

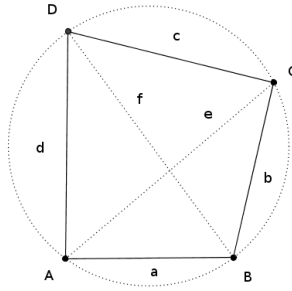
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**Intrinsic geometry
of cyclic heptagons/octagons
via new Brahmagupta's formula**

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0 Preliminaries



We first recall some basic formulas for cyclic quadrilaterals $ABCD$ with sides and diagonals of lengths $a = |AB|$, $b = |BC|$, $c = |CD|$, $d = |DA|$ and $e = |AC|$, $f = |BD|$ whose vertices lie on a circle of radius R .

- **Ptolemy's conditions** (convex case):

$$ef = ac + bd \quad (0.1)$$

- **Dual Ptolemy's conditions:**

$$(ab + cd)e = (ad + bc)f \quad (0.1')$$

- **Diagonal equation:**

$$(ab + cd)e^2 = (ac + bd)(ad + bc) \quad (0.2)$$

- **Area equation** (Brahmagupta's formula):

$$16S^2 = 2(a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2) - a^4 - b^4 - c^4 - d^4 + 8abcd \quad (0.3)$$

which, in a more popular form reads as

$$16S^2 = (-a+b+c+d)(a-b+c+d)(a+b-c+d)(a+b+c-d) \quad (0.3')$$

- **Circumradius equation:**

$$R^2 = \frac{(ab + cd)(ac + bd)(ad + bc)}{(-a + b + c + d)(a - b + c + d)(a + b - c + d)(a + b + c - d)} \quad (0.4)$$

- **Area times circumradius equation:**

let $Z = 4SR$, then

$$Z^2 = (ab + cd)(ac + bd)(ad + bc) \quad (0.5)$$

which in a case of a triangle ($d = 0$) reduces to the well known relation

$$4SR = abc. \quad (0.5')$$

The following lemma will be crucial in all our subsequent calculations concerning elimination of diagonals in cyclic polygons.

Key Lemma: (Intermediate Brahmagupta's formula)

In any cyclic quadrilateral we have

$$8Sd_a = 2bcd + (b^2 + c^2 + d^2 - a^2)a \quad (0.6)$$

where d_a denotes the distance from the center of the circumcircle to the side of length a .

Proof of the Key Lemma. By noticing that $d_a^2 = R^2 - (a/2)^2$ and using both Brahmagupta's formulas (0.3) and (0.4) we easily check the corresponding identity

$$64S^2d_a^2 = (2bcd + (b^2 + c^2 + d^2 - a^2)a)^2.$$

In the case of a nonconvex quadrilaterals we can formally obtain all the relations by simply allowing side lengths to be negative (e.g. by replacing a with $-a$).

Corollary 0.1 *From Lemma we get a new Brahmagupta formula*

$$16SS_a = a^2(b^2 + c^2 + d^2 - a^2) + 2abcd$$

where S_a denotes the area of a characteristic triangle $\triangle OAB$ determined by the side AB (of length a) and circumcenter O of a cyclic quadrilateral $ABCD$.

Note that by adding all four such formulas we get the original Brahmagupta's formula.

For general quadrilaterals in a plane we have:

- **Bretschneider's formula (1842) or Staudt's formula (1842):**

$$16S^2 = 4e^2f^2 - (a^2 - b^2 + c^2 - d^2)^2. \quad (0.7)$$

For cyclic quadrilaterals, in view of (0.1), it gives another form of (0.3):

$$16S^2 = 4(ac + bd)^2 - (a^2 - b^2 + c^2 - d^2)^2. \quad (0.3'')$$

The formula (0.7) is the simplest formula for the area of the quadrilaterals in terms of its sides and diagonals. But there are infinitely many other ways to do so, since these 6 quantities satisfy Euler's four point relation

$$\begin{aligned} e^2f^2(a^2 + b^2 + c^2 + d^2 - e^2 - f^2) = \\ = e^2(a^2 - b^2)(d^2 - c^2) + f^2(a^2 - d^2)(b^2 - c^2) + \\ + (a^2 - b^2 + c^2 - d^2)(a^2c^2 - b^2d^2) \end{aligned} \quad (0.8)$$

This is only a quadratic equation with respect to a square of each parameter.

