure 4, right). Remember that the angle at u is $\min\{\Theta, \frac{\pi}{4}\}$. Then the ratio between the radii of the circumcircle and the inscribed circle is

$$\frac{\rho'}{\rho} = \frac{1}{\sin(\frac{1}{2} \cdot \min\{\Theta, \frac{\pi}{4}\})} \leq \frac{1}{\sin(\frac{\delta}{2})} =: \alpha,$$

which is a constant. Without loss of generality we assume $\delta \leq \frac{\pi}{4}$.

The main Theorem in Efrat et al. [10] states, that the combinatorial complexity of the union of a collection \mathcal{F} of α -fat objects, whose boundary intersect pairwise in at most s points, is $O(|\mathcal{F}|^{1+\varepsilon})$, for any $\varepsilon > 0$, where the constant of proportionality depends on ε , α , and s .

It is already shown that α is a constant and that the objects in \mathcal{F} are α -fat. The boundary of each convex object $F_a \in \mathcal{F}$ has always three edges: two line segments and a circular arc. Therefore the boundary of each pair of objects in \mathcal{F} intersect in at most $s = 10$ points. As we said above $|\mathcal{C}| \in O(n)$, and hence $|\mathcal{F}| \in O(n)$, because Θ is bounded from below by a constant δ . Therefore the construction fulfills all preconditions to apply the Theorem of Efrat et al. which completes the proof. \square

Now we show that the complexity of the Θ -guarded region is linear for angles Θ at least $\frac{\pi}{2}$.

Theorem 7 *The complexity of the Θ -region is $O(n)$ for $\frac{\pi}{2} \leq \Theta < \pi$.*

Proof. Let \mathcal{J} be a set of m Jordan curves, i.e. simply-closed curves. Kedem et al. [11] proved that if any two curves in \mathcal{J} intersect in at most two points then the complexity of their union is $O(m)$.

For each set U_i we define a Jordan curve J_i . Let J_i be the curved boundary of U_i from l_0 to r_0 connected with an auxiliary half circle $C_{r_0, l_0}^{\frac{\pi}{2}}$, i.e.

$$J_i := \partial \left(\bigcap_{(l,r) \in E_i} D_{l,r}^\Theta \cap h_i \right) \cup C_{r_0, l_0}^{\frac{\pi}{2}}.$$

Note that the auxiliary half-circle lies in T_i because Θ is obtuse. Note further that J_i is the boundary of a convex region and that J_i lies inside the circle that supports $C_{r_0, l_0}^{\frac{\pi}{2}}$. We repeat this construction for all tunnels T_i , with $i \in I$, and collect all curves J_i in \mathcal{J} . We state that any two curves in \mathcal{J} intersect at most twice. Now assume that there are two curves $J_i, J_j \in \mathcal{J}$ which intersect

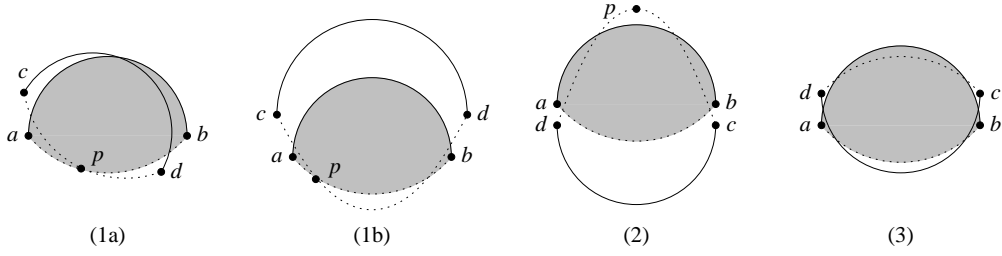


Figure 5. (Proof of Theorem 7.) The cases that could imply more than just two intersection points between the curves J_i and J_j . The dotted parts are the sequence of arcs of U_i and U_j from a to b and from c to d , respectively, and the solid arcs are the auxiliary half-circles.

in more than two points. We distinguish the following cases as they are shown in Figure 5. Let A_i (resp. A_j) denote the sequence of circular arcs of U_i (resp. U_j).

