Impact of Initial Conditions on the Biggest Acceptable Integration Step Used for Solving the Lorenz's Equations

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Abstract:
In the paper the model of Lorenz's equations is realized in Matlab/Simulink as an electrical circuit built using models available in SimPowerSystems Library. The simulations were carried out using five numerical methods available in Matlab/Simulink as solver option and varying the values of initial conditions for each numerical method in order to determine the biggest acceptable integration step regarding the values of initial conditions. Obtained impact of initial conditions increases with the increasing order of numerical method.

Key-Words: Lorenz's equations, Numerical method, Integration step, Initial conditions, Chaos, Attractor

1 Introduction

Lorenz's equations are comprehended as the inception of analysis of chaotic systems and, hence, investigated in detail since then [1].

The Lorenz's equations are good starting point for the analysis of chaos, because apparently simple equations reveal complex behavior including sensitivity to initial conditions and chaotic behavior in general.

Lorenz's equations describe a simple mathematical model of atmospheric instabilities:

$$\begin{aligned} \frac{dx}{dt} &= \sigma(y - x) \\ \frac{dy}{dt} &= \rho x - y - xz \\ \frac{dz}{dt} &= xy - b z \end{aligned} \tag{1}$$

where parameter values $\sigma = 10$, $\rho = 28$, and $b = 8/3$ are typical for chaotic steady-state [2, 3], and will be used in rest of the paper.

Nonlinear systems with chaotic behavior are usual in everyday world of electrical engineering. Characteristic examples are the ferroresonant circuit [4] and the DC/DC converter [5]. In order to bring the chaotic systems closer to electrical engineers it is helpful to present Lorenz's equations using equivalent electrical circuit as a simulation model [3] or physical model of Lorenz's system [6].

Here will be used a simulation model realized in Matlab/Simulink that presents Lorenz's system as an electrical circuit built using models available in SimPowerSystems Library, Fig. 1, where variables x, y and z are presented as capacitor voltages u_{C1}, u_{C2} and u_{C3}, respectively [3]. Thereby, all components are assumed to be ideal, e.g. the capacitance has no losses. Mathematical operations like addition and multiplication could be realized using electronic circuits with operational amplifiers.

One of the peculiarities of chaotic system is the sensitivity to initial conditions [7]. In the paper the sensitivity will be investigated as impact of initial conditions on the biggest acceptable integration step used for solving the equations.

2 Step of integration and initial conditions

The error of the solution of Lorenz's equations depends on the type of numerical method and on the step of integration [8].

Values calculated numerically for each moment t_0 can be comprehended as initial values for the following interval, $t > t_0$. Different methods and step of integration cause

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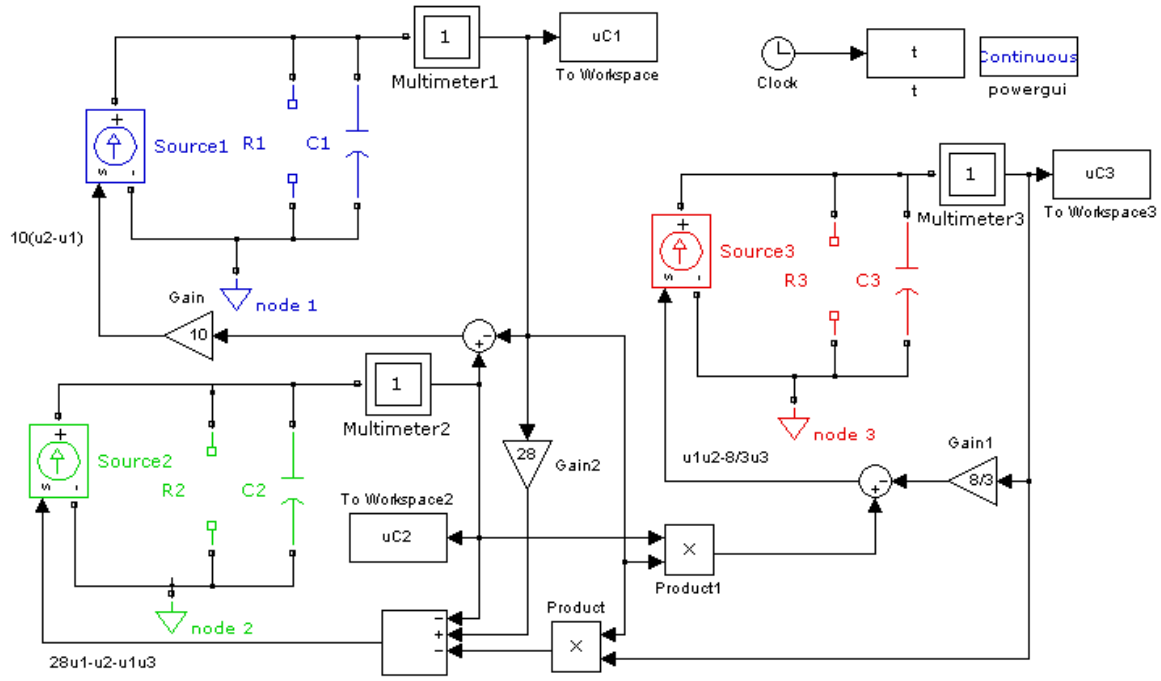

