Semiclassical limit

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Introduction

The Schrödinger equation:

$$ih\partial_t\psi = H\psi \ ,$$

where $H = -\frac{h^2}{2}\Delta + V$ is *the Schrödinger operator*.
Connection between quantuum and classical mechanics on the limit $h \to 0$.
Aim: asymptotic behaviour of the operator H.
We use *semiclassical measures*.

- introduced by Patrick Gérard
- have one characteristic lenght (*H-measures* have none)
- Luc Tartar: Variant of H-measure
- P.-L. Lions and T. Paul: *Wigner measures* (using the Wigner transform)

H-measures

- Definition
- Localisation principle

Semiclassical measures

- Construction: Tartar's concept

Semiclassical limit

- The Wigner transform
- Semiclassical limit
- Open problems

H-measures

- Luc Tartar and Patrick Gérard, around 1990
- weakly convergent sequence in L^2: $\mathsf{u}_n \xrightarrow{\mathrm{L}^2} 0$, $\mathsf{u}_n^2 \xrightarrow{*} \xi \neq 0$ in $\mathcal{M}_{\mathrm{b}}$; e.g.:

$$\sin nx \rightharpoonup 0\,, \text{ but } \sin^2 nx \xrightarrow{*} \frac{1}{2} \neq 0^2$$

- Radon measure (the limit of square terms of L^2 functions)

Theorem. (Existence of H-measures) *If $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2(\mathbf{R}^d; \mathbf{C}^r)$, then there exist a subsequence $(\mathsf{u}_{n'})$ and $\boldsymbol{\mu} \in \mathcal{M}_{\mathrm{b}}(\mathbf{R}^d \times \mathrm{S}^{d-1}; \mathrm{M}_{\mathrm{r}}(\mathbf{C}))$ such that for every $\varphi_1, \varphi_2 \in \mathrm{C}_0(\mathbf{R}^d)$ and $\psi \in \mathrm{C}(\mathrm{S}^{d-1})$ we have:*

$$\lim_{n'} \int_{\mathbf{R}^d} \widehat{\varphi_1 \mathsf{u}_{n'}}(\boldsymbol{\xi}) \otimes \widehat{\varphi_2 \mathsf{u}_{n'}}(\boldsymbol{\xi}) \psi\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right) d\boldsymbol{\xi} = \langle \boldsymbol{\mu}, \varphi_1 \bar{\varphi}_2 \boxtimes \psi \rangle$$
$$= \int_{\mathbf{R}^d \times \mathrm{S}^{d-1}} \varphi_1(\mathbf{x}) \bar{\varphi}_2(\mathbf{x}) \psi(\boldsymbol{\xi})\, d\boldsymbol{\mu}(\mathbf{x}, \boldsymbol{\xi})\ .$$

The bounded Radon measure $\boldsymbol{\mu}$ we call the H-measure *corresponding to the subsequence $(\mathsf{u}_{n'})$.*

∎

H-measures

- Luc Tartar and Patrick Gérard, around 1990
- weakly convergent sequence in L^2: $\mathsf{u}_n \xrightarrow{\mathrm{L}^2} 0$, $\mathsf{u}_n^2 \xrightarrow{\ *\ } \xi \neq 0$ in $\mathcal{M}_{\mathrm{b}}$; e.g.:

$$\sin nx \rightharpoonup 0\,, \text{ but } \sin^2 nx \xrightarrow{\ *\ } \frac{1}{2} \neq 0^2$$

- Radon measure (the limit of square terms of L^2 functions)

Theorem. (Existence of H-measures) *If* $\mathsf{u}_n \rightharpoonup 0$ *in* $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d; \mathbf{C}^r)$*, then there exist a subsequence* $(\mathsf{u}_{n'})$ *and* $\boldsymbol{\mu} \in \mathcal{M}(\mathbf{R}^d \times \mathrm{S}^{d-1}; \mathrm{M}_{\mathrm{r}}(\mathbf{C}))$ *such that for every* $\varphi_1, \varphi_2 \in \mathrm{C}_{\mathrm{c}}(\mathbf{R}^d)$ *and* $\psi \in \mathrm{C}(\mathrm{S}^{d-1})$ *we have:*

