Sveučilište u Zagrebu

Prirodoslovno - matematički fakultet

Matematički odjel

Milena Sošić

Računanje konstanti u multiparametarskim quonskim algebrama

Disertacija

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Voditelj disertacije:

dr. sc. Dragutin Svrtan, red. prof.

Zagreb, 2009.

Ova se disertacija predaje na ocjenu Prirodoslovno - matematičkom fakultetu, Matematičkom odjelu Sveučilišta u Zagrebu u svrhu stjecanja znanstvenog stupnja doktora prirodnih znanosti iz područja matematike.

"Možda se moje bavljenje matematikom najbolje može opisati kao ulaženje u jednu tamnu kuću. Uđete u prvu prostoriju i otkrijete da je tu potpuno mračno. Zapinjete za namještaj. No, postupno doznajete gdje se svaki komad nalazi. Konačno, nakon približno šest mjeseci, nalazite prekidač i uključite svjetlo. Sve postaje obasjano, tako da napokon možete vidjeti gdje ste. Zatim ulazite u sljedeću mračnu prostoriju ..." [1]

[1] prof. Andrew Wiles, citirano iz: A. D. Aczel: *Posljednji Fermatov teorem*, Izvori, Zagreb, 2001.

" Znaj sijati rasipno sve što imaš i što jesi
da bi mogao biti radost onih koji će sutra žeti."

Zahvaljujem voditelju rada prof. dr. sc. Dragutinu Svrtanu na svim korisnim savjetima i izvanrednom vođenju u svim fazama izrade ovog rada.

Također zahvaljujem svima, a posebno mojoj obitelji te svim studenticama i studentima, koji su mi iskazivali moralnu podršku.

Predgovor

Ova doktorska disertacija izrađena je pod vodstvom prof. dr. sc. Dragutina Svrtana, redovitog profesora na Matematičkom odjelu Prirodoslovno-matematičkog fakulteta Sveučilišta u Zagrebu.

Disertacija se sastoji od četiri poglavlja:

1. Aranžmani hiperravnina,
2. Slobodna asocijativna algebra $\mathcal{B}$ s jedinicom,
3. Zakrenuta grupovna algebra $\mathcal{A}_n$,
4. Reprezentacija zakrenute grupovne algebre $\mathcal{A}_n$ na težinskim potprostorima $\mathcal{B}_{l_1 l_2 \cdots l_n}$ algebre $\mathcal{B}$.

U prvom poglavlju razmatraju se orijentirani aranžmani hiperravnina i njima pridruženi pojmovi u n-dimenzionalnom afinom prostoru $\mathbf{V}^n$. Pritom se posebno promatra orijentirani diskriminantni aranžman $\mathbf{A}_{n-1}$ (braid arrangement), Varchenko-va matrica tog aranžmana i njena determinanta.

Predmet proučavanja drugog poglavlja je slobodna asocijativna $\mathbb{C}$-algebra $\mathcal{B}$ s jedinicom $\mathcal{B} := \mathbb{C}\langle e_{i_1}, e_{i_2}, ..., e_{i_N} \rangle$ sa skupom generatora $\{e_{i_s}\}_{1 \le s \le N}$. Pritom je $\mathcal{N} = \{i_1, i_2, ..., i_N\}$ fiksirani podskup skupa nenegativnih cijelih brojeva $\mathbb{N}_0 = \{0, 1, 2, ...\}$. Ako stavimo da je stupanj svakog generatora e_{i_s} jednak jedan i pišemo $\mathrm{st}(e_{i_s}) = 1$, $1 \le s \le N$, onda je algebra $\mathcal{B}$ prirodno $\mathbb{N}_0$-graduirana $\mathcal{B} = \bigoplus_{n \ge 0} \mathcal{B}(n)$, gdje je $\mathcal{B}(0) = \mathbb{C}$, a $\mathcal{B}(n)$ se sastoji od svih homogenih nekomutativnih polinoma ukupnog stupnja n u varijablama $e_{i_1}, e_{i_2}, ..., e_{i_N}$. Pritom dalje imamo rastav algebre $\mathcal{B}$ na direktnu sumu $\mathcal{B} = \mathcal{B}^{\mathrm{gen}} \oplus \mathcal{B}^{\mathrm{deg}}$, odnosno rastav svakog $\mathcal{B}(n)$ na direktnu sumu $\mathcal{B}(n) = \mathcal{B}(n)^{\mathrm{gen}} \oplus \mathcal{B}(n)^{\mathrm{deg}}$, gdje $\mathcal{B}(n)^{\mathrm{gen}}$ (generički potprostor) označava potprostor generiran svim multilinearnim monomima ukupnog stupnja $n\,(\le N)$, a $\mathcal{B}(n)^{\mathrm{deg}}$ (degenerirani potprostor) označava potprostor polinoma ukupnog stupnja n, nelinearnih u bar jednoj varijabli (generatoru).

Preciznije, rastav od $\mathcal{B}^{\text{gen}}$ i $\mathcal{B}^{\text{deg}}$ po multihomogenim komponentama glasi ovako:

$$\mathcal{B}^{\text{gen}} = \bigoplus_{n\geq 0} \mathcal{B}(n)^{\text{gen}} = \bigoplus_{\substack{n\geq 0 \\ k_1<k_2<\ldots<k_n,\ k_i\in\mathcal{N}}} \mathcal{B}_{k_1k_2\ldots k_n},$$

$$\mathcal{B}^{\text{deg}} = \bigoplus_{n\geq 0} \mathcal{B}(n)^{\text{deg}} = \bigoplus_{\substack{n\geq 0 \\ \sum_{s=1}^{N} m_s = n,\ \exists i,\ m_i\geq 2}} \mathcal{B}_{i_1^{m_1} i_2^{m_2}\ldots i_N^{m_N}},$$

gdje je:

$$\mathcal{B}_{k_1k_2\ldots k_n} = span_{\mathbb{C}}\left\{ e_{j_1} e_{j_2} \cdots e_{j_n} \mid j_1j_2\ldots j_n = \text{permutacija skupa } \{k_1, k_2, \ldots, k_n\} \right\},$$

$$\mathcal{B}_{i_1^{m_1} i_2^{m_2}\ldots i_N^{m_N}} = span_{\mathbb{C}}\left\{ e_{j_1} e_{j_2} \cdots e_{j_n} \mid j_1j_2\ldots j_n = \text{permutacija multiskupa } \{i_1^{m_1}, i_2^{m_2}, \ldots, i_N^{m_N}\} \right\}.$$

Općenito će rastav algebre $\mathcal{B}$ po multihomogenim komponentama glasiti:

$$\mathcal{B} = \bigoplus_{\substack{n\geq 0 \\ l_1\leq l_2\leq\ldots\leq l_n,\ l_j\in\mathcal{N}}} \mathcal{B}_{l_1l_2\ldots l_n}.$$

Pritom svaki potprostor $\mathcal{B}_Q = \mathcal{B}_{l_1l_2\ldots l_n}$ zovemo težinski potprostor algebre $\mathcal{B}$ pridružen multiskupu $Q = \{l_1 \leq l_2 \leq \ldots \leq l_n\}$, $l_p \in \mathcal{N}$, kojeg ćemo pisati i ovako $Q = l_1 l_2 \ldots l_n$. Specijalno, ako je $l_1 < l_2 < \ldots < l_n$, $l_1, l_2, \ldots, l_n \in \mathcal{N}$, onda je $Q = \{l_1, l_2, \ldots, l_n\}$ skup i njemu pridruženi potprostor će biti generički težinski potprostor $\mathcal{B}_Q$ algebre $\mathcal{B}$. Za multiskupove Q koji nisu skupovi će pripadni težinski potprostor $\mathcal{B}_Q$ biti degenerirani.

Neka su q_{ij}, $i, j \in \mathcal{N} = \{i_1, i_2, \ldots, i_N\}$ zadani (kompleksni) parametri. Tada se na algebri $\mathcal{B}$ mogu promatrati linearni operatori $\partial_{i_1}, \partial_{i_2}, \ldots, \partial_{i_N} : \mathcal{B} \to \mathcal{B}$ ("deformirane parcijalne derivacije") zadani ovako: $\partial_i(1) = 0$,

$$\partial_i(e_j) = \delta_{ij},$$

i dalje induktivno: $\quad \partial_i(e_j\, x) = \delta_{ij}\, x + q_{ij}\, e_j\, \partial_i\, x \quad$ za svaki $x \in \mathcal{B}$,

$i, j \in \mathcal{N} = \{i_1, i_2, \ldots, i_N\}$, gdje je $\delta_{ij} = 1$ za $i = j$, $\delta_{ij} = 0$ za $i \neq j$.

Napomena: Kada bi svi q_{ij} bili jednaki jedan, onda bi ∂_i bila obična i-ta parcijalna derivacija po varijabli e_i.

U algebri $\mathcal{B}$ definira se **konstanta** kao element $C \in \mathcal{B}$ takav da je $\partial_i C = 0$ za svaki $i \in \mathcal{N} = \{i_1, i_2, \ldots, i_N\}$.

Ako na algebri $\mathcal{B}$ uvedemo operator stupnja $\partial : \mathcal{B} \to \mathcal{B}$, formulom $\partial := \sum_{s=1}^{N} \mathrm{e}_{\mathrm{i}_s} \partial_{\mathrm{i}_s}$, onda se konstanta u $\mathcal{B}$ može definirati kao bilo koji element $C \in \mathcal{B}$ za koji je $\partial C = 0$.

Osnovni problem koji će se u ovoj radnji proučavati je nalaženje prostora konstanti u algebri $\mathcal{B}$, tj. problem određivanja jezgre operatora stupnja ∂.
Lako se vidi da operator stupnja $\partial : \mathcal{B} \to \mathcal{B}$ čuva rastav algebre $\mathcal{B}$ na težinske potprostore, tj. da je $\partial\left(\mathcal{B}_{\mathrm{Q}}\right) \subset \mathcal{B}_{\mathrm{Q}}$ za svaki multiskup Q nad $\mathcal{N} = \{\mathrm{i}_1, \mathrm{i}_2, ..., \mathrm{i}_{\mathrm{N}}\}$, stoga se problem nalaženja konstanti u algebri $\mathcal{B}$ svodi na traženje konstanti u pojedinim težinskim potprostorima $\mathcal{B}_{\mathrm{Q}}$ algebre $\mathcal{B}$ ili ekvivalentno na traženje jezgre operatora $\partial|_{\mathcal{B}_{\mathrm{Q}}} : \mathcal{B}_{\mathrm{Q}} \to \mathcal{B}_{\mathrm{Q}}$. U monomijalnoj bazi težinskog potprostora $\mathcal{B}_{\mathrm{Q}}$ operatoru $\partial|_{\mathcal{B}_{\mathrm{Q}}} : \mathcal{B}_{\mathrm{Q}} \to \mathcal{B}_{\mathrm{Q}}$ će biti pridružena matrica koju ćemo označiti sa B_{Q}. Red matrice B_{Q} je jednak $\dim \mathcal{B}_{\mathrm{Q}} = \mathrm{Card}\, \widehat{\mathrm{Q}}$, gdje je $\mathrm{Card}\, \widehat{\mathrm{Q}} = \#$ permutacija multiskupa Q.
Netrivijalna konstanta u $\mathcal{B}_{\mathrm{Q}}$ će postojati ako i samo ako je $\det \mathrm{B}_{\mathrm{Q}} = 0$, (tj. B_{Q} je singularna matrica), što nam daje određene uvjete na parametre q_{ij}, $\mathrm{i}, \mathrm{j} \in \mathcal{N}$.
Pritom će se proučavati uvjet kocikličnosti, odnosno takav izbor vrijednosti parametara q_{ij}, $\mathrm{i}, \mathrm{j} \in \mathcal{N}$ za koje postoji netrivijalna konstanta u $\mathcal{B}_{\mathrm{Q}}$, ali takovih nema u $\mathcal{B}_{\overline{\mathrm{Q}}}$ za svaki pravi podskup $\overline{\mathrm{Q}} \subset \mathrm{Q}$. Pokazati će se, takvi uvjeti kocikličnosti se dobivaju pomoću djeljitelja izraza: $1 - \prod_{1 \le a \ne b \le \mathrm{n}} \mathrm{q}_{l_a l_b}$, gdje produkt prolazi po svim parovima (l_a, l_b) elemenata multiskupa Q, $\mathrm{n} = \mathrm{Card}\, \mathrm{Q}$.

U odjeljku 2.2 detaljno će se izračunati netrivijalne bazične konstante u težinskim potprostorima $\mathcal{B}(1)$, $\mathcal{B}(2)$, $\mathcal{B}(3)$ i $\mathcal{B}(4)$ algebre $\mathcal{B}$ te će se pokazati da u $\mathcal{B}(1)$ nema netrivijalnih konstanti. Također će se pokazati da konstante u bilo kojem degeneriranom težinskom potprostoru možemo konstruirati iz konstanti nekog generičkog težinskog prostora u određenoj identifikaciji elemenata u skupu Q, što dovodi do zaključka da je pri izračunavanju netrivijalnih konstanti u $\mathcal{B}(\mathrm{n})$, $\mathrm{n} \ge 2$ dovoljno izračunati konstante u generičkim težinskim potprostorima.

U trećem poglavlju provode se razmatranja po uzoru na [MS1] u terminologiji zakrenute grupovne algebre (twisted group algebra).

Definira se općenitija grupovna algebra simetrične grupe S_n s koeficijentima u prstenu polinoma $R_n := \mathbb{C}[X_{km} \mid 1 \le k, m \le n]$, pri čemu se koristi djelovanje $S_n \times R_n \to R_n$ simetrične grupe S_n na R_n, definirano pravilom

$$g.p(\ldots, X_{km}, \ldots) = p(\ldots, X_{g(k)g(m)}, \ldots), \tag{1}$$

za $g \in S_n$, koje je inducirano s prirodnim djelovanjem $S_n \times X \to X$ simetrične grupe S_n na skupu $X = \{X_{km} \mid 1 \le k, m \le n\}$:

$$(g, X_{km}) \mapsto g.X_{km} = X_{g(k)g(m)}.$$

Tako definirana općenitija grupovna algebra naziva se zakrenuta grupovna algebra simetrične grupe S_n s koeficijentima u R_n i označava se sa: $\mathcal{A}_n = R_n \rtimes \mathbb{C}[S_n]$.

