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Linear Matrix Inequalities Based H_∞ Control of Gantry Crane using Tensor Product Transformation

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Abstract: This paper describes a H_∞ controller design procedure for tensor product based model of gantry crane augumented with friction model in order to minimize friction effects. The Tensor Product (TP) model transformation is a recently proposed technique for transforming given Linear Parameter Varying (LPV) state-space models into polytopic model form, namely, to parameter varying convex combination of Linear Time Invariant (LTI) systems. H_∞ controller guarantee stability and L_2 norm bound constraint on disturbance attenuation. H_∞ controller is found using relaxed LMIs which have proof of asymptotic convergence to the global optimal controller under quadratic stability. Control algorithm is experimentally tested on single pendulum gantry (SPG).

Keywords: Parallel Distributed Compensation, Linear Matrix Inequalities, Tensor Product (TP) model transformation, gantry crane control, friction compensation.

1. INTRODUCTION

In modern industrial system, gantry cranes are widely used for the heavy loads transfer. Fast load positioning and load swinging minimization are conflicting requirements imposed to the traveling crane control systems.

For the position and anti-sway control of travelling cranes, there are several solutions, i.e., by fuzzy control, optimal control, pole placement, etc. and each of them is reported to be effective (Popadić et al., 2005), (Nalley and Trabia, 2000), (Omar, 2003). All these approaches are based on linear/linearized model of gantry crane.

Recently, nonlinear control approach based on tensor product model representation (TP) of the process is proposed (Baranyi et al., 2003), (Petres, 2006) and successfully applied to control of Single Pendulum Gantry process (Kolonić et al., 2006). The TP model represents the Linear Parameter Varying (LPV) state-space models by the parameter varying combination of Linear Time Invariant (LTI) models.

However, none of these approaches takes into account friction effect, which is unavoidable in real applications (Olsson et al., 1998). This effect may seriously degrade the performance of the control system, specially when high precision positioning is required. As a consequence steady state error due to static friction is common for all these approaches. Introducing integral action in control loop, to eliminate steady state error due to friction effects, may result in limit cycling called hunting phenomenon, see Hensen et al. (2003).

One way to cope with friction phenomena is to introduce friction compensator based on friction model. The friction model may be a-priory known or learned in an on-line manner (Lee and Tomizuka, 1996), (Huang et al., 2000), (Matuško et al., 2010), (Boras et al., 2010).

Another approach consider the friction effect as an unknown disturbance and design the robust controller. In Cheang and Chen (2000), Burul et al. (2010), H_∞ controller for linearised model is synthesized using loop-shaping methodology, and experimentally proved effective.

In this paper a combination of these two approaches is used. It is assumed that friction model is partially known, and nominal model is identified, while its associated uncertainty is considered as a disturbance. In order to cope with nonlinear nature of the gantry crane model, as well as friction model, non-linear model of gantry crane augmented with nominal friction model is rewritten in LPV form, suitable for TP transformation. As a result of TP model transformation, polytopic model of process is obtained. For such model representation, LMI control approach is used to design a controller (Tanaka and Wang, 2001), (Boyd et al., 1994), (Gahinet et al., 2002). In this paper such approach is used to synthesize H_∞ controller.

The paper is organized as follows: Section II discusses the theoretical background of TP model transformation-based control design. Section III introduces the LPV model augmented with friction and its TP transformation. Section IV discuses the LMI relaxations used in controller design. Section V presents the experimental results obtained on the single pendulum gantry (SPG) experimental model, and Section VI concludes this paper.

2. TENSOR PRODUCT MODEL TRANSFORMATION-BASED CONTROL DESIGN METHODOLOGY

Consider the linear parameter-varying state-space model

$$\begin{pmatrix} \dot{x}(t) \\ y(t) \end{pmatrix} = S(p(t)) \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \tag{1}$$

with input $u(t) \in R^k$, output $y(t) \in R^l$ and state vector $x(t) \in R^m$. The system matrix

$$S(p(t)) = \begin{pmatrix} A(p(t)) & B(p(t)) \\ C(p(t)) & D(p(t)) \end{pmatrix} \in R^{(m+k)\times(m+l)} \tag{2}$$

is a parameter-varying object, where $p(t) \in \Omega$ is time varying parameter vector, where Ω is a closed hypercube in R^N, $\Omega = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_N, b_N]$. Parameter $p(t)$ can also include the elements of the state vector $x(t)$, therefore LPV system given in Eq. (1) is considered in the class of non-linear dynamic state space models.

The main idea of TP model transformation is to discretize the given LPV model given in Eq. (1) over hyper rectangular grid M in Ω, then via executing Higher Order Singular Value Decomposition, the tensor product structure of given model is obtained. By ignoring singular values, TP model of reduced complexity and accuracy can be obtained. For more details se Petres (2006) and Tikk et al. (2004).

Tensor product structure can be written as follows

$$\begin{aligned} S(p(t)) &= \mathcal{S} \mathop{\boxtimes}_{n=1}^{N} w_n(p_n) \\ &= \sum_{i_1=1}^{I_1} \sum_{i_2=1}^{I_2} \cdots \sum_{i_n=1}^{I_N} \prod_{n=1}^{N} w_{n,i_n}(p_n) S_{i_1,i_2\cdots,i_N}, \end{aligned} \tag{3}$$

where $\mathcal{S} \in R^{I_1\times I_2\times\cdots I_N\times(m+k)\times(m+l)}$ denotes obtained tensor, I_n denotes number of LTI systems in n-th dimension Ω, $\boxtimes$ denotes multiple n-mode product of a tensor by a matrix, w_n is row vector containing $w_{n,i_n}(p_n) \in [0,1]$ which is corresponding one variable weighting function defined on the n-th dimension of Ω and

$$S_{i_1,i_2\cdots i_N} = \begin{pmatrix} A_{i_1,i_2\cdots i_N} & B_{i_1,i_2\cdots i_N} \\ C_{i_1,i_2\cdots i_N} & D_{i_1,i_2\cdots i_N} \end{pmatrix}, \tag{4}$$

is LTI system matrix obtained by TP model transformation.