In Case 1, the sequence of circular arcs A_i and A_j intersect in point p as is shown in Pictures (1a) or (1b). That means that for each tunnel T_i and T_j there exists an empty Θ -cone with apex in p . Therefore the angle $\angle apb$ has to be at least 2Θ which is at least π . That is a geometrical contradiction.

In Case 2, we consider a point p that lies on the sequence of arcs A_j outside J_i as shown in Picture (2). The angle $\angle cpd$ is at least Θ . By construction the angle $\angle bpa$ is larger than $\angle cpd$ and hence $\angle apb > \Theta$. It is a contradiction that p does not lie inside J_i .

In Case 3, we consider an empty Θ -cone with apex c through tunnel T_j . Assume this cone passes between a and b . Then the angle $\angle bca$ is at least Θ and hence c has to lie inside J_i . This is a contradiction. In case that the empty Θ -cone c does not pass between a and b , but b and c , or a and d , similar geometric contradictions can be shown.

Other cases are excluded since no guard can lie inside J_i or J_j . This completes the proof. \square

5 Lower Bound on the Worst-Case Complexity

In the following we show that there is a sequence of inputs such that the asymptotic bound on the complexity of their Θ -guarded region is $\Omega(n^2)$. For this purpose we give a generic construction for point sets G_i with n_i guards and angles Θ_i for all $i \in \mathbb{N}$, such that the complexity of the Θ_i -region of the point set G_i is lower bounded by $c \cdot n_i^2$ for some constant c and $\lim_{i \rightarrow \infty} n_i = \infty$. In fact n_i is a linear function in i , and Θ_i is of order $\frac{1}{i}$. Therefore the complexity

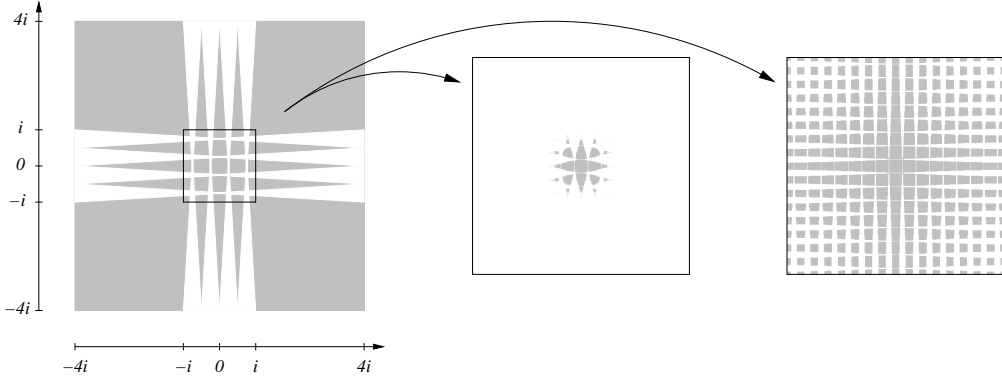


Figure 6. The location of the fragmented Θ_i -region is located at the centre of the convex hull (left, $i = 2$). Some connected components of the center are erased by unwanted tunnels (middle, $i = 8$). All connected components of the center are protected against unwanted tunnels (right, $i = 8$).

bound can also be interpreted as $\Omega(\frac{n}{\Theta})$.

First we motivate the construction for a given $i \in \mathbb{N}$. To achieve the desired complexity, we construct the point set G_i in such a way that the Θ_i -region is fragmented into $c \cdot n_i^2$ connected components, each of constant complexity. Figure 6 illustrates the idea of the construction. The area, where the Θ_i -region is highly fragmented is at the center of the convex hull. The decomposition is forced by long, thin tunnels that enter this area ‘axis parallel’ from above, below, left, and right; more precisely the medial axis of the cones that enter these tunnels deepest are parallel to the principal axes. In the *first step* of the construction we determine the tunnels that force the fragmentation and implicitly determine the area (bounded by the box in Figure 6) that contains these connected component. Unfortunately the same guards, that define these tunnels, define an even larger number of unwanted tunnels which can enter this area as well. Therefore we have to place additional guards in the *second step* with the intention to prevent unwanted tunnels from entering this area, because they could erase some of the connected components, hence reducing the total complexity (see Figure 6, middle and right). We show how to place a linear number of additional guards as obstacles in the plane to keep out a squared number of tunnels from this area. We note that because of the construction in the second step the convex hull can be huge compared to the box in which we count the connected components. For simplicity reasons we disregard the shape of the Θ_i -region outside this box. The construction details are given below.