Figure 1. Lorenz's equations as a model of electrical circuit built using SimPowerSystems Library

different calculated values in each moment t_0, i.e. they cause different initial values for the following interval, $t > t_0$. This kind of comprehension of impact of numerical methods on chaotic steady-state will be analyzed in the rest of the paper.

For a chosen numerical method the increasing of integration step decreases the duration of calculation as well as the memory needs. However, too big integration step causes erroneous solution that could be unacceptable for particular application, i.e. it could be technically unacceptable.

For a chaotic steady-state the solution is ambiguous and as a criterion for a technically acceptable error will be used attractor, i.e. solution will be comprehended as correct if the steady-state solution tends to an attractor.

Simulation will be carried out for following numerical methods available in Matlab/Simulink as solver option [9]:

- Euler's method (ode1)
- Heun's method (ode2)
- Bogacki-Shanpine's method (ode3)
- Runge-Kutta's method (ode4)
- Dormand-Prince's method (ode5)

The step of integration will be varied with step of 0,010 for each of numerical methods, in order to identify for which value the error becomes technically unacceptable.

Furthermore, the solutions will be obtained for values of initial conditions varied with step 5 V within the range -50 V < $u_{Cj}(0)$< 50 V for each capacitance, j = 1, 2, 3. The chosen range of initial conditions corresponds roughly to the range of possible instantaneous values of capacitance voltages in steady-states.

For each set of parameter values the behaviour of the electrical circuit is simulated for 100 seconds long, and it is assumed that solution has reached to its steady-state values after 100 seconds at the latest.

3 Results of simulation

Figures 2 shows examples of correct and erroneous solution obtained with Bogacki-Shanpine's method (ode 3) and Runge-Kutta's method (ode4) using different values of integration step and initial conditions $u_{Cm}(0)$, m = 1, 2, 3. Namely, figures 2b) and 2d) show solutions that do not tend to the attractor and therefore are not acceptable. The erroneous solutions, as those shown on Figs. 2b) and 2d), were not calculated to the end of simulation time because the calculation stops earlier with the error message of too high values of equation variables.