$$\lim_{n'} \int_{\mathbf{R}^d} \widehat{\varphi_1 \mathsf{u}_{n'}}(\boldsymbol{\xi}) \otimes \widehat{\varphi_2 \mathsf{u}_{n'}}(\boldsymbol{\xi}) \psi\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right) d\boldsymbol{\xi} = \langle \boldsymbol{\mu}, \varphi_1 \bar{\varphi}_2 \boxtimes \psi \rangle$$

$$= \int_{\mathbf{R}^d \times \mathrm{S}^{d-1}} \varphi_1(\mathbf{x}) \bar{\varphi}_2(\mathbf{x}) \psi(\boldsymbol{\xi})\, d\boldsymbol{\mu}(\mathbf{x}, \boldsymbol{\xi})\ .$$

The distribution of the zero order $\boldsymbol{\mu}$ *we call* the H-measure *corresponding to the subsequence* $(\mathsf{u}_{n'})$.

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Sketch of the proof

$(\widehat{\varphi_1 \mathsf{u}_{n'}}) \otimes (\widehat{\varphi_2 \mathsf{u}_{n'}})$ bounded in $\mathrm{L}^1(\mathbf{R}^d; \mathrm{M}_{\mathrm{r}}(\mathbf{C}))$, so there exists $\boldsymbol{\mu}_{\varphi_1,\varphi_2} \in \mathcal{M}_{\mathrm{b}}(\mathbf{R}^d; \mathrm{M}_{\mathrm{r}}(\mathbf{C}))$,

$$\lim_{n'} \int_{\mathbf{R}^d} (\widehat{\varphi_1 \mathsf{u}_{n'}})(\boldsymbol{\xi}) \otimes (\widehat{\varphi_2 \mathsf{u}_{n'}})(\boldsymbol{\xi}) \psi\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right) d\boldsymbol{\xi} = \langle \boldsymbol{\mu}_{\varphi_1,\varphi_2}, \psi \rangle \ .$$

Next step:

$$\Big(\exists\, \boldsymbol{\mu} \in \mathcal{M}(\mathbf{R}^d \times \mathrm{S}^{d-1}; \mathrm{M}_{\mathrm{r}}(\mathbf{C}))\Big) \quad \langle \boldsymbol{\mu}_{\varphi_1,\varphi_2}, \psi \rangle = \langle \boldsymbol{\mu}, \varphi_1 \bar{\varphi}_2 \boxtimes \psi \rangle$$

First commutation lemma

(Soboloev) multiplier M_b

$$M_b \mathsf{u}(\mathbf{x}) := b\mathsf{u}(\mathbf{x}) \ ,$$

Fourier multiplier P_a

$$\widehat{P_a \mathsf{u}}(\boldsymbol{\xi}) := a\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right)\widehat{\mathsf{u}}(\boldsymbol{\xi}) \ ,$$

where $a \in \mathrm{C}(\mathrm{S}^{d-1})$ and $b \in \mathrm{C}_0(\mathbf{R}^d)$.

Lemma. (First commutation lemma)

$$C := [P_a, M_b] = P_a M_b - M_b P_a$$

is a compact operator on $\mathrm{L}^2(\mathbf{R}^d)$ *(*$C \in \mathcal{K}(\mathrm{L}^2(\mathbf{R}^d))$*).* ∎

Localisation principle

The H-measure corresponding to a strongly convergent sequence is trivial ($\boldsymbol{\mu} = \mathbf{0}$).

Theorem. *(**Localisation principle for H-measures**) If a sequence* (u_n) *defines H-measure* $\boldsymbol{\mu}$*, and:*

$$\sum_{k=1}^{d} \partial_k(\mathbf{A}^k \mathsf{u}_n) \longrightarrow 0 \qquad \text{in } \mathrm{H}^{-1}_{\mathrm{loc}}(\Omega; \mathbf{R}^r) \ ,$$

where $\mathbf{A}^k$ *are continuous matrix functions in an open* $\Omega \subseteq \mathbf{R}^d$*, then*

$$\mathbf{P}\boldsymbol{\mu} = \mathbf{0} \ ,$$

where $\mathbf{P}(\mathbf{x}, \boldsymbol{\xi}) := \sum_{k=1}^{d} \xi_k \mathbf{A}^k(\mathbf{x})$ *on* $\mathbf{R}^d \times \mathrm{S}^{d-1}$*.* ∎

This property can give useful restrictions on components of $\boldsymbol{\mu}$.

