Generalization of 2D SLAM observability condition

Andreja Kitanov* Ivan Petrović*
*Faculty of Electrical Engineering and Computing, Department of Control and Computer Engineering, University of Zagreb, Zagreb, Croatia

***Abstract*— In this paper, we deduce general observability condition of all types of SLAM solutions regardless what states they consider directly in estimation, i.e. whether SLAM is feature based or pose (graph) based. This result comes from doing nonlinear observability analysis of the pose-based SLAM. We consider general vehicle motion model whose control inputs are translational and rotational velocity of the vehicle's body, and relative vehicle's pose measurements that come, for example but not necessarily, from stereo camera image registration. We conclude that, for the SLAM to be observable, one vehicle pose must be known apriori, or must be exactly reconstructable from observed features. This approach avoids extra non-singularity conditions, i.e. connection of the localization algorithm and a robot control law in observability analysis. Finally, we demonstrate the theory on the common pose estimation problem solved with the Extended Kalman Filter.**

***Index Terms*— SLAM, nonlinear observability, estimation**

I. Introduction

When Simultaneous Localization And Map building (SLAM) is considered as a control system, three of the main difficulties are interconnected: the localization of the vehicle with respect to the environment, the construction of the map of the environment itself, and the control of the vehicle to desired postures relative to the environment [2]. In literature, different approaches, considering first two aspects, have been proposed with main differences in motion and observation models they employ, states they consider directly in estimation (feature based or pose based), and representation of state uncertainty and filtering algorithm used (EKF-SLAM [13], SEIF [16], FastSLAM [11], ESDSF [5], CI-SLAM [9] etc.). However, structurally they all can be related one to another which means they all contain the same information - global map of the environment and the robot's pose in it, whether directly or not. The third aspect of SLAM is not often considered, as in most cases, vehicles are controlled in open loop. A serious flaw could be made if SLAM is assumed essentially observable like in most of the proposed solutions.

Observability of a control system is its structural property defined as being able to deduce state of the system from observing its input-output behavior. It provides understanding of the fundamental limits of every estimation method regardless of process and measurement noises. In literature, feature based 2D world centric SLAM observability analysis was done involving odometry inputs for robot speed and range/bearing measurements to features of the environment [10, 1]. Observability analysis of feature based SLAM comes to the same conditions like in this paper, but implies that robot must move on a certain observable trajectory. For example, in [1] mobile robot's pose and obstacle positions are unknown (more generally , known up to some apriori probability distribution) and the task is to reconstruct such information from angular measurements. If a robot moves along straight line which passes through 2 obstacles, it cannot localize itself.

In this paper, we generalize observability condition on the basis of observability analysis of the pose SLAM which implies that one robot's pose must be apriori known or reconstructable from measurements. It unifies all types of SLAM solutions and avoids extra non-singularity conditions, i.e. connection of the localization algorithm and a robot control law in observability analysis. Observability of the pose based 2D world centric SLAM involving odometry inputs for robot translational and rotational speed and stereo camera ego motion measurements is analyzed, with special case of 2D pose SLAM model with differentially-driven wheeled robot with stereo camera head. This kind of analysis ends up with condition for the given system to be observable.

World centric SLAM is a nonlinear and inherently coupled system. In general, control inputs to nonlinear coupled systems can not be neglected in the observability analysis contrary to the linear systems where structural properties do not depend on inputs. It was shown that the linearized feature based 2D SLAM system does not have the same structural properties as the nonlinear one [1, 10, 12], particularly, conditions which linearized SLAM observability analysis implies do not agree with common logic of possibility of reconstructing the pose by triangulation from at least two known features. Therefore, pose SLAM observability analysis must be done directly on the nonlinear model.

Our SLAM observability analysis is based on the nonlinear systems observability theory studied and described by [6, 8, 14, 15]. Applied approach is from a differential geometry point of view. Hermann and Krener [6] related the concept of observability to the concept of indistinguishability of states with respect to the inputs emphasizing effect of inputs to nonlinear systems observability. Explicit algebraic observability test is derived for the class of nonlinear systems affine in control inputs, as the SLAM systems belong to that class.

The rest of the paper is structured as follows. Section II briefly reviews observability theory of nonlinear systems affine in control inputs and Section III applies it to the observability analysis of the 2D pose SLAM. The observability conditions are demonstrated on the common pose estimation problem solved with the Extended Kalman Filter (EKF) in Section IV. Section V gives the conclusions of the paper.

II. Nonlinear systems observability

A. Control systems affine in control inputs

Here, we consider only the nonlinear Multi-Input Multi-Output (MIMO) control system Σ affine in control inputs to which pose SLAM, as will be shown later, can be reduced to:

$$\Sigma \begin{cases} \dot{x} = F(x,u) = f(x) + \sum_{i=1}^{m} g_i(x)u_i \\ y = h(x) = \begin{pmatrix} h_1(x) \\ h_2(x) \\ \vdots \\ h_p(x) \end{pmatrix}, \end{cases} \quad (1)$$

where $x \in M$ (an open subset of $\mathbb{R}^n$), is the state, $u \in \Omega \subseteq \mathbb{R}^m$ the control input and $y \in \mathbb{R}^p$ observable output of the system. $f, g_1, \ldots, g_m$ and h are smooth vector fields of adequate dimension defined on M.

