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# ENTIRELY CIRCULAR QUARTICS IN THE PSEUDO-EUCLIDEAN PLANE

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**Abstract.** A curve in the pseudo-Euclidean plane is circular if it passes through at least one of the absolute points. If it does not share any point with the absolute line except the absolute points, it is said to be entirely circular.

In this paper we construct all types of entirely circular quartics by using projectively linked pencils of conics.

## 1. Introduction

The *pseudo-Euclidean plane* is a projective plane where the metric is induced by a real line $f$ and two real points $F_1$ and $F_2$ incidental with it [5]. The line $f$ is called the *absolute line* and the points $F_1$, $F_2$ are the *absolute points.* All straight lines through the absolute points are called *isotropic lines,* and all points incidental with $f$ are called *isotropic points.*

Every curve $k$ of order $n$ intersects the absolute line in $n$ points. If one of them coincides with the absolute point, the curve is said to be *circular.* If $F_1$ is an intersection point of $k$ and $f$ with the intersection multiplicity $r$ and $F_2$ is an intersection point of $k$ and $f$ with the intersection multiplicity $t$, then $k$ is said to be a curve with the *type of circularity* $(r,t)$ and its *degree of circularity* is defined as $r+t$ [4]. If $n=r+t$, the curve is *entirely circular.*

In further classification we will not distinguish the type $(r,t)$ from the type $(t,r)$ since the possibility of constructing one of them implies the possibility of constructing the other. A reflection will transform one to another.

The conics are classified in [4] into: non-circular conics (ellipses, hyperbolas, parabolas), special hyperbolas (circularity of type $(1,0)$), special parabolas (circularity of type $(2,0)$) and circles (circularity of type $(1,1)$).

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The type of circularity of the cubics can be $(1,0)$, $(2,0)$, $(1,1)$, $(3,0)$ and $(2,1)$, while the type of circularity of quartics can be $(1,0)$, $(2,0)$, $(3,0)$, $(4,0)$, $(1,1)$, $(2,1)$, $(3,1)$ and $(2,2)$.

The entirely circular cubics are classified into:

• Type of circularity $(2,1)$

$(2,1)_1$ The cubic touches the absolute line $f$ at $F_1$ and passes through $F_2$.

$(2,1)_2$ The cubic has a double point (node, isolated double point or cusp) in $F_1$ and passes through $F_2$.

• Type of circularity $(3,0)$

$(3,0)_1$ The cubic osculates $f$ at $F_1$.

$(3,0)_2$ The cubic touches $f$ at its double point $F_1$.

The *entirely circular quartics* are classified into:

• Type of circularity $(2,2)$

$(2,2)_1$ The quartic touches $f$ at $F_1$ and $F_2$.

$(2,2)_2$ The quartic has double points in $F_1$ and $F_2$.

$(2,2)_3$ The quartic touches $f$ at $F_1$ and has a double point in $F_2$.

• Type of circularity $(3,1)$

$(3,1)_1$ The quartic osculates $f$ at $F_1$ and passes through $F_2$.

$(3,1)_2$ The quartic touches $f$ at its double point $F_1$ and passes through $F_2$.

$(3,1)_3$ The quartic has a triple point in $F_1$ and passes through $F_2$.

• Type of circularity $(4,0)$

$(4,0)_1$ The quartic hyperosculates $f$ at $F_1$.

$(4,0)_2$ The quartic osculates $f$ at $F_1$ being a double point.

$(4,0)_3$ The quartic has a double point in $F_1$ at which both tangents coincide with $f$.

$(4,0)_4$ The quartic touches $f$ at its triple point $F_1$.

The aim of this paper is to construct the entirely circular quartics by using two projectively linked pencils of conics and to classify them according to their position with respect to the absolute figure. In [3] entirely circular cubics and quartics were obtained by using the automorphic inversion, introduced in [4], and where it was shown that it was possible to construct all types of circular cubics by that method, while the same did not hold for the quartics. The new results related to the previous paper [3] have been obtained.

The projectivity between two pencils of conics has already been studied in [1]. The results were stated for a projective plane and then their isotropic interpretation was given. The pseudo-Euclidean interpretation of the same projective results will be given in this paper.