Semiclassical measures

Motivation: v periodic, $\mathsf{u}_n(\mathbf{x}) := \mathsf{v}(\varepsilon_n \mathbf{x})$, $\varepsilon_n \searrow 0$;
All significant values of $\widehat{\varphi \mathsf{u}_n}$ are $1/\varepsilon_n$ from the origin.

- TARTAR's concept: Variant of H-measures
  $\mathsf{u}_n \rightharpoonup 0$ u $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$,

$$\mathsf{v}_n(\mathbf{x}, x^{d+1}) = \mathsf{u}_n(\mathbf{x}) e^{\frac{2\pi i x^{d+1}}{\varepsilon_n}} ,$$

$\mathsf{v}_n \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega \times \mathbf{R}; \mathbf{C})$; defines H-measure $\boldsymbol{\mu}$: *a variant of H-measures with one characteristic lenght* associated to the (sub)sequence (u_n)

Theorem. *$\boldsymbol{\mu}$ is independent of the last variable x^{d+1}.* ∎

Lemma. *If for $T \in \mathcal{D}'(\mathbf{R}^{d+1})$*

$$(\forall\, h \in \mathbf{R}) \quad \tau_{h\mathsf{e}_{d+1}} T = T ,$$

then there exists $T_0 \in \mathcal{D}'(\mathbf{R}^d)$ for which

$$\langle T, \varphi \rangle = \langle T_0, \varphi_0 \rangle ,$$

where $\varphi_0(x^1, \dots, x^d) := \int_{\mathbf{R}} \varphi(x^1, \dots, x^d, x^{d+1})\, dx^{d+1}$. ∎

Existence of semiclassical measures

$(\exists\, \boldsymbol{\mu}_0 \in \mathcal{M}(\mathbf{R}^d \times \mathrm{S}^d))$

$$\langle \boldsymbol{\mu}, \varphi \boxtimes \psi \rangle = \langle \boldsymbol{\mu}_0, \varphi_0 \boxtimes \psi \rangle \ , \quad \varphi_0(\mathbf{x}) := \int_{\mathbf{R}} \varphi(\mathbf{x}, x^{d+1})\, dx^{d+1} \ .$$

$\boldsymbol{\mu}$ does not depend on x^{d+1}!

Theorem. *If* $\mathsf{u}_n \xrightarrow{\mathrm{L}^2} 0$, *then there exist a subsequence* $(\mathsf{u}_{n'})$ *and a hermitian nonnegative Radon measure* $\boldsymbol{\mu}_{sc}$ *on* $\Omega \times \mathbf{R}^d$ *such that for every* $\varphi \in \mathrm{C}_c^\infty(\Omega)$ *and* $\psi \in \mathcal{S}(\mathbf{R}^d)$*:*

$$\lim_{n'} \int_{\mathbf{R}^d} \mathcal{F}(\varphi \mathsf{u}_{n'}) \otimes \mathcal{F}(\varphi \mathsf{u}_{n'}) \psi(\varepsilon_{n'} \boldsymbol{\xi})\, d\boldsymbol{\xi} = \langle \boldsymbol{\mu}_{sc}, |\varphi|^2 \boxtimes \psi \rangle \ .$$

∎

Comparison to variant H-measures

The variant of H-measures with one characteristic lenght and the semiclassical measure are similar, but not identical objects.

Example. $\eta_n \to 0$, $\mathsf{e} \in \mathrm{S}^{d-1}$, $u_n(\mathbf{x}) := e^{\frac{2\pi i \mathbf{x}\cdot\mathsf{e}}{\eta_n}}$
λ Lebesgue measure on $\mathbf{R}^d$

Semiclassical measure:

- *if* $\frac{\varepsilon_n}{\eta_n} \to \infty$, $\mu_{sc} = 0$,
- *if* $\frac{\varepsilon_n}{\eta_n} \to 0$, $\mu_{sc} = \lambda \boxtimes \delta_0$,
- *if* $\frac{\varepsilon_n}{\eta_n} \to \kappa \in \langle 0, \infty\rangle$, $\mu_{sc} = \lambda \boxtimes \delta_{\kappa\mathsf{e}}$.

