TP transformation based control of rotary pendulum

Šandor Ileš* Jadranko Matuško* Fetah Kolonić*
*Department of Electrical Machines, Drives and Automation, Faculty of Electrical Engineering and Computing, University of Zagreb, Croatia

***Abstract*— The Tensor Product (TP) model transformation is a recently proposed technique for transformation of a given Linear Parameter Varying (LPV) state-space model into polytopic model form, namely, to parameter varying convex combination of Linear Time Invariant (LTI) systems. The main advantage of the TP model transformation is that it is executable in a relatively short time and the Linear Matrix Inequality (LMI)-based control design frameworks can immediately be applied to the resulting polytopic models. In this paper, such control approach of nonlinear systems is applied to the control of rotary pendulum gantry. Pendulum in hanging position represents simplified model of the real industry crane application. On the other hand, inverted pendulum is a classic problem in dynamics and control theory and widely used as a benchmark for testing control algorithms. In order to reach upright position, self erecting technique is employed. LMI control algorithm obtained for both positions is merged using self erecting control algorithm and tested in real experimental setup.**

***Index Terms*— Parallel Distributed Compensation, Linear matrix inequalities, TP model transformation, Rotary Pendulum Gantry (RPG), position control, sway control, inverted pendulum**

I. Introduction

In modern industrial system, gantry cranes as well as rotary cranes are widely used for the heavy loads transfer. For the anti-sway control of travelling cranes, there are several solutions, i.e., by fuzzy control, optimal control, etc. and each of them is reported to be effective [1]–[5].

Inverted pendulum is a classic problem in dynamics and control theory that is widely used as a benchmark for testing control algorithms [6]. In [7] the swing-up problem for the Furuta pendulum is solved applying Fradkov's speed-gradient (SG) method and in [8] energy based swing up control algorithm is proposed. In [9] fuzzy swing up controller is proposed and is used in this paper.

The TP model representation belongs to the class of polytopic models. The TP model represents the Linear Parameter Varying state-space models by the parameter varying combination of Linear Time Invariant (LTI) models. The TP model transformation was proposed as a uniform and automatic way to transform LPV model. The TP model transformation was introduced as the Higher Order Singular Value Decomposition (HOSVD) of Linear Parameter Varying (LPV) state-space models, and the result of the TP model transformation was defined as the HOSVD-based canonical form of LPV models [10], [11].

In [12] trade-off techniques between accuracy and complexity of TP form are proposed.

Furthermore, the TP model transformation offers options to satisfy various convexity constrains on the type of the resulting parameter varying combination, which is suitable, for instance for the Linear Matrix Inequality-based control designs [10], [13].

TP based control of gantry crane is given in [14]. In [15] and [16] LMI control methodology is presented.

The main contributions of this paper are verification of TP model of the rotary pendulum gantry in both upright and hanging position, and investigation the performance of the TP model transformation-based control design in a real experimental setup.

II. Tensor Product Model Transformation-based Control Design Methodology

Consider the linear parameter-varying state-space model

$$\begin{pmatrix}\dot{x}(t)\\ y(t)\end{pmatrix} = S(p(t))\begin{pmatrix}x(t)\\ u(t)\end{pmatrix} \tag{1}$$

with input $u(t) \in R^k$, output $y(t) \in R^l$ and state vector $x(t) \in R^m$. The system matrix

$$S(p(t)) = \begin{pmatrix}A(p(t)) & B(p(t))\\ C(p(t)) & D(p(t))\end{pmatrix} \in R^{(m+k)\times(m+l)} \tag{2}$$

is a parameter-varying object, where $p(t) \in \Omega$ is time varying parameter vector, where Ω is a closed hypercube in R^N, $\Omega = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_N, b_N]$. Parameter $p(t)$ can also include the elements of the state vector $x(t)$, therefore LPV system given in Eq. (1) is considered in the class of non-linear dynamic state space models.

The main idea of TP model transformation is to discretize the given LPV model given in Eq. (1) over hyper rectangular grid M in Ω, then via executing Higher Order Singular Value Decomposition, the tensor product structure of given model is obtained. By ignoring singular values, TP model of reduced complexity and accuracy can be obtained. For more details se [10] and [12].

Tensor product structure can be written as follows

$$\begin{aligned} S(p(t)) &= \mathcal{S} \mathop{\boxtimes}_{n=1}^{N} w_n(p_n) \\ &= \sum_{i_1=1}^{I_1}\sum_{i_2=1}^{I_2}\cdots\sum_{i_n=1}^{I_N}\prod_{n=1}^{N} w_{n,i_n}(p_n) S_{i_1,i_2\cdots,i_N}, \end{aligned} \tag{3}$$

where $\mathcal{S} \in R^{I_1\times I_2\times\cdots I_N\times(m+k)\times(m+l)}$ denotes obtained tensor, I_n denotes number of LTI systems in n-th dimension of Ω, $\boxtimes$ denotes multiple n-mode product of a tensor by a matrix, w_n is row vector containing $w_{n,i_n}(p_n) \in [0,1]$ which is corresponding one variable weighting function defined on the n-th dimension of Ω and $S_{i_1,i_2\cdots i_N}$ is LTI system matrix obtained by TP model transformation. By using i as linear index, equivalent to the multilinear array index with the size of $I_1 \times I_2 \times \cdots I_N$, TP model (3) can be rewritten in standard polytopic form

$$S(p) = \sum_{i=1}^{N} w_i(p) S_i, \tag{4}$$

where S_i denotes

$$S_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix}, \tag{5}$$

and w_i is corresponding weighting function.