Controller is determined in same form as TP model. Control signal is given by

$$u = -\sum_{i_1=1}^{I_1} \sum_{i_2=1}^{I_2} \cdots \sum_{i_n=1}^{I_N} \prod_{n=1}^{N} w_{n,i_n}(p_n) K_{i_1,i_2\cdots,i_N} x \tag{5}$$

where the $K_{i_1,i_2\cdots,i_N}$ are corresponding LTI feedback gains.

By using i as linear index, equivalent to the multilinear array index with the size of $I_1 \times I_2 \times \cdots I_N$, TP model (3) and control signal (5) can be rewritten in standard polytopic form

$$S(p) = \sum_{i=1}^{R} w_i(p) S_i, \tag{6}$$

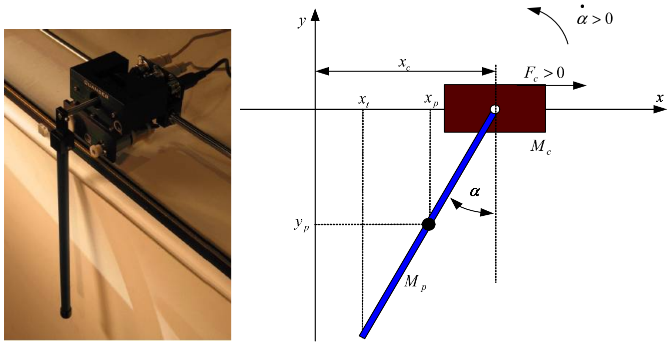
$$u = -\sum_{i=1}^{R} w_i(p) K_i, \tag{7}$$

where $R = I_1 + I_2 + \cdots + I_N$ and $w_i(p)$ is corresponding weighting function.

3. TP MODEL-BASED CONTROLLER DESIGN APPLIED TO THE SINGLE PENDULUM GANTRY CRANE EXPERIMENTAL MODEL

3.1 Mathematical model of the gantry crane

Experimental laboratory model of Single Pendulum Gantry (SPG) is used to emulate industrial crane application, see Fig 1.



(a) (b)

Fig. 1. SPG photo in mechatronics laboratory a), and schematics of the model b)

Non-linear model of single pendulum gantry (SPG) can be described by following equations [1] :

$$\begin{aligned} &(M_c + M_p)\ddot{x}_c + M_p l_p \ddot{\alpha}\cos(\alpha) - M_p l_p \dot{\alpha}^2 \sin(\alpha) = \\ &= F_m - F_f - B_{eq}\dot{x}_c, \\ &(I_p + M_p l_p^2)\ddot{\alpha} + M_p l_p \ddot{x}_c \cos(\alpha) + M_p g l_p \sin(\alpha) = -B_p\dot{\alpha}, \end{aligned} \tag{8}$$

where F_m is DC motor force while F_f is the friction force. Driving force obtained from DC motor is given by:

$$F_m = \frac{\eta_g \eta_m K_g K_t}{r_{mp}} \frac{1}{R_m} U_m. \tag{9}$$

The meanings and the values of other parameters in equations (8) and (9) are given in Table 1.

In this paper Stribeck friction model is used, see Olsson et al. (1998). It can be described by

$$F_f(\dot{x}_c) = \mathrm{sgn}(\dot{x}_c)\left[F_C + (F_S - F_C)e^{-(\frac{\dot{x}_c}{v_s})^\delta}\right], \tag{10}$$

[1] Derivation using Lagrangian formulation is omitted for brevity and can be found in (QUANSER User Manual, 1999)

where F_C is Coulomb friction, F_S is static friction force, v_s is Stribeck velocity, and δ is emprirical exponent.

The parameters of the model (10) are experimentally identified and following values are obtained:

$$F_C = 2.70, \qquad F_S = 3.10, \qquad v_s = 0.10, \qquad \delta = 2. \tag{11}$$

3.2 LPV model of single pendulum gantry (SPG)

Letting $x = [x_1\ x_2\ x_3\ x_4]^T = [x_c\ \dot{x}_c\ \alpha\ \dot{\alpha}]^T$, $u = U_m$ and w norm bounded disturbance, the equations of motion in linear parameter varying state space form are:

$$\begin{aligned} \dot{x} &= A(x(t))x + B(x(t))u + E(x(t))w, \\ y &= C(x(t))x + D(x(t))u + F(x(t))w. \end{aligned} \tag{12}$$

The system matrix in LPV form for the model (12) can be written as:

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & a_1/a_x & a_2/a_x & a_3/a_x & b_{11}/a_x & e_1/a_x \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & a_4/a_x & a_5/a_x & a_6/a_x & b_{12}/a_x & e_2/a_x \\ 1 & 0 & c_1 & 0 & 0 & 0 \end{pmatrix}, \tag{13}$$

where :