## 2. Projectively linked pencils of conics

A curve of order four in the projective plane can be defined as a locus of the intersections of pairs of corresponding conics in projectively linked pencils of conics. Let $A$, $B$, $C$ and $D$ be the conics (symmetric $3\times3$-matrices) in the projective plane and let $\pi : [A, B] \mapsto [C, D]$ be the projective mapping of the conics $A + \lambda B$ to the conics $C + \lambda D$ for all $\lambda \in \mathbb{R} \cup \infty$. Some calculation delivers the following equation of the quartic $k$:

$$\mathrm{F}(\vec{x}) \equiv \vec{x}^\top B\vec{x} \cdot \vec{x}^\top C\vec{x} - \vec{x}^\top A\vec{x} \cdot \vec{x}^\top D\vec{x} = 0.$$

The quartic $k$ in algebraic sense passes through four basic points of the pencil $[A, B]$, four basic points of the pencil $[C, D]$, four intersection points of $A$ and $C$ and four intersection points of $B$ and $D$.

We will now determine the conditions which the pencils and projectivity have to fulfill in order to obtain a circular quartic of a certain type.

We will start with a point $\vec{y}$ on the quartic $k$. We can assume that it is an intersection point of the basic conics $A$ and $C$: we have $\vec{y}^\top A\vec{y} = 0$, $\vec{y}^\top C\vec{y} = 0$. The tangential behavior in this point on $k$ is usually being studied by observing the intersections of $k$ with arbitrary straight lines through $\vec{y}$. Such a line $q$ will be spanned by $\vec{y}$ and a further point $\vec{z}$. It can be parametrized by

$$q \quad \ldots \quad \vec{y} + t\vec{z}, \quad t \in \mathbb{R} \cup \infty.$$

The intersections of $k$ and $q$ belong to the zeros of the following polynomial of degree 4 in $t$:

$$\text{(1)} \quad \mathrm{p}(t) = \mathrm{F}(\vec{y} + t\vec{z}) = 2t\mathrm{F}_1(\vec{y}, \vec{z}) + t^2\mathrm{F}_2(\vec{y}, \vec{z}) + 2t^3\mathrm{F}_3(\vec{y}, \vec{z}) + t^4\mathrm{F}_4(\vec{y}, \vec{z}),$$

where

$$\mathrm{F}_1(\vec{y}, \vec{z}) = \vec{y}^\top B\vec{y} \cdot \vec{y}^\top C\vec{z} - \vec{y}^\top A\vec{z} \cdot \vec{y}^\top D\vec{y},$$

$$\mathrm{F}_2(\vec{y}, \vec{z}) = \vec{y}^\top B\vec{y} \cdot \vec{z}^\top C\vec{z} + 4\vec{y}^\top B\vec{z} \cdot \vec{y}^\top C\vec{z} - 4\vec{y}^\top A\vec{z} \cdot \vec{y}^\top D\vec{z} - \vec{z}^\top A\vec{z} \cdot \vec{y}^\top D\vec{y},$$

$$\mathrm{F}_3(\vec{y}, \vec{z}) = \vec{y}^\top B\vec{z} \cdot \vec{z}^\top C\vec{z} + \vec{z}^\top B\vec{z} \cdot \vec{y}^\top C\vec{z} - \vec{y}^\top A\vec{z} \cdot \vec{z}^\top D\vec{z} - \vec{z}^\top A\vec{z} \cdot \vec{y}^\top D\vec{z},$$

$$\mathrm{F}_4(\vec{y}, \vec{z}) = \vec{z}^\top B\vec{z} \cdot \vec{z}^\top C\vec{z} - \vec{z}^\top A\vec{z} \cdot \vec{z}^\top D\vec{z}.$$

The equation of the tangent of $k$ at the *regular (simple) point* $\vec{y}$ is $\mathrm{F}_1(\vec{y}, \vec{z}) = 0$. A necessary condition to gain $\vec{y}$ as a *double point* of $k$ is $\mathrm{F}_1(\vec{y}, \vec{z}) = 0$ for every point $\vec{z}$. The equations of the tangents of $k$ at such a double point $\vec{y}$ are determined by $\mathrm{F}_2(\vec{y}, \vec{z}) = 0$. A necessary condition to gain $\vec{y}$ as a *triple point* is $\mathrm{F}_1(\vec{y}, \vec{z}) = 0$, $\mathrm{F}_2(\vec{y}, \vec{z}) = 0$ for every point $\vec{z}$. The equations of the tangents of $k$ at such a triple point $\vec{y}$ are determined by $\mathrm{F}_3(\vec{y}, \vec{z}) = 0$.

