SOME DIOPHANTINE TRIPLES AND QUADRUPLES FOR QUADRATIC POLYNOMIALS

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Abstract. In this paper, we give some new examples of polynomial $D(n)$-triples and quadruples, i.e. sets of polynomials with integer coefficients, such that the product of any two of them plus a polynomial $n \in \mathbb{Z}[X]$ is a square of a polynomial with integer coefficients. The examples illustrate various theoretical properties and constructions for a quadratic polynomial n which appeared in recent papers. One of the examples gives a partial answer to the question about number of distinct $D(n)$-quadruples if n is an integer that is the product of twin primes.

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1. Introduction

Let n be a nonzero integer. A set $\{a_1, a_2, ..., a_m\}$ of m distinct positive integers is called a Diophantine m-tuple with the property $D(n)$, or simply $D(n)$-m-tuple, if $a_i a_j + n$ is a perfect square for all $1 \le i < j \le m$. An improper $D(n)$-m-tuple is an m-tuple with the same property, but with relaxed condition that the elements a_i can be integers and need not be distinct. Diophantus found the quadruple $\{1, 33, 68, 105\}$ with the property $D(256)$. The first Diophantine quadruple with the property $D(1)$, the set $\{1, 3, 8, 120\}$, was found by Fermat (see [4]).

Diophantine triples and quadruples with the property $D(n)$ can be classified as regular or irregular, depending on whether they satisfy the conditions given in next two definitions. Let $\{a, b, c\}$ be a $D(n)$-triple and

$$ab + n = r^2, \; ac + n = s^2, \; bc + n = t^2, \tag{1}$$

where r, s, t are positive integers.

Definition 1. *A $D(n)$-triple $\{a, b, c\}$ is called regular if it satisfies the condition*

$$(c - b - a)^2 = 4(ab + n). \tag{2}$$

Equation (2) is symmetric under permutations of a, b, c. Also, from (2), using (1), we get

$$c_{\pm} = a + b \pm 2r,$$

and we have

$$ac_{\pm} + n = (a \pm r)^2, \; bc_{\pm} + n = (b \pm r)^2.$$

Definition 2. *A $D(n)$-quadruple $\{a, b, c, d\}$ is called regular if it satisfies the condition*

$$n(d + c - a - b)^2 = 4(ab + n)(cd + n). \tag{3}$$

Equation (3) is symmetric under permutations of a, b, c, d. Since the right hand side of (3) is a square, it follows that a regular $D(n)$-quadruple can exist only if n is a square, whereas regular $D(n)$-triples exist for all n. When n is a square, the quadruple formed by adding a zero to a regular triple is an improper regular quadruple. Equation (3) is a quadratic equation in d with roots

$$d_{\pm} = a + b + c + \frac{2}{n}(abc \pm rst),$$

and we have

$$ad_{\pm} + n = \frac{1}{n}(rs \pm at)^2,\ bd_{\pm} + n = \frac{1}{n}(rt \pm bs)^2,\ cd_{\pm} + n = \frac{1}{n}(st \pm cr)^2.$$

An irregular $D(n)$-tuple is one that is not regular. A semi-regular $D(n)$-quadruple is one which contains a regular triple, and a twice semi-regular $D(n)$-quadruple is one that contains two regular triples.

For $n = 1$, we have the following conjecture (see [10]).

Conjecture 1. *If $\{a, b, c, d\}$ is a $D(1)$-quadruple and $d > \max\{a, b, c\}$, then $d = d_+$.*

It is clear that Conjecture 1 implies that there does not exist a $D(1)$-quintuple. Baker and Davenport [1] proved Conjecture 1 for the triple $\{a, b, c\} = \{1, 3, 8\}$ with the unique extension $d = 120$. Dujella and Pethő [16] proved it for all triples of the form $\{1, 3, c\}$. Conjecture 1 was recently verified for all triples of the form $\{k-1, k+1, c\}$ (see [7, 18, 3]). Dujella [10] proved that there are only finitely many $D(1)$-quintuples and there does not exist a $D(1)$-sextuple. Fujita [19] proved that any $D(1)$-quintuple contains a regular Diophantine quadruple, i.e. if $\{a, b, c, d, e\}$ is a $D(1)$-quintuple and $a < b < c < d < e$, then $d = d_+$.

Polynomial variant of the above problem was first studied by Jones [22, 23] and it was for the case $n = 1$.

Definition 3. *Let $n \in \mathbb{Z}[X]$ and let $\{a_1, a_2, ..., a_m\}$ be a set of m different nonzero polynomials with integer coefficients. We assume that there does not exist a polynomial $p \in \mathbb{Z}[X]$ such that $\frac{a_1}{p}, \ldots, \frac{a_m}{p}$ and $\frac{n}{p^2}$ are integers. The set $\{a_1, a_2, ..., a_m\}$ is called a polynomial $D(n)$-m-tuple if for all $1 \le i < j \le m$ the following holds: $a_i a_j + n = b_{ij}^2$ where $b_{ij} \in \mathbb{Z}[X]$.*

The assumption concerning the polynomial p means that not all elements of $\{a_1, a_2, ..., a_m\}$ are allowed to be constant. Since the constructions from Definitions 1 and 2 are obtained using only algebraic manipulations, they are also valid in the polynomial case.

Dujella and Fuchs [11] proved that all polynomial $D(1)$-quadruples in $\mathbb{Z}[X]$ are regular. Note that this is not true in $\mathbb{C}[X]$, as Dujella and Jurasić showed in [15] by giving the following example

$$\left\{\frac{\sqrt{-3}}{2}, -\frac{2\sqrt{-3}}{3}(X^2-1), \frac{-3+\sqrt{-3}}{3}X^2 + \frac{2\sqrt{-3}}{3}, \frac{3+\sqrt{-3}}{3}X^2 + \frac{2\sqrt{-3}}{3}\right\}$$

of a $D(1)$-quadruple in $\mathbb{Q}(\sqrt{-3})[X]$ which is not regular. In [15], it is shown that there does not exist a Diophantine 8-tuple in $\mathbb{C}[X]$ (see also [12]).

For polynomial $D(n)$-m-tuples with n constant, from [15, Theorem 1] it follows that $m \le 7$ for all $n \in \mathbb{Z}\backslash\{0\}$. Dujella and Fuchs, jointly with Tichy [13] and later with Walsh [14], considered the case $n = \mu_1 X + \mu_0$ with integers $\mu_1 \neq 0$ and μ_0. Let

$$L = \sup\{|S| : S \text{ is a polynomial } D(\mu_1 X+\mu_0)\text{-}m\text{-tuple for some } \mu_1 \neq 0, \mu_0 \in \mathbb{Z}\}$$

where $|S|$ denotes the number of elements in the set S, and let L_k be the number of polynomials of degree k in a polynomial $D(\mu_1 X + \mu_0)$-m-tuple S. The results from [14] are sharp bounds $L_0 \leq 1$, $L_1 \leq 4$, $L_k \leq 3$ for all $k \geq 2$, and finally $L \leq 12$. Jurasić [24] considered the case where n is a quadratic polynomial in $\mathbb{Z}[X]$. Let

$$Q = \sup\{|S| : S \text{ is a polynomial } D(\mu_2 X^2 + \mu_1 X + \mu_0)\text{-}m\text{-tuple} \\ \text{for some } \mu_2 \neq 0, \mu_1, \mu_0 \in \mathbb{Z}\}$$

and let Q_k be the number of polynomials of degree k in a polynomial $D(\mu_2 X^2 + \mu_1 X + \mu_0)$-$m$-tuple S. In [24], it was proved that $Q \leq 98$ and that if a polynomial $D(n)$-m-tuple for a quadratic n contains only polynomials of odd degree, then $m \leq 18$. Moreover, $Q_0 \leq 2$, $Q_1 \leq 4$, $Q_2 \leq 81$, $Q_3 \leq 5$, $Q_4 \leq 6$, and $Q_k \leq 3$ for $k \geq 5$. The bounds for Q_0, Q_1 and Q_k for all $k \geq 5$ are sharp.