Variant of H-measures:

- *if* $\frac{\varepsilon_n}{\eta_n} \to \infty$, $\mu = \lambda \boxtimes \delta_{\mathsf{e}}$,
- *if* $\frac{\varepsilon_n}{\eta_n} \to 0$, $\mu = \lambda \boxtimes \delta_{\mathsf{e}_{d+1}}$,
- *if* $\frac{\varepsilon_n}{\eta_n} \to \kappa \in \langle 0, \infty\rangle$, $\mu = \lambda \boxtimes \delta_{\mathsf{m}_\kappa}$, $\mathsf{m}_\kappa = \frac{\kappa\mathsf{e} + \mathsf{e}_{d+1}}{\sqrt{\kappa^2+1}}$. ∎

Compatification of $\mathbf{R}^d \setminus \{0\}$

- $\mathrm{K}_{0,\infty}(\Omega)$; Σ_0, Σ_∞
- $\mathrm{C}(\mathrm{K}_{0,\infty}(\Omega))$; $(\exists\, f_0, f_\infty \in \mathrm{C}(\mathrm{S}^{d-1}))$,

$$f(\boldsymbol{\xi}) - f_0\Big(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\Big) \to 0, \ \text{when } |\boldsymbol{\xi}| \to 0\ ,$$
$$f(\boldsymbol{\xi}) - f_\infty\Big(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\Big) \to 0, \ \text{when } |\boldsymbol{\xi}| \to \infty\ .$$

Theorem. (Existence of the variant) *Let* $\mathsf{u}_n \rightharpoonup 0$ *in* $\mathrm{L}^2(\Omega; \mathbf{C}^r)$, $\varepsilon_n \to 0$. *Then there exist a subsequence* $(\mathsf{u}_{n'})$ *and an* $r \times r$ *hermitian matrix of Radon measures* $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}(\mathbf{R}^d)}$ *on* $\Omega \times \mathrm{K}_{0,\infty}(\mathbf{R}^d)$ *such that for every* $\varphi_1, \varphi_2 \in \mathrm{C}_c^\infty(\Omega)$ *and every* $\psi \in \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$ *we have:*

$$\lim_{n'} \int_{\mathbf{R}^d} \widehat{\varphi_1 \mathsf{u}_{n'}} \otimes \widehat{\varphi_2 \mathsf{u}_{n'}} \psi(\varepsilon_{n'} \boldsymbol{\xi}) d\boldsymbol{\xi} = \langle \boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}, \varphi_1 \bar{\varphi}_2 \boxtimes \psi \rangle\ .$$

∎

Sketch of the proof

$$\mathsf{v}_n(\mathbf{x}, x^{d+1}) = \mathsf{u}_n(\mathbf{x})e^{\frac{2\pi i x^{d+1}}{\varepsilon_n}}$$

$$\Phi_j(\mathbf{x}, x^{d+1}) = \varphi_j(\mathbf{x})\varphi(x^{d+1}) \ , \ j = 1, 2,$$

$$\Psi(\boldsymbol{\xi}, \xi^{d+1}) = \psi(\frac{\boldsymbol{\xi}}{\xi_{d+1}}) \ , \ \xi_{d+1} \neq 0, \text{ and } \Psi(\boldsymbol{\xi}, 0) = \psi_\infty(\boldsymbol{\xi}), \ \boldsymbol{\xi} \neq 0 \ .$$