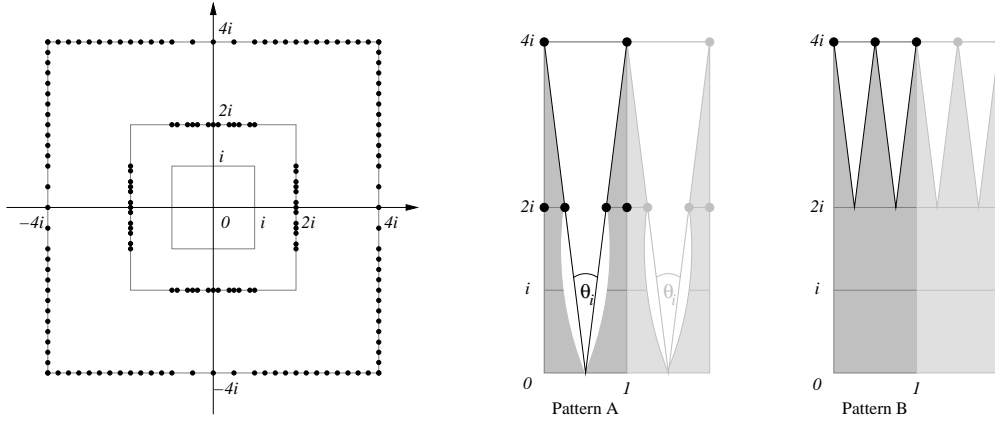


Figure 7. Placement of the guards in the first step for $i = 2$ (left). Guard patterns A and B (right).

5.1 First step: To determine the wanted tunnels.

We denote the square of edge length $2i$ that is centered at the origin and is oriented parallel to the principal axes with B_i . In this step guards are placed on the boundary of the boxes B_{4i} and B_{2i} . The area, in which we will count the connected components, is B_i (see Figure 7, left). The entire construction is symmetric to the origin as well as to the principal axes. For this reason we only give the construction for the upper half of box B_{4i} ; the constructions for the lower, left, and right half of this box are done analogously.

Now we introduce the guard patterns A and B (see Figure 7, right), that define two ways to place guards inside a cell of width 1 and height $4i$, which we will use later on to stamp the upper half of the box with. First we define Θ_i as the angle² between the rays emanating from $(\frac{1}{2}, 0)$ through the upper corners $(0, 4i)$ and $(1, 4i)$. To get guard pattern A we place four guards on the boundary of this cone: two with $y = 2i$ and two with $y = 4i$. These four guards define a wanted tunnel which is thin in the sense that the boundary of the tunnel stays in the box of width 1 for values $0 \leq y \leq i$; remember that we only care for the interior of B_i . For technical reasons we add guards at $(0, 2i)$ and $(1, 2i)$ to avoid unwanted tunnels between neighboring guard patterns A. In case we do not need a tunnel inside the cell we use pattern B: three guards that are placed equidistant on the top edge of the pattern make it impossible for any cone to enter this box from above deeper than $y = 2i$. Next we subdivide the upper half of the box B_{4i} in $8i$ cells of width 1. The medial quarter is stamped with pattern A, the remaining cells are stamped with pattern B (see Figure 8).

This way we can guarantee $2i$ wanted tunnels from above which intersect B_i

² Note that Θ and i are not independent because $\Theta = \arctan(\frac{1}{8i}) \leq \frac{1}{8i}$.