In Section 2 we give some examples of a polynomial $D(n)$-triples and quadruples, where n is a quadratic polynomial. These examples show that several auxiliary results from [24] are sharp, i.e. in the cases when the existence of Diophantine triples with certain properties cannot be excluded, such triples indeed exist. In Section 3 we study $D(n)$-quadruples if n is an integer that is the product of twin primes and we construct a polynomial Diophantine quadruple for the quadratic polynomial corresponding to that case. In Section 4 we give polynomial examples for all cases of Gibbs' construction of adjugates of a $D(n)$-quadruples. In some of them we use quadruples constructed in Sections 2 and 3.

2. Diophantine triples with elements of the same degree

Although the final bound from [24, Theorem 1], $Q \leq 98$, is likely very far from being optimal, the bounds from [24] for polynomials of fixed degree, i.e. bounds for Q_k, are sharp or almost sharp, except for $k = 2$. In particular, they are sharp for $k = 0$, $k = 1$ and $k \geq 5$, in view of the following examples given in [24]: the set $\{3, 5\}$ is a polynomial $D(9X^2 + 24X + 1)$-pair, the set

$$\{X, X + 8, 2X + 2, 5X + 20\}$$

is a polynomial $D(-(X + 9)(X - 1))$-quadruple, the set

$$\{X^{2l-1} + X, X^{2l-1} + 2X^l + 2X, 4X^{2l-1} + 4X^l + 5X\}$$

is a polynomial $D(-X^2)$-triple for any integer $l \geq 2$, and the set

$$\{X^{2l} + X^l, X^{2l} + X^l + 4X, 4X^{2l} + 4X^l + 8X\}$$

is a polynomial $D(4X^2)$-triple for any integer $l \geq 1$.

However, even in the cases $k = 2, 3, 4$, several auxiliary results from [24] are (almost) sharp, as it will be illustrated by concrete examples in this section.

The main role in the proofs of the upper bounds on Q_k, $k \geq 2$ has [13, Lemma 1]. It says that for a polynomial $D(n)$-triple $\{a, b, c\}$ for which (1) holds, there exist polynomials $e, u, v, w \in \mathbb{Z}[X]$ such that $ae + n^2 = u^2$, $be + n^2 = v^2$, $ce + n^2 = w^2$. More precisely,

$$e = n(a + b + c) + 2abc - 2rst, \tag{4}$$

$u = at - rs$, $v = bs - rt$, $w = cr - st$, and

$$c = a + b + \frac{e}{n} + \frac{2}{n^2}(abe + ruv). \tag{5}$$

We define

$$\overline{e} = n(a + b + c) + 2abc + 2rst, \tag{6}$$

and we easily get

$$e \cdot \overline{e} = n^2(c - a - b - 2r)(c - a - b + 2r). \tag{7}$$

Moreover, we have (see [24])

$$e = n(a + b - c) + 2rw, \tag{8}$$

$$e = n(a - b + c) + 2sv, \tag{9}$$

$$e = n(-a + b + c) + 2tu. \tag{10}$$

Let $\mathbb{Z}^+[X]$ denote the set of all polynomials with integer coefficients with positive leading coefficient. For $a, b \in \mathbb{Z}[X]$, $a < b$ means that $b - a \in \mathbb{Z}^+[X]$.

The polynomial e had an important role in getting upper bounds for Q_k, $k \geq 2$, and also in classifying various types of Diophantine triples with the same degree. In [24], all possibilities for e for a given quadratic polynomial n were investigated. Here we are interested to find examples of $D(n)$-triples which illustrate these possibilities. Trivial case is $e = 0$, for which we have a regular triple $\{a, b, a + b + 2r\}$. Let $\{a, b, c\}$, where $a < b < c$, be a polynomial $D(n)$-triple containing only quadratic polynomials and let $e \in \mathbb{Z}[X]$ be defined by (4). We are looking at extensions of $\{a, b\}$ to a polynomial $D(n)$-triple $\{a, b, c\}$ with $c > b$ and then at the corresponding $e \in \mathbb{Z}[X]$ defined by (4). We have (see [24, Lemmas 4-8]):

A2) If n and e have a common linear factor but $n \nmid e$ (we consider divisibility over $\mathbb{Q}[X]$), then $n = n_1 n_2$ where n_1, n_2 are linear polynomials over $\mathbb{Q}$ such that $n_1 \nmid n_2$. For fixed a and b, there exist at most two such e-s.

B2) If $n|e$, then $n = \lambda n_1^2$ where $\lambda \in \mathbb{Q}\backslash\{0\}$ and n_1 is a linear polynomial over $\mathbb{Q}$. For fixed a and b, there is at most one such e.

C2) If $e \in \mathbb{Z} \setminus \{0\}$, then, for fixed a and b, there is at most one such e.

D2) If e is a linear polynomial which does not divide n, then, for fixed a and b, there is at most one such e.

E2) Let e be a quadratic polynomial which does not have a common nonconstant factor with n. Then there is at most one polynomial $c' \neq c$ such that $\{a, b, c'\}$, $a < b < c'$, is a polynomial $D(n)$-triple and $f \in \mathbb{Z}[X]$, obtained by applying (4) on that triple ($f = n(a + b + c') + 2abc' - 2rs't'$, where $ac' + n = s'^2$, $bc' + n = t'^2$), is a quadratic polynomial which does not have a common nonconstant factor with n, or the analogous statement holds for $\{a, b', c\}$, $a < b' < c$, $b' \neq b$ or for $\{a', b, c\}$, $a' < b < c$, $a' \neq a$ (see [24, Lemma 8] for details).

In [24, Lemmas 9-11], the following results were proved for the sets of polynomials of degree $k \geq 3$. Let $\{a, b, c\}$, $a < b < c$, be a polynomial $D(n)$-triple of polynomials of degree k, then beside $e = 0$ we have the following possibilities. If $k = 3$:

A3) For fixed a and b, there is at most one $e \in \mathbb{Z}\backslash\{0\}$ defined by (4).

B3) Let $e \in \mathbb{Z}[X]$ be a linear polynomial defined by (4). Then $e \nmid n$. For fixed a and b, there exist at most two such e-s.

For $k = 4$, we have:

A4) Let $e \in \mathbb{Z}\backslash\{0\}$ be defined by (4). Then, for fixed a and b, there exist at most three such e-s.

For $k \geq 5$, we have only trivial situation $e = 0$ (see [24, Proposition 2 4.)]).