By the definition of H-measures:

$$\lim_{n'} \int_{\mathbf{R}^{d+1}} \widehat{\Phi_1 \mathsf{v}_{n'}}(\boldsymbol{\xi}, \xi_{d+1}) \otimes \widehat{\Phi_2 \mathsf{v}_{n'}}(\boldsymbol{\xi}, \xi_{d+1}) \Psi(\boldsymbol{\xi}, \xi_{d+1}) \, d\boldsymbol{\xi} d\xi_{d+1} = \langle \boldsymbol{\nu}, \Phi_1 \bar{\Phi}_2 \boxtimes \Psi \rangle \ .$$

$$\langle \boldsymbol{\nu}, \Phi_1 \bar{\Phi}_2 \boxtimes \Psi \rangle = \int_{\mathbf{R}} |\varphi|^2 dx^{d+1} \langle \boldsymbol{\nu}_0, \varphi_1 \bar{\varphi}_2 \boxtimes \Psi \rangle$$

$$\langle \boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}, \varphi_1 \bar{\varphi}_2 \boxtimes \psi \rangle = \langle \boldsymbol{\nu}_0, \varphi_1 \bar{\varphi}_2 \boxtimes \Psi \rangle$$

The Wigner transform

For $\mathsf{u} \in \mathrm{L}^2(\mathbf{R}^d; \mathbf{C}^r)$:

$$\mathbf{W}(\mathbf{x}, \boldsymbol{\xi}) := \int_{\mathbf{R}^d} \mathsf{u}\Big(\mathbf{x} + \frac{\mathbf{y}}{2}\Big) \otimes \mathsf{u}\Big(\mathbf{x} - \frac{\mathbf{y}}{2}\Big) e^{-2i\pi \mathbf{y}\cdot\boldsymbol{\xi}} d\mathbf{y} \ .$$

Aim: define the semiclassical measure using $\mathbf{W}_n$.

$$\mathbf{C}_n(\mathbf{x}, \mathbf{y}, \mathbf{z}) := \mathsf{u}_n(\mathbf{x} + \varepsilon_n \mathbf{y}) \otimes \mathsf{u}(\mathbf{x} + \varepsilon_n \mathbf{z}) \ , \quad \mathbf{C}_{n'} \xrightarrow{\ *\ } \mathbf{C}$$

$$(\forall\, \mathbf{h} \in \mathbf{R}^d) \qquad \tau_{0,\mathbf{h},\mathbf{h}} \mathbf{C} = \mathbf{C} \ .$$

Spliting of the space of functions:

$$\varphi_1 \sim \varphi_2 \iff (\exists\, \mathbf{h} \in \mathbf{R}^d) \quad \varphi_1 = \tau_{(\mathbf{h},\mathbf{h},0)} \varphi_2 \ .$$

$$\langle \mathbf{D}, \psi_\varphi \rangle := \langle \mathbf{C}, \varphi \rangle \ , \quad \psi_\varphi(\mathbf{x}, \mathbf{y}) := \int_{\mathbf{R}^d} \varphi(\mathbf{x}, \mathbf{y} + \mathbf{h}, \mathbf{h}) d\mathbf{h} \ .$$

$$\Big(\exists\, \boldsymbol{\mu} \in \mathcal{M}(\mathbf{R}^d \times \mathbf{R}^d; \mathrm{M_r}(\mathbf{C}))\Big) \quad \mathcal{F}_{\mathbf{y}} \mathbf{D} = \boldsymbol{\mu} \ ,$$

$$\mathbf{W}_{n'} \xrightarrow{\ *\ } \mathcal{F}_{\mathbf{y}} \mathbf{D} = \boldsymbol{\mu} \ .$$

Comparison to Wigner measure (1)

Wigner measure: $\varphi, \psi \in \mathcal{S}(\mathbf{R}^d)$,

$$\langle \boldsymbol{\mu}, |\varphi|^2 \boxtimes \psi \rangle = \lim_{n'} \int_{\mathbf{R}^d \times \mathbf{R}^d \times \mathbf{R}^d} \mathsf{u}_{n'}\Big(\mathbf{x} + \frac{\varepsilon_{n'}\mathbf{y}}{2}\Big) \otimes \mathsf{u}_{n'}\Big(\mathbf{x} - \frac{\varepsilon_{n'}\mathbf{y}}{2}\Big) e^{-2\pi i \mathbf{y}\cdot\boldsymbol{\eta}} \\ |\varphi(\mathbf{x})|^2 \psi(\boldsymbol{\eta})\, d\mathbf{x} d\mathbf{y} d\boldsymbol{\eta}\ .$$