To the set of all smooth vector fields $V(M)$ defined on M, the structure of a vector space over the set $\mathbb{R}$ of real numbers can be given - structure of a Lie algebra under the multiplication of vector fields f_1 and f_2 defined by their Lie bracket $[f_1, f_2]$ as:

$$[f_1, f_2] = \frac{\partial f_2}{\partial x}(x)f_1(x) - \frac{\partial f_1}{\partial x}(x)f_2(x),\ f_1, f_2 \in V(M). \quad (2)$$

$V(M)$ is an infinite dimensional real vector space whose elements are n-dimensional column vectors of real valued functions of x. Let $C^\infty(M)$ denotes the infinite dimensional real vector space of all smooth real valued functions on M. Elements of $V(M)$ act as linear operators on $C^\infty(M)$ by Lie differentiation. If $\tau \in V(M)$ and $\varphi \in C^\infty(M)$ then $L_\tau(\varphi)(x) \in C^\infty(M)$ is defined as a scalar product of the gradient of the function φ and vector τ:

$$L_\tau(\varphi)(x) = \frac{\partial \varphi}{\partial x}(x)\tau(x) = \langle d\varphi, \tau \rangle. \quad (3)$$

The gradient of the function φ is a row vector valued function:

$$d\varphi = \frac{\partial \varphi}{\partial x} = \begin{pmatrix} \frac{\partial \varphi}{\partial x_1} & \frac{\partial \varphi}{\partial x_2} & \ldots & \frac{\partial \varphi}{\partial x_n} \end{pmatrix}. \quad (4)$$

In order to study observability, we consider the smallest subspace of $C^\infty(M)$, $\mathcal{O}$, which contains output functions $h_1, \ldots, h_p$ and is closed under Lie differentiation along the vector fields $f, g_1, \ldots, g_m$. Subspace $\mathcal{O}$ will be called *observation space*. Element of $\mathcal{O}$ is a finite $\mathbb{R}$-linear combination LC of functions from the set S_0:

$$S_0 = \{\lambda \in C^\infty(M) : \lambda = h_j \text{ or } \lambda = L_{\tau_{i_1}} \ldots L_{\tau_{i_2}} h_j; \\ 1 \leqslant j \leqslant p, 1 \leqslant i_k \leqslant q, k \geqslant 1\}, \quad (5)$$

where vector fields $\{\tau_{i_k} : 1 \leqslant i_k \leqslant q\}$ belong to the set $\{f, g_1, \ldots, g_m\}$, i.e. all $\mathbb{R}$-linear combinations of the functions from the set S_0 form observation space, $LC(S_0) = \mathcal{O}$. With $\mathcal{O}$ we associate codistribution by setting

$$\Omega_{\mathcal{O}} = \text{span}\{d\lambda : \lambda \in \mathcal{O}\}. \quad (6)$$

Theorem 1: A sufficient condition for a control system Σ to be locally weakly observable on M is that

$$\dim(\Omega_{\mathcal{O}}(x)) = n \quad (7)$$

for all $x \in M$.

Proof: Proof is given in [8]. □

Above condition is called *observability rank condition*. Converse holds only partially.

Theorem 2: If control system Σ is locally weakly observable, then the observability rank condition is satisfied generically.

Proof: Proof is given in [6]. □

B. Lie algebra of analytic systems affine in control inputs

Observability of nonlinear control system does not imply that every input distinguishes points of M, but some generalizations can be drawn. Wang and Sonntag [4] proved that observation space of the control system affine in control inputs, Σ, defined in terms of piecewise constant inputs and of the same system defined in terms of analytic inputs are identical. Therefore, we can treat u as constant in testing observability rank of such systems and we can write:

$$L_F h(x) = \frac{\partial h(x)}{\partial x} F(x,u) \quad (8)$$

$$L_F\, \mathrm{dh}(x) = \mathrm{dL}_F\, h(x) = \\ = \frac{\partial h(x)}{\partial x}\frac{\partial F(x,u)}{\partial x} + F(x,u)^T \frac{\partial}{\partial x}\left(\frac{\partial h(x)}{\partial x}\right). \quad (9)$$

Analogously, differentials of repeated Lie derivatives can be given recursively [10]:

$$\mathrm{dL}_F^0\, h(x) = \frac{\partial h(x)}{\partial x} \quad (10)$$

$$\mathrm{dL}_F^i\, h(x) = \mathrm{dL}_F^{i-1}\, h(x)\frac{\partial F(x,u)}{\partial x} \ldots \\ + F(x,u)^T \frac{\partial}{\partial x}\left(\mathrm{dL}_F^{i-1}\, h(x)\right),\ i \geq 1. \quad (11)$$