U algebri $\mathcal{A}_n$ je definirano množenje relacijom: $(p'g_1) \cdot (p''g_2) := (p' \cdot g_1.p'')\, g_1 g_2$, gdje je $g_1.p''$ definirano pravilom (1), a $g_1 g_2$ je produkt od g_1 sa g_2 u S_n.

U četvrtom poglavlju promatra se reprezentacija zakrenute grupovne algebre $\mathcal{A}_n$ na bilo kojem težinskom potprostoru $\mathcal{B}_Q = \mathcal{B}_{l_1 l_2 \ldots l_n}$ algebre $\mathcal{B}$.

Uzimajući u obzir da se svaki degenerirani potprostor algebre $\mathcal{B}$ može dobiti iz generičkog potprostora specijalizacijom koja je inducirana poistovjećivanjem nekih elemenata u skupu Q, u nastavku ćemo se koncentrirati na proučavanje generičkih potprostora i reprezentacije algebre $\mathcal{A}_n$ na njima.

Posebno će se proučavati primjeri vrijednosti regularne reprezentacije $\mathcal{R}$ (izvjesnih elemenata $\widetilde{\alpha_n}$, $\widetilde{\beta_n}$ algebre $\mathcal{A}_n$, definiranih u trećem poglavlju) na općenitom generičkom potprostoru, ali isto tako i vrijednosti reprezentacije (tih elemenata) na općenitom degeneriranom težinskom potprostoru algebre $\mathcal{B}$.

Pokazati će se da regularna reprezentacija od $\widetilde{\alpha_n}$ na $\mathcal{B}_Q$ daje matricu koja je u direktnoj vezi s Varchenkovom matricom (prvo poglavlje). Također će se pokazati da regularna reprezentacija od $\widetilde{\beta_n}$ na $\mathcal{B}_Q$ daje matricu koja je u direktnoj vezi s odgovarajućom matricom B_Q (koja se dobiva pri izračunavanju konstanti u $\mathcal{B}_Q$).

Sadržaj

1. Aranžmani hiperravnina

U ovom poglavlju dati će se kratki prikaz osnovnih pojmova pridruženih aranžmanima hiperravnina u afinom prostoru **V**.

Neka je $\mathbf{V}$ n-dimenzionalni realni afini prostor. Hiperravnina H je afini potprostor od $\mathbf{V}$ kodimenzije 1, tj. $\dim H = n-1$. Možemo pretpostaviti da je $\mathbf{V} = \mathbb{R}^n$.

Definicija 1.1

Konačan skup međusobno različitih hiperravnina u afinom prostoru $\mathbb{R}^n$ se naziva *realni aranžman hiperravnina* (ili kraće *aranžman*) i označava se s $\mathcal{C}$.
Kažemo da je $\mathcal{C} = \{H_i \mid i = 1, 2, ..., k\}$ aranžman, koji se sastoji od k hiperravnina H_i u n-dimenzionalnom afinom prostoru.

Aranžman hiperravnina se naziva *centralni aranžman* ako je presjek svih hiperravnina iz aranžmana $\mathcal{C}$ neprazan.

Navedimo nekoliko primjera centralnih aranžmana.

U afinoj ravnini $\mathbb{R}^2$ centralni aranžman je pramen pravaca kroz neku točku T, tj. aranžman $\mathcal{C} = \{p_i \mid i = 1, 2, ..., k\}$, koji se sastoji od k pravaca p_i takvih da je $\bigcap_{i=1}^{3} p_i = \{T\}$.

U trodimenzionalnom afinom prostoru $\mathbb{R}^3$ centralni aranžmani su:

1) svezak ravnina kroz neki pravac p, tj. aranžman $\mathcal{C} = \{H_i \mid i = 1, 2, ..., k\}$, koji se sastoji od k ravnina H_i takvih da je $\bigcap_{i=1}^{k} H_i = \{p\}$, gdje je p zajednički pravac ravnina H_i.
2) snop ravnina kroz neku točku A, tj. aranžman $\mathcal{C} = \{H_i \mid i = 1, 2, ..., k\}$, koji se sastoji od k ravnina H_i takvih da je $\bigcap_{i=1}^{k} H_i = \{A\}$.

Slijede definicije osnovnih pojmova pridruženih aranžmanima hiperravnina.

Definicija 1.2

Brid aranžmana se naziva svaki neprazni presjek bilo koje familije hiperravnina iz aranžmana $\mathcal{C}$. Skup svih bridova aranžmana $\mathcal{C}$ označavati ćemo s $\mathcal{E}(\mathcal{C})$. Primijetimo: $\emptyset \notin \mathcal{E}(\mathcal{C})$.

Domena aranžmana je bilo koja komponenta povezanosti D prostora $\mathbb{R}^n \setminus \bigcup_{H \in \mathcal{C}} H$ = komplement unije svih hiperravnina H aranžmana $\mathcal{C}$.

Skup svih domena aranžmana $\mathcal{C}$ označavati ćemo s $\mathcal{P}(\mathcal{C})$.

Strana aranžmana se naziva bilo koja strana domene aranžmana, bilo koje dimenzije, a *vrh aranžmana* se naziva nul-dimenzionalna strana aranžmana.

Primjedba 1.3

Brid aranžmana $\mathcal{C}$ je i cijeli afini prostor $\mathbb{R}^n$, jer je $\mathbb{R}^n = \bigcap_{\alpha \in \emptyset} H_\alpha \neq \emptyset$ presjek praznog skupa hiperravnina. Svaka hiperravnina iz aranžmana $\mathcal{C}$ je također brid aranžmana $\mathcal{C}$, jer je $H_i = H_i \cap H_i \neq \emptyset$ za svaki $i \in \{1, 2, ..., k\}$.

U afinom prostoru $\mathbb{R}^n$ je domena aranžmana ujedno i n-dimenzionalna strana aranžmana.

Ako svakoj hiperravnini H iz aranžmana $\mathcal{C}$ pridružimo komutirajuću varijablu $a = a(H) = wt(H)$, koju zovemo *težina hiperravnine* $H \in \mathcal{C}$, onda se aranžman $\mathcal{C}$ naziva *otežani aranžman*.

Težina brida je jednaka produktu težina svih hiperravnina koje sadrže taj brid. Težina afinog prostora $\mathbb{R}^n$, kao brida aranžmana $\mathcal{C}$, jednaka je jedan i pišemo: $a(\mathbb{R}^n) = 1$.

Neka je $\mathcal{C}$ aranžman u afinom prostoru $\mathbb{R}^n$ i neka je $A_\mathcal{C} = \mathbb{Z}[a(H), H \in \mathcal{C}]$ prsten polinoma u varijablama $a(H)$, $H \in \mathcal{C}$.

Označimo sa $M_\mathcal{C}$ modul svih $A_\mathcal{C}$- linearnih kombinacija domena aranžmana $\mathcal{C}$. Tada se definira $A_\mathcal{C}$- bilinearna simetrična forma $\mathcal{B}$ na $A_\mathcal{C}$- modulu $M_\mathcal{C}$ na sljedeći način:

za bilo koje dvije domene P i Q aranžmana $\mathcal{C}$ stavimo:

$$\mathcal{B}(\mathrm{P},\mathrm{Q})=\prod_{\mathrm{H}\in\mathcal{C},\ \mathrm{H}\text{ odvaja P od Q}} \mathrm{a}(\mathrm{H}). \tag{1}$$

Pritom je produkt uzet po svim hiperravninama $\mathrm{H}\in\mathcal{C}$, koje odvajaju domenu P od domene Q.

$\mathcal{B}$ se naziva *kvantna bilinearna forma realnog otežanog aranžmana* $\mathcal{C}$ ili kraće *bilinearna forma* od $\mathcal{C}$.

$\mathcal{B}$ je simetrična bilinearna forma, tj. vrijedi: $\mathcal{B}(\mathrm{P},\mathrm{Q})=\mathcal{B}(\mathrm{Q},\mathrm{P}),\ \mathrm{P},\mathrm{Q}\in\mathcal{P}(\mathcal{C})$.

Primijetimo da je $\mathcal{B}(\mathrm{P},\mathrm{P})=1,\ \mathrm{P}\in\mathcal{P}(\mathcal{C})$.

Varchenkova matrica $\mathcal{B}$ (tj. matrica bilinearne forme realnog otežanog aranžmana $\mathcal{C}$) je simetrična kvadratna matrica, kojoj su retci i stupci indeksirani domenama aranžmana $\mathcal{C}$, a elementi su joj monomi oblika (1).

Teorem 1.4 (Varchenko [V1, Theorem 1.1])

Determinanta matrice bilinearne forme realnog otežanog aranžmana $\mathcal{C}$ dana je formulom:

$$\det\mathcal{B}=\prod_{\mathrm{L}\in\mathcal{E}(\mathcal{C})}\left(1-\mathrm{a}(\mathrm{L})^2\right)^{l(\mathrm{L})} \tag{2}$$

gdje je: $\mathrm{a}(\mathrm{L})$ težina brida L aranžmana $\mathcal{C}$,

$l(\mathrm{L})$ kratnost ili multiciplitet brida L aranžmana $\mathcal{C}$.

Metode izračunavanja kratnosti brida prikazane su u [V1], a detaljno su obrađene u magistarskoj radnji [So] .

Neka su $x_1,x_2,...,x_\mathrm{n}$ koordinate afinog prostora $\mathbb{R}^\mathrm{n}$. Tada u afinom prostoru $\mathbb{R}^\mathrm{n}$ imamo $\frac{\mathrm{n}(\mathrm{n}-1)}{2}$ dijagonalnih hiperravnina:

$$\mathrm{H}_{\mathrm{ij}}=\left\{(x_1,x_2,...,x_\mathrm{n})\in\mathbb{R}^\mathrm{n}\mid x_\mathrm{i}=x_\mathrm{j}\right\}\ \text{ za svaki } 1\le\mathrm{i}<\mathrm{j}\le\mathrm{n}.$$

Neka je $\mathbf{A}_{\mathrm{n}-1}=\left\{\mathrm{H}_{\mathrm{ij}}\mid 1\le\mathrm{i}<\mathrm{j}\le\mathrm{n}\right\}$ aranžman u afinom prostoru $\mathbb{R}^\mathrm{n}$, koji se sastoji od dijagonalnih hiperravnina H_{ij} za svaki $1\le\mathrm{i}<\mathrm{j}\le\mathrm{n}$ ($\mathbf{A}_{\mathrm{n}-1}$ korenspondira Dinkinom dijagramu $\mathrm{A}_{\mathrm{n}-1}$).

Aranžman $\mathbf{A}_{\mathrm{n}-1}=\left\{\mathrm{H}_{\mathrm{ij}}\mid 1\le\mathrm{i}<\mathrm{j}\le\mathrm{n}\right\}$ se naziva diskriminantni aranžman.

Primijetimo da je diskriminantni aranžman $\mathbf{A}_{n-1}$ centralni aranžman, pri čemu je centar dan sa: $\{(x_1, x_2, ..., x_n) \in \mathbb{R}^n \mid x_1 = x_2 = ... = x_n\}$.

Konkretno, u afinoj ravnini $\mathbb{R}^2$ imamo diskriminantni aranžman $\mathbf{A}_1 = \{H_{12}\}$, koji se sastoji od jednog pravca (dijagonala) $H_{12} = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = x_2\}$.
Pritom je pravac H_{12} ujedno i centar aranžmana $\mathbf{A}_1$.

U afinom prostoru $\mathbb{R}^3$ imamo diskriminantni aranžman $\mathbf{A}_2 = \{H_{12}, H_{13}, H_{23}\}$, koji se sastoji od tri dijagonalne ravnine:

$$H_{12} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = x_2\},$$

$$H_{13} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = x_3\},$$

$$H_{23} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_2 = x_3\}.$$

Centar aranžmana $\mathbf{A}_2$ je pravac

$$\{p\} = H_{12} \cap H_{13} \cap H_{23} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = x_2 = x_3\}.$$

Proučavanjem diskriminantnih aranžmana $\mathbf{A}_{n-1}$ u $\mathbb{R}^n$, $n \geq 1$ dobiva se veza između domena diskriminantnog aranžmana $\mathbf{A}_{n-1}$ i elemenata skupa S_n, gdje S_n označava skup svih permutacija prvih n prirodnih brojeva.
Broj domena diskriminantnog aranžmana $\mathbf{A}_{n-1}$ je jednak redu grupe S_n, tj. iznosi $n!$.
Time je svaka domena diskriminantnog aranžmana $\mathbf{A}_{n-1}$ indeksirana s točno jednom permutacijom iz skupa S_n, ali isto tako bilo koja permutacija $\sigma \in S_n$ indeksira točno jednu domenu diskriminantnog aranžmana $\mathcal{C}_{n-1}$ relacijom:

$$\sigma \leftrightarrow P_\sigma = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n \mid x_{\sigma(1)} < x_{\sigma(2)} < ... < x_{\sigma(n)}\}.$$

Neka je $\sigma = (\sigma(1), \sigma(2), ..., \sigma(n))$ neka permutacija iz skupa S_n, tada imamo:

$$x, y \in P_\sigma \quad \Leftrightarrow \quad x_{\sigma(1)} < x_{\sigma(2)} < ... < x_{\sigma(n)}, \quad y_{\sigma(1)} < y_{\sigma(2)} < ... < y_{\sigma(n)}.$$

Neka je $a_{ij} = wt(H_{ij})$ ($1 \leq i < j \leq n$) težina hiperravnine $H_{ij} \in \mathbf{A}_{n-1}$.

Za bilo koje dvije domene $P_\sigma, P_\tau \in \mathcal{P}(\mathbf{A}_{n-1})$ je (P_σ, P_τ)- ti element Varchenkove matrice $\mathcal{B} = \mathcal{B}(\mathbf{A}_{n-1})$ jednak produktu svih težina a_{ij} takvih da indeksi "i", "j" imaju svojstvo da se "i" pojavljuje lijevo od indeksa "j" u jednolinijskoj notaciji za σ, a desno od indeksa "j" u jednolinijskoj notaciji za τ.

✐ Uvodimo orijentaciju aranžmana, a samim time i orijentaciju diskriminantnog aranžmana $\mathbf{A}_{n-1}$. Najprije će se uvesti orijentacija bilo kojeg aranžmana u n-dimenzionalnom afinom prostoru.