$$\begin{aligned} a_1 &= -(I_p + M_p l_p^2) \cdot \left(\frac{\eta_g \eta_m K_g^2 K_t K_m}{r_{mp}^2 R_m} + B_{eq} \right) \\ a_2 &= \frac{M_p^2 l_p^2 g \sin(x_3) \cos(x_3)}{x_3} \\ a_3 &= (M_p^2 l_p^3 + l_p M_p l_p) \sin(x_3) x_4 + M_p l_p B_p \cos(x_3) \\ a_4 &= M_p l_p \cos(x_3) \cdot \left(B_{eq} - \frac{\eta_g \eta_m K_g^2 K_t K_m}{r_{mp}} \frac{1}{R_m} \right) \\ a_5 &= -\frac{(M_c + M_p) M_p l_p \sin(x_3)}{x_3} \\ a_6 &= -(M_c + M_p) B_p - M_p^2 l_p^2 \cos(x_3) \sin(x_3) x_4 \\ a_x &= (M_c + M_p) I_p + M_c M_p l_p^2 + M_p^2 l_p^2 \sin^2(x_3) \\ b_{11} &= -(I_p M_p l_p)^2 \mathrm{sgn}(\dot{x}_c) \left[F_C + (F_S - F_C) e^{-(\frac{\dot{x}_c}{v_s})^\delta} \right] \\ b_{12} &= -M_p l_p \cos(x_3) \mathrm{sgn}(\dot{x}_c) \left[F_C + (F_S - F_C) e^{-(\frac{\dot{x}_c}{v_s})^\delta} \right] \\ e_1 &= -(I_p M_p l_p)^2 \frac{\eta_g K_g \eta_m K_t}{R_m r_{mp}} \\ e_2 &= -M_p l_p \cos(x_3) \frac{\eta_g K_g \eta_m K_t}{R_m r_{mp}} \\ c_1 &= -0.614 \frac{\sin x_3}{x_3} \end{aligned} \tag{14}$$

3.3 Single pendulum ganty TP model representation

Operating area for single pendulum gantry, is selected as

$$\begin{aligned} \Omega &= [\dot{x}_{cmin}, \dot{x}_{cmax}] \times [\alpha_{min}, \alpha_{max}] \times [\dot{\alpha}_{min}, \dot{\alpha}_{max}] \\ &= [-0.6, 0.6] \times [-0.0873, 0.0873] \times [-0.8, 0.8]. \end{aligned} \tag{15}$$

Applying the TP transformation to the model (13) yield to the TP model representation consisting of 20 LTI models. The LTI system matrices of the TP model are:

$$\begin{aligned}
S_{1,1,1} &= \begin{pmatrix} 0 & 1.0 & 0 & 0 & 0 & 0 \\ 0 & -11.64 & 1.512 & 0.009566 & 1.529 & 1.153 \\ 0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & 26.71 & -26.05 & -0.09477 & -3.509 & -2.647 \\ 1.0 & 0 & 0.6132 & 0 & 0 & 0 \end{pmatrix} \\
S_{2,1,1} &= \begin{pmatrix} 0 & 1.0 & 0 & 0 & 0 & 0 \\ 0 & -11.64 & 1.512 & 0.009566 & 1.529 & -1.153 \\ 0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & 26.71 & -26.05 & -0.09477 & -3.509 & 2.647 \\ 1.0 & 0 & 0.6132 & 0 & 0 & 0 \end{pmatrix} \\
S_{1,2,1} &= \begin{pmatrix} 0 & 1.0 & 0 & 0 & 0 & 0 \\ 0 & -11.64 & 1.512 & 0.000189 & 1.529 & 1.153 \\ 0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & 26.71 & -26.05 & -0.07325 & -3.509 & -2.647 \\ 1.0 & 0 & 0.6132 & 0 & 0 & 0 \end{pmatrix} \\
S_{2,2,1} &= \begin{pmatrix} 0 & 1.0 & 0 & 0 & 0 & 0 \\ 0 & -11.64 & 1.512 & 0.000189 & 1.529 & -1.153 \\ 0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & 26.71 & -26.05 & -0.07325 & -3.509 & 2.647 \\ 1.0 & 0 & 0.6132 & 0 & 0 & 0 \end{pmatrix} \\
S_{1,3,1} &= \begin{pmatrix} 0 & 1.0 & 0 & 0 & 0 & 0 \\ 0 & -11.64 & 1.517 & 0.008731 & 1.529 & 1.154 \\ 0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & 26.78 & -26.08 & -0.0929 & -3.517 & -2.653 \\ 1.0 & 0 & 0.6136 & 0 & 0 & 0 \end{pmatrix}
\end{aligned} \tag{16}$$

$$\begin{aligned}
S_{2,3,1} &= \begin{pmatrix} 0 & 1.0 & 0 & 0 & 0 & 0 \\ 0 & -11.64 & 1.517 & 0.008731 & 1.529 & -1.154 \\ 0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & 26.78 & -26.08 & -0.0929 & -3.517 & 2.653 \\ 1.0 & 0 & 0.6136 & 0 & 0 & 0 \end{pmatrix} \\
S_{1,4,1} &= \begin{pmatrix} 0 & 1.0 & 0 & 0 & 0 & 0 \\ 0 & -11.65 & 1.517 & -0.001939 & 1.53 & 1.154 \\ 0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & 26.79 & -26.08 & -0.06834 & -3.518 & -2.654 \\ 1.0 & 0 & 0.6136 & 0 & 0 & 0 \end{pmatrix} \\
S_{2,4,1} &= \begin{pmatrix} 0 & 1.0 & 0 & 0 & 0 & 0 \\ 0 & -11.65 & 1.517 & -0.001939 & 1.53 & -1.154 \\ 0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & 26.79 & -26.08 & -0.06834 & -3.518 & 2.654 \\ 1.0 & 0 & 0.6136 & 0 & 0 & 0 \end{pmatrix} \\
S_{1,5,1} &= \begin{pmatrix} 0 & 1.0 & 0 & 0 & 0 & 0 \\ 0 & -11.66 & 1.525 & 0.006239 & 1.531 & 1.155 \\ 0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & 26.89 & -26.13 & -0.0872 & -3.532 & -2.665 \\ 1.0 & 0 & 0.6143 & 0 & 0 & 0 \end{pmatrix} \\
S_{2,5,1} &= \begin{pmatrix} 0 & 1.0 & 0 & 0 & 0 & 0 \\ 0 & -11.66 & 1.525 & 0.006239 & 1.531 & -1.155 \\ 0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & 26.89 & -26.13 & -0.0872 & -3.532 & 2.665 \\ 1.0 & 0 & 0.6143 & 0 & 0 & 0 \end{pmatrix}
\end{aligned} \tag{17}$$