In observability rank test, the rank of the linear space containing the gradients of all Lie derivatives of the output functions must be calculated. Since no bound is given for the number of Lie derivatives necessary for the calculation, the practical application of the test to other than the simplest examples is difficult. However, it can be shown that observability Lie algebra of the system Σ is spanned by gradients of the first n Lie derivatives of the output function components $d{L_F}^i h_j,\ i = 0, 1, \ldots, n-1$ for every $j = 1, 2, \ldots, p$. The system Σ satisfies the observability rank condition if any of the observability matrices are of rank n, i.e. full rank, where the observability matrices are given by [10]:

$$O_j = \begin{pmatrix} \mathrm{dL}_F^0\, h_j(x) & \mathrm{dL}_F^1\, h_j(x) & \ldots & \mathrm{dL}_F^{n-1}\, h_j(x) \end{pmatrix}^T \quad (12) \\ \forall j = 1, 2, \ldots, p.$$

It is also possible to use as an observability matrix any combination $\{i, j\}$ of n Lie derivatives $\mathrm{dL}_F^i h_j$ forming a square matrix of dimension n.

III. Observability analysis of 2D pose SLAM

In general scenario, a mobile robot autonomously navigates in an unknown (unstructured) three-dimensional space having all degrees of freedom in translational and rotational motion. For simplicity, we suppose that the environment is static because this assumption doesn't essentially change the SLAM problem. The motion of the features can easily be incorporated in the model and does not change SLAM's structural properties nor the estimation method. It, however, greatly complicates implementation because feature tracking must then be realized [17]. We only model the mobile robot's kinematics from the same reason of not influencing system's observability.

Let quaternion t denotes mobile robot's position (vector (x, y, z) in Cartesian coordinates) and quaternion q its orientation in the world coordinate system W:

$$\begin{aligned} q &= \begin{pmatrix} s & a & b & c \end{pmatrix}^T, \\ t &= \begin{pmatrix} 0 & x & y & z \end{pmatrix}^T, \end{aligned}$$

with the convention that the first component of quaternion is scalar (s and zero respectively). Let's say that we control a robot by applying inputs to its motors that generate translational velocity

$$v = \begin{pmatrix} 0 & v_x & v_y & v_z \end{pmatrix}^T$$

and rotational velocity

$$\omega = \begin{pmatrix} 0 & \omega_0 & \omega_1 & \omega_2 \end{pmatrix}^T$$

with respect to some coordinate system L attached to the robot body. Both v and ω are represented with quaternions where the vector part of quaternion corresponds to direction of motion or axis of rotation, respectively, and its length to the speed amplitude. Kinematics of such system can be described by:

$$\begin{pmatrix} 0 \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = q \begin{pmatrix} 0 \\ v_x \\ v_y \\ v_z \end{pmatrix} q^{-1} \tag{13}$$

$$\dot{q} = \frac{1}{2} q * \omega \tag{14}$$

$$\dot{q} = M \cdot q \tag{15}$$

$$\begin{pmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{\omega_0}{2} & -\frac{\omega_1}{2} & -\frac{\omega_2}{2} \\ \frac{\omega_0}{2} & 0 & \frac{\omega_2}{2} & -\frac{\omega_1}{2} \\ \frac{\omega_1}{2} & -\frac{\omega_2}{2} & 0 & \frac{\omega_0}{2} \\ \frac{\omega_2}{2} & \frac{\omega_1}{2} & -\frac{\omega_0}{2} & 0 \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix}. \tag{16}$$

In equation (14) "$*$" denotes quaternion multiplication while "$\cdot$" in eq. (15) denotes matrix multiplication which is then expanded in eq. (16). System (13)-(16) is analytic affine in control inputs for which observability rank test (12) is valid. Robot starts from some pose $x_0 = (t_0 \; q_0)^T$ and along its trajectory in time step i samples pose $x_i = (t_i \; q_i)^T$ and builds local map of the features in the environment (see Fig. 1). With stereo vision based SLAM, local map would be disparity map and pair of images taken from the left and right camera. When at least two poses are gathered, measurement can be generated by performing stereo image registration between corresponding poses. For clearness of the observability analysis, we will assume that the pose of the camera and the pose of the robot coincide. Then the measurement model is given in the form of the relative pose between robot's poses x_i and x_j in terms of rotation q_{ij}^C and

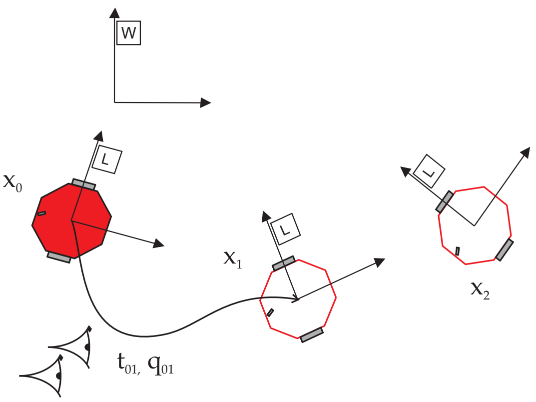

Fig. 1. Pose SLAM with stereo vision

translation t_{ij}^C of the stereo head between those poses with:

$$q_{\mathrm{ij}}^C = q_{ij} = q_i^{-1} q_j \tag{17}$$

$$t_{\mathrm{ij}}^C = t_{ij} = q_i^{-1} (t_j - t_i) q_i. \tag{18}$$

Based on the developed 3D robot's motion model, we will give in details observability analysis of 2D pose SLAM system with differentially-driven wheeled robot and stereo camera head mounted on it. We assume that the robot is moving in xOy plane of the world coordinate system W rotating only around z axis, with camera fixed to the robot. Again for clarity, we suppose that robot and camera poses coincide. Control inputs to this system are robot's body longitudinal velocity v and robot's body rotational velocity ω. With differential drive, only component of velocity v_x perpendicular to the base axle exists and the rotation is possible only around axis perpendicular to traversal plane through center of axle, i.e. $v = \begin{pmatrix} 0 & v_x & 0 & 0 \end{pmatrix}^T$ and $\omega = \begin{pmatrix} 0 & 0 & 0 & \omega_z \end{pmatrix}^T$. So, the SLAM system Σ in which the state vector consists of two 2D poses x_0 and x_1 of the trajectory and corresponding stereo vision measurement of their relative pose h_1 is:[1]

$$\begin{pmatrix} \dot{x_0} \\ \dot{x_1} \end{pmatrix} = F(x, u) = \begin{pmatrix} \mathbf{0}_{4x1} \\ v_x \left(s_1^2 - c_1^2 \right) \\ 2 v_x c_1 s_1 \\ -\frac{\omega_z c_1}{2} \\ \frac{\omega_z s_1}{2} \end{pmatrix} \tag{19}$$

$$h_1 (x_0, x_1) = \begin{pmatrix} (x_0 - x_1) c_0^2 + 2 s_0 (y_1 - y_0) c_0 + s_0^2 \; (x_1 - x_0) \\ (y_0 - y_1) c_0^2 + 2 s_0 (x_0 - x_1) c_0 + s_0^2 \; (y_1 - y_0) \\ c_0 c_1 + s_0 s_1 \\ c_1 s_0 - c_0 s_1 \end{pmatrix}. \tag{20}$$

Every observability matrix of this system is rank deficient so the system is not observable, e.g. if observability matrix O is formed like this

$$O = \begin{matrix} (\mathrm{dL}_F^0 \, x_{10} & \mathrm{dL}_F^0 \, y_{10} & \mathrm{dL}_F^0 \, s_{10} & \mathrm{dL}_F^0 \, c_{10} \; \ldots \\ \ldots \; \mathrm{dL}_F^1 \, x_{10} & \mathrm{dL}_F^1 \, y_{10} & \mathrm{dL}_F^1 \, s_{10} & \mathrm{dL}_F^1 \, c_{10} \quad)^T, \end{matrix} \tag{21}$$

where

$$\mathrm{dL}_F^0 \, x_{10} = \begin{pmatrix} c_0^2 - s_0^2 \\ -2 c_0 s_0 \\ 2 s_0 (x_1 - x_0) + 2 c_0 (y_1 - y_0) \\ 2 (c_0 (x_0 - x_1) + s_0 (y_1 - y_0)) \\ s_0^2 - c_0^2 \\ 2 c_0 s_0 \\ 0 \\ 0 \end{pmatrix}^T,$$

[1] For 2D pose x_i, only two non-zero elements of quaternion q_i, s_i and c_i, and x_i and y_i coordinates of t_i form the state vector, i.e its dimension is 4×1

$$\mathrm{dL}_F^0\, y_{10} = \begin{pmatrix} 2c_0 s_0 \\ c_0^2 - s_0^2 \\ 2\left(c_0\left(x_0 - x_1\right) + s_0\left(y_1 - y_0\right)\right) \\ 2\left(s_0\left(x_0 - x_1\right) + c_0\left(y_0 - y_1\right)\right) \\ -2c_0 s_0 \\ s_0^2 - c_0^2 \\ 0 \\ 0 \end{pmatrix}^T,$$

$$\mathrm{dL}_F^0\, s_{10} = \begin{pmatrix} 0 \\ 0 \\ s_1 \\ c_1 \\ 0 \\ 0 \\ s_0 \\ c_0 \end{pmatrix}^T, \mathrm{dL}_F^0\, c_{10} = \begin{pmatrix} 0 \\ 0 \\ c_1 \\ -s_1 \\ 0 \\ 0 \\ -c_0 \\ s_0 \end{pmatrix}^T,$$

$$\mathrm{dL}_F^1\, x_{10} = \begin{pmatrix} 0 \\ 0 \\ 2v_x\left(-s_0 c_1^2 + 2c_0 s_1 c_1 + s_0 s_1^2\right) \\ 2v_x\left(2c_1 s_0 s_1 + c_0\left(c_1^2 - s_1^2\right)\right) \\ 0 \\ 0 \\ 4v_x c_0 c_1 s_0 + 2v_x\left(s_0^2 - c_0^2\right) s_1 \\ 2v_x\left(c_1 c_0^2 + 2s_0 s_1 c_0 - c_1 s_0^2\right) \end{pmatrix}^T,$$