Neka je $C=\{H_i \mid i=1,2,...,k\}$ aranžman, koji se sastoji od k različitih hiperravnina u afinom prostoru $\mathbb{R}^n$. Orijentaciju aranžmana uvodimo na sljedeći način.
Označimo s $\overrightarrow{n_i}$ jediničnu normalu na hiperravninu $H_i \in C$ u proizvoljnoj točki te hiperravnine. Tada s obzirom na jediničnu normalu $\overrightarrow{n_i}$ hiperravnina H_i dijeli afini prostor $\mathbb{R}^n$ na tri dijela: na samu hiperravninu H_i, otvoreni poluprostor koji sadrži $\overrightarrow{n_i}$ i otvoreni poluprostor koji ne sadrži $\overrightarrow{n_i}$.

Neka je $H_i^+ \subseteq \mathbb{R}^n \setminus H_i$ otvoreni poluprostor koji sadrži jediničnu normalu $\overrightarrow{n_i}$, $H_i^- \subseteq \mathbb{R}^n \setminus H_i$ otvoreni poluprostor koji ne sadrži $\overrightarrow{n_i}$ i neka je $H_i^0 = H_i$, $1 \le i \le k$.

Definicija 1.5

Strana aranžmana je bilo koji neprazni presjek oblika $\bigcap\limits_{1\le i\le k} H_i^{\varepsilon_i} \neq \emptyset$, gdje je $\varepsilon_i \in \{0,+,-\}$, $1 \le i \le k$.

Domena aranžmana je bilo koji neprazni presjek oblika $\bigcap\limits_{1\le i\le k} H_i^{\varepsilon_i} \neq \emptyset$, gdje je $\varepsilon_i \in \{+,-\}$, $1 \le i \le k$.

Primjer 1.6

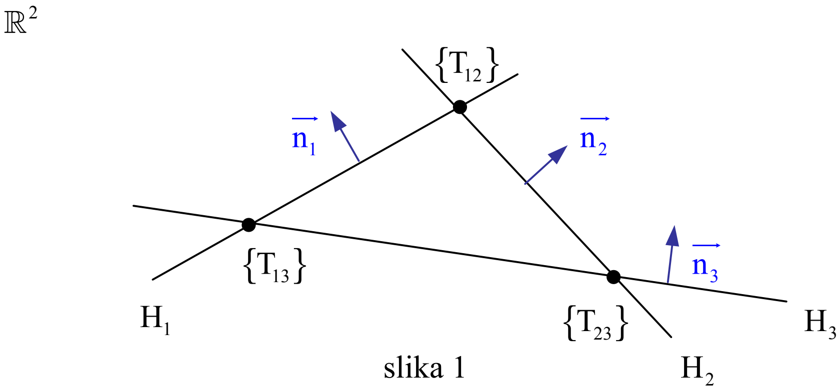
Promatrajmo necentralni aranžman $C=\{H_i \mid i=1,2,3\}$ u afinoj ravnini $\mathbb{R}^2$ $\left(\bigcap\limits_{i=1}^{3} H_i = \emptyset\right)$, koji se sastoji od tri pravca H_i takvih da je:

$$H_i \cap H_j = \begin{cases} H_i & \text{za } i=j \\ \{T_{ij}\} & \text{za } i \neq j \end{cases} \quad \text{za svaki } i,j \in \{1,2,3\}.$$

Pritom je $T_{ij} = T_{ji}$.

Neka je $\overrightarrow{n_i}$ jedinična normala na pravac $H_i \in \mathcal{C}$, $i = 1,2,3$ (slika 1).

Tada je $H_i^+ \subseteq \mathbb{R}^2 \setminus H_i$ otvoreni poluprostor koji sadrži $\overrightarrow{n_i}$, $H_i^- \subseteq \mathbb{R}^2 \setminus H_i$ otvoreni poluprostor koji ne sadrži $\overrightarrow{n_i}$. Pritom je $H_i^0 = H_i$, $i = 1,2,3$.



slika 1

Primjenom definicije 1.5 dobivamo:

(1) nul-dimenzionalne strane (tj. vrhove) aranžmana $\mathcal{C}$

$$\{T_{12}\} = H_1^0 \cap H_2^0 \cap H_3^+,$$

$$\{T_{13}\} = H_1^0 \cap H_2^- \cap H_3^0,$$

$$\{T_{23}\} = H_1^- \cap H_2^0 \cap H_3^0,$$

gdje je: $H_1^0 \cap H_2^0 \cap H_3^- = \varnothing$, $H_1^0 \cap H_2^+ \cap H_3^0 = \varnothing$, $H_1^+ \cap H_2^0 \cap H_3^0 = \varnothing$;

(2) jednodimenzionalne strane aranžmana $\mathcal{C}$

$$H_1^0 \cap H_2^+ \cap H_3^+ = E_6, \quad H_1^+ \cap H_2^0 \cap H_3^+ = E_7, \quad H_1^+ \cap H_2^+ \cap H_3^0 = \varnothing,$$

$$H_1^0 \cap H_2^+ \cap H_3^- = \varnothing, \quad H_1^+ \cap H_2^0 \cap H_3^- = \varnothing, \quad H_1^+ \cap H_2^- \cap H_3^0 = E_9,$$

$$H_1^0 \cap H_2^- \cap H_3^+ = E_8, \quad H_1^- \cap H_2^0 \cap H_3^+ = E_5, \quad H_1^- \cap H_2^+ \cap H_3^0 = E_4,$$

$$H_1^0 \cap H_2^- \cap H_3^- = E_1, \quad H_1^- \cap H_2^0 \cap H_3^- = E_3, \quad H_1^- \cap H_2^- \cap H_3^0 = E_2,$$

(3) domene (dvodimenzionalne strane) aranžmana $\mathcal{C}$

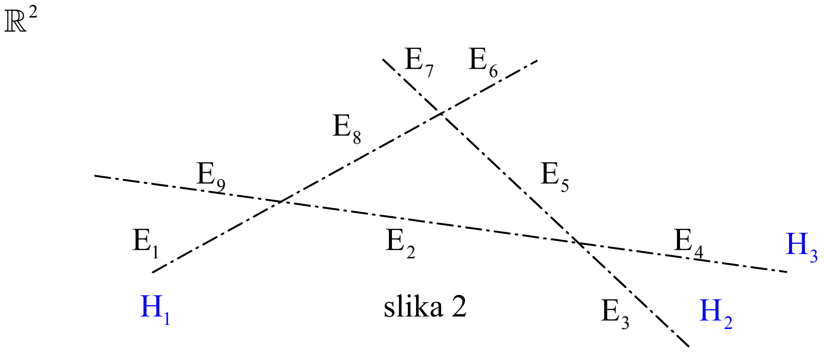
$$H_1^+ \cap H_2^+ \cap H_3^+ = D_4, \quad H_1^- \cap H_2^+ \cap H_3^+ = D_3,$$

$$H_1^+ \cap H_2^+ \cap H_3^- = \varnothing, \quad H_1^- \cap H_2^+ \cap H_3^- = D_2,$$

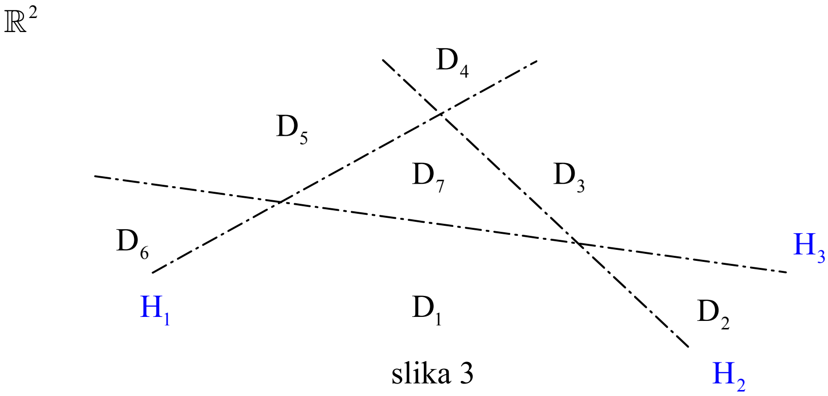
$$H_1^+ \cap H_2^- \cap H_3^+ = D_5, \quad H_1^- \cap H_2^- \cap H_3^+ = D_7,$$

$$H_1^+ \cap H_2^- \cap H_3^- = D_6, \quad H_1^- \cap H_2^- \cap H_3^- = D_1.$$

Oznake jednodimenzionalnih i dvodimenzionalnih strana aranžmana usklađene su sa njihovim prikazom na slici 2, odnosno 3.



slika 2



slika 3

Svakoj strani aranžmana $\mathcal{C}$ može se pridružiti niz oblika $(\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \{0,+,-\}^3$, gdje je $\varepsilon_{\mathrm{i}} = sign\left(\mathrm{H}_{\mathrm{i}}^{\varepsilon_{\mathrm{i}}}\right)$, $\mathrm{i} = 1,2,3$. U tom kontekstu imamo:

$$\{\mathrm{T}_{12}\} \mapsto (0,0,+), \qquad \{\mathrm{T}_{13}\} \mapsto (0,-,0), \qquad \{\mathrm{T}_{23}\} \mapsto (-,0,0),$$

$$\mathrm{E}_1 \mapsto (0,-,-), \qquad \mathrm{E}_2 \mapsto (-,-,0), \qquad \mathrm{E}_3 \mapsto (-,0,-),$$

$$\mathrm{E}_4 \mapsto (-,+,0), \qquad \mathrm{E}_5 \mapsto (-,0,+), \qquad \mathrm{E}_6 \mapsto (0,+,+),$$

$$\mathrm{E}_7 \mapsto (+,0,+), \qquad \mathrm{E}_8 \mapsto (0,-,+), \qquad \mathrm{E}_9 \mapsto (+,-,0),$$

$$\mathrm{D}_1 \mapsto (-,-,-), \qquad \mathrm{D}_2 \mapsto (-,+,-), \qquad \mathrm{D}_3 \mapsto (-,+,+),$$

$$\mathrm{D}_4 \mapsto (+,+,+), \qquad \mathrm{D}_5 \mapsto (+,-,+), \qquad \mathrm{D}_6 \mapsto (+,-,-), \qquad \mathrm{D}_7 \mapsto (-,-,+).$$

Ako se svakom otvorenom poluprostoru $\mathrm{H}_{\mathrm{i}}^{+} \subseteq \mathbb{R}^2 \setminus \mathrm{H}_{\mathrm{i}}$, $\mathrm{H}_{\mathrm{i}}^{-} \subseteq \mathbb{R}^2 \setminus \mathrm{H}_{\mathrm{i}}$, $\mathrm{i} = 1,2,3$ pridruži težina $\mathrm{a}_{\mathrm{i}}^{+} = wt\left(\mathrm{H}_{\mathrm{i}}^{+}\right)$, $\mathrm{a}_{\mathrm{i}}^{-} = wt\left(\mathrm{H}_{\mathrm{i}}^{-}\right)$ (primjedba 1.8), onda se dobiva da je Varchenkova matrica $\mathcal{B}$ otežanog aranžmana $\mathcal{C} = \{\mathrm{H}_{\mathrm{i}} \mid \mathrm{i} = 1,2,3\}$ oblika

$$
\mathcal{B} = \begin{array}{c} \\ D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \\ D_7 \end{array}
\begin{array}{c}
\begin{array}{ccccccc} D_1 & D_2 & D_3 & D_4 & D_5 & D_6 & D_7 \end{array} \\
\begin{bmatrix}
1 & a_2^- & a_2^-a_3^- & a_1^-a_2^-a_3^- & a_1^-a_3^- & a_1^- & a_3^- \\
a_2^+ & 1 & a_3^- & a_1^-a_3^- & a_1^-a_2^+a_3^- & a_1^-a_2^+ & a_2^+a_3^- \\
a_2^+a_3^+ & a_3^+ & 1 & a_1^- & a_1^-a_2^+ & a_1^-a_2^+a_3^+ & a_2^+ \\
a_1^+a_2^+a_3^+ & a_1^+a_3^+ & a_1^+ & 1 & a_2^+ & a_2^+a_3^+ & a_1^+a_2^+ \\
a_1^+a_3^+ & a_1^+a_2^-a_3^+ & a_1^+a_2^- & a_2^- & 1 & a_3^+ & a_1^+ \\
a_1^+ & a_1^+a_2^- & a_1^+a_2^-a_3^- & a_2^-a_3^- & a_3^- & 1 & a_1^+a_3^- \\
a_3^+ & a_2^-a_3^+ & a_2^- & a_1^-a_2^- & a_1^- & a_1^-a_3^+ & 1
\end{bmatrix}
\end{array}
$$

pri čemu je njena determinanta jednaka

$$\det \mathcal{B} = \left(1 - a_1^+ a_1^-\right)^3 \left(1 - a_2^+ a_2^-\right)^3 \left(1 - a_3^+ a_3^-\right)^3 .$$

Primjedba 1.7

Neka je $\mathcal{C} = \{H_i \mid i = 1, 2, ..., k\}$ bilo koji aranžman, koji se sastoji od k hiperravnina u n-dimenzionalnom afinom prostoru $\mathbb{R}^n$.

Tada vrijedi sljedeće tvrdnje:

(1) $\dim H_i = n - 1$ za svaki $1 \le i \le k$.

(2) Ako je $D_j \subseteq \mathbb{R}^n \setminus \bigcup_{i=1}^{k} H_i$ domena aranžmana $\mathcal{C}$, onda je $\dim D_j = n$.

(3) Svaka domena aranžmana je n-dimenzionalna strana aranžmana.

(4) Svakoj domeni aranžmana pridružen je niz oblika $(\varepsilon_1, \varepsilon_2, ..., \varepsilon_k)$, gdje je

$\varepsilon_i \in \{+, -\}$ za svaki $i \in \{1, 2, ..., k\}$.

(5) Svakoj (n–1)-dimenzionalnoj strani aranžmana pridružen je niz oblika $(\varepsilon_1, \varepsilon_2, ..., \varepsilon_k)$ takav da je:

(a) $\exists !\ i \in \{1, 2, ..., k\}$ sa svojstvom: $\varepsilon_i = 0$,

(b) $\varepsilon_j \in \{+, -\}$ za svaki $j \in \{1, 2, ..., k\} \setminus \{i\}$.