The proofs of [24, Lemmas 4-8] give us methods for a construction of the examples for the cases where $k = 2$. The first polynomial $D(n)$-triple is an example for the case **A2)**, obtained using [24, Lemma 5]. Let $n = n_1n_2$ and $e = e_1e_2$, where n_i, e_i for $i = 1, 2$ are linear polynomials over $\mathbb{Q}$. In the proof of [24, Lemma 4] we have $a = A^2(X - \phi_1)(X - \phi_2)$, where $A \in \mathbb{N}$ and $\phi_1, \phi_2 \in \mathbb{Q}$. Analogously, $b = B^2(X - \chi_1)(X - \chi_2)$, $c = C^2(X - \lambda_1)(X - \lambda_2)$, where $B, C \in \mathbb{N}$ and $\chi_i, \lambda_i \in \mathbb{Q}$ for $i = 1, 2$. From [13, Lemma 1] we obtain $b\nu + n_2^2 = v_1^2$, where $v = n_1v_1$ and $c\nu + n_2^2 = w_1^2$, where $w = n_1w_1$ and $v_1, w_1 \in \mathbb{Q}[X]$, $\deg(v_1), \deg(w_1) \leq 1$, $\nu \in \mathbb{Q}\backslash\{0\}$. Assume that at least one of the polynomials u_1, v_1 and w_1 has a degree equal to 0. Since $\deg(e) = \deg(n) = \deg(r) = 2$, if $\deg(w) = 1$, then by (8) $\deg(a + b - c) = 1$. Hence, $A^2 + B^2 = C^2$. Analogously, if $\deg(v) = 1$, then by (9) we obtain $C^2 = -A^2 + B^2 < B^2$, a contradiction. Also, if $\deg(u) = 1$, then by (10) we get $C^2 = A^2 - B^2 < A^2$, a contradiction. Therefore only w_1 can have a degree equal to 0. From [24, Lemma 4], we have

$$n_2 = \frac{\varepsilon_2 - \varepsilon_1}{2}X + \frac{\varepsilon_1\phi_1 - \varepsilon_2\phi_2}{2},$$

$$u_1 = \frac{\varepsilon_1 + \varepsilon_2}{2}X - \frac{\varepsilon_1\phi_1 + \varepsilon_2\phi_2}{2},$$

where $A^2\nu = \varepsilon_1\varepsilon_2$ and $\varepsilon_1, \varepsilon_2 \in \mathbb{Q}\backslash\{0\}$. Analogously, for b and c, we get $n_2 = \frac{\tau_2-\tau_1}{2}X + \frac{\tau_1\chi_1-\tau_2\chi_2}{2}$, $v_1 = \frac{\tau_1+\tau_2}{2}X - \frac{\tau_1\chi_1+\tau_2\chi_2}{2}$ and $n_2 = \frac{\psi_2-\psi_1}{2}X + \frac{\psi_1\lambda_1-\psi_2\lambda_2}{2}$, $w_1 = \frac{\psi_1+\psi_2}{2}X - \frac{\psi_1\lambda_1+\psi_2\lambda_2}{2}$, where $B^2\nu = \tau_1\tau_2$, $C^2\nu = \psi_1\psi_2$ and $\tau_i, \psi_i \in \mathbb{Q}\backslash\{0\}$ for $i = 1, 2$. We conclude that $\varepsilon_2 - \varepsilon_1 = \tau_2 - \tau_1 = \psi_2 - \psi_1 := \eta$ and $\psi_1 + \psi_2 = 0$. Hence, $\psi_1 = -\frac{\eta}{2}$, $\psi_2 = \frac{\eta}{2}$ so $C^2\nu = -\frac{\eta^2}{4}$. Note that $\nu = -Q^2$, for some $Q \in \mathbb{Q} \setminus \{0\}$. Multiplying $A^2+B^2 = C^2$ by ν, we get $\varepsilon_1(\varepsilon_1+\eta)+\tau_1(\tau_1+\eta) = -\frac{\eta^2}{4}$, from which we obtain $\varepsilon_1 = -\frac{\eta}{2} \pm BQ$ and $\tau_1 = -\frac{\eta}{2} \pm AQ$. If we choose, for example, $A = 3$, $B = 4$, $Q = 1$, $\eta = 10$, $\phi_2 = 0$, $\chi_1 = 1$, $\chi_2 = 2$ using the equations $\varepsilon_1\phi_1 - \varepsilon_2\phi_2 = \tau_1\chi_1 - \tau_2\chi_2$, $ab + n = r^2$ and (5), we obtain a polynomial $D(36(5X - 6)(29X - 54))$-triple

$$\{9X(X - 12), 16(X - 1)(X - 2), (5X + 14)(5X - 26)\}.$$

Here $e = -1296(29X - 54)^2$.

Similarly we find the example for the case **B2)**, described in [24, Lemma 4]. From (7) it follows that $n|\bar{e}$ and from (4) and (6) we get $\bar{e} - e = 4rst$, so $n|rst$. Therefore, by (4) $n|abc$. Since in this case $n = \lambda n_1^2$, where $\lambda \in \mathbb{Q}\backslash\{0\}$ and n_1 is a linear polynomial over $\mathbb{Q}$, we may assume that $n_1|a$. Now, from [13, Lemma 1] we get that $n_1|u_1$, where $u_1 \in \mathbb{Q}[X]$ and $u = n_1u_1$. Hence, $n_1^2|u_1^2$ so $n_1^2|a$. For $A = 3$, $B = 4$, $Q = 1$, $\eta = 10$, $\phi_1 = \phi_2 = 0$, $\chi_2 = 1$, analogously as in the previous case, we obtain a polynomial $D(900X^2)$-triple

$$\{9X^2, 16(X + 4)(X - 1), 25(X - 2)(X + 2)\}.$$

Here $e = -32400X^2$. This triple can be extended to a regular $D(n)$-quadruple

$$\{9X^2, 16(X + 4)(X - 1), 25(X - 2)(X + 2), d\}$$

in two ways. From (3) we obtain $d = -36$ and $d = 4(2X+1)(2X-3)(X+3)(X+1)$.

For the case **C2)**, we follow the strategy from the proof of [24, Lemma 6]. Let $a = A^2(X^2 + \alpha_1 X + \alpha_0)$, $b = B^2(X^2 + \beta_1 X + \beta_0)$ and $c = C^2(X^2 + \gamma_1 X + \gamma_0)$, where $\alpha_i, \beta_i, \gamma_i \in \mathbb{Q}$ for $i = 0, 1$. By [13, Lemma 1], there exist $u, v, w \in \mathbb{Z}[X]$, such that $ae + n^2 = u^2$, $be + n^2 = v^2$ and $ce + n^2 = w^2$. Here we have $\deg(u) = \deg(v) = \deg(w) = 2$ and the leading coefficients of the polynomials u, v, w are equal to the leading coefficient of n, up to a sign. Denote by μ, ϑ, ν, ω the leading coefficients of n, u, v, w, respectively. By (8) $\omega = \mu$, by (10) $\vartheta = -\mu$ and by (9) $\nu = -\mu$. From $ae + n^2 = u^2$ it follows that

$$\begin{aligned} u + n &= \varepsilon_1, \\ u - n &= \varepsilon_2(X^2 + \alpha_1 X + \alpha_0), \end{aligned} \tag{11}$$

where $\varepsilon_1\varepsilon_2 = A^2 e$ and $\varepsilon_1, \varepsilon_2 \in \mathbb{Q}\backslash\{0\}$. From (11), we obtain

$$-n = \frac{\varepsilon_2}{2}X^2 + \frac{\varepsilon_2\alpha_1}{2}X + \frac{\varepsilon_2\alpha_0 - \varepsilon_1}{2},$$
$$u = \frac{\varepsilon_2}{2}X^2 + \frac{\varepsilon_2\alpha_1}{2}X + \frac{\varepsilon_2\alpha_0 + \varepsilon_1}{2}.$$