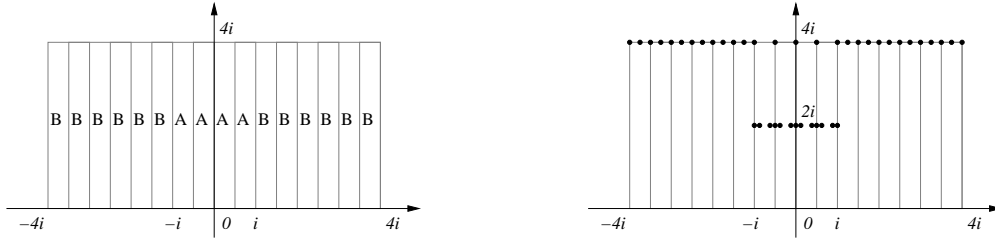


Figure 8. Subdivision of the upper half of B_{4i} in $8i$ cells (left). The guard pattern for the upper half (right).

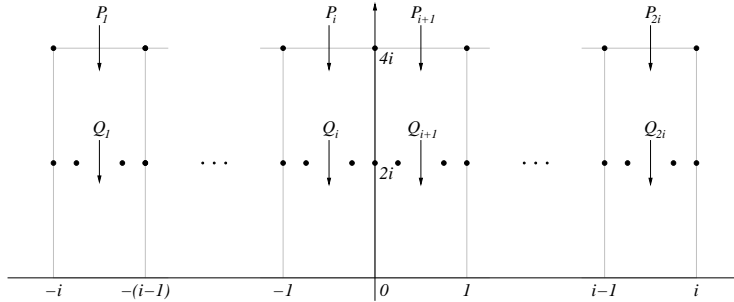


Figure 9. All possible guard pairs through which empty cones can reach B_i from above.

and touch the x -axis. After repeating this construction for the lower, left, and right half of B_{4i} , tunnels from above and below touch at the x -axis as well as tunnels from the left and right touch at the y -axis. This follows immediately from the symmetric construction. Removing these tunnels from the box B_i , yield to a fragmentation into $(2i + 1)^2$ connected components.

Finally we remark that we can save up to 2 guards per stamped pattern if guards overlap with the neighboring pattern. Hence the number of guards, that are placed in the entire first step, sum up to $80i + 4$.

5.2 Second step: To exclude the unwanted tunnels.

Again we start with the construction for the upper half of B_{4i} . Figure 9 shows the situation of the $2i$ neighboring guard patterns A. We denote the guard pairs on the line $y = 4i$ with P_1, \dots, P_{2i} and the guard pairs on the line $y = 2i$ with Q_1, \dots, Q_{2i} . An empty cone that enters B_i from above therefore has to pass through P_k and Q_ℓ for some $k, \ell \in \{1, \dots, 2i\}$. If $k = \ell$, the tunnel is wanted. So the task is to hinder all cones through tunnels with $k \neq \ell$ to intersect the box B_i . We introduce a new notion to reformulate the problem.

Definition 8 Let t be a tunnel through P_k and Q_ℓ . We say that an empty cone enters tunnel t the deepest if the y -value of its apex is minimal among all empty cones in t .

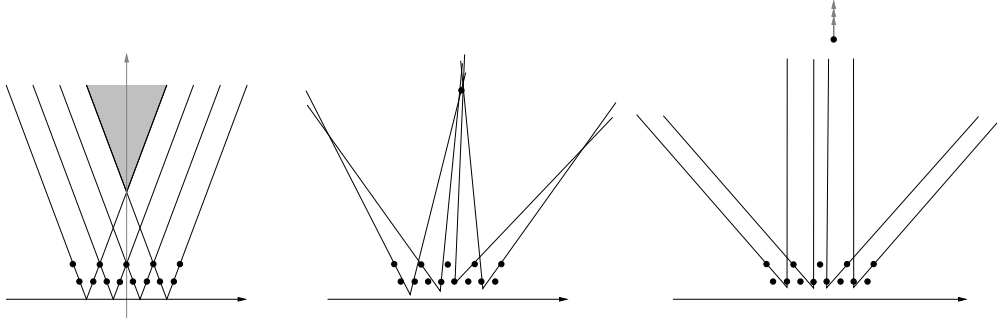


Figure 10. The deepest cones c_j used in Observation 9. Here the vertical arrow marks the direction of their medial axis (left). The deepest cones that are tangent to the new vertex (middle). The deepest cones, in the case that the new guard is pulled far out in the direction of the former medial axis (right).