Semiclassical measure: $\varphi, \psi \in \mathrm{C}_0(\mathbf{R}^d)$,

$$\begin{aligned} \langle \boldsymbol{\mu}_{sc}, |\varphi|^2 \boxtimes \psi \rangle &= \lim_{n'} \int_{\mathbf{R}^d} \mathcal{F}(\varphi \mathsf{u}_{n'})(\boldsymbol{\xi}) \otimes \mathcal{F}(\varphi \mathsf{u}_{n'})(\boldsymbol{\xi}) \psi(\varepsilon_{n'}\boldsymbol{\xi}) d\boldsymbol{\xi} \\ &= \lim_{n'} \int_{\mathbf{R}^d \times \mathbf{R}^d \times \mathbf{R}^d} e^{-2\pi i \mathbf{y}\cdot\boldsymbol{\eta}} \varphi\Big(\mathbf{x} + \frac{\varepsilon_{n'}\mathbf{y}}{2}\Big) \overline{\varphi\Big(\mathbf{x} - \frac{\varepsilon_{n'}\mathbf{y}}{2}\Big)} \mathsf{u}_{n'}\Big(\mathbf{x} + \frac{\varepsilon_{n'}\mathbf{y}}{2}\Big) \\ &\qquad \otimes \mathsf{u}_{n'}\Big(\mathbf{x} - \frac{\varepsilon_{n'}\mathbf{y}}{2}\Big) \psi(\varepsilon_{n'}\boldsymbol{\xi}) d\mathbf{x}' d\mathbf{y}' d\boldsymbol{\xi}\ , \end{aligned}$$

Comparison to Wigner measure (1)

Variant: $\varphi, \psi \in \mathcal{S}(\mathbf{R}^d)$,

$$\langle \boldsymbol{\mu}, |\varphi|^2 \boxtimes \psi \rangle = \lim_{n'} \int_{\mathbf{R}^d \times \mathbf{R}^d \times \mathbf{R}^d} \mathsf{u}_{n'}\Big(\mathbf{x} + \frac{\varepsilon_{n'}\mathbf{y}}{2}\Big) \otimes \mathsf{u}_{n'}\Big(\mathbf{x} - \frac{\varepsilon_{n'}\mathbf{y}}{2}\Big) e^{-2\pi i \mathbf{y}\cdot\boldsymbol{\eta}} |\varphi(\mathbf{x})|^2 \psi(\boldsymbol{\eta})\, d\mathbf{x} d\mathbf{y} d\boldsymbol{\eta}\ .$$

Semiclassical measure: $\varphi, \psi \in \mathrm{C}_0(\mathbf{R}^d)$,

$$\begin{aligned}\langle \boldsymbol{\mu}_{sc}, |\varphi|^2 \boxtimes \psi \rangle &= \lim_{n'} \int_{\mathbf{R}^d} \mathcal{F}(\varphi \mathsf{u}_{n'})(\boldsymbol{\xi}) \otimes \mathcal{F}(\varphi \mathsf{u}_{n'})(\boldsymbol{\xi}) \psi(\varepsilon_{n'}\boldsymbol{\xi}) d\boldsymbol{\xi} \\ &= \lim_{n'} \int_{\mathbf{R}^d \times \mathbf{R}^d \times \mathbf{R}^d} e^{-2\pi i \mathbf{y}\cdot\boldsymbol{\eta}} \varphi\Big(\mathbf{x} + \frac{\varepsilon_{n'}\mathbf{y}}{2}\Big) \overline{\varphi\Big(\mathbf{x} - \frac{\varepsilon_{n'}\mathbf{y}}{2}\Big)} \mathsf{u}_{n'}\Big(\mathbf{x} + \frac{\varepsilon_{n'}\mathbf{y}}{2}\Big) \\ &\qquad \otimes \mathsf{u}_{n'}\Big(\mathbf{x} - \frac{\varepsilon_{n'}\mathbf{y}}{2}\Big) \psi(\varepsilon_{n'}\boldsymbol{\xi}) d\mathbf{x}' d\mathbf{y}' d\boldsymbol{\xi}\ ,\end{aligned}$$

$\mathbf{y}$ unbounded!