For b and c we have equations analogous to (11). Let $B^2 e = \tau_1\tau_2$ and $C^2 e = \psi_1\psi_2$, where $\tau_i, \psi_i \in \mathbb{Q}\backslash\{0\}$ for $i = 1, 2$. It follows that $-n = \frac{\tau_2}{2}X^2 + \frac{\tau_2\beta_1}{2}X + \frac{\tau_2\beta_0 - \tau_1}{2}$, $v = \frac{\tau_2}{2}X^2 + \frac{\tau_2\beta_1}{2}X + \frac{\tau_2\beta_0 + \tau_1}{2}$ and $n = \frac{\psi_2}{2}X^2 + \frac{\psi_2\gamma_1}{2}X + \frac{\psi_2\gamma_0 - \psi_1}{2}$, $w = \frac{\psi_2}{2}X^2 + \frac{\psi_2\gamma_1}{2}X + \frac{\psi_2\gamma_0 + \psi_1}{2}$. From the expressions for n, we conclude that $\varepsilon_2 = \tau_2 = -\psi_2$ and $\alpha_1 = \beta_1 = \gamma_1$. Using this and the fact that the discriminants of the polynomials $ae+n^2$, $be+n^2$ and $ce+n^2$ must be equal to 0, we obtain a system of equations with infinitely many solutions. One of that solutions is $A = 2$, $B = 4$, $e = -3(\alpha_0 - \beta_0)$, $\varepsilon_2 = 3$, $\varepsilon_1 = -\alpha_0 + \beta_0$, $\gamma_2 = \frac{8\beta_0 - 5\alpha_0}{3}$. The equations (1) are satisfied for $a = 4X^2$, $b = 16X^2 - 2$, $c = 36X^2 - 12$, $n = 24X^2 + 1$. Hence,

$$\{4X^2, 16X^2 - 2, 36X^2 - 12\}$$

is a polynomial $D(24X^2+1)$-quadruple, where $e = -24$. This triple can be extended to a twice semi-regular polynomial $D(24X^2 + 1)$-quadruple

$$\{4X^2, 16X^2 - 2, 36X^2 - 12, 4X^2 - 4\}. \tag{12}$$

For the triple $\{4X^2, 36X^2 - 12, 4X^2 - 4\}$, we have $e = 180X^2 - 30$. For $a = 4X^2$ and $b = 16X^2 - 2$ we also have $e = 640X^2 - 40$, for which we get a polynomial $D(24X^2+1)$-triple $\{4X^2, 16X^2 - 2, 324X^2 - 24\}$. For $e = 0$ we have a regular polynomial $D(24X^2 + 1)$-triple $\{4X^2, 16X^2 - 2, 36X^2\}$. Observe that quadruple (12) can be obtained by taking $k = 1$ and $m = 4X^2$ in [5, formula (15)].

Let us now find an example for the case **D2)**, using the proof of [24, Lemma 7]. Using (8), we obtain

$$1 - 2r\xi = \sigma n, \tag{13}$$

where $\xi, \sigma \in \mathbb{Q}\backslash\{0\}$. We further obtain

$$e = \frac{1}{\xi^2 + \sigma}(a + b + 2r - 2n\xi).$$

Since e must be a linear polynomial, from the coefficient with X^2 on the right side of the previous equation we conclude that $\xi = \frac{(A+B)^2}{2\mu}$, where μ is a leading coefficient of n. Now, from (13) it follows that $\sigma = \frac{-AB(A+B)^2}{\mu^2}$. For $a = X^2 + X$, $B = 2$ and $\mu = 6$, using the relation $ab + n = r^2$ and (5), we obtain a polynomial $D(6X^2 + 3X + 1)$-triple

$$\{X(X+1), (2X-1)(2X+1), 9X^2 - 9X - 3\}.$$

Here $e = -24X - 8$. Notice that, for $a = X(X+1)$ and $b = (2X-1)(2X+1)$, from $e = \frac{1}{\xi^2+\sigma}(a + b - 2r + 2n\xi)$ we obtain $e = 8(5X+3)(4X-1)$ and then $c = 81X^2 + 27X - 15$.

We will now illustrate the case **E2)**, described in [24, Lemma 8 1)]. Let $a = X^2$, $b = 16X^2 + 8$ and $n = 16X^2 + 9$. We obtain $\psi = \frac{2}{3}$, $\phi = -\frac{1}{3}$ and then $e = 273X^2 + 126$, $f = 33X^2 + 18$. For e we get $c = 100X^2 + 44$, and for f we get $c' = 36X^2 + 20$ so we have two polynomial $D(n)$-triples $\{a, b, c\}$ and $\{a, b, c'\}$. The second of them is a part of a twice semi-regular polynomial $D(n)$-quadruple $\{X^2, 16X^2 + 8, 25X^2 + 14, 36X^2 + 20\}$ (this quadruple can be obtained by taking $k = X^2$ in the formula for a $D(16k + 9)$-quadruple from [5, Section 6]).

For the cases where $k = 3$ and $k = 4$, we find the examples using a method of undetermined coefficients and then fixing some of the coefficients to 0 or to 1. Let us first find the example for the case **A3)**. Let $a = X^3 + \alpha_2 X^2 + \alpha_1 X + \alpha_0$, $n = \mu_2 X^2 + \mu_0$, $u = \mu_2 X^2 + \vartheta_1 X + \vartheta_0$ be a polynomials over $\mathbb{Q}$. From $ae + n^2 = u^2$, we obtain $e = 2\vartheta_1\mu_2$, $\alpha_0 = \dfrac{\vartheta_0^2 - \mu_0^2}{2\vartheta_1\mu_2}$, $\alpha_1 = \dfrac{\vartheta_0}{\mu_2}$ and $\alpha_2 = -\dfrac{2\mu_0\mu_2 - 2\vartheta_0\mu_2 - \vartheta_1^2}{2\vartheta_1\mu_2}$. Let $b = X^3 + \beta_2 X^2 + \beta_1 X + \beta_0$ and $v = \mu_2 X^2 + \nu_1 X + \nu_0$ be a polynomials over $\mathbb{Q}$. From $be + n^2 = v^2$ and the obtained expression for e, we get $\vartheta_1 = \nu_1$. Also, $\beta_2 = -\dfrac{2\mu_0\mu_2 - 2\nu_0\mu_2 - \vartheta_1^2}{2\vartheta_1\mu_2}$, $\beta_1 = \dfrac{\nu_0}{\mu_2}$, and $\beta_0 = \dfrac{\nu_0^2 - \mu_0^2}{2\vartheta_1\mu_2}$. We now equate the expression $(X^3 + \alpha_2 X^2 + \alpha_1 X + \alpha_0)(X^3 + \beta_2 X^2 + \beta_1 X + \beta_0) + \mu_2 X^2 + \mu_0$ with the square of a polynomial of degree 3. It is easy to obtain equal coefficients of X^6, X^5, X^4, X^3 at both sides of the equation. Then, comparing the coefficients of X^2, X^1 and X^0, we get three equations in unknowns $\mu_2, \mu_0, \vartheta_1, \vartheta_0, \nu_0$. If $\mu_0 = 0$, the system has a nontrivial rational solution for $\nu_0 = 27\tau^6$, $\tau \in \mathbb{Q}$. Namely, $\mu_2 = 9\tau^4$, $\vartheta_1 = 9\tau^5$, $\vartheta_0 = -9\tau^6$. By taking $\tau = 1$ and multiplying the elements of the obtained rational triple by 2, we finally get a polynomial $D(36X^2)$-triple

$$\{(X-1)(X+1)(2X-1), (X+3)(2X^2+X+3), 4(2X+3)(X^2+2X+3)\},$$

with $e = 1296$. We can extend this triple to the regular quadruple in $\mathbb{Q}[X]$ by taking the fourth element $d = \frac{4}{9}X^2(X+2)(2X+5)(2X+1)(2X^2+3X+4)$.