Controller is determined in same polytopic form as TP model. Control signal is given by

$$u = -\sum_{i=1}^{N} w_i(p) F_i x, \tag{6}$$

where the F_i are corresponding LTI feedback gains.

III. Controller Design

A. Linear Matrix Inequalities

A class of numerical optimization problems called linear matrix inequality LMI problems has received significant attention. These optimization problems can be solved in polynomial time and hence are tractable, at least in a theoretical sense. Interior-point methods developed for these problems have been found to be extremely efficient in practice. For systems and control, the importance of LMI optimization stems from the fact that a wide variety of system and control problems can be recast as LMI problems. Except for a few special cases, these problems do not have analytical solutions. However, the main point is that through the LMI framework they can be efficiently solved numerically.

A linear matrix inequality (LMI) has the form

$$F(x) = F_0 + \sum_{i=1}^{m} x_i F_i > 0, \tag{7}$$

where $x \in R^m$ is the variable and the symmetric matrices $F_i = F_i^T$ are given. The inequality symbol in (7) means that $F(x)$ is positive definite i.e.

$$z^T F(x) z > 0, \forall z \neq 0. \tag{8}$$

B. Control objective

The control objective is to find stabilizing controller, with prescribed decay rate with minimal overshoot and constrained control signal.

In order to obtain stabilizing controller, Lyapunov stability condition is considered.

If there exist candidate quadratic Lyapunov function V(x) defined on some open set $D \in R^N$, containing the origin, such that

$$V(x) = x^T P x > 0, \tag{9}$$

and there exist derivation

$$\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x} < 0, \tag{10}$$

then origin of system $\dot{x} = f(x)$ is stable equilibrium point.

The speed of response is related to decay rate, that is, the largest Lyapunov exponent α [16] (Stability corresponds to positive decay rate) such that

$$\lim_{t\to\infty} e^{-\alpha t} \|x(t)\| = 0 \tag{11}$$

A sufficient condition for desired decay rate can be written as

$$\dot{V}(x) \leq -2\alpha V(x), \tag{12}$$

for any initial point. [16].

From (12) it follows that the equilibrium of the continuous system in polytopic form (4) is globally asymptotically stable if there exists a common positive definite matrix P such that

$$A_i^T P + P A_i + 2\alpha P < 0; \forall i \in (1, r). \tag{13}$$

Next, let us consider the stability of the closed-loop system (4) with control algorithm given in (6), which is globally asymptotically stable, with decay rate less than α, if there exists a common positive definite matrix P such that

$$\begin{gathered} G_{ii}^T P + P G_{ii} + 2\alpha P < 0, \\ \left(\frac{G_{ij} + G_{ji}}{2}\right)^T P + P\left(\frac{G_{ij} + G_{ji}}{2}\right) + 2\alpha P \leq 0, i < j. \end{gathered} \tag{14}$$

where

$$G_{ij} = A_i + B_i F_j, \tag{15}$$

denotes closed loop state matrix.

The largest possible decay rate can be found by solving generalized eigenvalue minimization problem (GEVP).

maximize α

subject to

$$\begin{aligned} & X > 0 \\ & -XA_i^T - A_i X + M_i^T B_i^T + B_i M_i + 2\alpha X > 0 \\ & -XA_i^T - A_i X - XA_j^T - A_j X + M_j^T B_i^T \\ & + B_i M_j + M_i^T B_j^T + B_j M_i - 4\alpha X \geq 0, \end{aligned} \tag{16}$$

where $X = P^{-1}$ and $M_i = F_i X$.

In sequel we use predescribed value of α.

In order to satisfy the constraints on control value and output constraints, the following LMIs are added to the Eq (16) [15]:

Constraint on the control value:

Assume that initial condition $x(0)$ is unknown, but its upper bound $\|x(0)\| \leq \phi$ is known, which can be recast as following LMI

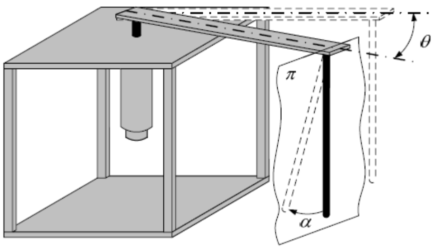


Fig. 1. Rotary pendulum gantry (RPG) model, variables notation

Fig. 2. Rotary pendulum gantry (RPG) laboratory model

$$\phi^2 I \leq X, \tag{17}$$

the constraint $\|u\|_2 \leq \mu$ is enforced $\forall t \geq 0$ if the following LMI holds

$$\begin{pmatrix} X & M_i^T \\ M_i & \mu^2 I \end{pmatrix} \geq 0. \tag{18}$$

Constraint on the output:

Assume that condition (17) is satisfied, the constraint $\|y(t)\|_2 \leq \lambda$ is enforced, $\forall t \geq 0$, if the following LMI holds

$$\begin{pmatrix} X & XC_i^T \\ C_i X & \lambda^2 I \end{pmatrix} \geq 0. \tag{19}$$

Furthermore, controller is obtained as follows:

$$F_r = M_r X^{-1}. \tag{20}$$

IV. TP Model-based Controller Design to the Rotary Pendulum

A. *Modelling the rotary pendulum plant*

Non-linear model of rotary pendulum gantry (RPG) system can be described by following equations:

$$\begin{aligned} \ddot{\theta} &= \frac{1}{A_x(\alpha)}(bc\sin\alpha\dot{\alpha}^2 + \frac{1}{2}bd\sin 2\alpha - ce\dot{\theta} + cfV_m), \\ \ddot{\alpha} &= \frac{1}{A_x(\alpha)}(-ad\sin\alpha - \frac{1}{2}b^2\sin 2\alpha\dot{\alpha}^2 + b\cos\alpha(e\dot{\theta} - fV_m)), \end{aligned} \tag{21}$$

where

$$\begin{aligned} a &= J_{eq} + m\cdot r^2 + n_g \cdot K_g^2 \cdot J_m, \\ b &= m \cdot L \cdot r, \\ c &= 4/3 \cdot m \cdot L^2, \\ d &= m \cdot g \cdot L, \\ e &= Beq + n_m \cdot n_g \cdot K_t \cdot K_g^2 \cdot K_m / R_m, \\ f &= n_m \cdot n_g \cdot K_t \cdot K_g / R_m, \end{aligned} \tag{22}$$

and

$$A_x(\alpha) = ac - b^2\cos^2\alpha. \tag{23}$$

Its derivation using Lagrangian formulation is omitted for the sake of brevity and can be found in [17]. In Fig. 1 model of rotary pendulum is shown. In Fig. 2 laboratory model of rotary pendulum is shown.

B. *LPV model of rotary pendulum*

Letting $x = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T = \begin{bmatrix} \theta & \dot{\theta} & \alpha & \dot{\alpha} \end{bmatrix}^T$, the equations of motion in linear parameter varying state space form is

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & A_1/A_x & A_2/A_x & A_3/A_x & B_1/A_x \\ 0 & 0 & 0 & 1 & 0 \\ 0 & A_4/A_x & A_5/A_x & A_6/A_x & B_2/A_x \end{pmatrix}, \tag{24}$$

where:

$$\begin{aligned} A_x &= ac - b^2\cos^2\alpha, \\ A_1 &= -ce, \\ A_2 &= bd\cos x \mathrm{sinc}\alpha, \\ A_3 &= bc\sin\alpha\dot{\alpha}, \\ A_4 &= be\cos\alpha, \\ A_5 &= -ad\mathrm{sinc}\alpha, \\ A_6 &= -b^2\cos\alpha\sin\alpha\dot{\alpha}, \\ B_1 &= cf, \\ B_2 &= -bf\cos\alpha, \\ E &= ac - b^2. \end{aligned} \tag{25}$$

List of paramaters is given in Table I.

V. Results

A. *TP model representation of rotary pendulum in hanging position*

Operating area for hanging position is selected as $\Omega = [\alpha_{min}, \alpha_{max}] \times [\dot{\alpha}_{min}, \dot{\alpha}_{max}] = [\frac{-27}{180}\pi, \frac{27}{180}\pi] \times [-0.8, 0.8]$. The exact TP model (TP model obtained by keeping all singular values) representation in hanging position is given by 12 LTI systems. Weighting functions of the TP model in hanging position are given in Fig. 3

The LTI system matrices of the TP model are given in Eq. (26).