Note that the deepest cone in a tunnel is unique and is tangent to at least one guard on each ray. We define the *slope of a cone* as the slope of its medial axis. Because of the regular structure of the cells with guard pattern A we can make the following Observation. Informally it states that the slope of a deepest cone through P_j and Q_{j+h} is independent of j and implicitly given by h .

Observation 9 *Let $h \in \{0, \dots, 2i - 1\}$. Let c_j be the deepest cone through P_j and Q_{j+h} and let d_j be the deepest cone through P_{j+h} and Q_j for all $j \in \{1, \dots, 2i - h\}$. Then all cones c_j have the same slope, and all cones d_j have the same slope.*

W.l.o.g. we concentrate on the cones c_j , but we can argue in the same way for the cones d_j . We derive from Observation 9 that for fixed h the intersection of all deepest cones c_j is again a cone with the same slope (see Figure 10, left). Assume we place a guard at a random position inside this intersection. Then none of the cones c_j is empty anymore. That means that there are new deepest cones with different slopes, since they have to be tangent to the new guard (see Figure 10, middle). Now we pull the new guard in the direction of the medial axis of the former deepest cones towards infinity. Then we observe, that while we move this point, the deepest cones are rotated and this way are pulled away from the x -axis. Since the limes of the rotation, compared to the slope of the original deepest cones c_j , has absolute value $\frac{\Theta_i}{2}$, we can force a rotation by an angle that is arbitrarily close to half of the apex angle, i.e. $\frac{\Theta_i}{2} - \varepsilon$ for any $\varepsilon > 0$ (see Figure 10, right). Note the generality of the above discussion for all $h = \{0, \dots, 2i - 1\}$. We will definitely *not* place a guard in the *union* of the deepest cones for $h = 0$, since these tunnels force the decomposition of the Θ_i -region in the center. In spite of this we have used exactly this case in the drawings in Figure 10, since it depicts the worst-case scenario we will consider later in the proof.

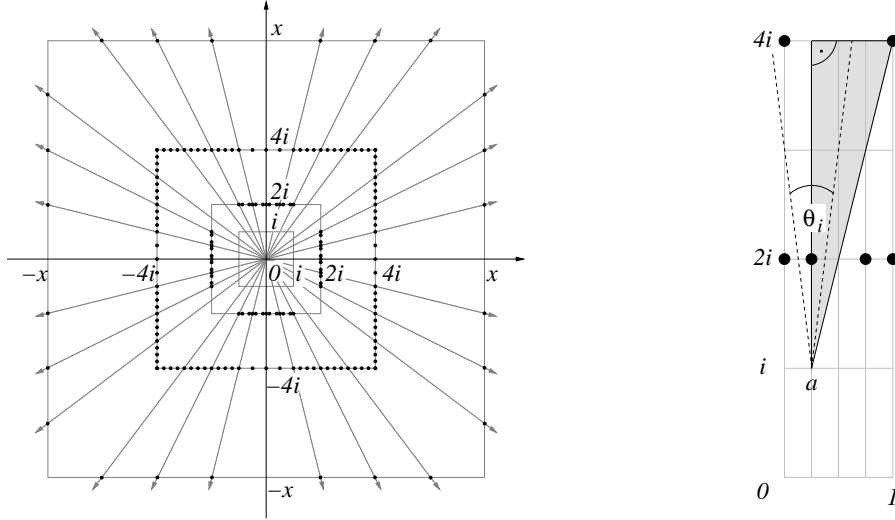


Figure 11. Construction of the second step for all parts: above, below, left, and right (left). Construction used in the proof of Lemma 10 (right).