Let us consider the case **B3)**. If we write $ae + n^2 = u^2$, where a, e, n, u are polynomials over $\mathbb{Q}$, in the form

$$(X^3 + \alpha_2 X^2 + \alpha_1 X + \alpha_0)((\varepsilon^2 - \varphi^2)X + \gamma) + (\varphi X^2)^2 = (\varepsilon X^2 + \vartheta_1 X + \vartheta_0)^2,$$

the coefficients of X^4 are equal on both sides of the equation. Comparing the other coefficients, we obtain

$$\alpha_2 = \frac{\gamma - 2\vartheta_1\varepsilon}{-\varepsilon^2 + \varphi^2}, \quad \alpha_1 = \frac{\vartheta_1^2\varepsilon^2 - \vartheta_1^2\varphi^2 + \gamma^2 - 2\gamma\vartheta_1\varepsilon - 2\varphi\varepsilon^2 + 2\varphi^3 + 2\vartheta_0\varepsilon^3 - 2\vartheta_0\varepsilon\varphi^2}{(-\varepsilon^2 + \varphi^2)^2},$$

$\alpha_0 = -\frac{-\gamma\vartheta_1^2\varepsilon^2+\gamma\vartheta_1^2\varphi^2-\gamma^3+2\gamma^2\vartheta_1\varepsilon+2\gamma\varphi\varepsilon^2-2\gamma\varphi^3-2\gamma\vartheta_0\varepsilon^3+2\gamma\vartheta_0\varepsilon\varphi^2+2\vartheta_0\vartheta_1\varepsilon^4-4\vartheta_0\vartheta_1\varepsilon^2\varphi^2+2\vartheta_0\vartheta_1\varphi^4}{(-\varepsilon^2+\varphi^2)^3}$, $\vartheta_1 = \frac{\vartheta_0\varepsilon^2\varphi-\vartheta_0\varepsilon\varphi^2+\gamma^2+\vartheta_0\varepsilon^3-\vartheta_0\varphi^3-\varepsilon^3-\varphi\varepsilon^2+\varepsilon\varphi^2+\varphi^3}{(\varphi+\varepsilon)\gamma}$. Similarly, from $be+n^2=v^2$ we get $\beta_2 = \dfrac{\gamma-2\nu_1\varepsilon}{-\varepsilon^2+\varphi^2}$, $\beta_1 = \dfrac{\nu_1^2\varepsilon^2-\nu_1^2\varphi^2+\gamma^2-2\gamma\nu_1\varepsilon-2\varphi\varepsilon^2+2\varphi^3+2\nu_0\varepsilon^3-2\nu_0\varepsilon\varphi^2}{(-\varepsilon^2+\varphi^2)^2}$, $\beta_0 = -\frac{-\gamma\nu_1^2\varepsilon^2+\gamma\nu_1^2\varphi^2-\gamma^3+2\gamma^2\nu_1\varepsilon+2\gamma\varphi\varepsilon^2-2\gamma\varphi^3-2\gamma\nu_0\varepsilon^3+2\gamma\nu_0\varepsilon\varphi^2+2\nu_0\nu_1\varepsilon^4-4\nu_0\nu_1\varepsilon^2\varphi^2+2\nu_0\nu_1\varphi^4}{(-\varepsilon^2+\varphi^2)^3}$, $\nu_1 = \frac{\nu_0\varepsilon^2\varphi-\nu_0\varepsilon\varphi^2+\gamma^2+\nu_0\varepsilon^3-\nu_0\varphi^3-\varepsilon^3-\varphi\varepsilon^2+\varepsilon\varphi^2+\varphi^3}{(\varphi+\varepsilon)\gamma}$, where $v = \varepsilon X^2+\nu_1X+\nu_0$ and $b = X^3+\beta_2X^2+\beta_1X+\beta_0$ are polynomials over $\mathbb{Q}$. We are left with the condition that $(X^3+\alpha_2X^2+\alpha_1X+\alpha_0)(X^3+\beta_2X^2+\beta_1X+\beta_0)+\varphi X^2$ is equal to a square of a polynomial r with degree 3. It is easy to obtain the equal coefficients of X^6, X^5, X^4, X^3 on both sides of the equation. Comparing the coefficients of X^2, X^1, X^0 on both sides of the equation, we get three equations in the remaining unknowns. For $\nu_0 = -\dfrac{\gamma^2}{\varepsilon(\varphi+\varepsilon)^2}$ and $\vartheta_0 = \dfrac{\gamma^2(\varphi+2\varepsilon)}{(\varphi+\varepsilon)^2\varepsilon\varphi}$ the equations for X^1 and X^0 are satisfied. Inserting those expressions for ν_0 and ϑ_0 into the equation for X^2, we obtain

$$\begin{aligned}&\varepsilon^2\varphi^9+6\varphi^8\varepsilon^3+15\varepsilon^4\varphi^7+20\varepsilon^5\varphi^6+15\varepsilon^6\varphi^5+6\varepsilon^7\varphi^4\\&+\varepsilon^8\varphi^3-4\gamma^4\varphi^2-16\gamma^4\varepsilon\varphi-16\varepsilon^2\gamma^4=0.\end{aligned} \tag{14}$$

One solution of the equation (14) is $\varepsilon=-3$, $\varphi=9$, $\gamma=54$, for which we have $a = X^3+\frac{5}{2}X^2-5X+\frac{3}{2}$, $b = X^3-\frac{3}{2}X^2-9X+\frac{27}{2}$, $n = 9X^2$, $r = X^3+\frac{1}{2}X^2-9X+\frac{9}{2}$, $e = -72X+54$, $u = 3X^2-21X+9$, $v = 3X^2+27X-27$, and $c = \frac{4}{9}X^3+\frac{10}{9}X^2+\frac{16}{9}X+\frac{2}{3}$. Multiplying those expressions by 18, we get integer coefficients. Indeed,

$$\{9(X-1)(2X^2+7X-3), 9(X+3)(X-3)(2X-3), 4(2X+1)(X^2+2X+3)\}$$

is a polynomial $D(2916X^2)$-triple with $e = -419904X+314928$. Extending this triple to the regular quadruple in $\mathbb{Q}[X]$ we get the fourth element $d = \frac{4}{9}X^2(X+2)(2X-5)(2X+7)(2X^2+X-12)$.