Similar studies were done in [1] and [2] for obtaining entirely circular quartics in an isotropic and a hyperbolic plane. In this paper we continue with observations in the third projective-metric plane, a pseudo-Euclidean plane. We will use the results from [1] stated for the projective plane and give them the pseudo-Euclidean interpretation.

## 3. Entirely circular quartics in the pseudo-Euclidean plane

**Entirely circular quartics with the type of circularity (2, 2).**
$(2,2)_1$ *The quartic touches $f$ at $F_1$ and $F_2$.* It was shown in ([1], p. 39) that if $\vec{y}$ is a basic point of one of the projectively linked pencils of conics, the tangent of the quartic $k$ at $\vec{y}$ is identical with the tangent of the conic of the first pencil linked to the conic of the second pencil which passes through $\vec{y}$. So we can state:

THEOREM 1. *Let $[A,B]$, $[C,D]$ be the pencils of special parabolas in the pseudo-Euclidean plane touching the absolute line $f$ at the absolute points $F_1$, $F_2$, respectively, and let $\pi : [A,B] \mapsto [C,D]$ be the projective mapping of the conics $A + \lambda B$ to the conics $C + \lambda D$ for all $\lambda \in \mathbb{R} \cup \infty$. The result of $\pi$ is an entirely circular quartic $k$ which touches $f$ at $F_1$ and $F_2$.*

Fig. 1 displays such an entirely circular quartic $k$ which touches $f$ at both absolute points. It is generated by the projectivity linked pencils of special parabolas $[A,B]$ and $[C,D]$. The projectivity is defined by three corresponding pairs: $A \leftrightarrow C$, $B \leftrightarrow D$, $V \leftrightarrow V'$, where $V$ is a singular conic formed by two lines, and the absolute line is one of them. The constructive determination of interlinked pairs of conics of pencils $[A,B]$, $[C,D]$ uses the induced projectivity between ranges of points on the lines $p$, $p'$.

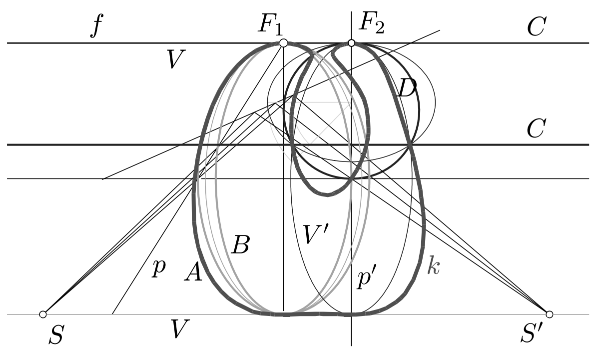


*Fig. 1*

$(2,2)_2$ *The quartic has double points in $F_1$ and $F_2$.* It was shown in ([1], p. 37) that if $\vec{y}$ is a basic point of both projectively linked pencils of conics, then it is a singular point of the quartic $k$. Thus, the following theorems are valid:

THEOREM 2. *Let $[A,B]$, $[C,D]$ be the pencils of special hyperbolas passing through the absolute point $F_1$ and let $\pi : [A,B] \mapsto [C,D]$ be the projective mapping of the conics $A + \lambda B$ to the conics $C + \lambda D$ for all $\lambda \in \mathbb{R} \cup \infty$. The result of $\pi$ is a circular quartic $k$ having a singular point in $F_1$.*

THEOREM 3. *Let $[A, B]$, $[C, D]$ be the pencils of circles and let $\pi : [A, B] \mapsto [C, D]$ be the projective mapping of the conics $A + \lambda B$ to the conics $C + \lambda D$ for all $\lambda \in \mathbb{R} \cup \infty$. The result of $\pi$ is an entirely circular quartic $k$ having double points in $F_1$ and $F_2$.*

The quartic $k$ shown in Fig. 2 is the result of projectively linked pencils of circles $[A, B]$, $[C, D]$ and therefore its type of circularity is $(2, 2)$ and it has two double points in the absolute points. The projectivity is defined by three corresponding pairs: $A \leftrightarrow C$, $B \leftrightarrow D$, $V \leftrightarrow V'$, where $V$ and $V'$ are the singular conics of the pencils.