Now we are able to complete the construction. First we compute the slope of the medial axes of the deepest cones c_j as well as d_j of Observation 9 for all $h = \{1, \dots, 2i - 1\}$. For each of these $4i - 2$ slopes we place a guard at the intersection point of the ray emanating from the origin having this slope and the boundary of a new box B_x (see Figure 11, left). Box B_x has *necessarily* to be large enough to guarantee, that each intersection point lies

- (1) outside the union of the wanted cones and
- (2) inside the intersection of the deepest cones of the given slope.

The existence of box B_x with these properties follows from the discussion above. But this is not *sufficient*. It remains to prove the following lemma.

Lemma 10 *Box B_x can be chosen large enough such that no empty Θ_i -cone, but the wanted cones, can intersect B_i .*

Proof. It is sufficient to prove the claim for the deepest cones with the minimum apex y -value amongst all deepest cones according to Observation 9. These are the cones for $h = 0$. (Please note that we do not block tunnels for $h = 0$ in practice; we just prove that we could even hinder cones through these tunnels from entering B_i .)

Remember that we can place the guard which blocks the deepest cones such that the deepest cones are rotated by an angle arbitrarily close to $\frac{\Theta_i}{2}$. W.l.o.g. we assume that the cone is rotated clockwise. Consider the empty cone c of maximum angle with apex at $a = (\frac{1}{4}, i)$ inside a cell with guard pattern A (shaded region in Figure 11, right). This cone touches the boundary of B_i and its left ray is vertical as it is the case for maximal rotated deepest cones. If we

can show that the angle of c is smaller than Θ_i it follows that c can not enter B_i . We crop c at the line $y = 4i$ to make it a rectangular triangle. Now we take the Θ_i -cone from pattern A, Figure 7, and move its apex to a . We divide the triangle along the right boundary of the Θ_i -cone through point $(\frac{5}{8}, 4i)$. Consequently the left sub-triangle has angle $\frac{\Theta_i}{2}$ at point a . Since the opposite leg of the entire triangle is 2-times the opposite leg of the left sub-triangle, the total angle of c at a is less than 2-times $\frac{\Theta_i}{2}$. \square

After repeating this construction for the lower, left, and right half we have placed $16i - 8$ additional guards. Together with the guards from the first step they define the set G_i with $n_i = 96i - 4$ guards in total. The generic example presented in this section proves the following Theorem.

Theorem 11 *There is a sequence of inputs $(\Theta_i, n_i, G_i)_{i \in \mathbb{N}}$ with $\lim_{i \rightarrow \infty} \Theta_i = 0$ such that the asymptotic bound on the complexity of their Θ -region is $\Omega(n^2)$ where n is the number of guards.*

6 Algorithm

Here we discuss a way to compute the boundary of the Θ -region. Note that we gave bounds on the worst-case complexity of the Θ -region above. Clearly, for any n and any Θ there are sets G for which the Θ -region is empty or extremely simple. Despite of this our algorithm will consider the $O(\frac{n}{\Theta})$ arcs in \mathcal{C} and hence can not be output-sensitive. We allow a simplification in the presentation of the algorithm: We will consider a set \mathcal{C}' of arcs which are longer on one side, i.e. $|\mathcal{C}| = |\mathcal{C}'|$ and $\cup \mathcal{C} \subset \cup \mathcal{C}'$.

First we compute the convex hull $CH(G)$ and add for each hull edge (u, v) the circular arc $C_{u,v}^\Theta$ to the set \mathcal{C}' . For each guard g , that is not a vertex of $CH(G)$, we compute all empty cones of maximal angle with apex at g together with two guards (witnesses) g_{\min} and g_{\max} per empty cone, which lie on its rays. (See the light shaded cone in Figure 12.) This can be done by using the algorithm of Avis et al. [3] in $O((\frac{n}{\Theta}) \log n)$ time and $O(n)$ space.