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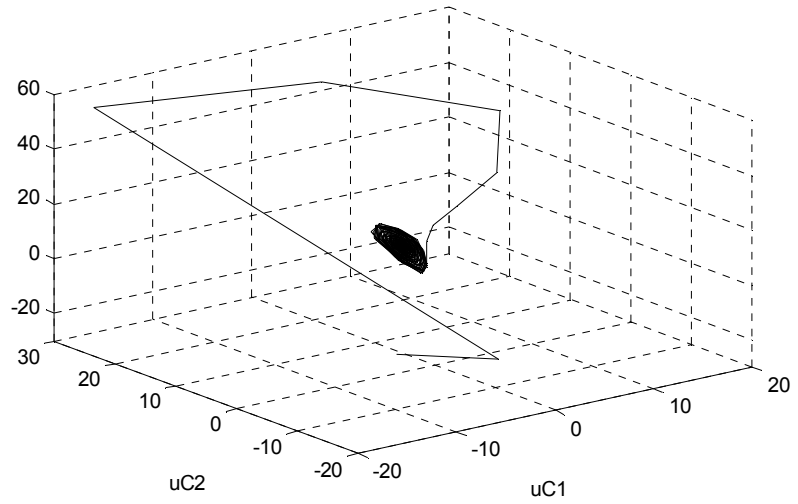
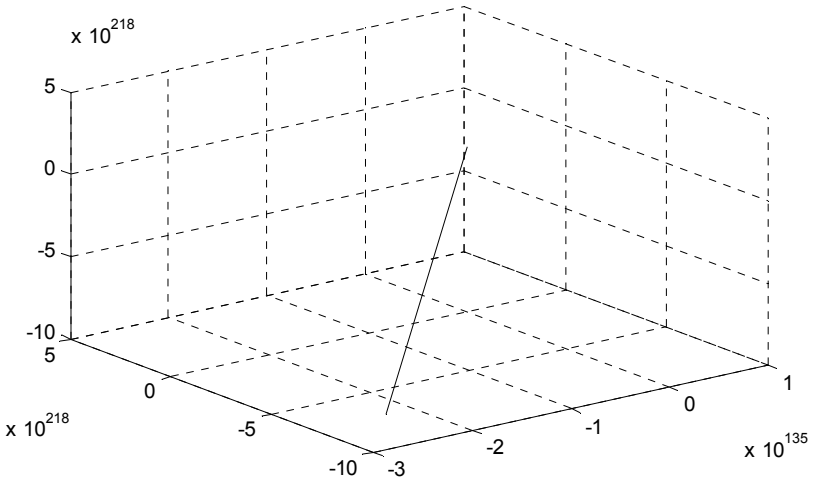
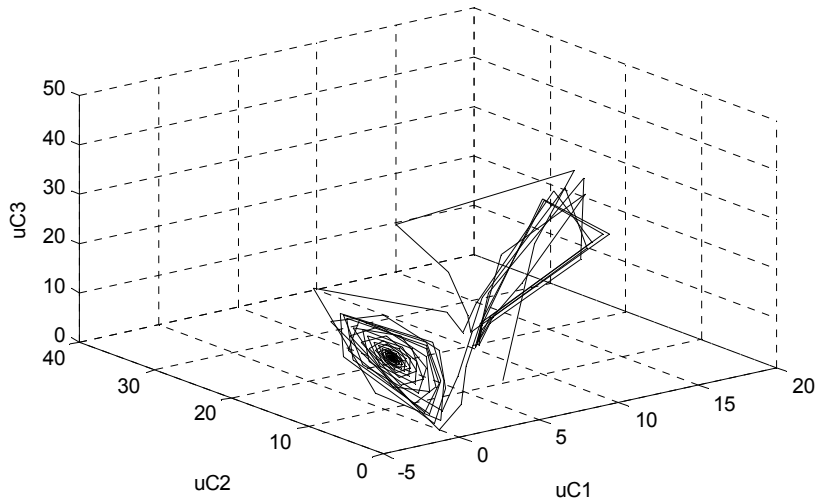


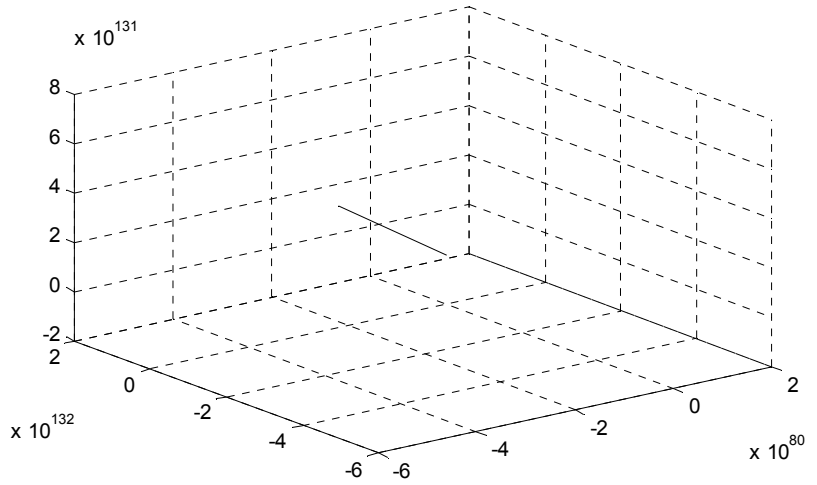
a) Bogacki-Shanpine's method (ode3); $u_{C1}(0) = u_{C2}(0) = u_{C3}(0) = -10$ V; integration step 0,080



