Generating functions for bi-wall directed polygons

Five-page abstract

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1 Introduction

Enumeration of polyominoes is a well-established topic in mathematics, physics and chemistry [6]. The two major open problems are how many polyominoes can be made up of n cells, and how many borders of polyominoes can be made up of m unit edges. Since both of these problems are discouragingly hard, it is a sound idea to confine attention to sets of polyominoes which do not contain all polyominoes, but anyway contain rather many of them. One example of such a set of polyominoes are directed animals. They have been enumerated by area; the area generating function is algebraic and satisfies a quadratic equation [2, 3]. On the other hand, the perimeter generating function of directed animals remains elusive. Directed animals, however, have two fairly large subsets for which the perimeter generating functions are known. Those two subsets are called column-convex directed polygons and diagonally convex directed polygons; each of them has an algebraic perimeter generating function satisfying a cubic equation (see pages 58–68 of [1] and the references given there). The area generating function of column-convex directed polygons is a rational function ([1], p. 58), while the area generating function of diagonally convex directed polygons is a q-series [4]. In this paper, we introduce a new model, called bi-wall directed polygons. The definition of a bi-wall directed polygon requires that, when the boundary of the figure is traversed clockwise (starting from the lower left corner), after the first leftward step there are no more upward steps. Bi-wall directed polygons are a subset of directed animals and a superset both of column-convex directed polygons and of diagonally convex directed polygons. Our results on bi-wall directed polygons can be seen below. The proofs of those results are somewhat too long for this five-page abstract. Of course, the journal version of the paper will include all the proofs.

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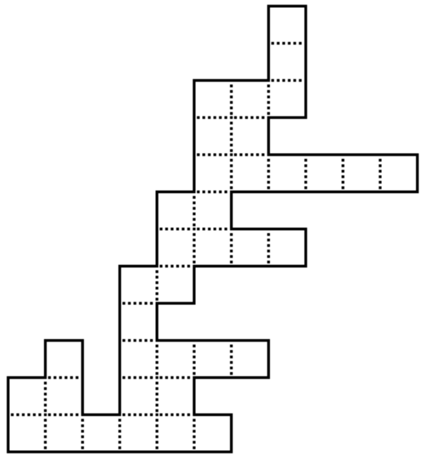

Figure 1: A bi-wall directed polygon.

2 Definitions and conventions

In this paper, a *cell* is a unit square whose vertices have integer coordinates. A plane figure P is a *polyomino* if P is a union of finitely many cells and the interior of P is connected. A polyomino P is a *polygon* if P has a connected border, and this border is a closed self-avoiding walk.

Let P and Q be two polyominoes. We consider P and Q to be equal if and only if there exists a translation f such that $f(P) = Q$.

A *directed animal* is such a polyomino P which has a cell (say a) with the following property: for every cell b of P, except for the cell a, there exist $k \in \mathbf{N}$ and cells $c_0,\ c_1, \ldots,\ c_k$ of P such that *(i)* $c_0 = a$ and $c_k = b$ and *(ii)* for $i \in \{1, \ldots,\ k\}$, the cell c_i is either the upper neighbour or the right neighbour of the cell c_{i-1}.

The above-mentioned cell a is called the *source cell* of the directed animal P.

A directed animal P is *column-convex* (resp. *diagonally convex*) if , for every $k \in \mathbf{Z}$, the intersection between P and the line $x = k + \frac{1}{2}$ (resp. $y = -x + k$) is a connected set. If a directed animal P is either column-convex or diagonally convex, then P is actually a polygon. A *wall polygon* (a.k.a. *bargraph polygon*) is a column-convex directed polygon in which all columns have the same minimal ordinate.

Let P be a directed polygon. (That is, P is both a directed animal and a polygon.) Let L be the lower left corner of the lowest cell of the leftmost column of P. By the clockwise tour of P we mean the closed walk which, after starting at L, goes along the boundary of P in the clockwise direction, and ends when the vertex L is reached again.

Let P be a directed polygon. Suppose that the clockwise tour of P has the property that, once the first leftward step has been made, upward steps occur never more. Then we call P a *bi-wall directed polygon*. See Figure 1. Obviously, every column-convex directed polygon is a bi-wall directed polygon. Every diagonally convex directed polygon is a bi-wall directed polygon, too (although this is slightly less obvious).

Let S be a set of directed animals. By the *full generating function* of S we mean the formal sum

$$\sum_{P\in S} q^{\text{area of } P} \cdot t^{\substack{\text{the length of the}\\ \text{bottom row of } P}} \cdot x^{\substack{\text{one half of the hori-}\\ \text{zontal perimeter of } P}} \cdot y^{\substack{\text{one half of the ver-}\\ \text{tical perimeter of } P}} .$$

The case $t = 1$ of the full generating function is the *perimeter and area generating function*, the case $q = 1$ of the perimeter and area generating function is the *two-variable perimeter generating function*, and the case $x = y$ of the two-variable perimeter generating function is the *one-variable perimeter generating function*. The case $x = y = 1$ of the perimeter and area generating function is the *area generating function*.

Let $A(t) = A(q, t, x, y)$ denote the full generating function for wall polygons, and let $B(t) = B(q, t, x, y)$ denote the full generating function for bi-wall directed polygons. Let $A_1 = [A(t)]_{with\ t=1}$ and $B_1 = [B(t)]_{with\ t=1}$.