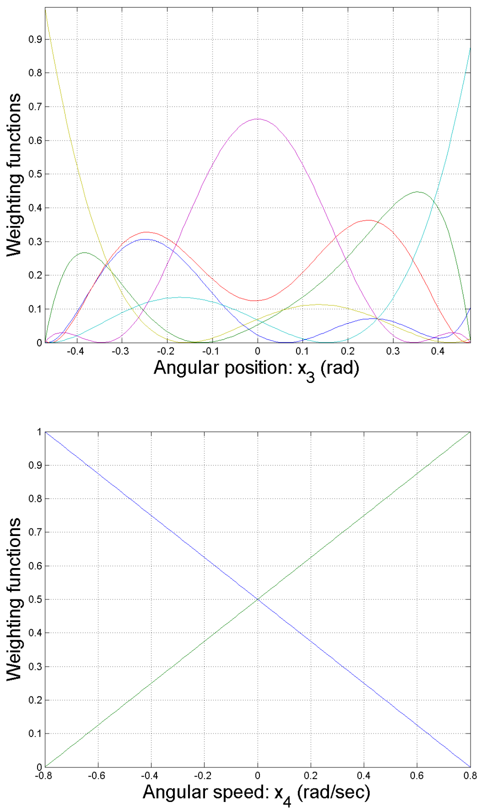


Fig. 3. Weighting functions of the TP model in hanging position

$$
\begin{aligned}
A_1 &= \begin{pmatrix} 0 & 1.0 & 0 & 0 \\ 0 & -10.18 & 25.85 & 1.224 \\ 0 & 0 & 0 & 1.0 \\ 0 & 9.329 & -66.85 & -1.134 \end{pmatrix} & B_1 &= \begin{pmatrix} 0 \\ 17.92 \\ 0 \\ -16.42 \end{pmatrix} \\
A_2 &= \begin{pmatrix} 0 & 1.0 & 0 & 0 \\ 0 & -10.18 & 25.85 & -1.224 \\ 0 & 0 & 0 & 1.0 \\ 0 & 9.329 & -66.85 & 1.134 \end{pmatrix} & B_2 &= \begin{pmatrix} 0 \\ 17.92 \\ 0 \\ -16.42 \end{pmatrix} \\
A_3 &= \begin{pmatrix} 0 & 1.0 & 0 & 0 \\ 0 & -9.93 & 24.45 & 0.1167 \\ 0 & 0 & 0 & 1.0 \\ 0 & 8.893 & -64.73 & -0.1078 \end{pmatrix} & B_3 &= \begin{pmatrix} 0 \\ 17.48 \\ 0 \\ -15.65 \end{pmatrix} \\
A_4 &= \begin{pmatrix} 0 & 1.0 & 0 & 0 \\ 0 & -9.93 & 24.45 & -0.1167 \\ 0 & 0 & 0 & 1.0 \\ 0 & 8.893 & -64.73 & 0.1078 \end{pmatrix} & B_4 &= \begin{pmatrix} 0 \\ 17.48 \\ 0 \\ -15.65 \end{pmatrix} \\
A_5 &= \begin{pmatrix} 0 & 1.0 & 0 & 0 \\ 0 & -11.0 & 30.38 & -0.7193 \\ 0 & 0 & 0 & 1.0 \\ 0 & 10.73 & -73.74 & 0.6675 \end{pmatrix} & B_5 &= \begin{pmatrix} 0 \\ 19.36 \\ 0 \\ -18.89 \end{pmatrix} \\
A_6 &= \begin{pmatrix} 0 & 1.0 & 0 & 0 \\ 0 & -11.0 & 30.38 & 0.7193 \\ 0 & 0 & 0 & 1.0 \\ 0 & 10.73 & -73.74 & -0.6675 \end{pmatrix} & B_6 &= \begin{pmatrix} 0 \\ 19.36 \\ 0 \\ -18.89 \end{pmatrix} \\
A_7 &= \begin{pmatrix} 0 & 1.0 & 0 & 0 \\ 0 & -9.47 & 21.79 & -0.3934 \\ 0 & 0 & 0 & 1.0 \\ 0 & 8.047 & -60.71 & 0.3475 \end{pmatrix} & B_7 &= \begin{pmatrix} 0 \\ 16.67 \\ 0 \\ -14.16 \end{pmatrix} \\
A_8 &= \begin{pmatrix} 0 & 1.0 & 0 & 0 \\ 0 & -9.47 & 21.79 & 0.3934 \\ 0 & 0 & 0 & 1.0 \\ 0 & 8.047 & -60.71 & -0.3475 \end{pmatrix} & B_8 &= \begin{pmatrix} 0 \\ 16.67 \\ 0 \\ -14.16 \end{pmatrix} \\
A_9 &= \begin{pmatrix} 0 & 1.0 & 0 & 0 \\ 0 & -11.14 & 31.18 & 0.1008 \\ 0 & 0 & 0 & 1.0 \\ 0 & 10.98 & -74.96 & -0.09343 \end{pmatrix} & B_9 &= \begin{pmatrix} 0 \\ 19.61 \\ 0 \\ -19.33 \end{pmatrix} \\
A_{10} &= \begin{pmatrix} 0 & 1.0 & 0 & 0 \\ 0 & -11.14 & 31.18 & -0.1008 \\ 0 & 0 & 0 & 1.0 \\ 0 & 10.98 & -74.96 & 0.09343 \end{pmatrix} & B_{10} &= \begin{pmatrix} 0 \\ 19.61 \\ 0 \\ -19.33 \end{pmatrix} \\
A_{11} &= \begin{pmatrix} 0 & 1.0 & 0 & 0 \\ 0 & -9.543 & 22.21 & 0.2187 \\ 0 & 0 & 0 & 1.0 \\ 0 & 8.18 & -61.35 & -0.1878 \end{pmatrix} & B_{11} &= \begin{pmatrix} 0 \\ 16.8 \\ 0 \\ -14.4 \end{pmatrix} \\
A_{12} &= \begin{pmatrix} 0 & 1.0 & 0 & 0 \\ 0 & -9.543 & 22.21 & -0.2187 \\ 0 & 0 & 0 & 1.0 \\ 0 & 8.18 & -61.35 & 0.1878 \end{pmatrix} & B_{12} &= \begin{pmatrix} 0 \\ 16.8 \\ 0 \\ -14.4 \end{pmatrix}
\end{aligned}
\tag{26}
$$

B. TP model representation of rotary pendulum in upright position

Operating area for upright position is selected as $\Omega = [\alpha_{min}, \alpha_{max}] \times [\dot{\alpha}_{min}, \dot{\alpha}_{max}] = [\frac{-135}{180}\pi, \frac{135}{180}\pi] \times [-0.8, 0.8]$. Exact TP model representation is given by 14 LTI models, however using exact model resulted in infeasible controller.