Comparison to Wigner measure (2)

$\varphi \in \mathrm{C}_c^\infty(\mathbf{R}^d)$, $\mathsf{u}_n \leftrightarrow \varphi_0 \mathsf{u}_n$

$$\mathbf{C}_n(\mathbf{x},\mathbf{y},\mathbf{z}) = \mathsf{u}_n(\mathbf{x}+\varepsilon_n\mathbf{y}) \otimes \mathsf{u}(\mathbf{x}+\varepsilon_n\mathbf{y})$$

$$\mathbf{C}_n \xrightharpoonup{*} \mathbf{C}$$

$$\langle |\varphi_0|^2\mathbf{D}, \psi_\varphi\rangle := \langle |\varphi_0|^2\mathbf{C}, \varphi\rangle \ , \quad \mathcal{F}_{\mathbf{y}}|\varphi_0|^2\mathbf{D} = |\varphi_0|^2\mathcal{F}_{\mathbf{y}}\mathbf{D} = |\varphi_0|^2\boldsymbol{\mu} \ ,$$

and

$$\boldsymbol{\mu}_{sc} \leftrightarrow |\varphi_0|^2\boldsymbol{\mu}_{sc} \ .$$

Take φ such that $\operatorname{supp}\varphi \subseteq \operatorname{supp}\varphi_0$ and we get

$$(\forall\, \varphi_0 \in \mathrm{C}_c^\infty(\mathbf{R})^d)(\forall\, \psi \in \mathcal{S}(\mathbf{R}^d)) \quad \langle \boldsymbol{\mu}, |\varphi_0|^2 \boxtimes \psi\rangle = \langle \boldsymbol{\mu}_{sc}, |\varphi_0|^2 \boxtimes \psi\rangle$$

$$\Longrightarrow \quad \boldsymbol{\mu} = \boldsymbol{\mu}_{sc}$$

Semiclassical limit

Theorem. *Let* (u_n) *be a sequence of solutions*

$$\partial_t u_n + i\kappa\varepsilon_n \Delta u_n = f_n \ ,$$

and assume $f_n \to 0$ *in* $\mathrm{L}^2_{\mathrm{loc}}(\langle 0,T\rangle \times \Omega)$, $\varepsilon_n \to 0$, $u_n \rightharpoonup 0$ *in* $\mathrm{L}^2_{\mathrm{loc}}(\langle 0,T\rangle \times \Omega)$.
Then the corresponding semiclassical measure μ_{sc} *of the sequence* (u_n) *satisfies*

$$\Big(\partial_t + 4\pi\kappa \sum_{j=1}^{d} \xi_j \partial_{x^j}\Big)\mu_{sc} = 0 \ .$$

∎

Proof.

$$\partial_t \overline{u_n(t,\mathbf{x})} - i\kappa\varepsilon_n \Delta \overline{u_n(t,\mathbf{x})} = \overline{f_n(t,\mathbf{x})} \ ,$$
$$\partial_t u_n(t,\mathbf{x}+\varepsilon_n\mathbf{y}) + i\kappa\varepsilon_n \Delta u_n(t,\mathbf{x}+\varepsilon_n\mathbf{y}) = f_n(t,\mathbf{x}+\varepsilon_n\mathbf{y}) \ .$$

$$\partial_{y^j} u_n(t,\mathbf{x}+\varepsilon_n\mathbf{y}) = \varepsilon_n \partial_{x^j} u_n(t,\mathbf{x}+\varepsilon_n\mathbf{y})$$
$$\partial_{y^j} u_n(t,\mathbf{x}) = 0 \ .$$

Schrödinger equation (1)

$$\Big(\partial_t - i\kappa\varepsilon_n\Delta + 2i\kappa\sum_{j=1}^{d}\partial_{x^j}\partial_{y^j}\Big)(u_n(t,\mathbf{x}+\varepsilon_n\mathbf{y})\overline{u_n(t,\mathbf{x})}) \to 0 \ ,$$

in $\mathrm{L}^2_{\mathrm{loc}}(\langle 0,T\rangle\times\Omega\times\mathbf{R}^d)$ strong.
Passing to the limit:

$$\Big(\partial_t + 2i\kappa\sum_{j=1}^{d}\partial_{x^j}\partial_{y^j}\Big)\mathcal{F}_{\boldsymbol{\xi}}\mu_{sc} = 0 \ ,$$

i.e.

$$\Big(\partial_t + 4\pi\kappa\sum_{j=1}^{d}\xi_j\partial_{x^j}\Big)\mu_{sc} = 0 \ .$$

Q.E.D.