$$\begin{aligned}
S_{1,1,2} &= \begin{pmatrix} 0 & 1.0 & 0 & 0 & 0 & 0 \\ 0 & -11.64 & 1.512 & 0.000189 & 1.529 & 1.153 \\ 0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & 26.71 & -26.05 & -0.07325 & -3.509 & -2.647 \\ 1.0 & 0 & 0.6132 & 0 & 0 & 0 \end{pmatrix} \\
S_{2,1,2} &= \begin{pmatrix} 0 & 1.0 & 0 & 0 & 0 & 0 \\ 0 & -11.64 & 1.512 & 0.000189 & 1.529 & -1.153 \\ 0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & 26.71 & -26.05 & -0.07325 & -3.509 & 2.647 \\ 1.0 & 0 & 0.6132 & 0 & 0 & 0 \end{pmatrix} \\
S_{1,2,2} &= \begin{pmatrix} 0 & 1.0 & 0 & 0 & 0 & 0 \\ 0 & -11.64 & 1.512 & 0.009566 & 1.529 & 1.153 \\ 0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & 26.71 & -26.05 & -0.09477 & -3.509 & -2.647 \\ 1.0 & 0 & 0.6132 & 0 & 0 & 0 \end{pmatrix} \\
S_{2,2,2} &= \begin{pmatrix} 0 & 1.0 & 0 & 0 & 0 & 0 \\ 0 & -11.64 & 1.512 & 0.009566 & 1.529 & -1.153 \\ 0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & 26.71 & -26.05 & -0.09477 & -3.509 & 2.647 \\ 1.0 & 0 & 0.6132 & 0 & 0 & 0 \end{pmatrix} \\
S_{1,3,2} &= \begin{pmatrix} 0 & 1.0 & 0 & 0 & 0 & 0 \\ 0 & -11.64 & 1.517 & 0.001048 & 1.529 & 1.154 \\ 0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & 26.78 & -26.08 & -0.07522 & -3.517 & -2.653 \\ 1.0 & 0 & 0.6136 & 0 & 0 & 0 \end{pmatrix}
\end{aligned} \tag{18}$$

Table 1. PARAMETERS OF THE SPG SYSTEM

Symbol	Description	Value	Unit
U_m	DC motor voltage		V
I_m	DC motor armature current		A
R_m	DC motor armature resistance	2.6	Ω
L_m	DC motor armature inductance	0.18	mH
K_t	Motor torque constant	0.007 67	$\mathrm{N\,m\,A^{-1}}$
η_m	Motor efficiency	1	
K_m	Back electro-motive force constant	0.007 67	V s/rad
J_m	Rotor moment of inertia	3.9001×10^{-7}	$\mathrm{kg\,m^2}$
I_p	Pendulum moment of inertia	0.007 883 8	$\mathrm{kg\,m^2}$
K_g	Planetary gearbox gear ratio	3.71	
η_g	Planetary gearbox efficiency	1	
M_c	Lumped mass of the cart system	1.0731	kg
l_p	Pendulum length from pivot to COG	0.3302	m
M_p	Pendulum mass	0.23	kg
r_{mp}	Motor pinion radius	0.006 35	m
B_{eq}	Equivalent viscous damping coeff.	5.4	$\mathrm{N\,m\,s\,rad^{-1}}$
B_p	Viscous damping coefficient	0.0024	$\mathrm{N\,m\,s\,rad^{-1}}$
F_c	Driving force		N
g	Gravitational constant of earth	9.81	$\mathrm{m/s^2}$
α	Pendulum angle		rad
x_c	Chart position		mm

$$
\begin{aligned}
S_{2,3,2} &= \begin{pmatrix} 0 & 1.0 & 0 & 0 & 0 & 0 \\ 0 & -11.64 & 1.517 & 0.001048 & 1.529 & -1.154 \\ 0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & 26.78 & -26.08 & -0.07522 & -3.517 & 2.653 \\ 1.0 & 0 & 0.6136 & 0 & 0 & 0 \end{pmatrix} \\
S_{1,4,2} &= \begin{pmatrix} 0 & 1.0 & 0 & 0 & 0 & 0 \\ 0 & -11.65 & 1.517 & 0.01172 & 1.53 & 1.154 \\ 0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & 26.79 & -26.08 & -0.09978 & -3.518 & -2.654 \\ 1.0 & 0 & 0.6136 & 0 & 0 & 0 \end{pmatrix} \\
S_{2,4,2} &= \begin{pmatrix} 0 & 1.0 & 0 & 0 & 0 & 0 \\ 0 & -11.65 & 1.517 & 0.01172 & 1.53 & -1.154 \\ 0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & 26.79 & -26.08 & -0.09978 & -3.518 & 2.654 \\ 1.0 & 0 & 0.6136 & 0 & 0 & 0 \end{pmatrix} \\
S_{1,5,2} &= \begin{pmatrix} 0 & 1.0 & 0 & 0 & 0 & 0 \\ 0 & -11.66 & 1.525 & 0.003582 & 1.531 & 1.155 \\ 0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & 26.89 & -26.13 & -0.08108 & -3.532 & -2.665 \\ 1.0 & 0 & 0.6143 & 0 & 0 & 0 \end{pmatrix} \\
S_{2,5,2} &= \begin{pmatrix} 0 & 1.0 & 0 & 0 & 0 & 0 \\ 0 & -11.66 & 1.525 & 0.003582 & 1.531 & -1.155 \\ 0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & 26.89 & -26.13 & -0.08108 & -3.532 & 2.665 \\ 1.0 & 0 & 0.6143 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}
\tag{19}
$$

Weighting functions of the TP model are given in Fig. 2.