As we did in the proof of Theorem 5, we find the arcs in \mathcal{C} via their end points. If we move an empty Θ -cone with apex g and its left ray through g_{\min} along the line through g and g_{\min} until a guard, say g_r , is tangent to the other ray, the new apex marks an end point p_r of two arcs in the set \mathcal{C} (see Figure 12). Since we do not know the second end points of the arcs, we add the piece of C_{g_{\min}, g_r}^Θ to \mathcal{C}' that ends in g_r and p_r , and we add the piece of C_{g, g_r}^Θ to \mathcal{C}' that ends in g and p_r . A similar construction for the line through g and g_{\max} will add another two arcs to \mathcal{C}' .

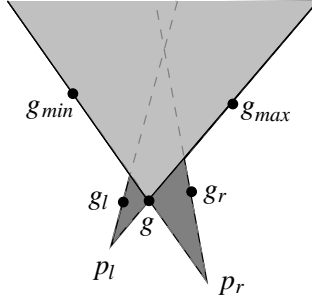


Figure 12. Guard g can contribute to at most two end points for each maximal empty cone with apex g .

Note that by fixing the line through gg_{\min} we can find guard g_r naively by simply inspecting all guards in G , and similarly we can find g_l for the line through gg_{\max} . However, one can compute guards g_r and g_l faster with the help of the well-know *Partition Theorem* that has been extensively used in the context of range searching. We cite the theorem for a planar point set.

Theorem 12 (*Partition Theorem [12].*) *Any set S of n points in the plain can be partitioned into $O(r)$ disjoint classes by a simplicial partition, such that every simplex (i.e. triangle) contains between $\frac{n}{r}$ and $\frac{2n}{r}$ points and every line crosses at most $O(r^{\frac{1}{2}})$ simplices (crossing number). Moreover, for any $\xi > 0$ such a simplicial partition can be constructed in $O(n^{1+\xi})$ time.*

Using this Theorem recursively one can construct a tree which is called a *partition tree* (e.g. the root of the tree, associated with S , has $O(r)$ children, each associated with a simplex from the first level, and so on). From now on we assume that r is a constant. Observe that if r is a constant, the partition tree is of $O(n)$ size and it can be constructed in $O(n^{1+\xi})$ time for any $\xi > 0$.

Lemma 13 *For any $\xi > 0$, there is a data structure of $O(n \log n)$ size and $O(n^{1+\xi})$ construction time such that for the given lines through gg_{\min} and gg_{\max} , corresponding guards g_l and g_r can be computed in additional $O(n^{\frac{1}{2}+\xi})$ time.*

Proof. Assume we are given a partition tree and suppose that we fix the line through gg_{\max} . Clearly by Theorem 12 we have the bound on the number of triangles that intersect the line which is $O(\sqrt{r})$. On those triangles we recur, which leads to a total of $O(\sqrt{n})$ triangles intersected by the line. But still there might be $O(r)$ triangles lying completely to the left of the line gg_{\max} . For those triangles we can precompute a convex hull for the points inside each triangle. This will increase the total space of the partition tree by a $O(\log n)$ factor since every level in the tree now will be of $O(n)$ size. However, this way we avoid recursing on the triangles that lie completely to the left of gg_{\max} . Namely, for every triangle that lies to the left of gg_{\max} , guard g_l can be found as an extreme point of the precomputed convex hull in the direction perpendicular

to the line that forms the Θ -cone with the line through gg_{\max} in $O(\log n)$ total time (see [14], Section 7.9). The case for the line through gg_{\min} is similar. \square

Therefore we can state the following Lemma.

Lemma 14 *For any $\xi > 0$, the set \mathcal{C}' can be computed in $O(n^{\frac{3}{2}+\xi}/\Theta)$ time and $O(n \log n)$ space.*

Next we discuss how to compute the Θ -guarded region from the set of circular arcs \mathcal{C}' . For each connected component of the Θ -region the algorithm outputs a sequence p_1, \dots, p_k of points in the plane and circular arcs incident with pairs p_{i-1}, p_i for $i = 2, \dots, k$ and p_k, p_1 as edges of the Θ -region.