The types $(2, 2)_1$ and $(2, 2)_2$ are the pseudo-Euclidean counterparts to the Euclidean entirely circular quartics [6].

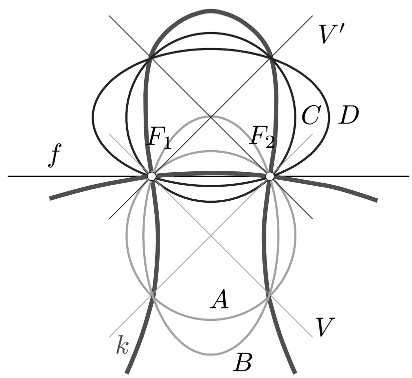


*Fig. 2*

$(2, 2)_3$ *The quartic touches $f$ at $F_1$ and has the double point in $F_2$.* By combining the statement of Theorem 2 and the statement mentioned preceding Theorem 1, we can construct the quartic touching the absolute line at one absolute point and having a double point in the other by projectively linked pencils (see Fig. 3).

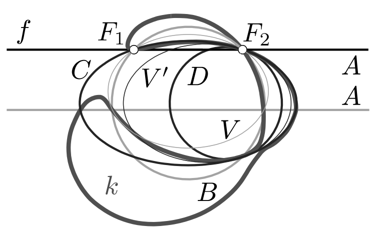


*Fig. 3*

**Entirely circular quartics with the type of circularity (3, 1).**

$(3,1)_1$ *The quartic osculates $f$ at $F_1$ and passes through $F_2$.* One of the results obtained in ([1], p. 39) is the following: If $\vec{y} \in A, B, C, \vec{y} \notin D$, and if $A$ is a singular conic, but $\vec{y}$ is not its singular point ($A = a \cup \overline{a}, \vec{y} \in a, \vec{y} \notin \overline{a}$), the tangent of $k$ at the regular point $\vec{y}$ is given by the equation $\vec{y}^\top A\vec{z} = 0$ (the equation of the line $a$). Furthermore, to gain $\vec{y}$ as the intersection of the tangent $a$ and the quartic $k$ with multiplicity 3 the following condition has to be fulfilled:

$$\mathrm{F}_1(\vec{y},\vec{z}) = 0 \quad \text{for every point} \quad \vec{z} \ \text{ of } \ a \quad (\vec{y}^\top A\vec{z} = 0).$$

It is satisfied iff one of the conics $B$, $C$ touches $a$.

A hyperosculation of $k$ and $a$ at $\vec{y}$ is characterized by the further condition:

$$\mathrm{F}_2(\vec{y},\vec{z}) = 0 \quad \text{for every point} \quad \vec{z} \ \text{ of } \ a \quad (\vec{y}^\top A\vec{z} = 0),$$

which is fulfilled iff both conics $B$, $C$ touch $a$.

By interpreting point $\vec{y}$ as one of the absolute points in the pseudo-Euclidean plane we can state the following Theorems 4 and 7.

THEOREM 4. *Let $[A,B]$ be a pencil of special parabolas touching $f$ at $F_1$ and $[C,D]$ be a pencil of conics and let $\pi : [A,B] \mapsto [C,D]$ be the projective mapping of the conics $A + \lambda B$ to the conics $C + \lambda D$ for all $\lambda \in \mathbb{R} \cup \infty$. The result of $\pi$ is a circular quartic $k$ touching $f$ at $F_1$. If a singular conic of the pencil $[A,B]$ containing $f$ as its part is mapped to a special hyperbola passing through $F_1$, the quartic $k$ is* 3*-circular osculating $f$ at $F_1$. Furthermore, if a singular conic of the pencil $[A,B]$ containing $f$ as its part is mapped to a circle of the pencil $[C,D]$, the quartic $k$ is entirely circular.*

Now we can construct the quartic shown in Fig. 4. It is the result of projectively linked pencils $[A,B]$, $[C,D]$ with the following properties: $[A,B]$ is a pencil of special parabolas and its singular conic $A$ containing $f$ as its part is linked to the circle $C$ from the second pencil.