4. CONTROLLER DESIGN

4.1 Linear Matrix Inequalities

Recently a class of numerical optimization problems called linear matrix inequality (LMI) problems has received significant attention. These optimization problems can be solved in polynomial time and hence are tractable, at least in a theoretical sense. Interior-point methods, developed for these problems, have been found to be extremely efficient in practice. For systems and control, the importance of LMI optimization stems from the fact that a wide variety of system and control problems can be recast as LMI problems. Except for a few special cases these problems do not have analytical solutions. However, the main point is that through the LMI framework they can be efficiently solved numerically in all cases. Therefore recasting a control problem as an LMI problem is equivalent to finding a solution to the original problem.

Generally, a linear matrix inequality (LMI) has the form

$$F(x) = F_0 + \sum_{i=1}^{m} x_i F_i > 0, \tag{20}$$

where $x \in R^m$ is the variable and the symmetric matrices $F_i = F_i^T$ are given. The inequality symbol in (20) means that $F(x)$ is positive definite

4.2 Control objective

In Kolonić et al. (2006) the control objective was to find stabilizing controller under quadratic stability (QS) with prescribed decay rate with minimal overshoot and constrained control signal. In order to obtain stabilizing controller Lyapunov stability approach is used, with Lyapunov function candidate given by:

$$V(x) = x^T P x > 0. \tag{21}$$

The speed of response is related to decay rate α, that is, the largest Lyapunov exponent. Therefore, the condition for desired decay rate can be written as

$$\dot{V}(x) \le -2\alpha V(x). \tag{22}$$

The equilibrium of the continuous system in polytopic form (3) is globally asymptotically stable if there exists a common positive definite matrix P such that

$$A_i^T P + P A_i + 2\alpha P < 0; i \in (1, R). \tag{23}$$

Next, let us consider the stability of the closed-loop control system of TP model of single pendulum gantry. It is globally asymptotically stable if there exists a common positive definite matrix P such that

$$
\begin{aligned}
G_{ii}^T P + P G_{ii} + 2\alpha P < 0, \\
\left(\frac{G_{ij} + G_{ji}}{2}\right)^T P + P\left(\frac{G_{ij} + G_{ji}}{2}\right) + 2\alpha P \le 0, i < j,
\end{aligned}
\tag{24}
$$

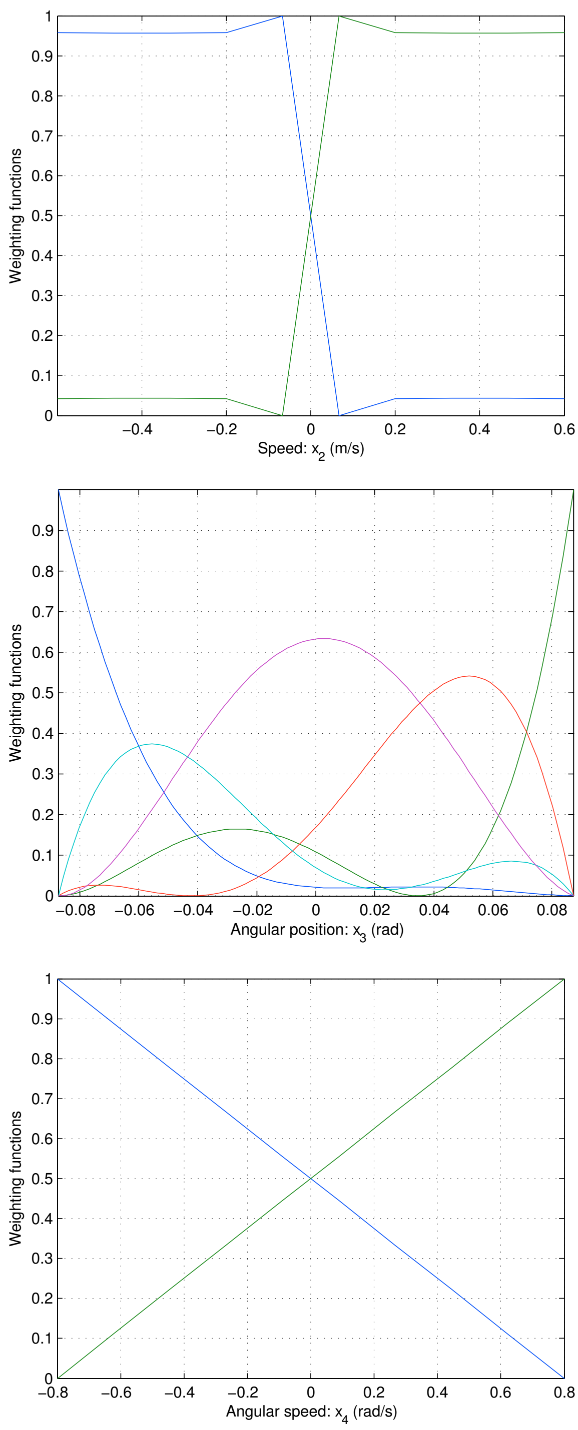


Fig. 2. Weighting functions of the TP model

where

$$G_{ij} = A_i + B_i F_j, \tag{25}$$

denotes closed loop state matrix.