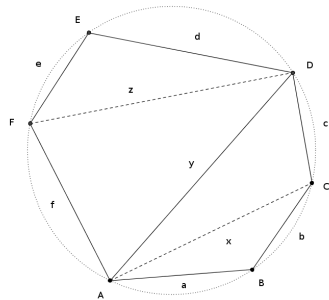
The Euler's four point relation follows from the Cayley–Menger determinant for the volume V of a tetrahedron with edges of lengths a, b, c, d, e, f if we set $V = 0$.

Remark 0.2 *In a solution of a problem by J.W.L.Glaisher: With four given straight lines to form a quadrilateral inscribable in a circle, A.Cayley (in 1874.) observed the following identity, equivalent to (0.8):*

$$\begin{aligned} & [(a^2 + b^2 + c^2 + d^2 - e^2 - f^2)(ef + ac + bd) - 2(ad + bc)(ab + cd)] (ef - ac - bd) = \\ & = [(ab + cd)e - (bc + ad)f]^2 \end{aligned} \tag{0.8'}$$

which directly shows that Ptolemy's relation (0.1) implies the dual Ptolemy's relation (0.1').

1 Cyclic hexagons



In this section we study geometry of cyclic hexagons $ABCDEF$ with sides of length $a = |AB|$, $b = |BC|$, $c = |CD|$, $d = |DE|$, $e = |EF|$, $f = |FA|$ with vertices A, B, C, D, E, F lying on a circle of radius R .

• Main diagonal equation

Let $y = |AD|$ denote the length of a main diagonal of the cyclic hexagon $ABCDEF$. Then we may think the hexagon $ABCDEF$ as made up of two quadrilaterals with a common side AD , both having the same circumradius R . Thus using the formula (0.4) twice we get equality

$$\begin{aligned}
 (R^2 =) & \frac{(de + fy)(df + ey)(ef + dy)}{(-d + e + f + y)(d - e + f + y)(d + e - f + y)(d + e + f - y)} = \\
 = & \frac{(ab + cy)(ac + by)(bc + ay)}{(-a + b + c + y)(a - b + c + y)(a + b - c + y)(a + b + c - y)} \quad (1.9)
 \end{aligned}$$

leading to a 7-th degree equation

$$(def - abc)y^7 + \dots = 0$$

for the length of the main diagonal y .

With substitutions

$$u = a^2 + b^2 + c^2, \quad v = a^2b^2 + a^2c^2 + b^2c^2, \quad w = abc \quad (1.10)$$

$$U = d^2 + e^2 + f^2, \quad V = d^2e^2 + d^2f^2 + e^2f^2, \quad W = def \quad (1.10')$$

we can express the area S' (resp. S'') of the quadrilateral $ABCD$ (resp. $ADEF$) as follows

$$16S'^2 = 4v - u^2 + 8wy + 2uy^2 - y^4, \quad 16S''^2 = 4V - U^2 + 8Wy + 2Uy^2 - y^4 \quad (1.11)$$

Then (1.9) becomes equivalent to

$$P_6^{\text{main diag.}} \equiv -S''^2 (wy^3 + vy^2 + uwy + w^2) + \\ + S'^2 (Wy^3 + Vy^2 + UWy + W^2) = 0 \quad (1.9')$$

$$\text{(i.e. } \boxed{(w - W)y^7 + (v - V)y^6 + \cdots + (4v - u^2)W^2 - (4V - U^2)w^2 = 0}) \quad (1.9'')$$

By letting $f = 0$ we obtain the diagonal equation for a cyclic pentagon $ABCDE$:

$$\boxed{P_5^{\text{diag.}} \equiv \\ abc y^7 + (a^2b^2 + a^2c^2 + b^2c^2 - d^2e^2)y^6 + \cdots + a^2b^2c^2(d^2 - e^2) = 0}$$

(cf. Bowman 1950's).

• Small diagonal equation

Let $x = |AC|$ denote the length of a "small" diagonal in the cyclic hexagon $ABCDEF$. By (0.2) we obtain the equation

$$(ab + cy)x^2 = (ac + by)(bc + ay)$$

by which we can eliminate y in our main diagonal equation (1.9"). This gives our small diagonal equation which has degree 7 in x^2 :

$$\begin{aligned} P_6^{\text{small diag.}} \equiv & (abc - def)(abd - cef)(abe - cdf)(abf - cde)x^{14} + (\dots)x^{12} + \dots + \\ & +(a^2 - b^2)^4(acd - bef)(ace - bdf)(acf - bde)(ade - bcf) \\ & (adf - bce)(aef - bcd) = 0 \end{aligned} \tag{1.12}$$

By letting $f = 0$ we obtain

$$P_6^{\text{small diag.}} \Big|_{f=0} = a^3 b^3 P_5^{\text{diag.}} \left(P_5^{\text{diag.}} \right)^* \quad (1.12')$$

where $P_5^{\text{diag.}} \equiv cde x^7 + \dots = 0$ and $\left(P_5^{\text{diag.}} \right)^*$ is obtained by changing sign of an odd number of side lengths c, d, e .