(6) Svaka strana A (bilo koje dimenzije) aranžmana $\mathcal{C}$ je dana izrazom:

$A = \bigcap_{1 \le i \le k} H_i^{\varepsilon_i} \ne \varnothing$, gdje je $\varepsilon_i \in \{0, +, -\}$, što povlači da je:

$$A = \bigcap_{1 \le i \le k} H_i^{\varepsilon_i} = \bigcap_{1 \le j \le r} H_{i_j}^{0} \bigcap_{r < s \le k} H_{i_s}^{\varepsilon_{i_s}} \ne \varnothing, \quad \varepsilon_{i_s} \in \{+, -\}.$$

Pritom je $H_{i_j}^0 = H_{i_j} \in \mathcal{C}$, $H_{i_s}^{\varepsilon_{i_s}} \subseteq \mathbb{R}^n \setminus H_{i_s}$.

(7) Ako je *A* strana aranžmana, koja se dobiva kao presjek r hiperravnina H_{i_j} iz aranžmana $\mathcal{C}$ i $(k-r)$ otvorenih poluprostora $H_{i_s}^{\varepsilon_{i_s}} \subseteq \mathbb{R}^n \setminus H_{i_s}$, onda je strani *A* pridružen niz oblika $(\varepsilon_1, \varepsilon_2, ..., \varepsilon_k)$ takav da vrijedi:

(*a*) $\exists!$ r indeksa $i_1, i_2, ..., i_r \in \{1, 2, ..., k\}$ sa svojstvom: $\varepsilon_{i_1} = \varepsilon_{i_2} = \cdots = \varepsilon_{i_r} = 0$

(*b*) $\varepsilon_j \in \{+, -\}$ za svaki $j \in \{1, 2, ..., k\} \setminus \{i_1, i_2, ..., i_r\}$.

(8) Ako je $\mathcal{C}$ centralni aranžman, onda je presjek svih hiperravnina iz aranžmana $\mathcal{C}$ neprazan. Neprazan presjek svih hiperravnina iz aranžmana $\mathcal{C}$ se *naziva centar aranžmana*.

Centru aranžmana $\bigcap_{i=1}^{k} H_i \neq \emptyset$ je pridružen niz oblika: $(\varepsilon_1, \varepsilon_2, ..., \varepsilon_k) = (0, 0, ..., 0)$.

Ako je centar aranžmana neka točka, tj. ako je $\bigcap_{i=1}^{k} H_i = \{T\}$, onda je centar aranžmana ujedno i nul-dimenzionalna strana aranžmana.

Primjedba 1.8

Kažemo da je aranžman $\mathcal{C}$ *otežani orijentirani aranžman hiperravnina* u afinom prostoru $\mathbb{R}^n$ ako je svakom otvorenom poluprostoru $H_i^+, H_i^- \subseteq \mathbb{R}^n \setminus H_i$, $1 \le i \le k$ pridružena komutirajuća varijabla $a_i^+ = wt(H_i^+)$, $a_i^- = wt(H_i^-)$ iz prstena $\mathbb{Z}[a_i^+, a_i^- \mid 1 \le i \le k]$. Pritom se a_i^+ naziva težina poluprostora H_i^+, odnosno a_i^- težina poluprostora H_i^-.

Težina brida orijentiranog aranžana je jednaka produktu težina svih otvorenih poluprostora afinog prostora $\mathbb{R}^n$, čiji zatvarači sadrže taj brid.

Konkretno, težina brida (tj. hiperravnine) $H_i = \mathrm{Cl}(H_i^+) \cap \mathrm{Cl}(H_i^-)$ je jednaka umnošku težina $a_i^+ a_i^-$ otvorenih poluprostora $H_i^+, H_i^- \subseteq \mathbb{R}^n \setminus H_i$, čiji zatvarači sadže brid H_i.

Elementi Varchenkove matrice $\mathcal{B}$ otežanog orijentiranog aranžmana $\mathcal{C}$ su monomi oblika

$$\mathcal{B}(P, Q) = \prod_{P \subset H_i^{\varepsilon_i},\, Q \not\subset H_i^{\varepsilon_i}} a_i^{\varepsilon_i}, \qquad \varepsilon_i \in \{+, -\}. \tag{3}$$

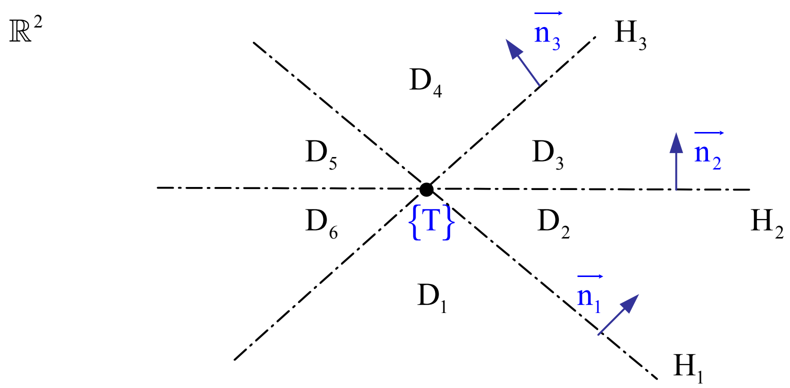
Pritom produkt prolazi po svim poluprostorima $H_i^{\varepsilon_i} \subseteq \mathbb{R}^n \setminus H_i$, $\varepsilon_i \in \{+, -\}$ koji sadrže domenu P ali ne sadrže domenu Q.

Primjer 1.9

Promatrajmo sada centralni aranžman $\mathcal{C}=\{H_1, H_2, H_3\}$, koji se sastoji od tri pravca u afinoj ravnini $\mathbb{R}^2$ takvih da je $\bigcap_{i=1}^{3} H_i = \{T\}$. Točka T je centar aranžmana $\mathcal{C}$.

Uvedimo orijentaciju aranžmana $\mathcal{C}$. Neka je $\overrightarrow{n_i}$ jedinična normala na pravac $H_i \in \mathcal{C}$, $i=1,2,3$. Tada u suglasnosti s gore navedenim imamo da je $H_i^+ \subseteq \mathbb{R}^2 \setminus H_i$ otvoreni poluprostor koji sadrži $\overrightarrow{n_i}$, a $H_i^- \subseteq \mathbb{R}^2 \setminus H_i$ otvoreni poluprostor koji ne sadrži $\overrightarrow{n_i}$. Primjenom definicije 1.5 dobiva se da aranžman $\mathcal{C}$ ima šest domena, koje ćemo označiti sa $D_1, D_2, \ldots, D_6$, gdje je:

$$
\begin{aligned}
&H_1^+ \cap H_2^+ \cap H_3^+ = D_4, \qquad && H_1^- \cap H_2^+ \cap H_3^+ = D_5,\\
&H_1^+ \cap H_2^+ \cap H_3^- = D_3, && H_1^- \cap H_2^+ \cap H_3^- = \varnothing,\\
&H_1^+ \cap H_2^- \cap H_3^+ = \varnothing, && H_1^- \cap H_2^- \cap H_3^+ = D_6,\\
&H_1^+ \cap H_2^- \cap H_3^- = D_2, && H_1^- \cap H_2^- \cap H_3^- = D_1.
\end{aligned}
$$



slika 4

Pritom je svakoj domeni aranžmana $\mathcal{C}$ pridružen odgovarajući niz oblika $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ (primjedba 1.7) i pišemo

$$
\begin{aligned}
&D_1 \mapsto (-,-,-), \qquad && D_2 \mapsto (+,-,-), \qquad && D_3 \mapsto (+,+,-),\\
&D_4 \mapsto (+,+,+), && D_5 \mapsto (-,+,+), && D_6 \mapsto (-,-,+).
\end{aligned}
$$

Svakom otvorenom poluprostoru $H_i^{\varepsilon_i} \subseteq \mathbb{R}^2 \setminus H_i$, $\varepsilon_i \in \{+,-\}$, $i \in \{1,2,3\}$ pridružuje se težina $a_i^{\varepsilon_i} = wt\left(H_i^{\varepsilon_i}\right)$ te se primjenom formule (3) (primjedba 1.8) dobiva da je Varchenkova matrica $\mathcal{B}$ aranžmana $\mathcal{C}$ dana sa

$$
\mathcal{B} \;=\; \begin{array}{c} \\ D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \end{array}
\begin{array}{c}
\begin{array}{cccccc} \;\;D_1 & \quad D_2 & \quad D_3 & \quad D_4 & \quad D_5 & \quad D_6 \end{array} \\
\begin{bmatrix}
1 & a_1^- & a_1^- a_2^- & a_1^- a_2^- a_3^- & a_2^- a_3^- & a_3^- \\
a_1^+ & 1 & a_2^- & a_2^- a_3^- & a_1^+ a_2^- a_3^- & a_1^+ a_3^- \\
a_1^+ a_2^+ & a_2^+ & 1 & a_3^- & a_1^+ a_3^- & a_1^+ a_2^+ a_3^- \\
a_1^+ a_2^+ a_3^+ & a_2^+ a_3^+ & a_3^+ & 1 & a_1^+ & a_1^+ a_2^+ \\
a_2^+ a_3^+ & a_1^- a_2^+ a_3^+ & a_1^- a_3^+ & a_1^- & 1 & a_2^+ \\
a_3^+ & a_1^- a_3^+ & a_1^- a_2^- a_3^+ & a_1^- a_2^- & a_2^- & 1
\end{bmatrix}
\end{array}
$$

Pritom je: $\det \mathcal{B} = \left(1 - a_1^+ a_1^-\right)^2 \left(1 - a_2^+ a_2^-\right)^2 \left(1 - a_3^+ a_3^-\right)^2 \left(1 - a_1^+ a_1^- a_2^+ a_2^- a_3^+ a_3^-\right).$

Proučavanjem otežanih orijentiranih aranžmana u afinom prostoru $\mathbb{R}^n$ zaključujemo sljedeće.

Primjedba 1.10

Neka je $\mathcal{C} = \{H_i \mid i = 1, 2, ..., k\}$ centralni otežani orijentirani aranžman u afinom prostoru $\mathbb{R}^n$, gdje je $\left\{\bigcap\limits_{i=1}^{k} H_i \mid \bigcap\limits_{i=1}^{k} H_i \neq \emptyset\right\}$ centar aranžmana $\mathcal{C}$.

U suglasnosti s definicijom 1.2 proizlazi da su hiperravnine H_i, $1 \le i \le k$ i centar $\left\{\bigcap\limits_{i=1}^{k} H_i\right\}$ ujedno bridovi aranžmana $\mathcal{C}$.

Iz svojstva da je težina brida orijentiranog aranžmana jednaka produktu težina svih otvorenih poluprostora čiji zatvarači sadrže taj brid, proizlazi da se determinanta Varchenkove matrice (centranog orijentiranog) aranžmana $\mathcal{C}$ izračunava po formuli:

$$\det \mathcal{B} = \left(1 - a_1^+ a_1^-\right)^2 \left(1 - a_2^+ a_2^-\right)^2 \cdots \left(1 - a_k^+ a_k^-\right)^2 \left(1 - a_1^+ a_1^- \cdots a_k^+ a_k^-\right)^{k-2}.$$

Pritom je: $a_i^+ a_i^-$ težina brida $H_i = \mathrm{Cl}\left(H_i^+\right) \cap \mathrm{Cl}\left(H_i^-\right)$, $i \in \{1, 2, ..., k\}$,

$a_1^+ a_1^- \cdots a_k^+ a_k^-$ težina brida (tj. centra) $\left\{\bigcap\limits_{i=1}^{k} H_i\right\} = \bigcap\limits_{i=1}^{k} \mathrm{Cl}\left(H_i^{\varepsilon_i}\right)$,

$\varepsilon_i \in \{+, -\}$, $1 \le i \le k$.

Na osnovu rečenog proizlazi da se Varchenkov teorem za otežane orijentirane aranžmane hiperravnina u afinom prostoru $\mathbb{R}^n$ iskazuje na sljedeći način:

✐ Determinanta matrice bilinearne forme realnog otežanog orijentiranog aranžmana $\mathcal{C}$ dana je formulom:

$$\det \mathcal{B} = \prod_{L \in \mathcal{E}(\mathcal{C})} \left(1 - a_L^+ a_L^-\right)^{l(L)} \tag{4}$$

gdje je: $a_L^+ a_L^-$ težina brida L aranžmana $\mathcal{C}$,

$\mathcal{E}(\mathcal{C})$ skup svih bridova aranžmana $\mathcal{C}$,

$l(\mathrm{L})$ kratnost ili multiciplitet brida L aranžmana $\mathcal{C}$.

Pritom je: $a_L^+ := \prod_{L \subset \mathrm{Cl}(H_i^+)} a_i^+$, $a_L^- := \prod_{L \subset \mathrm{Cl}(H_i^-)} a_i^-$, $a_i^+ = wt(H_i^+)$, $a_i^- = wt(H_i^-)$.

Primjedba 1.11

Analogno gore navedenom uvodi se orijentacija diskriminantnog aranžmana $\mathbf{A}_{n-1} = \{H_{ij} \mid 1 \le i < j \le n\}$ u afinom prostoru $\mathbb{R}^n$. Pritom se sa $\overrightarrow{n_{ij}}$ označava jedinični vektor normale $\overrightarrow{n_{ij}}$ na hiperravninu $H_{ij} \in \mathbf{A}_{n-1}$ u proizvoljnoj točki te hiperravnine.

S obzirom na jediničnu normalu $\overrightarrow{n_{ij}}$ (dijagonalna) hiperravnina

$$H_{ij} = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n \mid x_i = x_j\} \in \mathbf{A}_{n-1},\ 1 \le i < j \le n$$

dijeli afini prostor $\mathbb{R}^n$ na samu hiperravninu $H_{ij}^0 = H_{ij}$, na otvoreni poluprostor $H_{ij}^+ \subseteq \mathbb{R}^n \setminus H_{ij}$ koji sadrži $\overrightarrow{n_{ij}}$ i na otvoreni poluprostor $H_{ij}^- \subseteq \mathbb{R}^n \setminus H_{ij}$ koji ne sadrži $\overrightarrow{n_{ij}}$. Pritom je $H_{ij}^+ = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n \mid x_i > x_j\}$,

$H_{ij}^- = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n \mid x_i < x_j\}$,

$1 \le i < j \le n$.