Largest possible decay rate can be found by solving generalized eigenvalue minimization problem (GEVP):

maximize α
subject to

$$\begin{aligned} & X > 0 \\ & -XA_i^T - A_iX + M_i^T B_i^T + B_i M_i + 2\alpha X > 0 \\ & -XA_i^T - A_iX - XA_j^T - A_jX + M_j^T B_i^T \\ & + B_i M_j + M_i^T B_j^T + B_j M_i - 4\alpha X \geq 0 \end{aligned} \tag{26}$$

where $X = P^{-1}$ and $M_i = F_i X$.

In order to satisfy the constraints on control input and output constraints, the following LMIs are added to the (26).

Constraint on the control value:

Assume that initial condition $x(0)$ is unknown, but its upper bound $\|x(0)\| \leq \phi$ is known, which can be recast as following LMI

$$\phi^2 I \leq X, \tag{27}$$

the constraint $\|u\|_2 \leq \mu$ is enforced $\forall t \geq 0$ if the following LMI holds

$$\begin{pmatrix} X & M_i^T \\ M_i & \mu^2 I \end{pmatrix} \geq 0. \tag{28}$$

Constraint on the output:

Assume that condition (27) is satisfied, the constraint $\|y(t)\|_2 \leq \lambda$ is enforced, $\forall t \geq 0$, if the following LMI holds

$$\begin{pmatrix} X & XC_i^T \\ C_i X & \lambda^2 I \end{pmatrix} \geq 0. \tag{29}$$

LMI conditions (26) - (29) guarantee stability, constrained control signal and constrained output, however since friction effects were neglected it resulted in steady state error. In order to minimize steady state error, H_∞ norm was minimized since it is related to the capacity of the closed-loop system to reject energy bounded disturbance.

If there exist Lyapunov function (21) such that

$$\dot{V}(x) + y^T y - \gamma^2 w^T w \leq 0, \tag{30}$$

closed loop system has guaranteed H_∞ disturbance attenuation less than γ,

$$\|y(t)\|_2 \leq \gamma \|w(t)\|_2, \tag{31}$$

besides being quadratically stable. Condition (30) can be rewritten as

$$\sum_{i=1}^{R} w_i^2(p) T_{ii} + \sum_{i=1}^{R-1} \sum_{j=i+1}^{R} w_i(p) w_j(p) (T_{ij} + T_{ji}) < 0, \tag{32}$$

where

$$T_{ij} = \begin{pmatrix} \begin{pmatrix} A_i X + X A_i^T + B_i M_j + \\ + M_j^T B_i^T + E_i E_i^T \end{pmatrix} & \star \\ C_i X + F_i E_i^T + D_i M_j & F_i F_i^T - \gamma^2 I \end{pmatrix}, \tag{33}$$

which is in literature known as LMI representation of bounded real lemma (Boyd et al., 1994), (Fridman and Shaked, 2001).

Since γ is related to the level of disturbance attenuation it is of great interest to compute controller which ensures minimum value of γ.

$$\begin{aligned} &\text{minimize } \gamma \\ &\text{subject to (33)} \end{aligned} \tag{34}$$

Classical LMI conditions test the negative definiteness of (33), by imposing that the coefficients $T_{ii} < 0$ and $T_{ij} + T_{ji} < 0$, which is obviously only sufficient condition. Instead we use condition with proved convergence towards (33) in following theorem

Theorem 1: The TP model in (3) is quadratically stabilizable by means of all linear parameter-dependent state feedback control gain $K(p) = \sum_{i=1}^{R} w_i(p)K_i$ with an $\mathcal{H}_\infty$ guaranteed cost $\gamma > 0$ if and only if there exist a symmetric positive definite matrix $X \in \mathbb{R}^{n\times n}$, matrices $M_i \in \mathbb{R}^{m\times n}$, $i = 1, ..., N$, matrices $X_k \in \mathbb{R}^{2n+p+q\times n+q}$, $k \in \mathcal{N}(g)$, a degree $g \geq 1$, $g \in \mathbb{N}$, and a sufficiently large $d \in \mathbb{N}$ such that

$$\sum_{\substack{k'\in\mathcal{N}(d)\\ k\succeq k'}} \left(\sum_{i\in\{1,\ldots,N\} k_i > k'_i} \frac{d!}{\pi(k')} \left(X_{k-k'-e_i}\mathcal{B}_i + \mathcal{B}_i^T X_{k-k'-e_i}^T\right) + \right.$$
$$+ \sum_{\substack{i,j\in\{1,\ldots,N\}\\ k-k'-e_i-e_j\succeq 0}} \frac{d!}{\pi(k')} \cdot \frac{(g-1)!}{\pi(k-k'-e_i-e_j)}$$
$$\left. \times \begin{bmatrix} B_iM_j + M_j^T B_i^T & X & 0 & M_i^T D_j^T \\ \star & 0 & 0 & 0 \\ \star & \star & I & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} \right) < 0,$$
$$\forall k \in \mathcal{N}(g+d+1), \tag{35}$$

with

$$\mathcal{B}_i = \begin{bmatrix} A_i^T & -I & 0 & C_i^T \\ E_i^T & 0 & -I & F_i^T \end{bmatrix}, \tag{36}$$

where $\mathcal{N}(g)$ is set of N-tuples obtained as all possible combinations of nonnegative integers k_i, $i \in [1, N]$, such that $\sum_{i=1}^{N} k_i = g.$, for N-tuples k, k', comparation, summation and substraction are defined componentwise, N-tuple e_i denotes N-tuple with all components equal 0, except i-th component which equals 1, and where $\pi(k)$ is defined as $\pi(k) = (k_1!)(k_2!) \ldots (k_N!)$.

Minimising γ subject to LMI constraints proposed in Theorem 1., have the asymptotic convergence to the minimum $\mathcal{H}_\infty$ guaranteed cost under quadratic stabilizability, for more details and proof of Theorem 1 see Montagner et al. (2009), `http://www.dt.fee.unicamp.br/~ricfow/robust.htm`.