• **Area equation:** Naive approach

A naive approach to get the area equation of cyclic hexagon would be to write the area S of our hexagon as

$$S = S' + S'' \quad (1.13)$$

where S'^2 and S''^2 are given by Brahmagupta's formula (1.11).

Then by rationalizing the equation (1.13) we obtain an equation of degree 4 in y :

$$(S^2 + S'^2 - S''^2)^2 - 4S^2 S'^2 = 0 \quad (1.14)$$

More explicitly, in terms of the squared area $A = (4S)^2$ we have

$$\begin{aligned} Q \equiv & (A + 4(v - V) + U^2 - u^2 + 8(w - W)y + 2(u - U)y^2)^2 - \\ & - 4A(4v - u^2 + 8wy + 2uy^2 - y^4) = 0 \end{aligned} \quad (1.14')$$

By computing the resultant of this equation and the main diagonal equation (1.9') w.r.t. y we obtain a degree 14 polynomial in A .

$$\text{Resultant} \left(Eq(1.14'), P_6^{\text{main diag.}}, y \right) = F_1 F_2$$

both of whose factors have degree 7 in A :

$$F_1 = (w - W)^2 A^7 + \dots$$

$$F_2 = A^7 + (7u^2 + 7U^2 - 10uU - 24v - 24V)A^6 + \dots$$

The true equation (obtained first by Robbins in 1994. by undetermined coefficients method) is given by F_2 (it has 2042 monomials), and the extraneous factor F_1 (which has 8930 monomials) is 4 time bigger¹.

- **Area equation:** new approach leading to an intrinsic proof.

The complications with the extraneous factor in the previous proof were probably caused by using squaring operation twice in order to get the equation (1.14) (or (1.14')). So we are searching a simpler equation relating the area

¹The computation with MAPLE 9.5 on a PC with 2GHz and 2GB RAM took ≈ 300 hours (in year 2004). Nowadays with MAPLE 12 on a 64-bit PC with 8GB it takes ≈ 3 hours.

S and the main diagonal. After a long struggle we obtained an extraordinary simple relationship given in the following

Key Lemma. The area S of the cyclic hexagon $ABCDEF$ and areas S' and S'' of the cyclic quadrilaterals $ABCD$ and $ADEF$ obtained by subdivision with the main diagonal of length $y = |AD|$ satisfy the following relations:

$$\begin{aligned} \text{a)} \quad & (y^3 - (a^2 + b^2 + c^2)y - 2abc)S'' + (y^3 - (d^2 + e^2 + f^2)y - 2def)S' = 0 \\ \text{b)} \quad & (y^3 - (a^2 + b^2 + c^2)y - 2abc)S + ((a^2 + b^2 + c^2 - d^2 - e^2 - f^2)y + 2(abc - def))S' = 0 \end{aligned}$$

Proof. a) Let $x = |AC|$, $y = |AD|$, $z = |DF|$. Let S'_1, S'_2, S''_1 and S''_2 be the areas of triangles ABC, ACD, ADF and DEF respectively. Then, by (0.5') we have $4S'_1 R = abx$, $4S'_2 R = cxy$, $4S''_1 R = fyz$, $4S''_2 R = dez$. So we have $4S' R = (ab + cy)x$, $4S'' R = (fy + de)z$. This implies

$$\frac{S''}{S'} = \frac{fy + de}{ab + cy} \cdot \frac{z}{x}$$

The diagonal equation for the main diagonal $y = |AD|$ in the middle quadrilateral $ACDF$: $(cx + fz)y^2 = (cf + xz)(fx + cz)$ can be rewritten as

$$cx(y^2 - f^2 - z^2) = fz(-y^2 + c^2 + x^2)$$

Now we have

$$\frac{S''}{S'} = \frac{fy + de}{ab + cy} \cdot \frac{y^2 - f^2 - z^2}{x^2 + c^2 - y^2} \cdot \frac{c}{f} = \frac{c(fy + de)(y^2 - f^2) - (fy + de)z^2}{f(ab + cy)(c^2 - y^2) + (ab + cy)x^2}$$