Svakom otvorenom poluprostoru $H_{ij}^+ \subseteq \mathbb{R}^n \setminus H_{ij}$, $H_{ij}^- \subseteq \mathbb{R}^n \setminus H_{ij}$, $1 \le i < j \le n$ prostora $\mathbb{R}^n$ pridružuje se odgovarajuća težina, koju ćemo označiti sa q_{ij} ili q_{ji} u ovisnosti o tome da li otvoreni poluprostor sadrži ili ne sadrži $\overrightarrow{n_{ij}}$.

Neka je

$q_{ij} = wt(H_{ij}^+)$ težina otvorenog poluprostora H_{ij}^+,

$q_{ji} = wt(H_{ij}^-)$ težina otvorenog poluprostora H_{ij}^-,

gdje je: $q_{ij} \ne q_{ji}$ za svaki $1 \le i < j \le n$.

U nastavku će se $\mathbf{A}_{n-1}$ zvati *otežani orijentirani diskriminantni aranžman* ili kraće *orijentirani aranžman* $\mathbf{A}_{n-1}$.

Neka su $P_\sigma, P_\tau \in \mathcal{P}(\mathbf{A}_{n-1})$ bilo koje dvije domene orijentiranog aranžmana $\mathbf{A}_{n-1}$ i neka $\mathcal{B}(\sigma,\tau)$ označava (P_σ, P_τ)-ti element Varchenkove matrice $\mathcal{B}=\mathcal{B}(\mathbf{A}_{n-1})$.

Tada je

$$\mathcal{B}(\sigma,\tau)=\prod_{(r,s)\in I(\tau^{-1}\sigma)} q_{\sigma(r)\sigma(s)}, \tag{5}$$

gdje je $I(\sigma)=\{(r,s) \mid r<s, \sigma(r)>\sigma(s)\}$ skup svih inverzija permutacije $\sigma\in S_n$.

Primjer 1.12

U afinom prostoru $\mathbb{R}^3$ imamo diskriminantni aranžman $\mathbf{A}_2=\{H_{12},H_{13},H_{23}\}$, koji se sastoji od tri dijagonalne ravnine $H_{12}=\{(x_1,x_2,x_3)\in\mathbb{R}^3 \mid x_1=x_2\}$, $H_{13}=\{(x_1,x_2,x_3)\in\mathbb{R}^3 \mid x_1=x_3\}$ i $H_{23}=\{(x_1,x_2,x_3)\in\mathbb{R}^3 \mid x_2=x_3\}$.

Neka je $\overrightarrow{n_{ij}}$ jedinična normala na hiperravninu $H_{ij}\in\mathbf{A}_2$, $1\le i<j\le 3$.

Tada je $H_{ij}^+=\{(x_1,x_2,x_3)\in\mathbb{R}^3 \mid x_i>x_j\}$ otvoreni poluprostor koji sadrži $\overrightarrow{n_{ij}}$, $H_{ij}^-=\{(x_1,x_2,x_3)\in\mathbb{R}^3 \mid x_i<x_j\}$ otvoreni poluprostor koji ne sadrži $\overrightarrow{n_{ij}}$, $1\le i<j\le 3$.

Svakom otvorenom poluprostoru pridružuje se težina $q_{ij}=wt(H_{ij}^+)$, $q_{ji}=wt(H_{ij}^-)$, $q_{ij}\ne q_{ji}$, $1\le i<j\le 3$ te se dobiva da je Varchenkova matrica $\mathcal{B}=\mathcal{B}(\mathbf{A}_2)$ oblika

$$\mathcal{B}=\begin{array}{c} \\ P_{123} \\ P_{132} \\ P_{312} \\ P_{321} \\ P_{231} \\ P_{213} \end{array}\begin{array}{c}\begin{array}{cccccc} P_{123} & P_{132} & P_{312} & P_{321} & P_{231} & P_{213}\end{array}\\ \begin{bmatrix} 1 & q_{23} & q_{13}q_{23} & q_{12}q_{13}q_{23} & q_{12}q_{13} & q_{12} \\ q_{32} & 1 & q_{13} & q_{12}q_{13} & q_{12}q_{13}q_{32} & q_{12}q_{32} \\ q_{31}q_{32} & q_{31} & 1 & q_{12} & q_{12}q_{32} & q_{12}q_{31}q_{32} \\ q_{21}q_{31}q_{32} & q_{21}q_{31} & q_{21} & 1 & q_{32} & q_{31}q_{32} \\ q_{21}q_{31} & q_{21}q_{31}q_{23} & q_{21}q_{23} & q_{23} & 1 & q_{31} \\ q_{21} & q_{21}q_{23} & q_{21}q_{13}q_{23} & q_{13}q_{23} & q_{13} & 1 \end{bmatrix}\end{array} \tag{6}$$

kojoj je determinanta dana sa

$$\det\mathcal{B}=(1-q_{12}q_{21})^2(1-q_{13}q_{31})^2(1-q_{23}q_{32})^2(1-q_{12}q_{21}q_{13}q_{31}q_{23}q_{32}).$$

S druge strane imamo da je Varchenkova matrica (neorijentiranog) diskriminantnog aranžmana $\mathbf{A}_2$, koju ćemo u ovom slučaju označiti sa $\mathcal{B}' = \mathcal{B}'(\mathbf{A}_2)$, dana sa

$$\mathcal{B}' = \begin{array}{c} \\ P_{123} \\ P_{132} \\ P_{312} \\ P_{321} \\ P_{231} \\ P_{213} \end{array} \begin{array}{c} \begin{array}{cccccc} P_{123} & P_{132} & P_{312} & P_{321} & P_{231} & P_{213} \end{array} \\ \begin{bmatrix} 1 & q_{23} & q_{13}q_{23} & q_{12}q_{13}q_{23} & q_{12}q_{13} & q_{12} \\ q_{23} & 1 & q_{13} & q_{12}q_{13} & q_{12}q_{13}q_{23} & q_{12}q_{23} \\ q_{13}q_{23} & q_{13} & 1 & q_{12} & q_{12}q_{23} & q_{12}q_{13}q_{23} \\ q_{12}q_{13}q_{23} & q_{12}q_{13} & q_{12} & 1 & q_{23} & q_{13}q_{23} \\ q_{12}q_{13} & q_{12}q_{13}q_{23} & q_{12}q_{23} & q_{23} & 1 & q_{13} \\ q_{12} & q_{12}q_{23} & q_{12}q_{13}q_{23} & q_{13}q_{23} & q_{13} & 1 \end{bmatrix} \end{array} \tag{7}$$

Pritom je

$$\det \mathcal{B}' = \left(1-q_{12}^2\right)^2 \left(1-q_{13}^2\right)^2 \left(1-q_{23}^2\right)^2 \left(1-q_{12}^2 q_{13}^2 q_{23}^2\right).$$

Uspoređivanjem matrice $\mathcal{B}'$ s matricom $\mathcal{B}$ proizlazi da se matrica $\mathcal{B}'$ može dobiti iz matrice $\mathcal{B}$ specijalizacijom njenih elemenata uz uvjet $q_{ij} = q_{ji}$ za svaki $1 \le i < j \le 3$.

Zaključujemo:

Specijalno, ako je $q_{ij} = q_{ji}$ za svaki $1 \le i < j \le n$, onda Varchenkova matrica (neorijentiranog) diskriminantnog aranžmana proizlazi iz Varchenkove matrice orijentiranog diskriminantnog aranžmana.

Od posebne je važnosti Varchenkova matrica orijentiranog diskriminantnog aranžmana $\mathbf{A}_{n-1}$ u afinom prostoru $\mathbb{R}^n$. U nastavku, tj. u poglavljima 2 i 4 pokazati će se da je matrica orijentiranog diskriminantnog aranžmana $\mathbf{A}_{n-1}$ u direktnoj vezi s matricom koja se dobiva pri izračunavanju konstanti u težinskom potprostoru $\boldsymbol{\mathcal{B}}_{l_1 l_2 \cdots l_n}$, $l_1 < l_2 < \ldots < l_n$, $n \ge 2$, $l_1, l_2, \ldots, l_n \in \mathcal{N} = \{i_1, i_2, \ldots, i_N\}$ slobodne asocijativne $\mathbb{C}$-algebre s jedinicom $\boldsymbol{\mathcal{B}} = \mathbb{C}\langle e_{i_1}, e_{i_2}, \ldots, e_{i_N} \rangle$ u kojoj je uvedena $\mathbf{q}$-diferencijalna struktura.

2. Slobodna asocijativna algebra $\mathcal{B}$ s jedinicom

Neka je $\mathbb{N}_0 = \{0,1,2,...\}$ skup svih nenegativnih cijelih brojeva i neka je $\mathcal{N} = \{\mathrm{i}_1, \mathrm{i}_2, ..., \mathrm{i}_\mathrm{N}\}$ fiksirani podskup od $\mathbb{N}_0$, gdje je $\mathrm{N} \in \mathbb{N}_0$.

Neka je $\{\mathrm{e}_{\mathrm{i}_1}, \mathrm{e}_{\mathrm{i}_2}, ..., \mathrm{e}_{\mathrm{i}_\mathrm{N}}\}$ skup nekomutirajućih varijabli.

Označimo sa $\mathcal{B} = \mathbb{C}\langle \mathrm{e}_{\mathrm{i}_1}, \mathrm{e}_{\mathrm{i}_2}, ..., \mathrm{e}_{\mathrm{i}_\mathrm{N}} \rangle$ slobodnu asocijativnu $\mathbb{C}$-algebru s jedinicom sa skupom generatora $\{\mathrm{e}_{\mathrm{i}_s}\}_{1 \le s \le \mathrm{N}}$. Ako stavimo da je stupanj svakog generatora $\mathrm{e}_{\mathrm{i}_s}$ jednak jedan i pišemo $\mathrm{st}(\mathrm{e}_{\mathrm{i}_s}) = 1$, $1 \le s \le \mathrm{N}$, onda je algebra $\mathcal{B}$ prirodno $\mathbb{N}_0$-graduirana $\mathcal{B} = \bigoplus_{\mathrm{n} \ge 0} \mathcal{B}(\mathrm{n})$, gdje je $\mathcal{B}(0) = \mathbb{C}$, a $\mathcal{B}(\mathrm{n})$ se sastoji od svih homogenih nekomutativnih polinoma ukupnog stupnja n u N varijabli $\mathrm{e}_{\mathrm{i}_1}, \mathrm{e}_{\mathrm{i}_2}, ..., \mathrm{e}_{\mathrm{i}_\mathrm{N}}$.

Neka je $\mathrm{N} \in \mathbb{N}_0 \setminus \{0\}$. Tada prostore multihomogenih polinoma n-tog stupnja generiranih varijablama $\mathrm{e}_{\mathrm{i}_1}, \mathrm{e}_{\mathrm{i}_2}, ..., \mathrm{e}_{\mathrm{i}_\mathrm{N}}$, gdje je $\mathrm{n} \le \mathrm{N}$, možemo parametrizirati pomoću

(a) skupova: $\{k_1, k_2, ..., k_\mathrm{n}\} \subseteq \mathcal{N}$ takvih da je $k_1 < k_2 < ... < k_\mathrm{n}$,

(b) multiskupova: $\{\mathrm{i}_1^{m_1}, \mathrm{i}_2^{m_2}, ..., \mathrm{i}_\mathrm{N}^{m_\mathrm{N}}\}$ takvih da vrijedi:

(b_1) $m_s \in \mathbb{N}_0$ za svaki $1 \le s \le \mathrm{N}$, $\mathrm{N} \in \mathbb{N}_0$,

(b_2) postoji barem jedan i takav da je $m_i \ge 2$,

(b_3) $\sum_{s=1}^{\mathrm{N}} m_s = \mathrm{n}$.

Nenegativni cijeli broj m_s je kratnost, tj. broj ponavljanja elementa i_s, $1 \le s \le \mathrm{N}$.

Označimo sa M_n skup koji sadrži sve skupove oblika (a), odnosno sa M^n skup koji sadrži sve multiskupove oblika (b):

$$\mathrm{M}_\mathrm{n} = \{\{k_1, k_2, ..., k_\mathrm{n}\} \subseteq \mathcal{N} \mid k_1 < k_2 < ... < k_\mathrm{n}\}, \tag{1}$$

$$\mathrm{M}^\mathrm{n} = \left\{ \{\mathrm{i}_1^{m_1}, \mathrm{i}_2^{m_2}, ..., \mathrm{i}_\mathrm{N}^{m_\mathrm{N}}\} \mid \sum_{s=1}^{\mathrm{N}} m_s = \mathrm{n}, \ \exists\, i, \ m_i \ge 2 \right\}. \tag{2}$$

Neka je $Q=\{k_1,k_2,...,k_n\}\in M_n$, $k_1<k_2<...<k_n$ bilo koji n-člani podskup od $\mathcal{N}$. Identificiramo li skup Q s nizom $k_1k_2...k_n$ i označimo sa

$$\widehat{Q}=S_nQ=\{\sigma(k_1k_2...k_n)\,|\,\sigma\in S_n\}$$

skup svih permutacija skupa Q, tada je: $\mathrm{Card}\,\widehat{Q}=n!$.

Analogno, neka je $Q=\{i_1^{m_1},i_2^{m_2},...,i_N^{m_N}\}$, $\sum_{s=1}^{N}m_s=n$ bilo koji n-člani multiskup nad $\mathcal{N}$ u kojem se element i_s pojavljuje m_s puta, $1\le s\le N$ te identificirajmo multiskup Q s nizom $i_1^{m_1}\,i_2^{m_2}...\,i_N^{m_N}=\underbrace{i_1...i_1}_{m_1}\,\underbrace{i_2...i_2}_{m_2}...\,\underbrace{i_N...i_N}_{m_N}$.