$$\mathrm{dL}_F^1\, y_{10} = \begin{pmatrix} 0 \\ 0 \\ 2v_x\left(2c_1 s_0 s_1 + c_0\left(c_1^2 - s_1^2\right)\right) \\ 2v_x\left(s_0 c_1^2 - 2c_0 s_1 c_1 - s_0 s_1^2\right) \\ 0 \\ 0 \\ -2v_x\left(c_1 c_0^2 + 2s_0 s_1 c_0 - c_1 s_0^2\right) \\ 4v_x c_0 c_1 s_0 + 2v_x\left(s_0^2 - c_0^2\right) s_1 \end{pmatrix}^T,$$

$$\mathrm{dL}_F^1\, s_{10} = \begin{pmatrix} 0 \\ 0 \\ -\frac{\omega_z c_1}{2} \\ \frac{\omega_z s_1}{2} \\ 0 \\ 0 \\ \frac{\omega_z c_0}{2} \\ -\frac{\omega_z s_0}{2} \end{pmatrix}^T, \mathrm{dL}_F^1\, c_{10} = \begin{pmatrix} 0 \\ 0 \\ \frac{\omega_z s_1}{2} \\ \frac{\omega_z c_1}{2} \\ 0 \\ 0 \\ \frac{\omega_z s_0}{2} \\ \frac{\omega_z c_0}{2} \end{pmatrix}^T,$$

its rank is only 4. Adding new unknown poses to the state vector, would not change the system's observability. However, the SLAM system can become observable if we know at least one trajectory pose, let's say x_2, and add a displacement measurement h_2 between that pose and some other, unknown, pose:

$$h_2\left(x_1, x_2\right) = \begin{pmatrix} \left(x_1 - x_2\right) c_1^2 + 2s_1\left(y_2 - y_1\right) c_1 + s_1^2\ \left(x_2 - x_1\right) \\ \left(y_1 - y_2\right) c_1^2 + 2s_1\left(x_1 - x_2\right) c_1 + s_1^2\ \left(y_2 - y_1\right) \\ c_1 c_2 + s_1 s_2 \\ c_2 s_1 - c_1 s_2 \end{pmatrix}. \tag{22}$$

By choosing O to be

$$O = \begin{pmatrix} \mathrm{dL}_F^0\, x_{10} & \mathrm{dL}_F^0\, y_{10} & \mathrm{dL}_F^0\, s_{10} & \mathrm{dL}_F^0\, c_{10} & \dots \\ \dots & \mathrm{dL}_F^2\, x_{21} & \mathrm{dL}_F^2\, y_{21} & \mathrm{dL}_F^1\, s_{21} & \mathrm{dL}_F^1\, c_{21} \end{pmatrix}^T, \tag{23}$$

where

$$\mathrm{dL}_F^0\, x_{10} = \begin{pmatrix} c_0^2 - s_0^2 \\ -2c_0 s_0 \\ 2s_0\left(x_1 - x_0\right) + 2c_0\left(y_1 - y_0\right) \\ 2\left(c_0\left(x_0 - x_1\right) + s_0\left(y_1 - y_0\right)\right) \\ s_0^2 - c_0^2 \\ 2c_0 s_0 \\ 0 \\ 0 \end{pmatrix}^T,$$

$$\mathrm{dL}_F^0\, y_{10} = \begin{pmatrix} 2c_0 s_0 \\ c_0^2 - s_0^2 \\ 2\left(c_0\left(x_0 - x_1\right) + s_0\left(y_1 - y_0\right)\right) \\ 2\left(s_0\left(x_0 - x_1\right) + c_0\left(y_0 - y_1\right)\right) \\ -2c_0 s_0 \\ s_0^2 - c_0^2 \\ 0 \\ 0 \end{pmatrix}^T,$$

$$\mathrm{dL}_F^2\, x_{21} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \omega_z^2\left(s_1^2 - c_1^2\right) \\ 2\omega_z^2 c_1 s_1 \\ 2\omega_z^2\left(s_1\left(x_1 - \mathrm{x}2\right) + c_1\ \left(y_1 - y_2\right)\right) \\ 2\omega_z^2\left(c_1\left(x_2 - x_1\right) + s_1\ \left(y_1 - y_2\right)\right) \end{pmatrix}^T,$$

$$\mathrm{dL}_F^2\, y_{21} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -2\omega_z^2 c_1 s_1 \\ \omega_z^2\left(s_1^2 - c_1^2\right) \\ 2\omega_z\left(2v_x s_1 c_1^2 + \omega_z\left(x_2 - x_1\right) c_1 + s_1\ \left(2v_x s_1^2 + \omega_z\left(y_1 - y_2\right)\right)\right) \\ 2\omega_z\left(2v_x c_1^3 + \left(2v_x s_1^2 + \omega_z\ \left(y_2 - y_1\right)\right) c_1 + \omega_z s_1\ \left(x_2 - x_1\right)\right) \end{pmatrix}^T,$$

$$\mathrm{dL}_F^0\, s_{10} = \begin{pmatrix} 0 \\ 0 \\ s_1 \\ c_1 \\ 0 \\ 0 \\ s_0 \\ c_0 \end{pmatrix}^T, \quad \mathrm{dL}_F^0\, c_{10} = \begin{pmatrix} 0 \\ 0 \\ c_1 \\ -s_1 \\ 0 \\ 0 \\ -c_0 \\ s_0 \end{pmatrix}^T,$$

$$\mathrm{dL}_F^1\, s_{21} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{q_2 \omega_z}{2} \\ -\frac{s_2 \omega_z}{2} \end{pmatrix}^T, \quad \mathrm{dL}_F^1\, c_{21} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{s_2 \omega_z}{2} \\ -\frac{q_2 \omega_z}{2} \end{pmatrix}^T,$$

observability rank condition is satisfied.