Finally we use the diagonal equations for small diagonals x and z in respective quadrilaterals

$$(ab + cy)x^2 = (ac + by)(bc + ay), \quad (fy + de)z^2 = (df + ey)(ef + dy)$$

and by simplifying we get

$$\frac{S''}{S'} = \frac{y^3 - (d^2 + e^2 + f^2)y - 2def}{2abc + (a^2 + b^2 + c^2)y - y^3}$$

b) follows from a) by substituting $S'' = S - S'$. ■

By writing the equation b) in Key Lemma with shorthand notations (1.10) and (1.10')

$$(y^3 - uy - 2w)S + ((u - U)y + 2(w - W))S' = 0$$

and multiplying it by $2S$, $2S'$ respectively and using the relation

$$2SS' = S^2 + S'^2 - S''^2$$

obtained from (1.13) by squaring, we obtain the following

KEY EQUATIONS:

$$Q1 := 2(y^3 - uy - 2w)S^2 + ((u - U)y + 2(w - W))(S^2 + S'^2 - S''^2) = 0$$

$$Q2 := (y^3 - uy - 2w)(S^2 + S'^2 - S''^2) + 2((u - U)y + 2(w - W))S'^2 = 0$$

where S'^2 and S''^2 are given by Brahmagupta's formulas (1.11).

MAIN THEOREM. The resultant of the Key Equations with respect to y gives the minimal degree 7 equation for the squared area $A = (4S)^2$ of cyclic hexagon.

Proof. The minimal polynomial

$$\alpha_6 = \text{Resultant}(Q_1, Q_2, y)/C = A^7 + (7(u^2 + U^2) - 10uU - 24(v + V))A^6 + \dots$$

where $C = 4 [4(W - w)^3 + (u - U)^3(wU - uW)]$.

Remark. Observe that $16Q_1 = [2A + 2(u - U)^2]y^3 + \dots$. Similarly the polynomial Q in equation (1.14') has the form

$$Q = [4A + 2(u - U)^2] y^4 + \dots$$

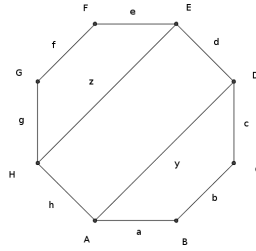
If we define

$$\begin{aligned} Q_3 &:= Q - 2 \cdot 16Q_1 \\ &= 4(-uy^2 - 6wy - 4v + u^2)A + (4(v + w - V - W) + U^2 - u^2 + A) \cdot \\ &\quad \cdot (A + 4(v - V) + U^2 - u^2 + 8(w - W)y + 2(u - U)y^2) \end{aligned}$$

then we also get

$$\alpha_6 = \text{Resultant}(Q_3, 16Q_1, y)/(-8A^2)$$

2 Area equations of cyclic octagons (and heptagons)



We trisect cyclic octagon $ABCDEFGH$, by two diagonals AD and EH into three quadrilaterals $ABCD$, $ADEH$ and $EFGH$ whose areas we denote by S_1 , S_2 and S_3 respectively. The area S of $ABCDEFGH$ is then equal to

$$S = S_1 + S_2 + S_3 \quad (2.15)$$

By Key Lemma a) applied to hexagons $ABCDEH$ and $ADEFGH$ we obtain the following equations:

$$(2jz + (i + z^2)y - y^3)S_1 + (2w + uy - y^3)S_2 = 0 \quad (2.16)$$

$$(2jy + (i + y^2)z - z^3)S_3 + (2W + Uz - z^3)S_2 = 0 \quad (2.17)$$

where we have used the following abbreviations:

$$\begin{aligned} y &= |AD|, z = |EH| \\ u &= a^2 + b^2 + c^2, v = a^2b^2 + a^2c^2 + b^2c^2, w = abc \\ U &= e^2 + f^2 + g^2, V = e^2f^2 + e^2g^2 + f^2g^2, w = efg \\ i &= d^2 + h^2, j = dh \end{aligned} \quad (2.18)$$

Furthermore the Brahmagupta formulas for the 16 times squared areas $A_i = 16S_i^2$, $i = 1, 2, 3$ can be written now as follows:

$$A_1 = 4v - u^2 + 8wy + 2uy^2 - y^4 \quad (2.19)$$

$$A_2 = 4(j + yz)^2 - (y^2 + z^2 - i)^2 \quad (2.20)$$

$$A_3 = 4V - U^2 + 8Wz + 2Uz^2 - z^4 \quad (2.21)$$

(For A_1 , A_3 cf. (1.11), for A_2 cf. (0.3'') from Preliminaries!)