Označimo s $\widehat{Q}=S_nQ=\left\{\sigma\left(\underbrace{i_1...i_1}_{m_1}\,\underbrace{i_2...i_2}_{m_2}...\,\underbrace{i_N...i_N}_{m_N}\right)|\,\sigma\in S_n\right\}$

skup svih permutacija multiskupa Q. Tada je:

$$\mathrm{Card}\,\widehat{Q}=\binom{n}{m_1,m_2,...,m_N}=\frac{n!}{\prod_{s=1}^{N}m_s!},\ \text{gdje je}\ \sum_{s=1}^{N}m_s=n.$$

Uzimajući u obzir gore navedeno imamo sljedeće:

(*i*) Bilo kojem skupu $Q\in M_n$ pridružuje se generički težinski potprostor $\mathcal{B}_Q$ u $\mathcal{B}(n)$. Baza potprostora $\mathcal{B}_Q$ je sastavljena od svih monoma oblika $e_{j_1j_2\cdots j_n}:=e_{j_1}e_{j_2}\cdots e_{j_n}$ za svaki $j_1\,j_2\,...j_n\in\widehat{Q}=S_nQ$, stoga je dimenzija potprostora $\mathcal{B}_Q$ jednaka $\mathrm{Card}\,\widehat{Q}=n!$.

(*ii*) Bilo kojem multiskupu $Q\in M^n$ pridružuje se degenerirani težinski potprostor $\mathcal{B}_Q$ u $\mathcal{B}(n)$, kojemu bazu čine monomi $e_{j_1j_2\cdots j_n}:=e_{j_1}e_{j_2}\cdots e_{j_n}$, $j_1\,j_2\,...j_n\in\widehat{Q}=S_nQ$.

Dimenzija potprostora $\mathcal{B}_Q$ je jednaka $\mathrm{Card}\,\widehat{Q}=\dfrac{n!}{\prod_{s=1}^{N}m_s!}$.

Pritom je $\sum_{s=1}^{N}m_s=n$.

Drugim rječima, ako je Q skup, onda će njemu pridruženi potprostor $\mathcal{B}_Q \subset \mathcal{B}(n)$ biti generički težinski potprostor, a ako je Q multiskup koji nije skup, onda će njemu pridruženi potprostor $\mathcal{B}_Q \subset \mathcal{B}(n)$ biti degenerirani težinski potprostor. Pritom je $\dim \mathcal{B}_Q = \mathrm{Card}\, \widehat{Q}$.

Na osnovu rečenog proizlazi rastav od $\mathcal{B}(n)$ na direktnu sumu

$$\mathcal{B}(n) = \mathcal{B}(n)^{gen} \oplus \mathcal{B}(n)^{deg},$$

gdje je $\mathcal{B}(n)^{gen} = \bigoplus_{Q \in M_n} \mathcal{B}_Q$, $\mathcal{B}(n)^{deg} = \bigoplus_{Q \in M^n} \mathcal{B}_Q$.

Skupovi M_n i M^n su dani relacijama (1) i (2).

Pritom $\mathcal{B}(n)^{gen}$ (generički potprostor) označava potprostor generiran svim multilinearnim monomima ukupnog stupnja $n(\leq N)$, a $\mathcal{B}(n)^{deg}$ (degenerirani potprostor) označava potprostor polinoma ukupnog stupnja n, nelinearnih u bar jednoj varijabli (generatoru).

S druge strane, skup $\mathfrak{B} = \{ e_{j_1 j_2 \cdots j_n} := e_{j_1} e_{j_2} \cdots e_{j_n} \mid j_1, j_2, ..., j_n \in \mathcal{N},\ n \geq 0 \}$ je baza čitavog vektorskog prostora $\mathcal{B} = \bigoplus_{n \geq 0} \mathcal{B}(n)$. Označimo s

$$\mathfrak{B}' = \{ e_{j_1 j_2 \cdots j_n} := e_{j_1} e_{j_2} \cdots e_{j_n} \mid j_1, j_2, ..., j_n \in \mathcal{N} \text{ svi različiti, } n \geq 0 \} \subset \mathfrak{B},$$

$$\mathfrak{B}'' = \mathfrak{B} \setminus \mathfrak{B}' = \{ e_{j_1 j_2 \cdots j_n} := e_{j_1} e_{j_2} \cdots e_{j_n} \mid j_1, j_2, ..., j_n \in \mathcal{N} \text{ nisu svi različiti, } n \geq 0 \} \subset \mathfrak{B}.$$

Tada imamo rastav od $\mathcal{B}$ na direktnu sumu

$$\mathcal{B} = \mathcal{B}^{gen} \oplus \mathcal{B}^{deg},$$

pri čemu je:

$$\mathcal{B}^{gen} = span_{\mathbb{C}}\, \mathfrak{B}', \quad \mathcal{B}^{deg} = span_{\mathbb{C}}\, \mathfrak{B}''$$

pa je

$$\mathcal{B}^{gen} := \bigoplus_{n \geq 0} \mathcal{B}(n)^{gen}, \quad \mathcal{B}^{deg} := \bigoplus_{n \geq 0} \mathcal{B}(n)^{deg}.$$

Napomena:

U nastavku će se generički težinski potprostor $\mathcal{B}_Q \subset \mathcal{B}(n)^{gen}$ pridružen skupu $Q = \{k_1, k_2, ..., k_n\}$ zapisivati u obliku $\mathcal{B}_{k_1 k_2 \ldots k_n}$ ako je $k_1 < k_2 < ... < k_n$, $k_p \in \mathcal{N}$, $1 \leq p \leq n$.

Analogno će se degenerirani težinski potprostor $\mathcal{B}_{\mathrm{Q}} \subset \mathcal{B}(\mathrm{n})^{\mathrm{deg}}$ pridružen multiskupu $\mathrm{Q} = \left\{ \mathrm{i}_1^{m_1}, \mathrm{i}_2^{m_2}, ..., \mathrm{i}_{\mathrm{N}}^{m_{\mathrm{N}}} \right\}$, koji nije skup, zapisivati u obliku $\mathcal{B}_{\mathrm{i}_1^{m_1}\mathrm{i}_2^{m_2}\ldots\mathrm{i}_{\mathrm{N}}^{m_{\mathrm{N}}}}$, $\sum_{s=1}^{\mathrm{N}} m_s = \mathrm{n}$, $\exists\, i,\ m_i \geq 2$.

Time dobivamo:

$$\mathcal{B}^{\mathrm{gen}} := \bigoplus_{\substack{\mathrm{n}\geq 0 \\ k_1 < k_2 < \ldots < k_{\mathrm{n}},\ k_{\mathrm{p}} \in \mathcal{N}}} \mathcal{B}_{k_1 k_2 \ldots k_{\mathrm{n}}}, \tag{3}$$

$$\mathcal{B}^{\mathrm{deg}} := \bigoplus_{\substack{\mathrm{n}\geq 0 \\ \sum_{s=1}^{\mathrm{N}} m_s = \mathrm{n},\ \exists\, i,\ m_i \geq 2}} \mathcal{B}_{\mathrm{i}_1^{m_1}\mathrm{i}_2^{m_2}\ldots\mathrm{i}_{\mathrm{N}}^{m_{\mathrm{N}}}}, \tag{4}$$

pri čemu je:

$$\mathcal{B}_{k_1 k_2 \ldots k_{\mathrm{n}}} = span_{\mathbb{C}} \left\{ \mathrm{e}_{\mathrm{j}_1 \mathrm{j}_2 \ldots \mathrm{j}_{\mathrm{n}}} \mid \mathrm{j}_1 \mathrm{j}_2 \ldots \mathrm{j}_{\mathrm{n}} = \text{permutacija skupa } \{k_1, k_2, ..., k_{\mathrm{n}}\} \right\},$$

$$\mathcal{B}_{\mathrm{i}_1^{m_1}\mathrm{i}_2^{m_2}\ldots\mathrm{i}_{\mathrm{N}}^{m_{\mathrm{N}}}} = span_{\mathbb{C}} \left\{ \mathrm{e}_{\mathrm{j}_1 \mathrm{j}_2 \ldots \mathrm{j}_{\mathrm{n}}} \mid \mathrm{j}_1 \mathrm{j}_2 \ldots \mathrm{j}_{\mathrm{n}} = \text{permutacija multiskupa } \left\{ \mathrm{i}_1^{m_1}, \mathrm{i}_2^{m_2}, ..., \mathrm{i}_{\mathrm{N}}^{m_{\mathrm{N}}} \right\} \right\}.$$

Općenito će rastav algebre $\mathcal{B}$ po multihomogenim komponentama glasiti:

$$\mathcal{B} = \bigoplus_{\substack{\mathrm{n}\geq 0 \\ l_1 \leq l_2 \leq \ldots \leq l_{\mathrm{n}},\ l_{\mathrm{j}} \in \mathcal{N}}} \mathcal{B}_{l_1 l_2 \ldots l_{\mathrm{n}}}, \tag{5}$$

gdje svaki potprostor $\mathcal{B}_{l_1 l_2 \ldots l_{\mathrm{n}}}$ zovemo težinski potprostor algebre $\mathcal{B}$.

Pritom je težinski potprostor

$$\mathcal{B}_{\mathrm{Q}} = \mathcal{B}_{l_1 l_2 \ldots l_{\mathrm{n}}} = span_{\mathbb{C}} \left\{ \mathrm{e}_{\mathrm{j}_1 \mathrm{j}_2 \ldots \mathrm{j}_{\mathrm{n}}} \mid \mathrm{j}_1 \mathrm{j}_2 \ldots \mathrm{j}_{\mathrm{n}} = \text{permutacija multiskupa } \{ l_1 \leq l_2 \leq \ldots \leq l_{\mathrm{n}} \} \right\}$$

pridružen multiskupu $\mathrm{Q} = \{ l_1 \leq l_2 \leq \ldots \leq l_{\mathrm{n}} \}$, $l_1, l_2, ..., l_{\mathrm{n}} \in \mathcal{N}$, kojeg ćemo u nastavku pisati u obliku: $\mathrm{Q} = l_1 l_2 \ldots l_{\mathrm{n}}$.

Napomena:

Ako je $l_1 < l_2 < \ldots < l_{\mathrm{n}}$, $l_1, l_2, ..., l_{\mathrm{n}} \in \mathcal{N}$, onda je $\mathrm{Q} = \{ l_1, l_2, ..., l_{\mathrm{n}} \}$ skup i njemu pridruženi potprostor će biti $\mathcal{B}_{\mathrm{Q}}$ generički težinski potprostor algebre $\mathcal{B}$.

Za multiskupove Q koji nisu skupovi, će pripadni težinski potprostor $\mathcal{B}_{\mathrm{Q}}$ biti degenerirani.

2. 1. q-diferencijalna struktura na algebri $\mathcal{B}$

U slobodnoj asocijativnoj $\mathbb{C}$-algebri $\mathcal{B}$ s jedinicom uvesti će se **q**-diferencijalna struktura na sljedeći način.

Neka su q_{ij}, $i, j \in \mathcal{N} = \{i_1, i_2, ..., i_N\}$ zadani kompleksni parametri.

Ekvivalentno, označimo sa **q** pridruženo preslikavanje $\mathbf{q} : \mathcal{N} \times \mathcal{N} \to \mathbb{C}$ takvo da

$$(i, j) \mapsto q_{ij}, \quad i, j \in \mathcal{N} = \{i_1, i_2, ..., i_N\}.$$

Tada uvodimo niz linearnih operatora na algebri $\mathcal{B}$ (koji poopćuju operatore parcijalnih derivacija) $\partial_{i_1}, \partial_{i_2}, ..., \partial_{i_N} : \mathcal{B} \to \mathcal{B}$ zadanih sa:

$$\left.\begin{aligned} \partial_i(1) &= 0, \\ \partial_i(e_j) &= \delta_{ij}, \\ \partial_i(e_j\, x) &= \delta_{ij}\, x + q_{ij}\, e_j\, \partial_i\, x \quad \text{za svaki } x \in \mathcal{B}, \end{aligned}\right\} \tag{6}$$

$i, j \in \mathcal{N} = \{i_1, i_2, ..., i_N\}$.

Pritom je $\delta_{ij} = 1$ za $i = j$, $\delta_{ij} = 0$ za $i \neq j$.

Parametri q_{ij}, $i, j \in \mathcal{N}$ su kompleksni brojevi. Izbor parametara q_{ij} može se interpretirati kao izbor točke u parametarskom prostoru $\mathbb{C}^{N^2}$.

Primjedba 2.1.1

Relacijom (6) definirana je multiparametarski deformirana i-ta parcijalna derivacija, koja je analogon klasične i-te parcijalne derivacije, time da se u njenom opisu dodatno pojavljuju parametri q_{ij}.

Specijalno, kada bi svi q_{ij} bili jednaki jedan, tada bi ∂_i bila obična i-ta parcijalna derivacija po varijabli e_i.

Kažemo još da je relacijom (6) dana tzv. **q**-deformirana i-ta parcijalna derivacija.

U daljnjem ćemo algebru $\mathcal{B}$ promatrati zajedno s gore definiranom **q**-diferencijalnom strukturom.

Primjenom rekurzivne formule (6) lako dobivamo eksplicitnu formulu:

$$\begin{aligned}\partial_i\left(e_{j_1}e_{j_2}\cdots e_{j_n}\right) &= \sum_{\substack{1\le p\le n\\ j_p=i}} q_{ij_1}q_{ij_2}\cdots q_{ij_{p-1}}\, e_{j_1}e_{j_2}\cdots e_{j_{p-1}}e_{j_{p+1}}\cdots e_{j_n}\\ &= \sum_{\substack{1\le p\le n\\ j_p=i}}\left(\prod_{r=1}^{p-1} q_{ij_r}\right) e_{j_1}e_{j_2}\cdots e_{j_{p-1}}e_{j_{p+1}}\cdots e_{j_n}\ . \end{aligned}\tag{7}$$

Specijalno ako postoji samo jedan p takav da je $j_p = i$, (tj. u nizu $j_1 j_2 ... j_{p-1} j_p j_{p+1} ... j_n$ se i pojavljuje točno jednom), onda se formula (7) reducira na

$$\begin{aligned}\partial_{j_p}\left(e_{j_1}e_{j_2}\cdots e_{j_{p-1}}e_{j_p}e_{j_{p+1}}\cdots e_{j_n}\right) &= q_{j_pj_1}q_{j_pj_2}\cdots q_{j_pj_{p-1}}\, e_{j_1}e_{j_2}\cdots e_{j_{p-1}}e_{j_{p+1}}\cdots e_{j_n}\\ &= \prod_{r=1}^{p-1} q_{j_pj_r}\, e_{j_1}e_{j_2}\cdots e_{j_{p-1}}e_{j_{p+1}}\cdots e_{j_n}\ . \end{aligned}\tag{8}$$

Slično u slučaju $i = j_1 = j_2 = ... = j_{p-1} = j_p = j_{p+1} = ... = j_n$ iz formule (7) dobivamo:

$$\partial_i\left(e_i^n\right) = [n]_{q_{ii}}\, e_i^{n-1}, \tag{9}$$

pri čemu je

$$[n]_q := \sum_{r=0}^{n-1} q^r = 1 + q + q^2 + ... + q^{n-1}\ ,\ \ n \ge 1\ .$$

Primjer 2.1.2

Neka je $n = 6$, $j_1 = 1$, $j_2 = 3$, $j_3 = 1$, $j_4 = 2$, $j_5 = 1$, $j_6 = 2$.