$(3,1)_2$ *The quartic touches $f$ at its double point $F_1$ and passes through $F_2$.* In [1] we proved: If $\vec{y}$ is common basic point of the pencils $[A,B]$ and $[C,D]$ and if the conics of one pencil touch each other at $\vec{y}$, one of the tangents to the resulting quartic at that point coincides with the common tangent of the conics. The other tangent to the quartic coincides with the tangent of the conic from the second pencil that is linked to the singular conic with the singular point $\vec{y}$ from the first pencil. This fact allows us to state Theorem 5 and later Theorem 10 as well as to construct the quartics shown in Figs. 5 and 9.

THEOREM 5. *Let $[A,B]$ be a pencil of special parabolas and $[C,D]$ be a pencil of special hyperbolas with the common basic point in $F_1$ and let $\pi :$ $[A,B] \mapsto [C,D]$ be the projective mapping of the conics $A + \lambda B$ to the conics*

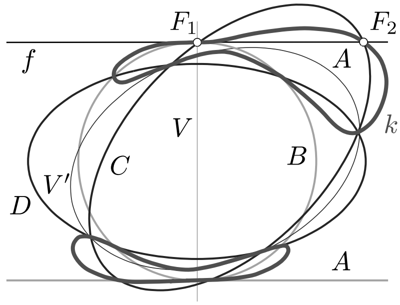


*Fig. 4*

*$C + \lambda D$ for all $\lambda \in \mathbb{R} \cup \infty$. The result of $\pi$ is a circular quartic $k$ touching $f$ at the double point $F_1$. Furthermore, if the singular conic from the pencil $[A, B]$ containing $f$ as its part is mapped to the circle from the pencil $[C, D]$, the quartic $k$ is entirely circular.*

The quartic $k$ presented in Fig. 5 is entirely circular as it is the result of the projectively linked pencil of special parabolas $[A, B]$ and the pencil of special hyperbolas $[C, D]$. The singular conic $V$ (containing $f$ as its part) is linked to the circle $V'$. The absolute point $F_1$ is a double point at which $k$ touches $f$. The other tangent at that point is the common tangent of the conics of the second pencil.

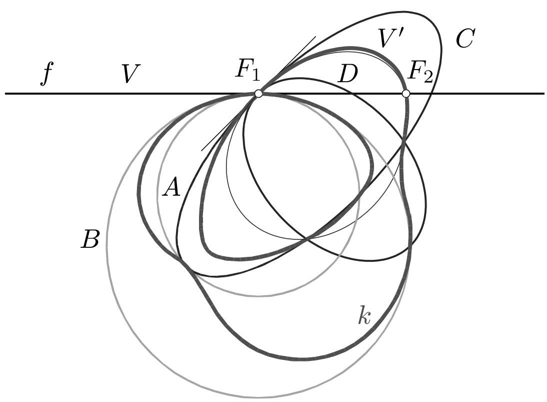


*Fig. 5*

$(3, 1)_3$ *The quartic has a triple point in $F_1$ and passes through $F_2$.* It was already stated that if $\vec{y}$ is a basic point of both projectively linked pencils of conics, it is a singular point of the quartic $k$. Furthermore, $\vec{y}$ is a triple point of the quartic $k$ iff the singular conic with the singular point $\vec{y}$ from

$[A, B]$ is mapped to the singular conic with the singular point $\vec{y}$ from $[C, D]$. If both pencils have a common tangent at $\vec{y}$, that tangent will be one of the tangents to $k$ at the triple point ([1], p. 47–48). By identifying $\vec{y}$ with one of the absolute points of the pseudo-Euclidean plane we conclude:

THEOREM 6. *Let $[A, B]$, $[C, D]$ be two pencils of conics touching at the absolute point $F_1$ and let $\pi : [A, B] \mapsto [C, D]$ be the projective mapping of the conics $A + \lambda B$ to the conics $C + \lambda D$ for all $\lambda \in \mathbb{R} \cup \infty$. The result of $\pi$ is a circular quartic $k$ having the triple point in $F_1$ iff the singular conic with the singular point $F_1$ from $[A, B]$ is mapped to the singular conic with the singular point $F_1$ from $[C, D]$. Furthermore, if a circle from $[A, B]$ is mapped to a circle from $[C, D]$, the quartic $k$ is entirely circular.*

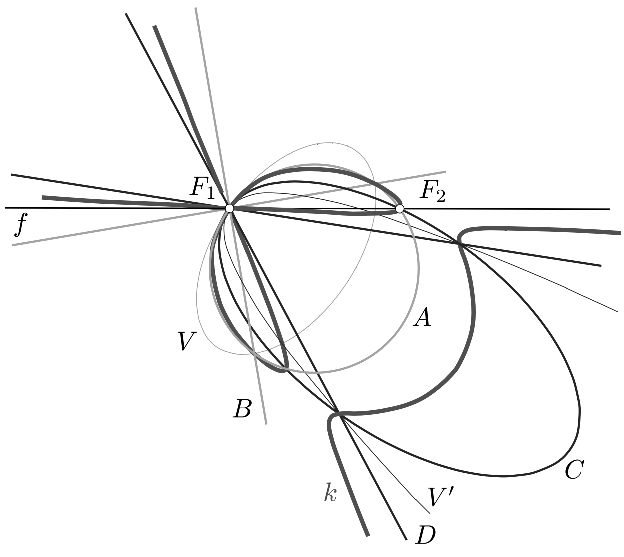


*Fig. 6*

Fig. 6 presents the entirely circular quartic $k$ having the triple point in one and passing through the other absolute point. It is generated by the projectivity mapping a pair of isotropic lines and a circle from one pencil to a pair of isotropic lines and a circle from the other, respectively. As conics from both pencils have the same tangent at the common point, that line is also the tangent to $k$. By exchanging pencils of special hyperbolas with the pencils of special parabolas, we could obtain entirely circular quartics of type $(4, 0)$ touching $f$ at the triple point, but another construction will be presented here.

**Entirely circular quartics with the type of circularity (4, 0).**
$(4,0)_1$ *The quartic hyperosculates $f$ at $F_1$.*

THEOREM 7. *Let $[A,B]$ be a pencil of special parabolas and $[C,D]$ be a pencil of conics and let $\pi : [A,B] \mapsto [C,D]$ be the projective mapping of the conics $A+\lambda B$ to the conics $C+\lambda D$ for all $\lambda \in \mathbb{R} \cup \infty$. If the singular conic from the pencil $[A,B]$ containing $f$ as its part is mapped to a special parabola touching $f$ at the same absolute point, the result of $\pi$ is an entirely circular quartic $k$ hyperosculating $f$.*

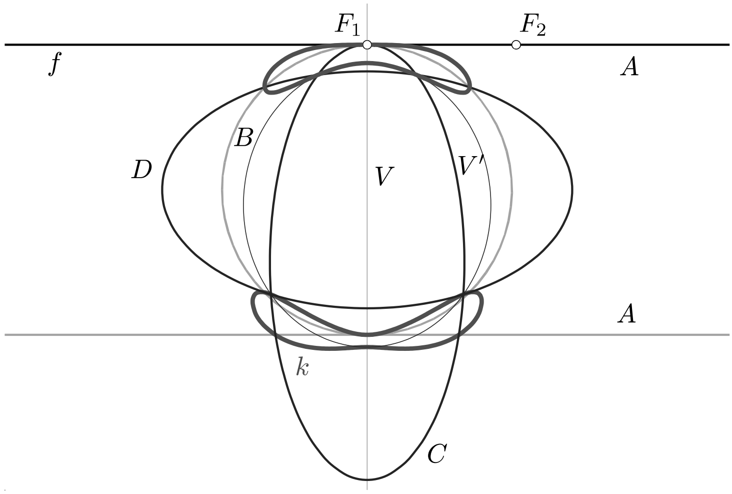


*Fig. 7*

The quartic $k$ shown in Fig. 7 hyperosculates the absolute line $f$ at the absolute point $F_1$. It is generated by the projectively linked pencils of conics $[A,B]$ and $[C,D]$ with the following properties: $[A,B]$ is a pencil of special parabolas and its singular conic $A$ is linked to a special parabola $C$.