b) Bogacki-Shanpine's method (ode3); $u_{C1}(0) = u_{C2}(0) = u_{C3}(0) = -10$ V; integration step 0,090



c) Runge-Kutta's method (ode4); $u_{C1}(0) = u_{C2}(0) = u_{C3}(0) = 5$ V; integration step 0,140



d) Runge-Kutta's method (ode4); $u_{C1}(0) = u_{C2}(0) = u_{C3}(0) = 5$ V; integration step 0,150

Figure 2. Examples of obtained solutions

Biggest integration steps as results of simulation can be presented as a 3-dimensional matrix $\boldsymbol{BIS}_{i,j,k}$ with dimension 21x21x21 for each numerical method (i, j, k = 1, 2, 3, ..., 20, 21). Each element of matrix $\boldsymbol{BIS}_{i,j,k}$ presents the value of biggest integration step that does not cause erroneous solution for initial conditions:

$$u_{C1}(0) = -50\,\text{V} + 5 \cdot (i-1)\,\text{V}$$
$$u_{C2}(0) = -50\,\text{V} + 5 \cdot (j-1)\,\text{V} \qquad (2)$$
$$u_{C3}(0) = -50\,\text{V} + 5 \cdot (k-1)\,\text{V}$$

Hence, five 3-dimensional matrices $\boldsymbol{BIS}_{i,j,k}$ comprise 5x(21x21x21)=46305 values of the biggest acceptable integration step for five used numerical methods.

Figures 3 show some of these results presented as colour coded squares where tones of grey were employed to represent different values of the biggest integration steps. The figures unveil the impact of initial conditions on the biggest acceptable integration step.

On Figures 2 colour coded squares for $i = j = 11$ are crossed because they should be neglected. Namely, for corresponding values of initial conditions $u_{C1}(0) = 0\,\text{V}$, $u_{C2}(0) = 0\,\text{V}$ the solutions are equal to zero constantly from the beginning of calculation, i.e. for these initial conditions there is no non-zero solution, and therefore there is no impact of numerical method and integration step.

Table 1. shows the range of biggest integration steps that do not cause erroneous solution for each used numerical method depending on the value of initial conditions within the range, (2). Thereby, the lower bound is the sufficient one for obtaining the correct solution independent on the values of initial conditions within the range.

Table 1: Range of biggest acceptable integration step for the range of initial conditions, (2)

Numerical method	Integration step
ode1	between 0,004 and 0,030
ode2	between 0,020 and 0,080
ode3	between 0,030 and 0,160
ode4	between 0,050 and 0,200
ode5	between 0,040 and 0,230

Using Euler's method (ode1) for some values of initial condition the solution is erroneous even for the smallest assumed integration step of 0,010. In these cases, additional simulations were carried out varying the step of integration with step of 0,001.

Results of simulation presented in Table 1 show that the range of the biggest integration step, i.e. impact of initial conditions, increases with the increasing order of numerical method.