Let us take a closer look at equation (14). It leads to the condition $4\varphi^3\varepsilon^2(\varphi+\varepsilon)^6(\varphi+2\varepsilon)^2 = U^4$. By substituting $\varphi = T^2$, $\varepsilon = Z$, we get the elliptic surface (elliptic curve over $\mathbb{Q}(T)$) $2TZ(T^2+Z)(T^2+2Z) = V^2$. Now, substituting $4TZ = X$, we obtain the elliptic surface in the Weierstrass form

$$X^3+6T^3X^2+8T^6X = Y^2. \tag{15}$$

The rank of curve (15) over $\mathbb{Q}(T)$ is equal to 0. But standard conjectures predict that for infinitely many specializations $T = t$, the specialized curve has positive rank over $\mathbb{Q}$. We list some small solutions $(\varepsilon, \varphi, \gamma)$ of equation (14): $(-180, 900, 1296000)$, $(-160, 1600, 3456000)$, $(-153, 289, 235824)$, $(-136, 289, 265302)$, $(-135, 225, 60750)$, $(-108, 144, 7776)$, $(-96, 144, 13824)$, $(-90, 100, 750)$, $(-90, 225, 91125)$, $(-48, 144, 27648)$, $(-36, 144, 23328)$, $(-18, 9, 81)$, $(-10, 100, 6750)$, $(-6, 9, 27)$, $(-3, 9, 54)$, $(72, 144, 46656)$, $(200, 1600, 4320000)$.

Finally, let us find the example for the case **A4)**. For $a, e, n, u \in \mathbb{Q}[X]$, it must hold $ae+n^2=u^2$. Thus, let us find an identity of the form

$$(X^4+\alpha_3X^3+\alpha_2X^2+\alpha_1X+\alpha_0)(\vartheta_2^2-\mu_2^2)+(\mu_2X^2+\mu_0)^2 = (\vartheta_2X^2+\vartheta_1X+\vartheta_0)^2. \tag{16}$$

From (16), we obtain $\alpha_3 = 2\frac{\vartheta_2\vartheta_1}{\vartheta_2^2-\mu_2^2}$, $\alpha_2 = \frac{\vartheta_1^2-2\mu_2\mu_0+2\vartheta_2\vartheta_0}{\vartheta_2^2-\mu_2^2}$, $\alpha_1 = 2\frac{\vartheta_1\vartheta_0}{\vartheta_2^2-\mu_2^2}$ and $\alpha_0 = \frac{\vartheta_0^2-\mu_0^2}{\vartheta_2^2-\mu_2^2}$. Similarly, from $be+n^2=v^2$, where $b = X^4+\beta_3X^3+\beta_2X^2+\beta_1X+\beta_0$ and $v = \nu_2X^2+\nu_1X+\nu_0$ are polynomials over $\mathbb{Q}$, we see that $\nu_2 = \vartheta_2$, and then we get $\beta_3 = 2\frac{\vartheta_2\nu_1}{\vartheta_2^2-\mu_2^2}$, $\beta_2 = \frac{\nu_1^2-2\mu_2\mu_0+2\vartheta_2\nu_0}{\vartheta_2^2-\mu_2^2}$, $\beta_1 = 2\frac{\nu_1\nu_0}{\vartheta_2^2-\mu_2^2}$, $\beta_0 = \frac{\nu_0^2-\mu_0^2}{\vartheta_2^2-\mu_2^2}$. From the condition that $(X^4+\alpha_3X^3+\alpha_2X^2+\alpha_1X+\alpha_0)(X^4+\beta_3X^3+\beta_2X^2+\beta_1X+\beta_0)+\mu_2X^2+\mu_0$ is a square of a polynomial of degree 4, we have a system of three equations. For $\vartheta_1=\nu_1=\mu_0=0$, one of the equations has a rational solution if $\vartheta_0\nu_0$ is a square. Thus, we assume that $\nu_0 = \vartheta_0\tau^2$, where $\tau\in\mathbb{Q}$, and then $\vartheta_2 = \frac{\mu_2(1+\tau^2)}{2\tau}$. It remains to solve the equation $32\tau^9\vartheta_0^3+\mu_2^4\tau^8+64\tau^7\vartheta_0^3-4\mu_2^4\tau^6+32\tau^5\vartheta_0^3+6\mu_2^4\tau^4-4\mu_2^4\tau^2+\mu_2^4=0$, which has a solution if $-2\vartheta_0^3\tau(1+\tau^2)^2$ is a fourth power. Therefore, we take $\vartheta_0 = -2\tau(1+\tau^2)^2$. We obtain $\mu_2 = 4\tau^2\frac{\tau^4+2\tau^2+1}{(\tau-1)(\tau+1)}$, and then $a = X^4-2\frac{1+\tau^2}{\tau^2-1}X^2+1$, $b = X^4-2\tau^2\frac{1+\tau^2}{\tau^2-1}X^2+\tau^4$, $e = 4\tau^2(\tau^4+2\tau^2+1)^2$, $n = 4\tau^2\frac{\tau^4+2\tau^2+1}{(\tau-1)(\tau+1)}X^2$. For $\tau=2$ we obtain polynomials over $\mathbb{Q}$ and multiplying them by 12, we get that

$$\{4(3X^2-1)(X^2-3), 4(X^2-12)(3X^2-4), 25(3X^2-25)(X^2-3)\}$$

is a polynomial $D(19200X^2)$-triple with $e = 17280000$. Extending this triple to the regular quadruple in $\mathbb{Q}[X]$ we get the fourth element $d = \frac{1}{12}X^2(X-3)(X+3)(3X^2-31)(3X^2-19)(3X^2-7)$.

3. Diophantine quadruples for products of twin primes

By considering congruences modulo 4, it is easy to prove that if n is an integer of the form $n = 4k+2$, then there does not exist a Diophantine quadruple with the property $D(n)$ [2, 21, 25]. On the other hand, it was proved in [5] that if an integer n does not have the form $4k+2$ and $n \notin S_1 = \{-4,-3,-1,3,5,8,12,20\}$, then there exists at least one Diophantine quadruple with the property $D(n)$. Moreover, if $n \not\equiv 2 \pmod 4$ and $n \notin S_2$, where $S_2 = S_1 \cup \{-15,-12,-7,7,13,15,21,24,28,32,48,60,84\}$, then there exist at least two distinct $D(n)$-quadruples. Thus, it seems natural to ask whether similar results hold for integers n with at least ℓ distinct $D(n)$-quadruples, where $\ell \geq 3$. An experimental search for quadruples with elements of small size, together with some theoretical observations from [8, 9], suggest that the answer is negative.

Conjecture 2. *The set S_3 of all integers n, not of the form $4k+2$, with the property that there exist at most two different $D(n)$-quadruples is infinite.*

By the results from [8] we know that if an integer n is not of the form $4k+2$, $|n| > 48$, and there exist at most two distinct $D(n)$-quadruples, then n has one of the following forms: $4k+3$, $16k+12$, $8k+5$, $32k+20$. Since multiplying all elements of $D(4k+3)$ and $D(8k+5)$-quadruples by 2 we obtain $D(16k+12)$ and $D(32k+20)$-quadruples respectively (by [5, Remark 3], all $D(16k+12)$-quadruples can be obtained on this way), we may restrict our attention to the integers of the forms $4k+3$ and $8k+5$. We now quote main result of [9].