IV. Example

We demonstrate the observability theory on the common pose estimation problem solved with EKF. Non-holonomic differentially driven mobile robot was placed in origin of the world coordinate system $[x(0)\ y(0)\ \varphi(0)] = [0\ 0\ 0]$ and driven with constant control inputs $v_x = 1.1$ m/s and $\omega_z = 0.5$ rad/s. Robot's nonlinear kinematics model is:

$$\left.\begin{aligned} \dot{x} &= v_x \cos(\varphi) \\ \dot{y} &= v_x \sin(\varphi) \\ \dot{\varphi} &= \omega_z \end{aligned}\right\}, \tag{24}$$

where x and y denote position and φ orientation coordinate in Euler's representation. Let the measured velocity inputs, robot receives from its encoders, be corrupted with white noise. Standard deviation of the linear velocity is set to $\sigma_{v_x} = 0.15$ m/s and of the angular velocity to $\sigma_{\omega_z} = 7$ deg/s. Exact solution of the system (24) is the trajectory:

$$x(t) = x(0) + \frac{v_x}{\omega_z}\sin\left[\omega_z t + \varphi(0)\right] \tag{25}$$

$$y(t) = y(0) + \frac{v_x}{\omega_z}\left\{1 - \cos\left[\omega_z t + \varphi(0)\right]\right\} \tag{26}$$

$$s(t) = \cos\frac{\omega_z t + \varphi(0)}{2} \tag{27}$$

$$c(t) = \sin\frac{\omega_z t + \varphi(0)}{2}, \tag{28}$$

where, this time, orientation is expressed with quaternion $[0\ s(t)\ 0\ c(t)]^T$ as in theoretic model. True robot's trajectory is shown with green line in Fig. 2 and Fig 3. We have simulated several possible trajectories with given noise in velocity inputs

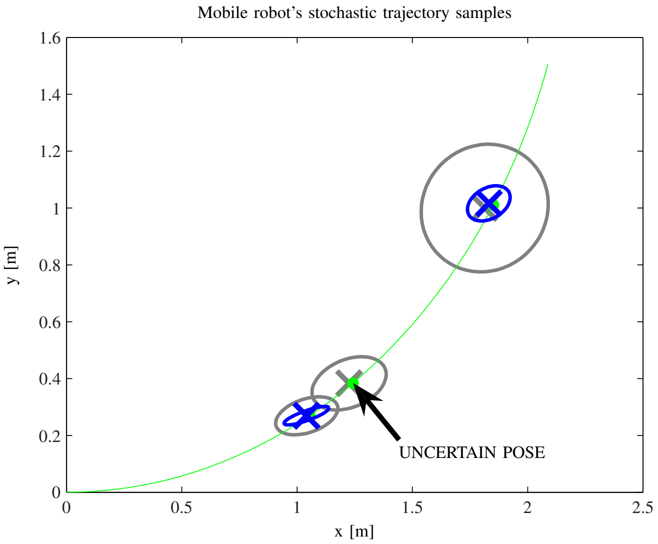

Fig. 2. CASE 1: green - true trajectory and pose samples, gray - initial mean and mean 1-sigma covariances of sampled poses, blue - mean and 1-sigma covariances of sampled poses after N measurements

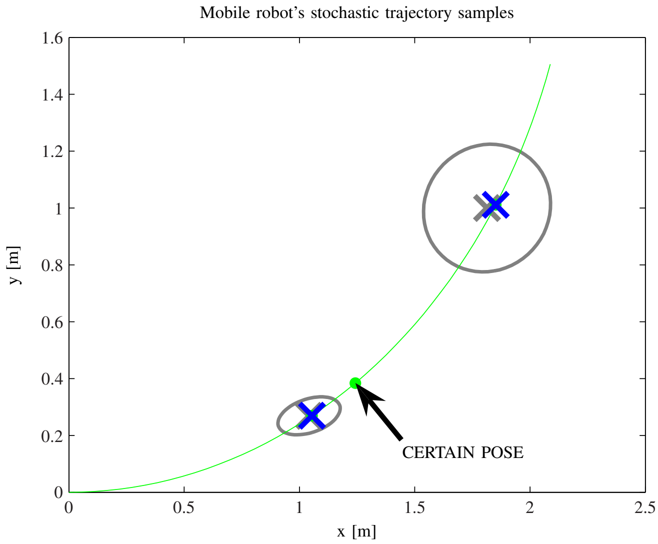

Fig. 3. CASE 2: green - true trajectory and pose samples, gray - initial mean and mean 1-sigma covariances of sampled poses, blue - mean and 1-sigma covariances of sampled poses after N measurements

using Monte-Carlo (MC) simulation and computed statistics of the robot's poses during motion in three different times. MC mean and covariances were used to correctly initialize EKF applied later for update of the robot's poses after N observations of their relative displacements (see Figs 2 and 3).