$(4,0)_2$ *The quartic osculates $f$ at $F_1$ being a double point.* If $\vec{y}$ is a basic point of only one of the projectively linked pencils of conics $[A,B]$, $[C,D]$, e.g. $\vec{y} \in A,B,C, \vec{y} \notin D$, $\vec{y}$ is a singular point of the quartic $k$ iff the conic $A$ from the first pencil mapped to the conic $C$ from the second pencil passing through $\vec{y}$, is a singular conic with a singular point $\vec{y}$, i.e. $A = a \cup \overline{a}$, $\vec{y} \in a, \overline{a}$ ([1], p. 45). In that case $\vec{y}$ is the base point of the pencil $[A,B]$ with multiplicity 2 and therefore, conics of the pencil $[A,B]$ touch each other at $\vec{y}$. If $B$ touches $a$ (i.e. the pencil $[A,B]$ contains conics osculating each other at $\vec{y}$), one of the tangents of the quartic $k$ at the double point $\vec{y}$ coincides with $a$. This tangent osculates $k$ iff $C$ also touches $a$. Giving the pseudo-Euclidean interpretation to this observation, we get the following results:

Theorem 8. *Let $A$, $B$, $C$ be the conics passing through the absolute point $F_1$ and $D$ not passing through it and let $\pi : [A, B] \mapsto [C, D]$ be the projective mapping of the conics $A + \lambda B$ to the conics $C + \lambda D$ for all $\lambda \in \mathbb{R} \cup \infty$. The result of $\pi$ is a circular quartic $k$ having a double point in $F_1$ iff $A$ is a pair of isotropic lines.*

Theorem 9. *Let $[A, B]$ be a pencil of special parabolas osculating at the absolute point $F_1$ and $[C, D]$ be a pencil of conics and let $\pi : [A, B] \mapsto [C, D]$ be the projective mapping of the conics $A + \lambda B$ to the conics $C + \lambda D$ for all $\lambda \in \mathbb{R} \cup \infty$. The result of $\pi$ is an entirely circular quartic $k$ osculating $f$ at the double point $F_1$ iff the singular conic $A$ from $[A, B]$ is mapped to the special parabola $C$ through $F_1$ from $[C, D]$.*

This theorem provides the construction of the entirely circular quartic $k$ of type $(4, 0)$ shown in Fig. 8.

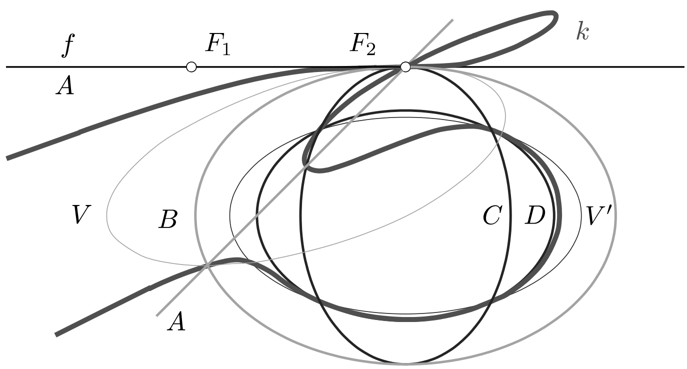


*Fig. 8*

$(4, 0)_3$ *The quartic has a double point in $F_1$ at which both tangents coincide with $f$.* The considerations for Theorem 5 can be extended to the following

Theorem 10. *Let $[A, B]$ and $[C, D]$ be pencils of special parabolas with a common basic point in $F_1$ and let $\pi : [A, B] \mapsto [C, D]$ be the projective mapping of the conics $A + \lambda B$ to the conics $C + \lambda D$ for all $\lambda \in \mathbb{R} \cup \infty$. The result of $\pi$ is an entirely circular quartic $k$ having a double point in $F_1$ at which both branches touch the absolute line $f$.*

It should be noted that the singular conics with the singular point $F_1$ should not be linked to each other as otherwise the quartic would have a triple point in $F_1$.