Obtained singular values are:

$$\sigma = \begin{bmatrix} 1.2955 \\ 0.0386 \\ 0.0051 \\ 0.0018 \\ 0.0000 \\ 0.0000 \end{bmatrix} \cdot 10^3. \tag{27}$$

By keeping only first four singular values, only 10 LTI models are obtained. LTI models are given in Eq. (28). Obtained weighting functions in upright position is given in Fig. 4.

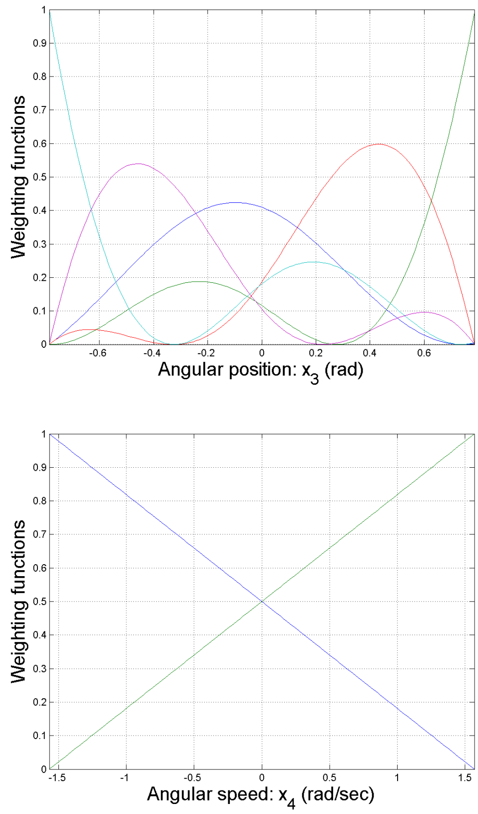


Fig. 4. Weighting functions of the TP model in upright position

The LTI system matrices of the TP model are given in Eq. (28):

$$
\begin{aligned}
A_1 &= \begin{pmatrix} 0 & 1.0 & 0 & 0 \\ 0 & -14.02 & 47.61 & -0.08115 \\ 0 & 0 & 0 & 1.0 \\ 0 & -16.16 & 99.84 & -0.08906 \end{pmatrix} & B_1 &= \begin{pmatrix} 0 \\ 24.69 \\ 0 \\ 28.45 \end{pmatrix} \\
A_2 &= \begin{pmatrix} 0 & 1.0 & 0 & 0 \\ 0 & -14.02 & 47.61 & 0.08115 \\ 0 & 0 & 0 & 1.0 \\ 0 & -16.16 & 99.84 & 0.08906 \end{pmatrix} & B_2 &= \begin{pmatrix} 0 \\ 24.69 \\ 0 \\ 28.45 \end{pmatrix} \\
A_3 &= \begin{pmatrix} 0 & 1.0 & 0 & 0 \\ 0 & -8.163 & 14.02 & 0.5705 \\ 0 & 0 & 0 & 1.0 \\ 0 & -5.534 & 49.0 & 0.3885 \end{pmatrix} & B_3 &= \begin{pmatrix} 0 \\ 14.37 \\ 0 \\ 9.742 \end{pmatrix} \\
A_4 &= \begin{pmatrix} 0 & 1.0 & 0 & 0 \\ 0 & -8.163 & 14.02 & -0.5705 \\ 0 & 0 & 0 & 1.0 \\ 0 & -5.534 & 49.0 & -0.3885 \end{pmatrix} & B_4 &= \begin{pmatrix} 0 \\ 14.37 \\ 0 \\ 9.742 \end{pmatrix} \\
A_5 &= \begin{pmatrix} 0 & 1.0 & 0 & 0 \\ 0 & -9.555 & 22.42 & 0.8248 \\ 0 & 0 & 0 & 1.0 \\ 0 & -8.27 & 61.63 & 0.7079 \end{pmatrix} & B_5 &= \begin{pmatrix} 0 \\ 16.82 \\ 0 \\ 14.56 \end{pmatrix} \\
A_6 &= \begin{pmatrix} 0 & 1.0 & 0 & 0 \\ 0 & -9.555 & 22.42 & -0.8248 \\ 0 & 0 & 0 & 1.0 \\ 0 & -8.27 & 61.63 & -0.7079 \end{pmatrix} & B_6 &= \begin{pmatrix} 0 \\ 16.82 \\ 0 \\ 14.56 \end{pmatrix} \\
A_7 &= \begin{pmatrix} 0 & 1.0 & 0 & 0 \\ 0 & -8.183 & 14.14 & -0.5617 \\ 0 & 0 & 0 & 1.0 \\ 0 & -5.571 & 49.17 & -0.3823 \end{pmatrix} & B_7 &= \begin{pmatrix} 0 \\ 14.4 \\ 0 \\ 9.806 \end{pmatrix} \\
A_8 &= \begin{pmatrix} 0 & 1.0 & 0 & 0 \\ 0 & -8.183 & 14.14 & 0.5617 \\ 0 & 0 & 0 & 1.0 \\ 0 & -5.571 & 49.17 & 0.3823 \end{pmatrix} & B_8 &= \begin{pmatrix} 0 \\ 14.4 \\ 0 \\ 9.806 \end{pmatrix} \\
A_9 &= \begin{pmatrix} 0 & 1.0 & 0 & 0 \\ 0 & -8.038 & 13.81 & -0.8126 \\ 0 & 0 & 0 & 1.0 \\ 0 & -5.564 & 48.59 & -0.6816 \end{pmatrix} & B_9 &= \begin{pmatrix} 0 \\ 14.15 \\ 0 \\ 9.794 \end{pmatrix} \\
A_{10} &= \begin{pmatrix} 0 & 1.0 & 0 & 0 \\ 0 & -8.038 & 13.81 & 0.8126 \\ 0 & 0 & 0 & 1.0 \\ 0 & -5.564 & 48.59 & 0.6816 \end{pmatrix} & B_{10} &= \begin{pmatrix} 0 \\ 14.15 \\ 0 \\ 9.794 \end{pmatrix}
\end{aligned}
\tag{28}
$$