In the affirmative case, local feedback gains are given by $K_i = M_i X^{-1}$, $i = 1, ..., N$.

In order to ensure constrained control input (27) and (28) are added to (35).

By using Yalmip [2] and Sedumi [3] 1.3, the following feasible solution and feedback gains are obtained:

$$\begin{aligned}
K_{1,1,1} &= \begin{pmatrix} 64.55 & 22.34 & -5.605 & 8.128 \end{pmatrix}^T \\
K_{2,1,1} &= \begin{pmatrix} 64.55 & 22.34 & -5.605 & 8.128 \end{pmatrix}^T \\
K_{1,2,1} &= \begin{pmatrix} 64.52 & 22.32 & -5.616 & 8.118 \end{pmatrix}^T \\
K_{2,2,1} &= \begin{pmatrix} 64.52 & 22.32 & -5.616 & 8.118 \end{pmatrix}^T \\
K_{1,3,1} &= \begin{pmatrix} 65.29 & 22.52 & -5.287 & 8.213 \end{pmatrix}^T \\
K_{2,3,1} &= \begin{pmatrix} 65.29 & 22.52 & -5.287 & 8.213 \end{pmatrix}^T \\
K_{1,4,1} &= \begin{pmatrix} 65.34 & 22.51 & -5.267 & 8.212 \end{pmatrix}^T \\
K_{2,4,1} &= \begin{pmatrix} 65.34 & 22.51 & -5.267 & 8.212 \end{pmatrix}^T \\
K_{1,5,1} &= \begin{pmatrix} 66.71 & 22.86 & -4.684 & 8.374 \end{pmatrix}^T \\
K_{2,5,1} &= \begin{pmatrix} 66.71 & 22.86 & -4.684 & 8.374 \end{pmatrix}^T \\
K_{1,1,2} &= \begin{pmatrix} 64.52 & 22.32 & -5.616 & 8.118 \end{pmatrix}^T \\
K_{2,1,2} &= \begin{pmatrix} 64.52 & 22.32 & -5.616 & 8.118 \end{pmatrix}^T \\
K_{1,2,2} &= \begin{pmatrix} 64.55 & 22.34 & -5.605 & 8.128 \end{pmatrix}^T \\
K_{2,2,2} &= \begin{pmatrix} 64.55 & 22.34 & -5.605 & 8.128 \end{pmatrix}^T \\
K_{1,3,2} &= \begin{pmatrix} 65.27 & 22.5 & -5.296 & 8.206 \end{pmatrix}^T \\
K_{2,3,2} &= \begin{pmatrix} 65.27 & 22.5 & -5.296 & 8.206 \end{pmatrix}^T \\
K_{1,4,2} &= \begin{pmatrix} 65.38 & 22.54 & -5.251 & 8.225 \end{pmatrix}^T \\
K_{2,4,2} &= \begin{pmatrix} 65.38 & 22.54 & -5.251 & 8.225 \end{pmatrix}^T \\
K_{1,5,2} &= \begin{pmatrix} 66.7 & 22.86 & -4.687 & 8.371 \end{pmatrix}^T \\
K_{2,5,2} &= \begin{pmatrix} 66.7 & 22.86 & -4.687 & 8.371 \end{pmatrix}^T
\end{aligned} \tag{37}$$

4.3 Results

Simulation and experimental results are shown on Fig 3. During simulation and experimental tests proposed H_∞ approach is compared to original TP model based approach (QS) described in Kolonić et al. (2006). It can be seen that proposed approach has successfully minimized steady state error of chart position from 14.55 mm to 2.4 mm in simulation, and from 7mm to 0.4 mm in experimental results. However, slower decay rate as well as oscillatory angle response is obtained due to increased control action needed to overcome static friction force.

5. CONCLUSION

TP transform (HOSVD) creates polytopic model suitable for PDC controller synthesis via LMIs. Presented results for gantry crane are obtained by using exact TP transform. Simulation and experimental results verify that H_∞ controller has successfully minimized steady state error due to friction effect, however it resulted in oscillatory angle response and slower decay rate. In order to minimize oscillations H_∞ controller could be extended with robust pole placement inside LMI regions.