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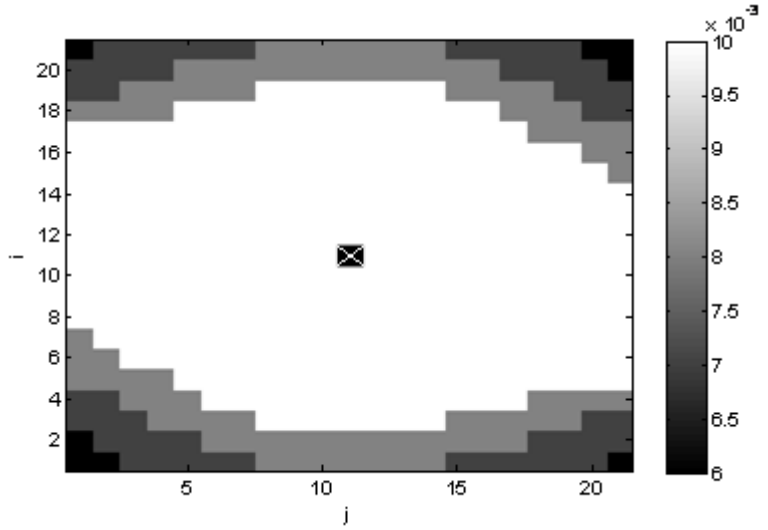
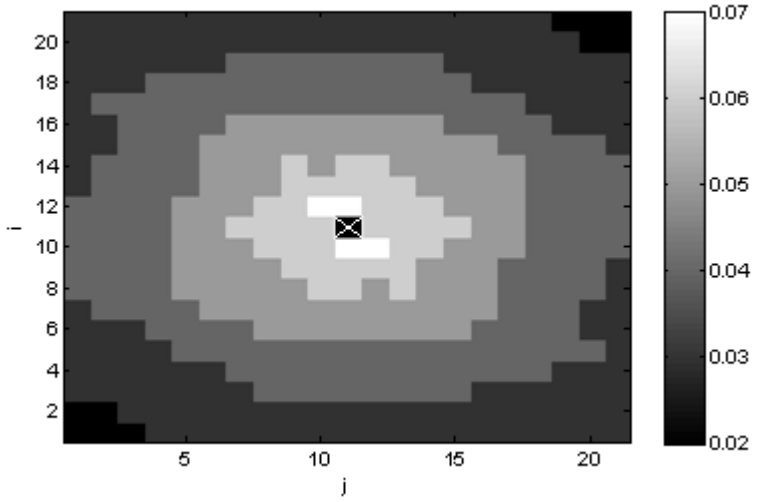
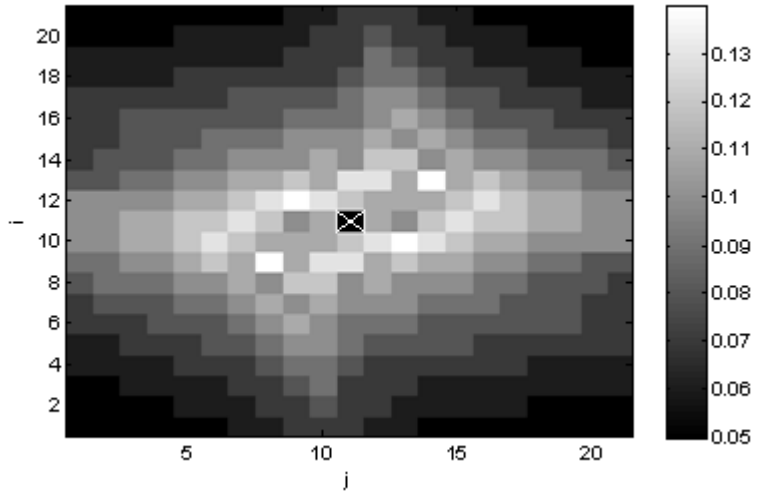


a) $\boldsymbol{BIS}_{i,i,6}$ for Euler's method (ode1)



b) $\boldsymbol{BIS}_{i,i,11}$ for Heun's method (ode2)



c) $\boldsymbol{BIS}_{i,j,21}$ for Dormand-Prince's method (ode5)

Figure 3. Examples of impact of initial conditions on the biggest acceptable integration step

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Thereby, the lower bound of the range (0,040 - 0,004 = 0,036) does not change significantly as the upper bound (0,230 - 0,030 = 0,200). As it was mentioned, the lower bound is the sufficient one and it must be used as the integration step in order to avoid the impact of initial conditions on the correctness of solution for values of initial conditions within the range, (2).

5 Conclusion

Model of Lorenz's equations is realized in Matlab/Simulink as an electrical circuit built using models available in SimPowerSystems Library.

For a chosen numerical method the increasing of integration step decreases the duration of calculation as well as the memory needs. However, too big integration step causes erroneous solution that could be technically unacceptable.

In order to determine biggest acceptable integration step regarding the values of initial conditions, the simulation was carried out varying the values of initial conditions for five numerical methods available in Matlab/Simulink as solver option.

There is a significant impact of initial conditions on the biggest acceptable integration step. The impact of initial conditions increases with the increasing of order of numerical method.

Future work will include the analysis of impact of initial conditions on technically acceptable solutions in a case of numerical methods with variable step of integration.

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