C. Control objective

The control objective is to find stabilizing controller, with prescribed decay rate $\alpha < 3$ with minimal overshoot and constrained control signal $V_m \leq 10V$.

By using the Yalmip [18] and Sedumi 1.3 [19] the following feasible solution of (16) - (19) and feedback gains are obtained.

D. Feedback gains for hanging position

$$
\begin{aligned}
F_1 &= \begin{pmatrix} 11.28 & 6.775 & -57.06 & 5.915 \end{pmatrix}^T \\
F_2 &= \begin{pmatrix} 11.39 & 6.847 & -57.67 & 5.837 \end{pmatrix}^T \\
F_3 &= \begin{pmatrix} 11.63 & 7.012 & -59.17 & 6.038 \end{pmatrix}^T \\
F_4 &= \begin{pmatrix} 11.65 & 7.02 & -59.25 & 6.031 \end{pmatrix}^T \\
F_5 &= \begin{pmatrix} 10.55 & 6.264 & -52.16 & 5.436 \end{pmatrix}^T \\
F_6 &= \begin{pmatrix} 10.48 & 6.224 & -51.99 & 5.474 \end{pmatrix}^T \\
F_7 &= \begin{pmatrix} 12.14 & 7.371 & -62.7 & 6.268 \end{pmatrix}^T \\
F_8 &= \begin{pmatrix} 12.04 & 7.306 & -62.18 & 6.259 \end{pmatrix}^T \\
F_9 &= \begin{pmatrix} 10.37 & 6.147 & -51.18 & 5.387 \end{pmatrix}^T \\
F_{10} &= \begin{pmatrix} 10.38 & 6.153 & -51.2 & 5.383 \end{pmatrix}^T \\
F_{11} &= \begin{pmatrix} 11.99 & 7.264 & -61.76 & 6.22 \end{pmatrix}^T \\
F_{12} &= \begin{pmatrix} 12.06 & 7.313 & -62.1 & 6.238 \end{pmatrix}^T
\end{aligned}
\tag{29}
$$

E. Feedback gains for upright position

$$
\begin{aligned}
F_1 &= \begin{pmatrix} -0.1689 & -0.7571 & 7.117 & 0.77 \end{pmatrix}^T \\
F_2 &= \begin{pmatrix} -0.1532 & -0.742 & 6.801 & 0.7144 \end{pmatrix}^T \\
F_3 &= \begin{pmatrix} -0.4763 & -1.066 & 14.14 & 1.954 \end{pmatrix}^T \\
F_4 &= \begin{pmatrix} -0.4909 & -1.076 & 14.36 & 1.971 \end{pmatrix}^T \\
F_5 &= \begin{pmatrix} -0.2819 & -0.8834 & 9.937 & 1.308 \end{pmatrix}^T \\
F_6 &= \begin{pmatrix} -0.2964 & -0.8975 & 10.17 & 1.297 \end{pmatrix}^T \\
F_7 &= \begin{pmatrix} -0.4855 & -1.07 & 14.19 & 1.962 \end{pmatrix}^T \\
F_8 &= \begin{pmatrix} -0.4554 & -1.047 & 13.59 & 1.856 \end{pmatrix}^T \\
F_9 &= \begin{pmatrix} -0.4682 & -1.056 & 13.82 & 1.909 \end{pmatrix}^T \\
F_{10} &= \begin{pmatrix} -0.4384 & -1.032 & 13.33 & 1.867 \end{pmatrix}^T
\end{aligned}
\tag{30}
$$

In Fig. 5 and 6 simulation and experimental results are shown. It can be seen that proposed methodology was suitable for both control designs. However it can be seen that in experimental results steady state error exists due to friction effects that were neglected during the controller synthesis.