Tada primjenom formule (7) dobivamo:

$$\partial_i\left(e_{j_1}e_{j_2}e_{j_3}e_{j_4}e_{j_5}e_{j_6}\right) = \partial_i\left(e_1e_3e_1e_2e_1e_2\right)$$

$$= \begin{cases} e_3e_1e_2e_1e_2 + q_{11}q_{13}\, e_1e_3e_2e_1e_2 + q_{11}^2q_{13}q_{12}\, e_1e_3e_1e_2e_2 & \text{za } i = 1,\\ q_{21}^2q_{23}\, e_1e_3e_1e_1e_2 + q_{21}^3q_{23}q_{22}\, e_1e_3e_1e_2e_1 & \text{za } i = 2,\\ q_{31}\, e_1e_1e_2e_1e_2 & \text{za } i = 3,\\ 0 & \text{za } i \ge 4. \end{cases}$$

Označimo sada sa $x = e_{j_1}e_{j_2}$, $y = e_{j_3}e_{j_4}e_{j_5}e_{j_6}$.

Tada je

$$\partial_i(x\cdot y) = \partial_i\left(e_{j_1}e_{j_2}e_{j_3}e_{j_4}e_{j_5}e_{j_6}\right),$$

odnosno

$$\partial_i(x\cdot y) = \partial_i\left(\left(e_{j_1}e_{j_2}\right)\cdot\left(e_{j_3}e_{j_4}e_{j_5}e_{j_6}\right)\right) = \partial_i\left(\left(e_1e_3\right)\cdot\left(e_1e_2e_1e_2\right)\right),$$

pri čemu se koristila pretpostavka da je $j_1=1$, $j_2=3$, $j_3=1$, $j_4=2$, $j_5=1$, $j_6=2$.

Imamo sljedeće:

za $i=1$

$$\begin{aligned}\partial_1\left((e_1e_3)\cdot(e_1e_2e_1e_2)\right) &= e_3e_1e_2e_1e_2 + q_{11}q_{13}\, e_1e_3e_2e_1e_2 + q_{11}^2q_{13}q_{12}\, e_1e_3e_1e_2e_2 \\ &= e_3\cdot(e_1e_2e_1e_2) + q_{11}q_{13}(e_1e_3)\cdot(e_2e_1e_2 + q_{11}q_{12}\, e_1e_2e_2) \\ &= \partial_1(e_1e_3)\cdot(e_1e_2e_1e_2) + q_{11}q_{13}(e_1e_3)\cdot\partial_1(e_1e_2e_1e_2)\end{aligned}$$

za $i=2$

$$\begin{aligned}\partial_2\left((e_1e_3)\cdot(e_1e_2e_1e_2)\right) &= q_{21}^2q_{23}\, e_1e_3e_1e_1e_2 + q_{21}^3q_{23}q_{22}\, e_1e_3e_1e_2e_1 \\ &= 0\cdot(e_1e_2e_1e_2) + q_{21}q_{23}(e_1e_3)\cdot\left(q_{21}\, e_1e_1e_2 + q_{21}^2q_{22}\, e_1e_2e_1\right) \\ &= \partial_2(e_1e_3)\cdot(e_1e_2e_1e_2) + q_{21}q_{23}(e_1e_3)\cdot\partial_2(e_1e_2e_1e_2)\end{aligned}$$

za $i=3$

$$\begin{aligned}\partial_3\left((e_1e_3)\cdot(e_1e_2e_1e_2)\right) &= q_{31}\, e_1e_1e_2e_1e_2 \\ &= q_{31}\, e_1\cdot(e_1e_2e_1e_2) + q_{31}q_{33}(e_1e_3)\cdot 0 \\ &= \partial_3(e_1e_3)\cdot(e_1e_2e_1e_2) + q_{31}q_{33}(e_1e_3)\cdot\partial_3(e_1e_2e_1e_2)\end{aligned}$$

za $i\geq 4$

$$\begin{aligned}\partial_i\left((e_1e_3)\cdot(e_1e_2e_1e_2)\right) &= 0 \\ &= 0\cdot(e_1e_2e_1e_2) + q_{i1}q_{i3}(e_1e_3)\cdot 0 \\ &= \partial_i(e_1e_3)\cdot(e_1e_2e_1e_2) + q_{i1}q_{i3}(e_1e_3)\cdot\partial_i(e_1e_2e_1e_2)\end{aligned}$$

Time možemo pisati

$$\partial_i\left((e_{j_1}e_{j_2})\cdot(e_{j_3}e_{j_4}e_{j_5}e_{j_6})\right) = \partial_i(e_{j_1}e_{j_2})\cdot(e_{j_3}e_{j_4}e_{j_5}e_{j_6}) + q_{ij_1}q_{ij_2}(e_{j_1}e_{j_2})\cdot\partial_i(e_{j_3}e_{j_4}e_{j_5}e_{j_6})$$

ili

$$\begin{aligned}\partial_i(x\cdot y) &= \partial_i(x)\cdot y + \prod_{r=1}^{2} q_{ij_r}\, x\cdot\partial_i(y) \\ &= \partial_i(x)\cdot y + q'\cdot x\cdot\partial_i(y)\end{aligned}$$

gdje je $q' = \prod_{r=1}^{2} q_{ij_r} \in \mathbb{C}$,

$x = e_{j_1}e_{j_2}$, $y = e_{j_3}e_{j_4}e_{j_5}e_{j_6}$.

Primjedba 2.1.3

Neka je $z = e_{j_1} e_{j_2} \cdots e_{j_m}$ takav da je $z = x \cdot y$,

pri čemu je $x = e_{j_1} e_{j_2} \cdots e_{j_k}$, $y = e_{j_{k+1}} e_{j_{k+2}} \cdots e_{j_m}$.

Tada primjenom formule (7) dobivamo:

$$\partial_i(z) = \partial_i(x \cdot y) = \partial_i\left(e_{j_1} e_{j_2} \cdots e_{j_m}\right) = \sum_{\substack{1 \le p \le m \\ j_p = i}} \prod_{r=1}^{p-1} q_{i j_r} \, e_{j_1} e_{j_2} \cdots e_{j_{p-1}} e_{j_{p+1}} \cdots e_{j_m}$$

$$\partial_i(x) = \partial_i\left(e_{j_1} e_{j_2} \cdots e_{j_k}\right) = \sum_{\substack{1 \le p \le k \\ j_p = i}} \prod_{r=1}^{p-1} q_{i j_r} \, e_{j_1} e_{j_2} \cdots e_{j_{p-1}} e_{j_{p+1}} \cdots e_{j_k}$$

$$\partial_i(y) = \partial_i\left(e_{j_{k+1}} e_{j_{k+2}} \cdots e_{j_m}\right) = \sum_{\substack{k+1 \le p \le m \\ j_p = i}} \prod_{r=k+1}^{p-1} q_{i j_r} \, e_{j_{k+1}} e_{j_{k+2}} \cdots e_{j_{p-1}} e_{j_{p+1}} \cdots e_{j_m}$$

gdje je $i \in \mathcal{N} = \{i_1, i_2, ..., i_N\}$.

Za svaki (fiksni) $1 \le k < m$ imamo da se suma

$$\partial_i(z) = \sum_{\substack{1 \le p \le m \\ j_p = i}} \prod_{r=1}^{p-1} q_{i j_r} \, e_{j_1} e_{j_2} \cdots e_{j_{p-1}} e_{j_{p+1}} \cdots e_{j_m}$$

može (na prirodan način) rastaviti na dva sumanda

(1) $$\sum_{\substack{1 \le p \le k \\ j_p = i}} \prod_{r=1}^{p-1} q_{i j_r} \, e_{j_1} e_{j_2} \cdots e_{j_{p-1}} e_{j_{p+1}} \cdots e_{j_k} \cdot e_{j_{k+1}} e_{j_{k+2}} \cdots e_{j_m}$$

$$= \underbrace{\left(\sum_{\substack{1 \le p \le k \\ j_p = i}} \prod_{r=1}^{p-1} q_{i j_r} \, e_{j_1} e_{j_2} \cdots e_{j_{p-1}} e_{j_{p+1}} \cdots e_{j_k} \right)}_{=\partial_i(x)} \cdot \underbrace{e_{j_{k+1}} e_{j_{k+2}} \cdots e_{j_m}}_{=y} = \partial_i(x) \cdot y$$

i

(2) $$\sum_{\substack{k+1 \le p \le m \\ j_p = i}} \prod_{r=1}^{p-1} q_{i j_r} \, e_{j_1} e_{j_2} \cdots e_{j_k} \cdot e_{j_{k+1}} e_{j_{k+2}} \cdots e_{j_{p-1}} e_{j_{p+1}} \cdots e_{j_m}$$

$$= \underbrace{e_{j_1} e_{j_2} \cdots e_{j_k}}_{=x} \cdot \left(\sum_{\substack{k+1 \le p \le m \\ j_p = i}} \prod_{r=1}^{p-1} q_{i j_r} \, e_{j_{k+1}} e_{j_{k+2}} \cdots e_{j_{p-1}} e_{j_{p+1}} \cdots e_{j_m} \right)$$

$$= x \cdot \left(\sum_{\substack{k+1 \le p \le m \\ j_p = i}} \prod_{r=1}^{k} q_{i j_r} \prod_{r=k+1}^{p-1} q_{i j_r} \, e_{j_{k+1}} e_{j_{k+2}} \cdots e_{j_{p-1}} e_{j_{p+1}} \cdots e_{j_m} \right)$$

$$=\prod_{r=1}^{k} q_{ij_r}\, x\cdot\underbrace{\left(\sum_{\substack{k+1\le p\le m\\ j_p=i}}\prod_{r=k+1}^{p-1} q_{ij_r}\, e_{j_{k+1}} e_{j_{k+2}}\cdots e_{j_{p-1}} e_{j_{p+1}}\cdots e_{j_m}\right)}_{=\partial_i(y)}$$

$$=\prod_{r=1}^{k} q_{ij_r}\, x\cdot\partial_i(y)$$

stoga dobivamo:

$$\sum_{\substack{1\le p\le m\\ j_p=i}}\prod_{r=1}^{p-1} q_{ij_r}\, e_{j_1} e_{j_2}\cdots e_{j_{p-1}} e_{j_{p+1}}\cdots e_{j_m}$$

$$=\left(\sum_{\substack{1\le p\le k\\ j_p=i}}\prod_{r=1}^{p-1} q_{ij_r}\, e_{j_1} e_{j_2}\cdots e_{j_{p-1}} e_{j_{p+1}}\cdots e_{j_k}\right)\cdot e_{j_{k+1}} e_{j_{k+2}}\cdots e_{j_m}$$

$$+e_{j_1} e_{j_2}\cdots e_{j_k}\cdot\left(\sum_{\substack{k+1\le p\le m\\ j_p=i}}\prod_{r=1}^{p-1} q_{ij_r}\, e_{j_{k+1}} e_{j_{k+2}}\cdots e_{j_{p-1}} e_{j_{p+1}}\cdots e_{j_m}\right)$$

ili

$$\partial_i(x\cdot y)=\partial_i(x)\cdot y+\prod_{r=1}^{k} q_{ij_r}\, x\cdot\partial_i(y),$$

gdje je $x=e_{j_1} e_{j_2}\cdots e_{j_k}$.

Propozicija 2.1.4

Neka je $x\in\mathcal{B}_Q=span_{\mathbb{C}}\left\{e_{j_1} e_{j_2}\cdots e_{j_n}\mid j_1 j_2 ... j_n=\text{permutacija multiskupa } Q\right\}$,

gdje je $\mathcal{B}_Q$ težinski potprostor (algebre $\mathcal{B}$) pridružen multiskupu $Q=\{l_1\le l_2\le ...\le l_n\}$

nad $\mathcal{N}=\{i_1, i_2, ..., i_N\}$ i neka je $y\in\mathcal{B}$.

Tada vrijedi formula

$$\partial_i(x\cdot y)=\partial_i(x)\cdot y+q_{il_1} q_{il_2}\cdots q_{il_n}\cdot x\cdot\partial_i(y) \tag{10}$$

za bilo koji $x\in\mathcal{B}_Q$.

Formula (10) je poopćena formula (6).

Propozicija 2.1.4 dokazuje se analogno provedenim razmatranjima u primjedbi 2.1.3.

Primjedba 2.1.5

U radovima fizičara [F2], [F3] i [FG] se operatori $\partial_i : \mathcal{B} \to \mathcal{B}$ interpretiraju kao operatori poništavanja (i-te čestice), a operatori množenja sa e_i, $e_i : \mathcal{B} \to \mathcal{B}$ kao operatori stvaranja i-te čestice, $i \in \mathcal{N}$.

Za svaki par (i, j), $i \neq j$ uvodi se po jedna kvantna Serre-ova relacija na osnovu kojih se definira Serre-ov ideal u generaliziranoj super Kac-Moody-jevoj algebri.

Nadalje, gore definirana algebra $\mathcal{B}$ je u direktnoj vezi s multiparametarskom quonskom algebrom $\mathcal{A} = \mathcal{A}^{(\mathbf{q})}$, definiranom u [MSP], [MS1], [MS2].

Ukratko, multiparametarska quonska algebra $\mathcal{A} = \mathcal{A}^{(\mathbf{q})}$ je asocijativna (kompleksna) algebra, koja je generirana sa skupom generatora $\{ a_i , a_i^+ , i \in I\}$ te u kojoj su jedine relacije oblika $a_i a_j^+ = q_{ij} a_j^+ a_i + \delta_{ij}$ za svaki $i, j \in I$.