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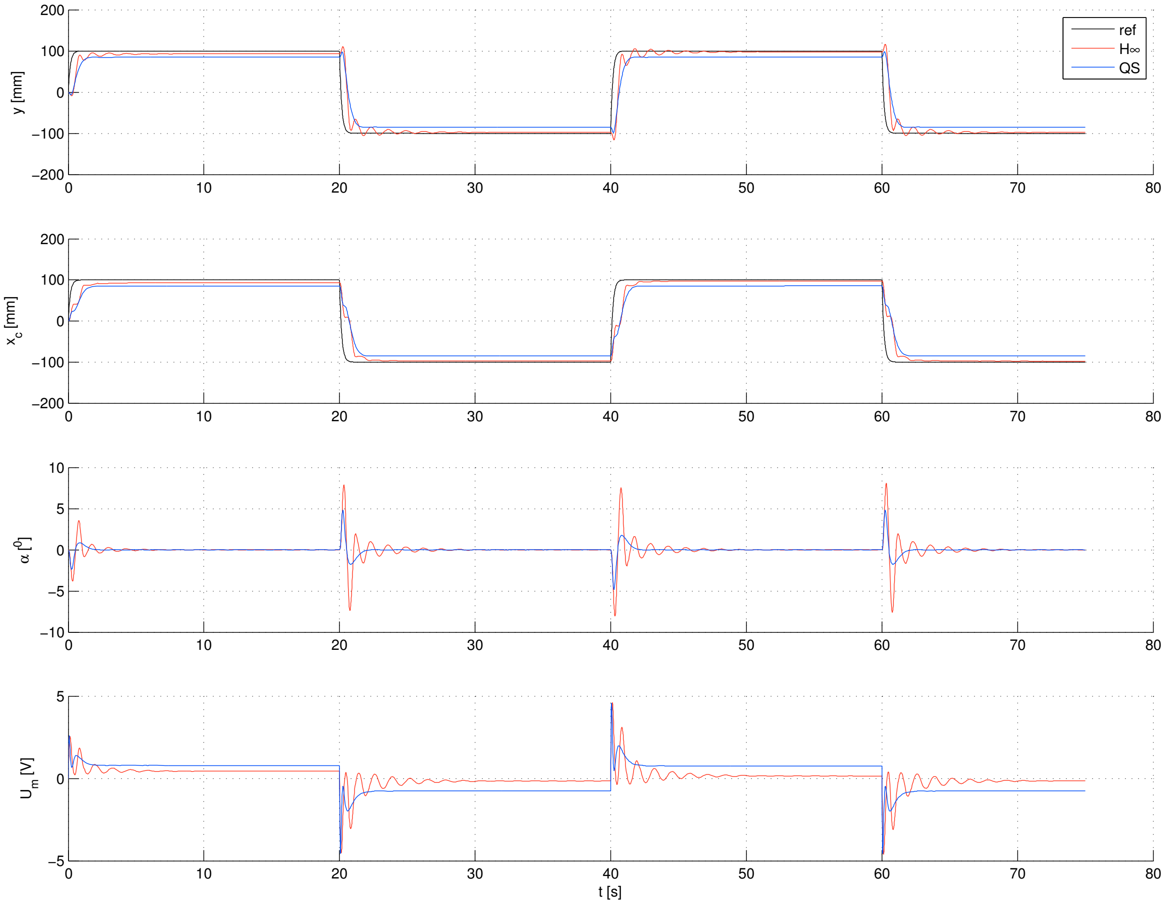
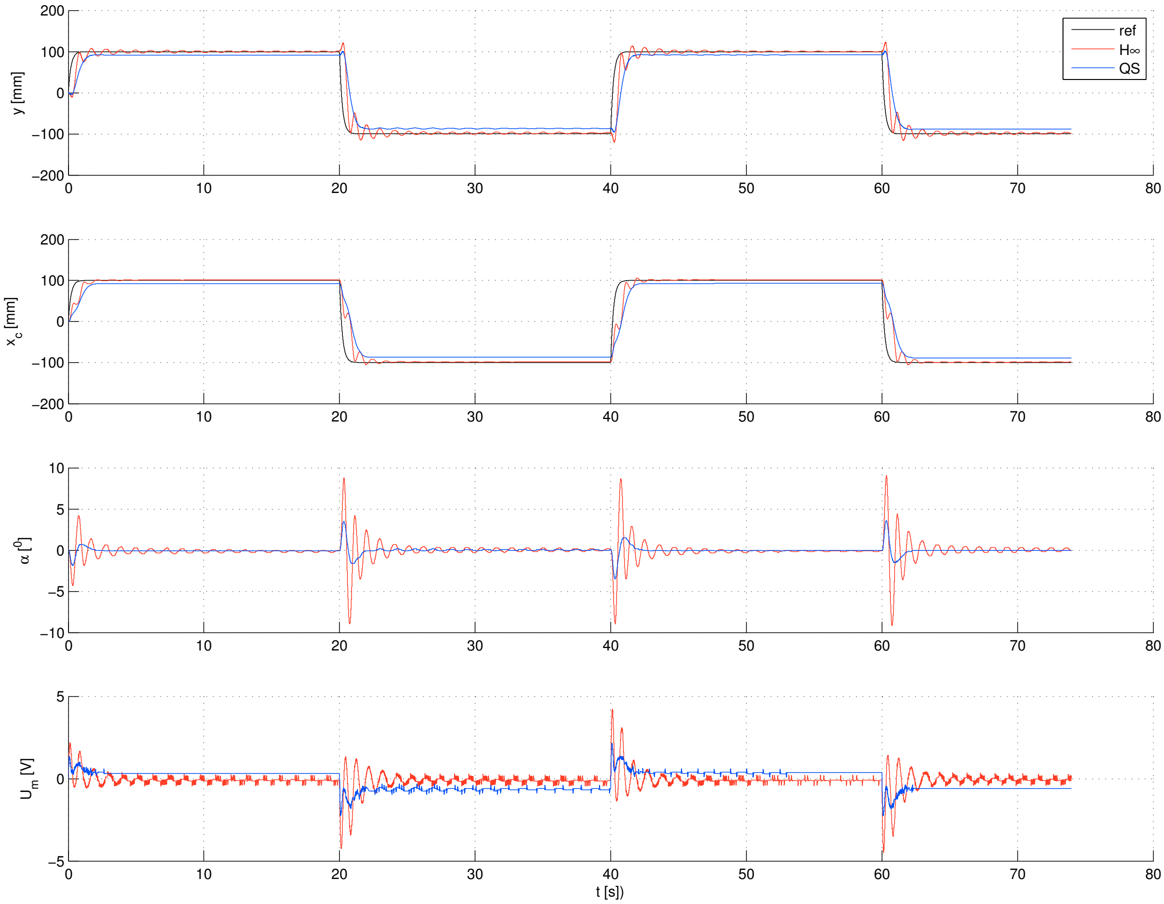
[2] YALMIP is a modelling language for advanced modeling and solution of convex and nonconvex optimization problems.
[3] SeDuMi: a Matlab toolbox for optimization over symmetric cones



(a) Simulation results



(b) Experimental results

Fig. 3. Simulation and experimental results after step reference position, where y is pendulum tip position, x_c is cart position, α is pendulum angle and U_m is control input, respectively

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