3 A functional equation for $B(t)$

Using the Temperley methodology ([1], pp. 63–68), we obtained the functional equation

$$\begin{aligned} B(t) &= \frac{qtxy}{1-qtx} \\ &+ [y + qtxy + qtx \cdot A(t)] \left[\frac{qt}{1-qt} B_1 - \frac{qt(1-x)}{(1-qt)(1-qtx)} B(qt) \right] . \end{aligned} \tag{1}$$

4 The perimeter generating function for bi-wall directed polygons

The case $q = 1$ of equation (1) can be written as

$$B(t) = \frac{txy(1-t) + t(1-tx)[y + txy + tx \cdot A(t)] \cdot B_1}{(1-t)(1-tx) + t(1-x)[y + txy + tx \cdot A(t)]} . \tag{2}$$

Wall polyominoes have been enumerated by perimeter some time ago [5], so it is known that

$$A(t) = \frac{1}{2tx} \cdot \left[1 - y - tx(1+y) - \sqrt{(1-tx)^2(1-y)^2 - 4txy(1-y)} \right] . \tag{3}$$

Now we substitute (3) into (2) and then look for those values of t for which the denominator of $B(t)$ equals zero. When the equation [denominator of $B(t)$] $= 0$ is suitably rewritten and then squared, the square root vanishes and the equation takes the form

$$\begin{aligned} &(x^3 + x^2y - x^3y)t^4 - (3x^2 + x^3 + x^2y - x^3y)t^3 \\ &+(3x + 3x^2 - xy + x^2y)t^2 - (1 + 3x - y + xy)t + 1 = 0. \end{aligned} \tag{4}$$

Using the so-called *kernel argument* ([1], pp. 63–68), from (4) we obtained the quartic equation

$$\begin{aligned} &B_1^4 + 4xy \cdot B_1^3 - (3xy - 5xy^2 - 5x^2y + x^2y^2) \cdot B_1^2 \\ &-(xy^2 + x^2y - xy^3 - 6x^2y^2 - x^3y + x^2y^3 + x^3y^2) \cdot B_1 \\ &+x^2y^3 + x^3y^2 - x^3y^3 = 0, \end{aligned} \tag{5}$$

as well as the Taylor expansion

$$\begin{aligned} B_1 &= xy + xy^2 + x^2y + xy^3 + 4x^2y^2 + x^3y + xy^4 + 9x^2y^3 + 9x^3y^2 + x^4y \\ &\quad +xy^5 + 16x^2y^4 + 38x^3y^3 + 16x^4y^2 + x^5y + \dots . \end{aligned}$$

Equation (5) cannot be factored. However, the case $x = y$ of equation (5) can be written as

$$(B_1 + x)^2 \cdot [B_1^2 - (2x - 4x^2) \cdot B_1 + 2x^3 - x^4] = 0.$$

Since $B_1 \neq -x$, it must be that

$$B_1^2 - (2x - 4x^2) \cdot B_1 + 2x^3 - x^4 = 0.$$

This quadratic equation has the following implications.

Theorem 1 *(a) The one-variable perimeter generating function for bi-wall directed polygons is* $B_1 = x \cdot (1 - 2x - \sqrt{1 - 6x + 5x^2})$.
(b) For $k \in \mathbf{N}$*, the number of bi-wall directed polygons with perimeter* $2k + 4$ *is*

$$\left[x^{k+2}\right] B_1 = \frac{3^{k+1}}{k \cdot 2^k} \cdot \sum_{i=0}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k}{i} \binom{2k-2i}{k-1} \left(-\frac{5}{9}\right)^i .$$

In the case $x = y$, the Taylor expansion of B_1 is

$$\begin{aligned} B_1 &= x^2 + 2x^3 + 6x^4 + 20x^5 + 72x^6 + 274x^7 + 1086x^8 \\ &\quad +4438x^9 + 18570x^{10} + 79174x^{11} + 342738x^{12} + \dots . \end{aligned}$$

5 The area generating function for bi-wall directed polygons

The area generating function for wall polyominoes is just $\frac{q}{1-2q}$. Thus, when we set $x = y = t = 1$, equation (1) comes down to

$$B_1 = \frac{q}{1-q} + \left(1 + q + q \cdot \frac{q}{1-2q}\right) \cdot \frac{q}{1-q} \cdot B_1 ,$$

from where it follows that

$$\begin{aligned} B_1 &= \frac{q(1-2q)}{1-4q+3q^2+q^3} \\ &= q+2q^2+5q^3+13q^4+35q^5+96q^6+266q^7+741q^8 \\ &\quad +2070q^9+5791q^{10}+16213q^{11}+45409q^{12}\dots . \end{aligned}$$

The coefficients of the above Taylor expansion appear as sequence A085810 in The On-Line Encyclopedia of Integer Sequences, maintained by N. J. A. Sloane.

6 The perimeter and area generating function for bi-wall directed polygons

The perimeter and area generating function for bi-wall directed polygons is rather complicated. To save space, here we shall only state the case $y = 1$ of that generating function. In that case, we have

$$B_1 = \frac{x \cdot \sum_{i=1}^{\infty} \frac{q^{\frac{i(i+1)}{2}}(x-1)^{i-1}(1-q-q^i x)}{\left[\prod_{k=1}^{i-1}(1-q^k)(1-q^k x)\right](1-q^i x)}}{1-q-qx-\sum_{i=1}^{\infty} \frac{q^{\frac{i(i+1)}{2}}(x-1)^{i-1}(1-q-q^{i+1}x)}{\left[\prod_{k=1}^{i-1}(1-q^k)(1-q^k x)\right](1-q^i)}} .$$

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