$(4, 0)_4$ *The quartic touches $f$ at its triple point $F_1$.* As all previous results regarding a projective plane, the following result comes from ([1], p. 47): If

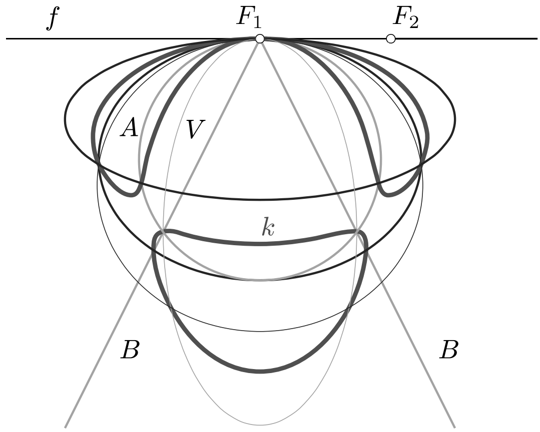


*Fig. 9*

$[A, B]$ is a pencil of degenerated conics (pairs of lines) with vertex $\vec{y}$ and $[C, D]$ is a pencil of conics touching at $\vec{y}$, the quartic $k$ has a triple point in $\vec{y}$. One of the tangents to $k$ at $\vec{y}$ is the common tangent of the conics from $[C, D]$. The other two tangents coincide with the lines from the first pencil that are linked to the singular conic with the singular point $\vec{y}$ from the second pencil. This leads us to:

THEOREM 11. *Let $[A, B]$ be a pencil of the pairs of isotropic lines and $[C, D]$ be a pencil of special parabolas through the same absolute point and let $\pi : [A, B] \mapsto [C, D]$ be the projective mapping of the conics $A + \lambda B$ to the conics $C + \lambda D$ for all $\lambda \in \mathbb{R} \cup \infty$. The result of $\pi$ is an entirely circular quartic $k$.*

Fig. 10 presents an entirely circular quartic $k$ having an intersection of multiplicity 4 with the absolute line $f$ in the absolute point $F_1$. It is generated by the projectively linked pencil of pairs of isotropic lines and a pencil of special parabolas.

## 4. Conclusion

In this article we used the results regarding the projectively linked pencils of conics in the projective plane that have already been stated in [1]. The pseudo-Euclidean interpretations of the projective situations have been presented. We have not observed all possible types of projectivity, just those sufficient for constructing all types of entirely circular quartics.

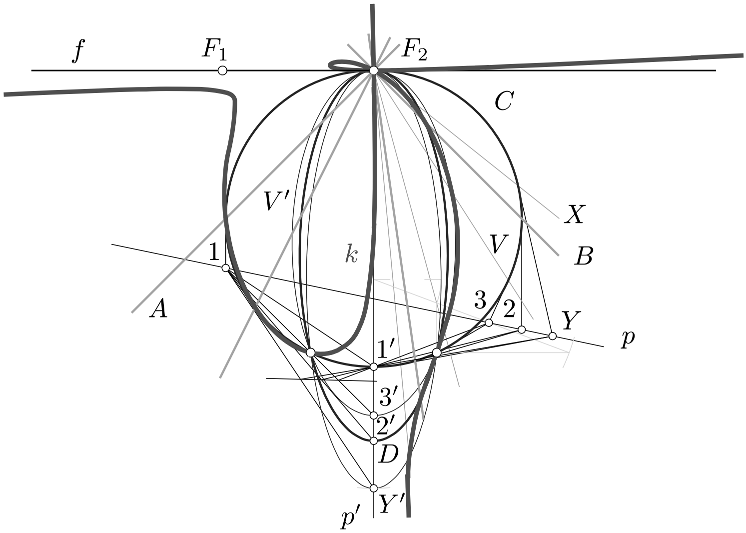


*Fig. 10*

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