We consider two cases of the trajectory estimation of the first three sampled robot's poses:

- CASE 1: filter does not have information of any of the sampled trajectory poses, or more precisely it has information known up to some apriori distribution (Fig. 2)
- CASE 2: filter has exact information of the second robot's pose, the other two are known up to same apriori distribution as in CASE 1 (Fig. 3)

Both filters were given almost correct, near zero covariance (because of numerical stability), relative pose measurements $h_1(x_0, x_1)$ and $h_2(x_1, x_2)$ N times ($N = 100$). While in CASE 1 robot's trajectory after N EKF update iterations converges only to vicinity of the true pose, in CASE 2 it converges to true pose. This can be better seen in figures comparing position coordinates in Fig. 4 and estimated angles of the two cases in Fig. 5 (only the first estimated robot's pose x_0 is considered because similar results are obtained for other poses). Traces of the pose error covariances differ by the order of magnitude (Fig. 6) and have almost reached their saturation.

From the above example, it can be concluded that even well initialized filter with exact relative measurements is not able to observe the system trajectory if none of the poses is known. To become observable, filter must know at least one pose, i.e. at least one pose should be *grounded*. This is not a limitation as the initial robot pose can be selected as absolutely known and the environment map and robot trajectory can be built relative to it.

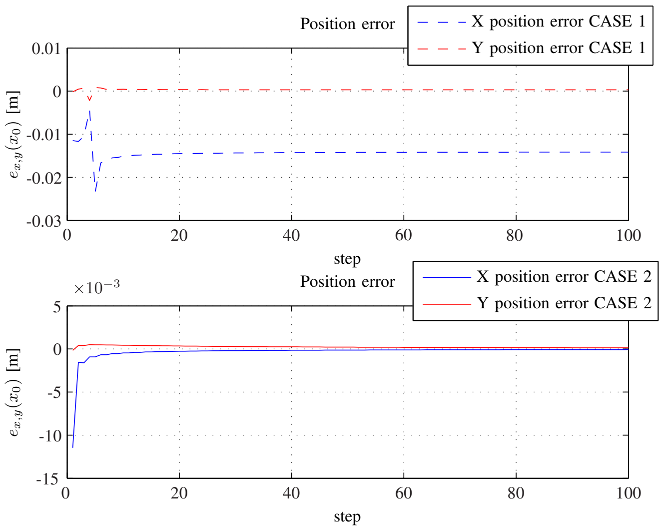

Fig. 4. Position errors

V. Conclusions

In this paper, we have derived general SLAM observability condition by applying nonlinear observability theory of control systems affine in control inputs to 2D pose SLAM. 2D pose SLAM with general kinematics model of the robot and relative pose measurements were sufficient in observability analysis of all types of 2D SLAM systems as they can all be reduced to that. Observability rank test was shown to be satisfied iff at least one known pose was given to the estimator. Pose SLAM analysis gives explicit observability condition without extra constraints on motion or relative position of robot and features. This conclusion, however, does not make SLAM infeasible, as the starting robot pose