Lemma 1.

(i) *Let n be an integer such that $n \equiv 3 \pmod 4$, $n \notin \{-9,-1,3,7,11\}$, and there exist at most two distinct $D(n)$-quadruples. Then $\frac{|n-1|}{2}$, $\frac{|n-9|}{2}$ and $\frac{|9n-1|}{2}$ are primes. Furthermore, either $|n|$ is prime or $n = pq$, where p and $q = p+2$ are twin primes.*

(ii) *Let n be an integer such that $n \equiv 5 \pmod 8$, $n \notin \{-27,-3,5,13,21,45\}$, and there exist at most two distinct $D(n)$-quadruples. Then the integers $|n|$, $\frac{|n-1|}{4}$, $\frac{|n-9|}{4}$ and $\frac{|9n-1|}{4}$ are primes.*

Note that by Dickson's conjecture on simultaneous prime values of linear polynomials, and its extension by Schinzel and Sierpiński [26], it is predicted that there are infinitely many integers n satisfying the primality conditions from Lemma 1. Note also that there is one notable difference between (i) and (ii) parts of Lemma 1, which concerns twin primes occurring in (i). However, the preliminary version of (ii) also contained similar condition concerning the numbers n of the form $n = pq$, where p and q are primes such that $q = p+4$. But, it was possible to remove this case because of the following polynomial formula: the set

$$\{2, 32X^2+32X+10, 2(12X+11)(12X^3+17X^2+7X+1),$$
$$2(12X+1)(12X^3+19X^2+9X+1)\}$$

has the property $D((6X+1)(6X+5))$. Thus, we may ask whether the condition concerning the product of twin primes can be removed from the statement (i) of Lemma 1. We will give partial positive answer to this question. Namely, we will show that the condition can be removed if $p \equiv 1 \pmod 4$.

Let us recall that in [5] it was proved that for an integer $k \notin \{-1,0,1,2\}$, the sets $\{1, k^2-2k-2, k^2+1, 4k^2-4k-3\}$ and $\{1, 9k^2+8k+1, 9k^2+14k+6, 36k^2+44k+13\}$ are two distinct $D(4k+3)$-quadruples.

If p and q are twin primes, then $p = 2k-1$, $q = 2k+1$, for a positive integer k, and $n = pq = 4k^2-1$. Thus, our goal is to find a new polynomial quadruple with property $D(4X^2-1)$. Motivated by the following example

$$\{1, 1667501, 1834262, 6999553\}$$

of a $D(41\cdot 43)$-quadruple (note that for $n = 41\cdot 43 = 1763$ the numbers $\frac{n-1}{2} = 881$, $\frac{n-9}{2} = 877$ and $\frac{9n-1}{2} = 7933$ are primes), we search for a polynomial quadruple of the form $\{1, a, b, a+b+2r\}$, where a and b are polynomials of degree 4. In search for more general result, we take $n = \alpha X^2 + \beta$. Since the polynomial n is even, we assume that $a(-X) = b(X)$. Let us put

$$a + n = (\alpha_2 X^2 - \alpha_1 X + \alpha_0)^2, \tag{17}$$
$$b + n = (\alpha_2 X^2 + \alpha_1 X + \alpha_0)^2. \tag{18}$$

We express a and b from (17) and (18). Then $ab+n$ is a polynomial of degree 8, which should be a perfect square. Thus, we put

$$ab + n = (\alpha_2^2 X^4 + (-\alpha + 2\alpha_0\alpha_2 - \alpha_1^2)X^2 + \gamma)^2. \tag{19}$$

This gives us three equations in $\alpha_0, \alpha_1, \alpha_2, \alpha, \beta, \gamma$. Let $r = \alpha_2^2 X^4 + (-\alpha + 2\alpha_0\alpha_2 - \alpha_1^2)X^2 + \gamma$. Consider the set $\{1, a, b, a+b+2r\}$. In order to satisfy the definition of a $D(n)$-quadruple, it remains to satisfy the condition that $a+b+2r+n$ is a perfect square. Namely, by (17) and (18), $1\cdot a+n$ and $1\cdot b+n$ are squares, and we have $ab+n=r^2$, $a(a+b+2r)+n=(a+r)^2$, $b(a+b+2r)+n=(b+r)^2$. From

$$a+b+2r+n = 4\alpha_2^2X^4+(-3\alpha+8\alpha_0\alpha_2)X^2-\beta+2\gamma+2\alpha_0^2 = \left(2\alpha_2X^2+\frac{-3\alpha+8\alpha_0\alpha_2}{4\alpha_2}\right)^2,$$

we get

$$\gamma = \frac{16\alpha_2^2\beta+32\alpha_2^2\alpha_0^2+9\alpha^2-48\alpha_2\alpha_0\alpha}{32\alpha_2^2}.$$

Inserting this in the three equations induced by (19), we get

$$\begin{aligned}
\alpha_0 &= -\frac{9\alpha_2^2+78\alpha_1^2\alpha_2+64\alpha_1^4}{36\alpha_1^2\alpha_2},\\
\alpha &= -\frac{24\alpha_2+16\alpha_1^2}{9},\\
\beta &= \frac{2(27\alpha_2^3+198\alpha_1^2\alpha_2^2+420\alpha_1^4\alpha_2+200\alpha_1^6)}{81\alpha_1^2\alpha_2^2}.
\end{aligned}$$

In that way, we obtained two parametric families of a polynomial quadruples of the desired type. It remains to choose the parameters α_1 and α_2 in order to obtain $D(4X^2-1)$-quadruples. We get the following pairs $(\alpha_1,\alpha_2)=(\pm 3,-\frac{15}{2})$, $(\pm\frac{3}{2},-3)$, $(\pm\frac{1}{2},-\frac{5}{3})$. The last task is to get integer elements of the quadruple. By taking $(\alpha_1,\alpha_2)=(\frac{3}{2},-3)$, we are almost done, since we get $a=9X^4-9X^3-\frac{3}{2}X+\frac{5}{4}X^2+\frac{5}{4}$, $b=9X^4+9X^3+\frac{3}{2}X+\frac{5}{4}X^2+\frac{5}{4}$, $a+b+2r=36X^4-4X^2+1$, i.e. the set

$$\{1, 9X^4-9X^3-\frac{3}{2}X+\frac{5}{4}X^2+\frac{5}{4}, 9X^4+9X^3+\frac{3}{2}X+\frac{5}{4}X^2+\frac{5}{4}, 36X^4-4X^2+1\}$$

is the Diophantine quadruple with the property $D(4X^2-1)$. For X odd, say $X=2k+1$, numbers a and b are integers. Therefore, we get that if $k\notin\{-1,0\}$ is an integer, then

$$\begin{aligned}
&\{1,\ 144k^4+216k^3+113k^2+20k+1,\ 144k^4+360k^3+329k^2+134k+22,\\
&\qquad 576k^4+1152k^3+848k^2+272k+33\} \qquad (20)
\end{aligned}$$

is a $D((4k+1)(4k+3))$-quadruple. Thus, we proved the following extension of Lemma 1 (i).