Schrödinger equation (2)

$V \in \mathrm{C}^1(\mathbf{R}^d;\mathbf{R})$, $\psi_0^h \in \mathrm{L}^2(\mathbf{R}^d)$; Cauchy problem

$$\begin{cases} ih\partial_t\psi^h = -\frac{h^2}{2}\Delta\psi^h + V\psi^h \\ \psi^h(0,\cdot) = \psi_0^h \in \mathrm{L}^2(\mathbf{R}^d) \end{cases},$$

has the unique solution $\psi^h \in \mathrm{C}(\mathbf{R};\mathrm{L}^2(\mathbf{R}^d))$ satisfying $\|\psi^h(t,\cdot)\|_{\mathrm{L}^2} = \|\psi_0^h\|_{\mathrm{L}^2}$.

Theorem. *The corresponding semiclassical measure μ_{sc} of the sequence of solutions (ψ^h) satisfies*

$$\begin{cases} \left(\partial_t + \boldsymbol{\xi}\cdot\nabla_{\mathbf{x}} - \frac{1}{2\pi}\nabla_{\mathbf{x}}V\cdot\nabla_{\boldsymbol{\xi}}\right)\mu_{sc} = 0 \\ \mu_{sc}{}_{|_{t=0}} = \mu_{sc}^0 \end{cases},$$

where μ_{sc}^0 is a semiclassical measure associate to the sequence of initial values (ψ_0^h).

∎

Proof.

$$\kappa = -1\,, \quad \varepsilon_n = \frac{h}{2}\,, \quad u_n := \psi^h\,, \quad f_n = 0\,,$$

$$\partial_t u_n + i\varepsilon_n\Delta u_n + \frac{i}{2\varepsilon_n}Vu_n = 0$$

Schrödinger equation (3)

$\varphi \in \mathrm{C}_c^\infty(\Omega)$,

$$i\int_{\mathbf{R}^d} \frac{V(\mathbf{x}+\varepsilon_n\mathbf{y})-V(\mathbf{x})}{\varepsilon_n}\varphi(\mathbf{x})u_n(t,\mathbf{x}+\varepsilon_n\mathbf{y})\overline{u_n(t,\mathbf{x})}\,d\mathbf{x}$$

$$\to i\langle \nabla_{\mathbf{y}} V \mathcal{F}_{\boldsymbol{\xi}}\mu_{sc}, \varphi\rangle$$

$$i\nabla V_{\mathbf{y}}\mathcal{F}_{\boldsymbol{\xi}}\mu_{sc} = \nabla V\cdot \mathbf{y}\mathcal{F}_{\boldsymbol{\xi}}\mu_{sc} = \frac{1}{2\pi}\sum_{k=1}^{d}\partial_k V\mathcal{F}_{\boldsymbol{\xi}}(\partial_{y^k}\mu_{sc}) = \frac{1}{2\pi}\nabla_{\mathbf{x}}V\cdot\nabla_{\mathbf{y}}\mu_{sc}$$

Two characteristic lenghts

This model can not be implemented to the problems with two or more characteristic lenghts:

$$u_n(x) := \begin{cases} \sqrt{n} & , \quad \frac{k}{n} < x < \frac{k}{n} + \frac{1}{n^2} \\ 0 & , \quad \text{otherwise} \end{cases} ,$$

For the characteristic lenght:

$$\varepsilon_n = \frac{1}{n} , \qquad \mu_{sc} = 0 ,$$
$$\varepsilon_n = \frac{1}{n^2} , \quad \mu_{sc} = \frac{\sin^2(\pi\xi)}{\pi^2\xi^2}\lambda .$$