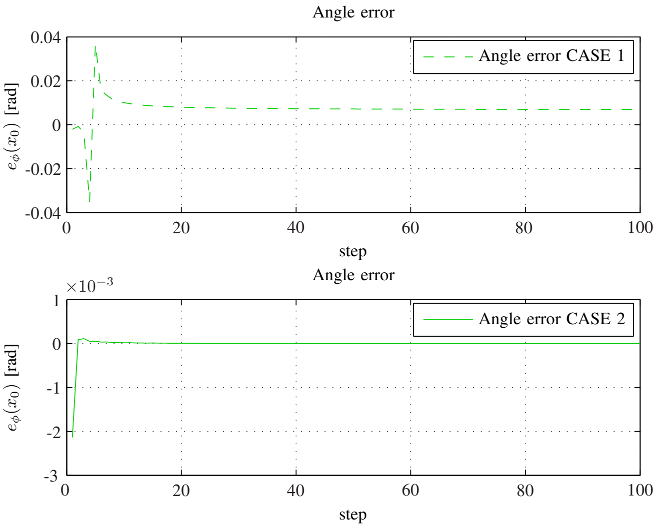

Fig. 5. Angular errors

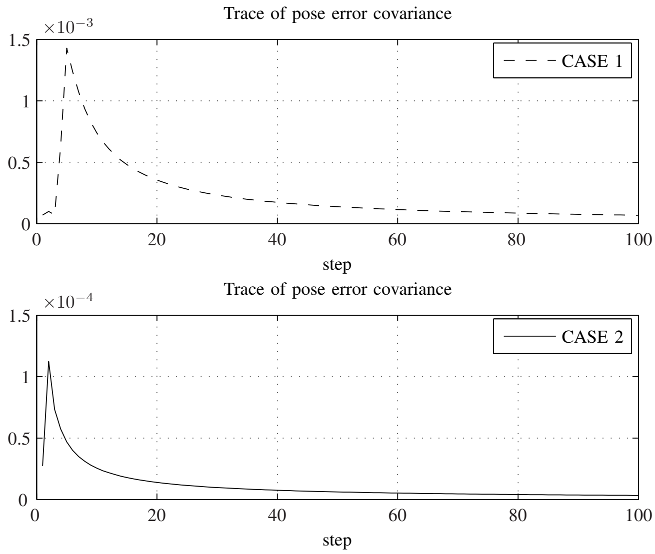

Fig. 6. Trace of the pose error covariances

could be selected as absolutely known and the map and trajectory be consistently built relative to that starting world coordinate system. Finally, the theory was demonstrated on the simulation example of pose estimation with Extended Kalman Filter (EKF) as a common way of designing an observer of nonlinear systems. Since EKF uses piecewise linear approximation of the system to observe its state and linear approximation of the SLAM is not observable under the given condition, it is reasonable to further investigate success of such approach, i.e. consistency of EKF SLAM. In [7] possible solutions were provided in which EKF linearization points are selected in a way that ensures that the resulting linearized system model has an observable subspace of appropriate dimension. Another approach would be to consider designing a nonlinear observer as in [3].

Acknowledgment

This work was supported by the Ministry of Science, Education and Sports of the Republic of Croatia under grant No. 036-0363078-3018.

References

[1] A. Bicchi, D. Prattichizzo, A. Marigo, and A. Balestrino. On the observability of mobile vehicle localization. In *Proc. of the the 6th IEEE Mediterranean Conf. on Control and Systems*, Italy, June 1998. in Invited Session on *Navigation of Mobile Robots in Unstructured Environment*.

[2] Antonio Bicchi, Federico Lorussi, Pierpaolo Murrieri, and Vincenzo Scordio. On the problem of simultaneous localization, map building, and servoing of autonomous vehicles. In *Advances in Control of Articulated and Mobile Robots*, volume 10 of *Springer Tracts in Advanced Robotics*. Springer Berlin / Heidelberg, 2004.

[3] Fabio Conticelli, Antonio Bicchi, and Aldo Balestrino. Observability and nonlinear observers for mobile robot localization. In *In IFAC Int. Symp. on Robot Control, SyRoCo 2000.*, 2000.

[4] Yuan Wang Eduardo, Yuan Wang, and Eduardo D. Sontag. On two definitions of observation spaces, 1989.

[5] R. M. Eustice, H. Singh, and J. J. Leonard. Exactly Sparse Delayed-State Filters for View-Based SLAM. *IEEE Transactions on Robotics*, 22(6):pp. 1100–1114, 2006.

[6] R. Hermann and A. Krener. Nonlinear controllability and observability. *Automatic Control, IEEE Transactions on*, 22(5):728 – 740, oct 1977.

[7] Guoquan P. Huang, Anastasios I. Mourikis, and Stergios I. Roumeliotis. Observability-based rules for designing consistent ekf slam estimators. *The International Journal of Robotics Research*, 29(5):502–528, 2010.

[8] A. Isidori. *Nonlinear control systems*. Springer Verlag, 3rd edition, 1995.

[9] Simon J. Julier and Jeffrey K. Uhlmann. Using covariance intersection for SLAM. *Robotics and Autonomous Systems*, 55(1):3 – 20, 2007. Simultaneous Localisation and Map Building.

[10] Kwang Wee Lee, W.S. Wijesoma, and I.G. Javier. On the observability and observability analysis of SLAM. In *Intelligent Robots and Systems, 2006 IEEE/RSJ International Conference on*, pages 3569 –3574, 9-15 2006.

[11] M. Montemerlo and S. Thrun. *FastSLAM: A Scalable Method for the Simultaneous Localization and Mapping Problem in Robotics*. Springer Tracts in Advanced Robotics. Springer-Verlag, 2007.

[12] L.D.L. Perera, A. Melkumyan, and E. Nettleton. On the linear and nonlinear observability analysis of the SLAM problem. In *Mechatronics, 2009. ICM 2009. IEEE International Conference on*, pages 1 –6, 14-17 2009.

[13] R. Smith, M. Self, and P. Cheeseman. Estimating uncertain spatial relationships in robotics. *Robotics and Automation. Proceedings. 1987 IEEE International Conference on*, 4:850 – 850, March 1987.

[14] E.D. Sontag and Y. Wang. I/O equations for nonlinear systems and observation spaces. In *Decision and Control, 1991., Proceedings of the 30th IEEE Conference on*, pages 720 –725 vol.1, 11-13 1991.

[15] Héctor J. Sussmann. Single-input observability of continuous-time systems. *Theory of Computing Systems*, 12(1):371–393, December 1978.

[16] S. Thrun, Y. Liu, D. Koller, A. Ng, Z. Ghahramani, and H. Durrant-Whyte. Simultaneous Localization and Mapping with Sparse Extended Information Filters. *Int. Journal of Robotics Research*, 23(7-8):pp. 693–716, July-August 2004.

[17] Chieh-Chih Wang. *Simultaneous Localization, Mapping and Moving Object Tracking*. PhD thesis, Robotics Institute, Carnegie Mellon University, Pittsburgh, PA, April 2004.