Theorem 1. *Let n be an integer such that $n\equiv 3 \pmod 4$, $n\notin\{-9,-1,3,7,11\}$, and there exist at most two distinct $D(n)$-quadruples. Then $\frac{|n-1|}{2}$, $\frac{|n-9|}{2}$ and $\frac{|9n-1|}{2}$ are primes. Furthermore, either $|n|$ is prime or $n=pq$, where $p\equiv 3 \pmod 4$ and $q=p+2$ are twin primes.*

4. Adjugates of $D(n)$-quadruples

In [20] Gibbs described the following construction by which from one Diophantine quadruple we can obtain up to eight new Diophantine quadruples. Given a (possibly improper) Diophantine quadruple $\{a_1,a_2,a_3,a_4\}$ with the property $D(n)$ such that $a_ia_j+n=x_{ij}^2$, an adjugate quadruple is obtained by constructing the symmetric matrix S with components $S_{ii}=a_i$ and $S_{ij}=x_{ij}$ for $i\neq j$ and forming its adjugate matrix T. The adjugate quadruple $\{A_1,A_2,A_3,A_4\}$ is a (possibly improper) Diophantine quadruple such that $A_iA_j+N=X_{ij}^2$, where $A_i=-T_{ii}$, $X_{ij}=-T_{ij}$ and $N=n\det(S)$. A quadruple has at most eight distinct adjugates which arise from different combinations of the signs of x_{ij}. Gibbs also stated the following relationships between properties of a quadruple and its adjugates:

A) If a quadruple is regular with property $D(1)$, then four of its eight adjugates are the regular quadruples formed by dropping one of its four components and replacing it with the other extension value.
B) If a quadruple is semi-regular, then four of its eight adjugates will be improper quadruples with zero component.
C) If a quadruple is twice semi-regular, then two of its adjugates have two zeros and four others have one zero component.
D) If a quadruple is improper with a zero component then all of its adjugates include a regular triple.
E) If a quadruple is improper with two duplicate components then it can have at most six distinct adjugates and four of them are improper with two duplicate components.
F) If a quadruple is improper with two zero components then all of its adjugates are twice semi-regular, but four of them are improper with two duplicate components and there are only two other distinct adjugates.

Gibbs gave numerical examples for each of the cases **A)**–**F)**. We will give polynomial examples for all cases. In some of them we will use quadruples constructed in the previous sections. We will give details on quadruples obtained by Gibbs' construction only in the cases where this construction produces Diophantine quadruples for quadratic polynomials.

A) The set $\{X-1, X+1, 4X(2X-1)(2X+1), 8X(2X-1)(2X+1)(2X^2-1)\}$ is a regular polynomial $D(1)$-quadruple (see [3]). By the above construction, we obtain:

- four regular $D(1)$-quadruples and
- irregular quadruples with the properties
  $D(256X^6-256X^5-128X^4+128X^3+16X^2-8X+1)$,
  $D(256X^6+256X^5-128X^4-128X^3+16X^2+8X+1)$,
  $D(4096X^8-5120X^6+1920X^4-224X^2+9)$ and
  $D(-16384X^{10}+28672X^8-16896X^6+3968X^4-320X^2+9)$.

B) By applying Gibbs' construction on the quadruple (20), which is a semi-regular polynomial $D((4X+1)(4X+3))$-quadruple

$$\{1, (12X+1)(12X^3+17X^2+8X+1), (12X+11)(12X^3+19X^2+10X+2), \\ (24X^2+16X+3)(24X^2+32X+11)\},$$

we obtain:

- four improper irregular quadruples and
- (proper) irregular quadruples with the properties
  $D((4X+1)^2(4X+3)^2(82944X^8+331776X^7+560448X^6+520128X^5 + 288280X^4+96752X^3+18928X^2+1944X+81))$,
  $D((2X+1)^2(4X+1)^2(4X+3)(331776X^7+1078272X^6+1453824X^5 + 1047744X^4+430528X^3+98800X^2+11356X+507))$,
  $D((2X+1)^2(4X+1)(4X+3)^2(331776X^7+1244160X^6+1951488X^5 + 1659456X^4+824512X^3+237776X^2+36284X+2161))$ and

$$D(-(4X+1)^2(4X+3)^2(11943936X^{10}+59719680X^9+133373952X^8 + 175177728X^7+149726016X^6+86878656X^5+34567880X^4 + 9272656X^3+1593200X^2+156264X+6471)).$$

C) The set $\{4X, 16X-2, 36X-12, 4X-4\}$, obtained from (12), is a twice semi-regular polynomial $D(24X+1)$-quadruple, from which we obtain:

– six improper irregular quadruples and
– irregular quadruples with the properties
$D((24X+1)(6144X^3-6656X^2+2616X-279))$ and
$D(-(24X+1)^2(6144X^3-5184X^2+816X-1))$.

D) In **A)** we obtained the set $\{X+1, 4(X+2), 3(3X+5), 0\}$, which is an improper regular polynomial $D(1)$-quadruple. For this set, using Gibbs' construction we obtain:

– four improper regular quadruples,
– a regular $D(1)$-quadruple and
– the following (proper) quadruples for quadratic polynomials:
$\{X+1, 8(2X+3), X+3, 4(2X+3)(3X+4)(6X+11)\}$ is a semi-regular $D(48X^2+136X+97)$-quadruple,
$\{5(5X+9), 4(X+2), X+3, 4(2X+3)(3X+4)(6X+11)\}$ is a semi-regular polynomial $D(96X^2+320X+265)$-quadruple,
$\{5(5X+9), 8(2X+3), 3(3X+5), 4(2X+3)(3X+4)(6X+11)\}$ is a semi-regular polynomial $D(-3(48X^2+152X+117))$-quadruple.

E) The set $\{-2, X, X+4, X+4\}$ is an improper irregular polynomial $D(2X+9)$-quadruple (see [5]), which includes three regular triples. By the above construction we obtain six improper irregular quadruples.

However, if we take the set

$$\{(X-1)(X+1)(5X^2+3)^2, (X-1)(X+1)(5X^2+3)^2, (3X-1)(3X+1)(5X^2+3)^2, 32X^2(7X^2+1)\},$$

which is an improper irregular polynomial $D(4X^2(5X^2+3)^4)$-quadruple (obtained using [6, Proposition 3]), the above construction gives:

– four improper irregular quadruples and
– irregular quadruples with the properties
$D(4X^2(5X^2+3)^2(225X^6+370X^4-23X^2+4)$ and
$D(4X^4(5X^2+3)^2(124X^6-63X^4-6X^2+9))$.

Let us mention that it is possible to find a polynomial quadruple with two pairs of equal elements. E.g.

$$\{4X^2, 4X^2, -(X^2-X-1)(X^2+X-1), -(X^2-X-1)(X^2+X-1)\}$$

is an improper twice semi-regular polynomial $D(4(X-1)^2X^2(X+1)^2)$-quadruple (see [17]). Here Gibbs' construction gives five improper irregular quadruples.

F) The set $\{-4(X+5), -3(X+6), 0, 0\}$, obtained in **E)**, is an improper irregular polynomial $D((2X+9)^2)$-quadruple. For this set we obtain:

– four improper twice semi-regular quadruples and
– the following (proper) quadruples for quadratic polynomials:
$\{-(7X+36), -2, -5(3X+16), (X+4)\}$ is a twice semi-regular polynomial $D((2X+9)(8X+41))$-quadruple,

$\{X, -2(4X+19), -5(3X+16), X+4\}$ is a twice semi-regular polynomial $D((2X+9)(12X+49))$-quadruple.

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