VI. CONCLUSION

Rotary pendulum is non-linear plant with two equilibrium positions, hanging and upright. The main point of this paper is that controller for both positions can be obtained, using TP transformation and LMI framework, simply by changing operating area of the plant. Trough LMI framework, stability of obtained TP model is guaranteed and common physical constraints are taken into account. Furthermore, self erecting algorithm is guaranteed to stabilize obtained TP model since swing up algorithm is taken into account trough initial state constraints. Presented results for rotary pendulum in hanging position are obtained by using exact TP transformation. By ignoring smaller singular values non-linear model can be approximated with less LTI models. Exact TP transform of inverted pendulum results in too complex model for PDC controller synthesis and results in infeasible solution of given LMIs. However by reducing number of singular values, feasible solution was obtained. By comparing simulation and experimental results, for both positions , steady state error due friction effects that were neglected in controller synthesis, can be noticed. Obtained PDC controller through LMI framework can guarantee only stability of polytopic model, obtained using TP transformation, but not of non-linear model itself. In order to guarantee stability of non-linear model, polytopic model should be expanded with uncertainties. That will be considered in the future work.

ACKNOWLEDGEMENT

This research is supported by the Ministry of Science, Education and Sports of the Republic of Croatia under Grant No. 036-0363078-1629.

REFERENCES

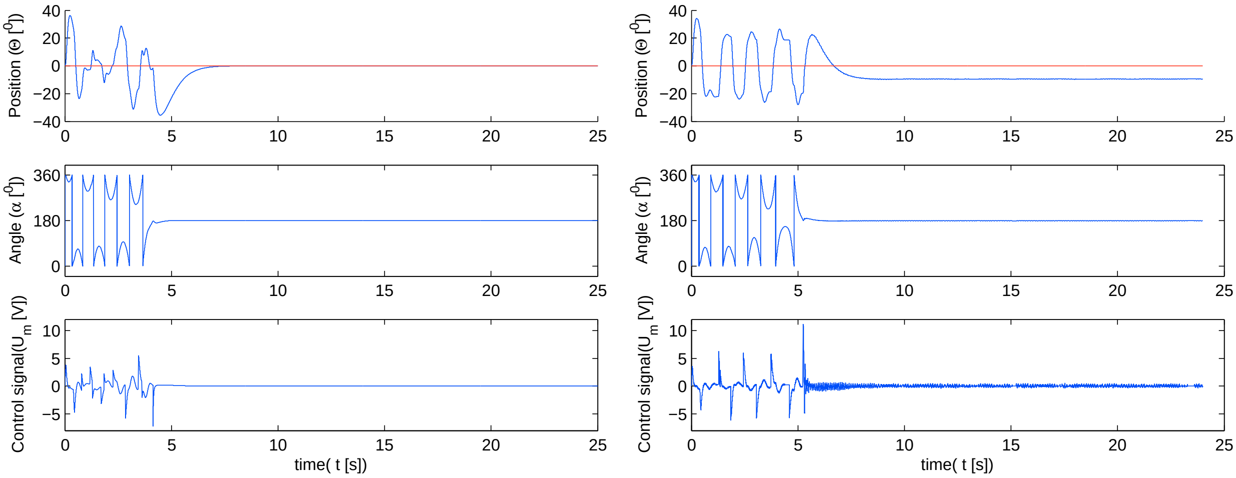
[1] T. Popadić, F. Kolonić, and A. Poljugan, "A fuzzy control scheme for the gantry crane position and load swing control," 2005.

[2] M. Nalley and M. Trabia, "Control of overhead cranes using a fuzzy logic controller," *Journal of Intelligent and Fuzzy systems*, vol. 8, no. 1, pp. 1–18, 2000.

[3] H. Omar, *Control of gantry and tower cranes*. PhD thesis, Virginia Polytechnic Institute and State University, 2003.

TABLE I
PARAMETERS OF THE RPG SYSTEM

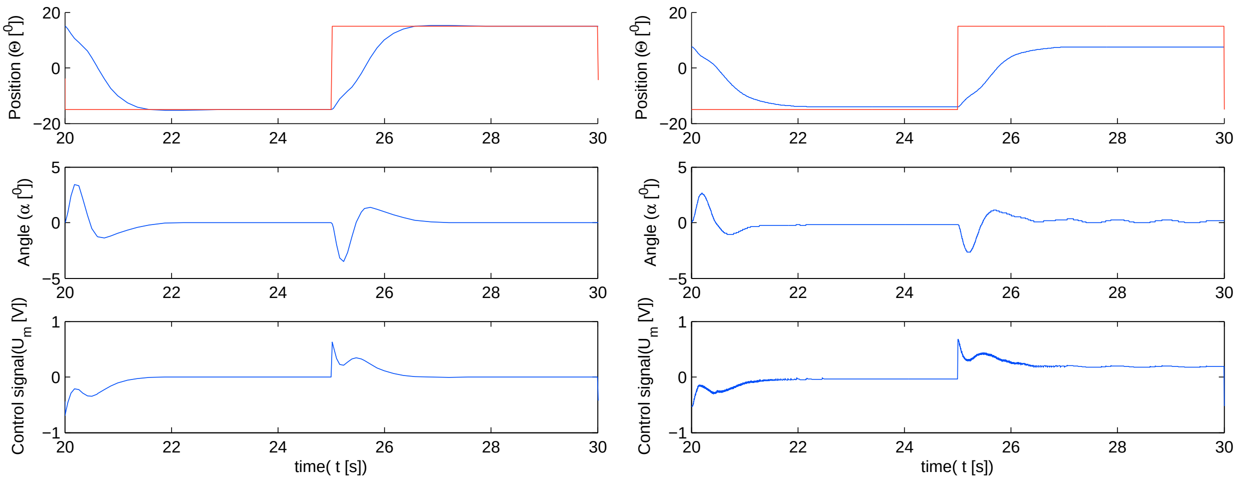
Symbol	Description	Value	Unit
V_m	Input voltage		V
m	Mass of pendulum	0.125	kg
r	Rotating arm length	0.215	m
L	Half length of the pendulum	0.1675	m
K_t	Motor torque constant	0.00767	$\frac{Nm}{A}$
R_m	Armature resistance	2.6	Ω
K_m	Back electro-motive force constant	0.00767	$\frac{Vs}{rad}$
J_m	Moment of inertia of the motor rotor	$3.9001 \cdot 10^{-7}$	$kg\,m^2$
K_g	System gear ratio (motor - load)	$70(14 \times 5)$	
n_g	Gearbox efficiency	0.9	
n_m	Motor efficiency	0.69	
B_{eq}	Equivalent viscous damping coeff.	0.004	$\frac{Nms}{rad}$
g	Gravitational constant of earth	9.81	$\frac{m}{s^2}$
α	Pendulum position		rad
θ	Load shaft position		rad