By equating the circumradius formulas for cyclic quadrilaterals $ABCD$ and $ADEH$ (resp. $ABCD$ and $EFGH$) we obtain two equations:

$$\boxed{EQ1 : (4v - u^2 + 8wy + 2uy^2 - y^4)(jzy^3 + (iz^2 + j^2)y^2 + (i + z^2)jzy + (jz)^2) - (4(j + yz)^2 - (y^2 + z^2 - i)^2)(wy^3 + vy^2 + uwy + w^2) = 0} \quad (2.22)$$

$$\boxed{EQ2 : (4v - u^2 + 8wy + 2uy^2 - y^4)(Wz^3 + Vz^2 + UWz + W^2) - (4V - U^2 + 8Wz + 2Uz^2 - z^4)(wy^3 + vy^2 + uwy + w^2) = 0} \quad (2.23)$$

Our next aim is to get one more equation (as simple as possible) relating the lengths y and z of diagonals and the squared area $A = 16S^2$ of our cyclic octagon. Here is a result of a many years long search:

Theorem 2.1 (*Fundamental equation involving area of cyclic octagons*)

Let $A = 16S^2$ be the squared area of any cyclic octagon. Then we have the following equation of degree 6 in y and z and linear in A :

$$EQ3 : \alpha\gamma(A + \eta) + 2(\alpha - \beta)(\delta - \gamma)A_2 = 0$$

(2.24)

where

$$\begin{aligned}\alpha &= 2jz + iy + yz^2 - y^3, & \beta &= 2w + uy - y^3 \\ \gamma &= 2jy + iz + y^2z - z^3, & \delta &= 2W + Uz - z^3 \\ \eta &= u^2 + U^2 - i^2 - 4v - 4V + 4j^2 - 8wj - 8Wz + 8jyz + \\ &\quad + 2(i - u)y^2 + 2(i - U)z^2 + 2y^2z^2\end{aligned}$$

Proof. We start by squaring the equation (2.15)

$$S^2 = S_1^2 + S_2^2 + S_3^2 + 2S_1S_2 + 2S_1S_3 + 2S_2S_3 \quad (2.25)$$

Solving (2.16) for S_1 and (2.17) for S_2 yields:

$$S_1 = -\frac{\beta}{\alpha}S_2, \quad S_3 = -\frac{\delta}{\gamma}S_2 \quad (2.26)$$

Then we substitute these only into the mixed terms of (2.25). This gives:

$$S^2 = S_1^2 + S_2^2 + S_3^2 + 2\left(-\frac{\beta}{\alpha} + \frac{\beta\delta}{\alpha\gamma} - \frac{\delta}{\gamma}\right)S_2^2$$

By multiplying the last equation by 16 and using that $A_i = 16S_i^2$, $A = 16S^2$ we obtain

$$\alpha\gamma(A - A_1 + A_2 - A_3) + 2(\alpha - \beta)(\delta - \gamma)A_2 = 0$$

and set

$$\eta = -A_1 + A_2 - A_3$$

and the result follows by (2.19), (2.20) and (2.21). $\blacksquare$

Remark 2.2 *By using Gröbner basis for $\{EQ1, EQ2, EQ3\}$ we get minimal equation α_7 (α_8) for squared area ($A = 16 Area^2$) of cyclic heptagons (octagons) in concrete instances very fast.*

Remark 2.3 *Maley M.F., Robins D.P. and Roskies J. [2005.] obtained explicit formulas for α_7 and α_8 in terms of elementary symmetric functions of sides lengths squared.*

$$\alpha_7 = \frac{2^{101} 5^5 Res(\tilde{F}, \tilde{G}, u_3)}{u_2^4 Res(\tilde{F}_1, \tilde{F}_2, u_3)}$$

We have recently expanded α_7 which has 955641 terms with up to 40-digits coefficients (approx. 5000 pages).

Remark 2.4 For ζ_7 , the $Z(= (4SR)^2)$ -polynomial, by a similar method, we obtained explicit formula with 31590 terms with up to 11 digits coefficients.

Remark 2.5 For $\rho_7 = R^2$ -equation of cyclic heptagon, by a different technique, we obtained a 15 pages output in a condensed (Pellian) form with up to 4 digits coefficients in terms of new quantities (which are certain linear combinations of elementary symmetric functions of side lengths squared).