We start with computing the arrangement $\mathcal{A}(\mathcal{C}')$ of set \mathcal{C}' . Let ψ denote the number of cells in $\mathcal{A}(\mathcal{C}')$ and let μ denote the total complexity of the arrangement $\mathcal{A}(\mathcal{C}')$, which upper bounds the complexity of the Θ -region. Edelsbrunner et al. [8] showed that μ is at most $O(\sqrt{\psi}(\frac{n}{\Theta})2^{\alpha(n)})$, where $\alpha(\cdot)$ is the inverse Ackerman function which is an extremely slow-growing function. Moreover, the arrangement $\mathcal{A}(\mathcal{C}')$ can be constructed in $O((n + \mu) \log n)$ time by the plane-sweep algorithm of Bentley and Ottman [4]

Since arcs in \mathcal{C}' are bounding circular segments from the Formula (3), cells in the arrangement $\mathcal{A}(\mathcal{C}')$ will have the property that they are either Θ -guarded or not Θ -guarded. Hence, if some point from the cell is Θ -guarded then the whole cell belongs to the Θ -region and opposite. Let P denote the set of ψ different points such that each point is taken from the interior of ψ different cells in $\mathcal{A}(\mathcal{C}')$. To detect the cells that belong to the Θ -region, we use the following lemma.

Lemma 15 (Avis et al. [3]) *Let G be a set of n guards and let P be a set of ψ query points in \mathbb{R}^2 . The Θ -unguarded points of P can be reported together with their witnesses g_{\min} and g_{\max} in $O(\frac{n+\psi}{\Theta} \log(n + \psi))$ time and $O(n)$ space.*

Proof. Avis et al. [3] presented an algorithm to compute all Θ -unguarded guards of G in $O(\frac{n}{\Theta} \log n)$ time and $O(n)$ space. So far the set of query points and the set of guards are the same. But since their algorithm actually distinguishes between query points and guards, it can be extended immediately: In Steps 2 and 3 of procedure *Unoriented Maxima* on page 284f. only guards are inserted into the convex hull constructions, while tangents to these convex hulls are only computed through query points. The running time of the algorithm is dominated by sorting the points in $G \cup P$ for $\frac{\pi}{\Theta}$ many directions which takes $O(\frac{n+\psi}{\Theta} \log(n + \psi))$ time. For more details see Section 2 and Section 6 (Appendix) in [3]. \square

At the end we collect all Θ -guarded cells and output the sequence of nodes and

angle Θ	worst-case complexity
$\pi \leq \Theta < 2\pi$	$ CH(G) $ vertices
$\frac{\pi}{2} \leq \Theta < \pi$	$O(n)$
$\delta < \Theta < \frac{\pi}{2}$, for constant $\delta > 0$	$O(n^{1+\varepsilon})$, for any $\varepsilon > 0$

Table 1

The worst-case complexity of the Θ -region in dependency on the angle Θ .

edges on the boundary of the union of them. We conclude with the following Theorem.

Theorem 16 *For any $\xi > 0$, the Θ -region for $\Theta < \pi$ can be computed in time $O(n^{\frac{3}{2}+\xi}/\Theta + \mu \log n)$, where μ denotes the complexity of the arrangement $\mathcal{A}(\mathcal{C}')$.*

7 Conclusion

In this paper we consider a point to be guarded, if it is guarded from 'all' sides by a given finite set of guards G . Our main goals were to analyze the shape and the complexity of the Θ -region, i.e. the set of all Θ -guarded points, and give a mathematical description of it.

As a result, we showed that the Θ -region is defined by a set of at most $O(\frac{n}{\Theta})$ many circular arcs. The difficulty in the complexity analysis of the Θ -region itself appeared while arguing about the complexity of the union of convex sets U_i which are bounded by these arcs (cf. Formula 3). In dependency on Θ we summarize our results on the worst-case complexity of the Θ -region in Table 1. Furthermore, we could give a series of inputs with decreasing angle and increasing number of guards whose asymptotic complexity is $\Omega(n^2)$. Finally we gave an algorithm to compute the Θ -region.

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