(a) Simulation results

(b) Experimental results

Fig. 5. Results in upright position



(a) Simulation results

(b) Experimental results

Fig. 6. Results in hanging position

[4] Y. Sakawa and A. Nakazumi, “Modeling and control of a rotary crane,” *Journal of Dynamic Systems, Measurement, and Control*, vol. 107, p. 200, 1985.

[5] I. Rastić, F. Kolonić, and A. Poljugan, “State Feedback Optimal Controller Design for the Rotational Electromechanical System,”

[6] K. Furuta, M. Yamakita, and S. Kobayashi, “Swing-up control of inverted pendulum using pseudo-state feedback,” *ARCHIVE: Proceedings of the Institution of Mechanical Engineers, Part I: Journal of Systems and Control Engineering 1991-1996 (vols 205-210)*, vol. 206, no. 49, pp. 263–269, 1992.

[7] F. Gordillo, J. Acosta, and J. Aracil, “A new swing-up law for the Furuta pendulum,” *International Journal of Control*, vol. 76, no. 8, pp. 836–844, 2003.

[8] K. Yoshida, “Swing-up control of an inverted pendulum by energy-based methods,” in *American Control Conference, 1999. Proceedings of the 1999*, vol. 6, pp. 4045–4047, IEEE, 2002.

[9] Q. Inc, “IPO2 Self Erecting Inverted Pendulum Users Guide, 1996.”

[10] Z. Petres, “Polytopic Decomposition of Linear Parameter-Varying Models by Tensor-Product Model Transformation Ph. D. Thesis Booklet,” 2006.

[11] P. Baranyi, D. Tikk, Y. Yam, and R. Patton, “From differential equations to PDC controller design via numerical transformation,” *Computers in Industry*, vol. 51, no. 3, pp. 281–297, 2003.

[12] D. Tikk, P. Baranyi, R. Patton, and J. Tar, “Approximation Capability of TP model forms,” *Australian Journal of Intelligent Information Processing Systems*, vol. 8, no. 3, pp. 155–163, 2004.

[13] P. Baranyi, “TP model transformation as a way to LMI-based controller design,” *IEEE Transactions on Industrial Electronics*, vol. 51, no. 2, pp. 387–400, 2004.

[14] F. Kolonić, A. Poljugan, and I. Petrović, “Tensor product model transformation-based controller design for gantry crane control system–an application approach,” *Acta Polytechnica Hungarica*, vol. 3, no. 4, 2006.

[15] K. Tanaka and H. Wang, *Fuzzy control systems design and analysis: a linear matrix inequality approach*. Wiley-Interscience, 2001.

[16] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear matrix inequalities in system and control theory*. Society for Industrial Mathematics, 1994.

[17] Q. Inc, “IPO2 Rotary Inverted Pendulum Student Handout, 1996.”

[18] J. Lofberg, “YALMIP: A toolbox for modeling and optimization in MATLAB,” in *Computer Aided Control Systems Design, 2004 IEEE International Symposium on*, pp. 284–289, IEEE, 2005.

[19] Y. Labit, D. Peaucelle, and D. Henrion, “SeDuMi interface 1.02: a tool for solving LMI problems with SeDuMi,” in *Computer Aided Control System Design, 2002. Proceedings. 2002 IEEE International Symposium on*, pp. 272–277, IEEE, 2002.

[20] S. Nagy, Z. Petres, and P. Baranyi, “TP Tool-a Matlab toolbox for TP model transformation,” in *Proc. of 8th International Symposium of Hungarian Researchers on Computational Intelligence and Informatics*, pp. 483–495, 2007.

[21] P. Baranyi, Y. Yam, D. Tikk, and R. Patton, “Trade-off between approximation accuracy and complexity: TS controller design via HOSVD based complexity minimization,” *Studies in fuzziness and soft computing*, vol. 128, pp. 249–277, 2003.