Pritom je $\mathbf{q} = \left\{ q_{ij} : i, j \in I, \overline{q_{ij}} = q_{ji} \right\}$ hermitska familija kompleksnih brojeva, tj. parametara, a I je konačan (ili beskonačan) skup indeksa.

U [MS1], [MS2] dana je Fock-ova reprezentacija algebre $\mathcal{A}^{(\mathbf{q})}$ na slobodnoj asocijativnoj algebri **f** (algebra nekomutirajućih polinoma u nekomutirajućim varijablama θ_i, $i \in I$) u kojoj se generator a_i reprezentira pomoću generalizirane q_{ij}-deformirane parcijalne derivacije po varijabli θ_i (i-ti operator poništavanja), a generator a_i^+ pomoću množenja s θ_i (i-ti operator stvaranja). Pritom je operator koji reprezentira a_j^+ hermitski adjungiran operatoru koji reprezentira a_i ako je $|q_{ij}| < 1$ za svaki $i, j \in I$.

Definicija 2.1.6

Za element $C \in \mathcal{B}$ kažemo da je **konstanta** ako je $\partial_{i_s} C = 0$ za svaki $1 \leq s \leq N$.

Propozicija 2.1.7

Konstante u algebri $\mathcal{B}$ generiraju obostrani ideal $\boldsymbol{I}$ u algebri $\mathcal{B}$.

Dokaz:

Neka je $\mathcal{C} \subset \mathcal{B}$ potprostor svih konstanti u algebri $\mathcal{B}$.

Za bilo koju konstantu $C \in \mathcal{C}$ je po definiciji $\partial_{\mathrm{i}}(C)=0$ za svaki $\mathrm{i} \in \mathcal{N}$, gdje su $\partial_{\mathrm{i}}: \mathcal{B} \rightarrow \mathcal{B}$ linearni operatori na $\mathcal{B}$ definirani relacijom (6).

Neka je $\boldsymbol{I}=\langle\mathcal{C}\rangle \subset \mathcal{B}$ obostrani ideal u algebri $\mathcal{B}$ generiran konstantama.

Prisjetimo se da je tipični element $\mathrm{x} \in \boldsymbol{I}$ oblika $\mathrm{x}=\sum_{\mathrm{j}=1}^{\mathrm{k}} b_{\mathrm{j}}^{'} \cdot c_{\mathrm{j}} \cdot b_{\mathrm{j}}^{''}$, $b_{\mathrm{j}}^{'}, b_{\mathrm{j}}^{''} \in \mathcal{B}$, $c_{\mathrm{j}} \in \mathcal{C}$.

Pokažimo sada da je ideal $\boldsymbol{I}$ invarijantni potprostor svakog operatora $\partial_{\mathrm{i}}: \mathcal{B} \rightarrow \mathcal{B}$, $\mathrm{i} \in \mathcal{N}$, tj. da je $\partial_{\mathrm{i}}(\boldsymbol{I}) \subset \boldsymbol{I}$ za svaki $\mathrm{i} \in \mathcal{N}$.

Treba dokazati da $\mathrm{x} \in \boldsymbol{I}$ povlači da je $\partial_{\mathrm{i}}(\mathrm{x}) \in \boldsymbol{I}$, $\mathrm{i} \in \mathcal{N}$.

Neka je $\mathrm{x} \in \boldsymbol{I}$, $\mathrm{x}=\sum_{\mathrm{j}=1}^{\mathrm{k}} b_{\mathrm{j}}^{'} \cdot c_{\mathrm{j}} \cdot b_{\mathrm{j}}^{''}$, $b_{\mathrm{j}}^{'}, b_{\mathrm{j}}^{''} \in \mathcal{B}$, $c_{\mathrm{j}} \in \mathcal{C}$.

Koristeći svojstvo da linearni operatori $\partial_{\mathrm{i}}: \mathcal{B} \rightarrow \mathcal{B}$, $\mathrm{i} \in \mathcal{N}$ zadovoljavaju svojstvo aditivnosti, dovoljno je pogledati djelovanje linearnih operatora na elementima

$$x_{\mathrm{j}}=b_{\mathrm{j}}^{'} \cdot c_{\mathrm{j}} \cdot b_{\mathrm{j}}^{''}, \quad b_{\mathrm{j}}^{'}, b_{\mathrm{j}}^{''} \in \mathcal{B},\ c_{\mathrm{j}} \in \mathcal{C}$$

pri čemu će se koristiti propozicija 2.1.4. Imamo

$$\begin{aligned}
\partial_{\mathrm{i}}\left(x_{\mathrm{j}}\right) &= \partial_{\mathrm{i}}\left(b_{\mathrm{j}}^{'} \cdot c_{\mathrm{j}} \cdot b_{\mathrm{j}}^{''}\right) \\
&= \partial_{\mathrm{i}}\left(b_{\mathrm{j}}^{'}\right) \cdot c_{\mathrm{j}} \cdot b_{\mathrm{j}}^{''}+q_{\mathrm{j}}^{'} \cdot b_{\mathrm{j}}^{'} \cdot \partial_{\mathrm{i}}\left(c_{\mathrm{j}} \cdot b_{\mathrm{j}}^{''}\right) \quad \left(\text{za neki } q_{\mathrm{j}}^{'} \in \mathbb{C}\right) \\
&= \partial_{\mathrm{i}}\left(b_{\mathrm{j}}^{'}\right) \cdot c_{\mathrm{j}} \cdot b_{\mathrm{j}}^{''}+q_{\mathrm{j}}^{'} \cdot b_{\mathrm{j}}^{'} \cdot\left(\underbrace{\partial_{\mathrm{i}}\left(c_{\mathrm{j}}\right)}_{=0} \cdot b_{\mathrm{j}}^{''}+q_{\mathrm{j}}^{''} \cdot c_{\mathrm{j}} \cdot \partial_{\mathrm{i}}\left(b_{\mathrm{j}}^{''}\right)\right) \quad \left(\text{za neki } q_{\mathrm{j}}^{''} \in \mathbb{C}\right) \\
&= \partial_{\mathrm{i}}\left(b_{\mathrm{j}}^{'}\right) \cdot c_{\mathrm{j}} \cdot b_{\mathrm{j}}^{''}+q_{\mathrm{j}}^{'} \cdot q_{\mathrm{j}}^{''} \cdot b_{\mathrm{j}}^{'} \cdot c_{\mathrm{j}} \cdot \partial_{\mathrm{i}}\left(b_{\mathrm{j}}^{''}\right) \\
&= \left(\partial_{\mathrm{i}}\left(b_{\mathrm{j}}^{'}\right) \cdot c_{\mathrm{j}} \cdot b_{\mathrm{j}}^{''}+q_{\mathrm{j}} \cdot b_{\mathrm{j}}^{'} \cdot c_{\mathrm{j}} \cdot \partial_{\mathrm{i}}\left(b_{\mathrm{j}}^{''}\right)\right) \in \boldsymbol{I}
\end{aligned}$$

gdje je $q_{\mathrm{j}}=q_{\mathrm{j}}^{'} \cdot q_{\mathrm{j}}^{''}$.

Primjedba 2.1.8

Na kvocjentnoj algebri $\overline{\mathcal{B}}=\mathcal{B} / \boldsymbol{I}=\{\mathrm{x}+\boldsymbol{I} \mid \mathrm{x} \in \mathcal{B}\}$ možemo definirati operatore deriviranja $\overline{\partial_{\mathrm{i}}}: \overline{\mathcal{B}} \rightarrow \overline{\mathcal{B}}$, $\mathrm{i} \in \mathcal{N}=\left\{\mathrm{i}_{1}, \mathrm{i}_{2}, \ldots, \mathrm{i}_{\mathrm{N}}\right\}$ na sljedeći način:

$$\overline{\partial_{\mathrm{i}}}(\mathrm{x}+\boldsymbol{I}):=\partial_{\mathrm{i}} \mathrm{x}+\boldsymbol{I}, \tag{11}$$

gdje je djelovanje linearnih **q**-diferencijalnih operatora $\{\partial_i\}_{i\in\mathcal{N}}$ na algebri $\boldsymbol{\mathcal{B}}$ dano formulom (6).

Želimo dokazati da je operator $\overline{\partial_i}$ dobro definiran, tj. da ne ovisi o izboru reprezentante.

Pretpostavimo da je $x\in\boldsymbol{\mathcal{B}}$ ekvivalentan s $x'\in\boldsymbol{\mathcal{B}}$, tj. $x+\boldsymbol{I}=x'+\boldsymbol{I}$ ili $x=x'+y$ za neki $y\in\boldsymbol{I}$.

Dokažimo da je $\partial_i x\in\boldsymbol{\mathcal{B}}$ ekvivalentan s $\partial_i x'\in\boldsymbol{\mathcal{B}}$ za svaki $i\in\mathcal{N}=\{i_1,i_2,...,i_N\}$.

Koristeći primjedbu 2.1.7 imamo da je ideal $\boldsymbol{I}$ invarijantni potprostor operatora $\partial_i:\boldsymbol{\mathcal{B}}\to\boldsymbol{\mathcal{B}}$, $i\in\mathcal{N}$, stoga iz pretpostavke

$$x=x'+y\,,\ y\in\boldsymbol{I}$$

proizlazi
$$\partial_i x=\partial_i(x'+y)=\partial_i x'+\partial_i y=\partial_i x'+y'\,,\quad y'\in\boldsymbol{I}$$

ili
$$\partial_i x+\boldsymbol{I}=\partial_i x'+\boldsymbol{I}$$

na osnovu čega zaključujemo da je $\partial_i x$ ekvivalentan s $\partial_i x'$ za svaki $i\in\mathcal{N}$.

Time smo pokazali da **q**-diferencijalna struktura $\{\partial_i\}$ algebre $\boldsymbol{\mathcal{B}}$ inducira **q**-diferencijalnu strukturu $\{\overline{\partial_i}\}$ na kvocjentnoj algebri $\overline{\boldsymbol{\mathcal{B}}}=\boldsymbol{\mathcal{B}}/\boldsymbol{I}$.

U [FG], [F2] se kvocjentna algebra $\overline{\boldsymbol{\mathcal{B}}}=\boldsymbol{\mathcal{B}}/\boldsymbol{I}$ dovodi u relaciju sa Yang-Baxter-ovom jednadžbom i kvantnim Kac-Moody-jevim algebrama.

Primjer 2.1.9

Ako za svaki $i\in\mathcal{N}$ je $q_{ii}=-1$, pri čemu je $\prod\limits_{1\le j\ne k\le m} q_{i_j i_k}\ne 1$, $i_j,i_k\in\mathcal{N}$, $m\in\{2,3,...,N\}$, onda je ideal $\boldsymbol{I}$ algebre $\boldsymbol{\mathcal{B}}$ generiran sa skupom $\{e_i^2\}_{i\in\mathcal{N}}$ i pišemo $\boldsymbol{I}=\langle\, e_i^2\mid i\in\mathcal{N}\rangle$. U tom je slučaju e_i^2 konstanta za svaki $i\in\mathcal{N}=\{i_1,i_2,...,i_N\}$.

Dokažimo to.

Primjenom formule (6) za svaki $j\in\mathcal{N}$ imamo

$$\begin{aligned}\partial_j\left(e_i^2\right)&=\partial_j\left(e_i\,e_i\right)=\left(\partial_j\,e_i\right)e_i+q_{ji}\,e_i\left(\partial_j\,e_i\right)\\&=\delta_{ji}\,e_i+q_{ji}\,e_i\,\delta_{ji}=\begin{cases}0 & \text{za } j\ne i\,,\\(1+q_{ii})e_i & \text{za } j=i\,.\end{cases}\end{aligned}$$

Dobivamo da $q_{ii} = -1$ povlači $\partial_j\left(e_i^2\right) = 0$ za svaki $j \in \mathcal{N}$. U protivnom, ako je $q_{ii} \neq -1$, onda je $\partial_j\left(e_i^2\right) \neq 0$ za $j = i$.

Time proizlazi da je e_i^2 ako je $q_{ii} = -1$ za svaki $i \in \mathcal{N} = \{i_1, i_2, ..., i_N\}$.

Napomena:

Ako je $q_{ii} = -1$ za svaki $i \in \mathcal{N}$ i ako postoji barem jedan $m \in \{2,3,...,N\}$ takav da je $\prod_{1 \leq j \neq k \leq m} q_{i_j i_k} = 1$, onda će ideal $\boldsymbol{I}$ algebre $\boldsymbol{\mathcal{B}}$ biti generiran sa skupom $\left\{e_i^2\right\}_{i \in \mathcal{N}}$, ali isto tako i sa nekim odgovarajućim skupom konstanti (koji se dobiva za $\prod_{1 \leq j \neq k \leq m} q_{i_j i_k} = 1$). Prirodno se nameće problem kako glasi taj skup konstanti.

U ovoj radnji će se detaljno proučavati nalaženje prostora konstanti u algebri $\boldsymbol{\mathcal{B}}$ te će se detaljno izračunati prostor konstanti u težinskim potprostorima $\boldsymbol{\mathcal{B}}(k)$, $1 \leq k \leq 4$ algebre $\boldsymbol{\mathcal{B}}$.

Uvodimo sljedeće oznake.

✐ <u>Definiramo</u>

$$\sigma_{ij} := q_{ij} q_{ji} \tag{12}$$

$i, j \in \mathcal{N} = \{i_1, i_2, ..., i_N\}$, $i \neq j$; $q_{ij} \in \mathbb{C}$. Pritom je $\sigma_{ji} = \sigma_{ij}$.

Analogno za $i, j, k \in \mathcal{N} = \{i_1, i_2, ..., i_N\}$, $i \neq j \neq k$ definira se

$$\sigma_{ijk} := \sigma_{ij} \sigma_{ik} \sigma_{jk} = q_{ij} q_{ji} q_{ik} q_{ki} q_{jk} q_{kj} \tag{13}$$

odnosno induktivno

$$\sigma_{i_1 i_2 \ldots i_N} := \prod_{1 \leq j < k \leq N} \sigma_{i_j i_k} = \prod_{1 \leq j \neq k \leq N} q_{i_j i_k} . \tag{14}$$

Specijalno za $j = i$ iz (12) proizlazi $\sigma_{ii} = q_{ii}^2$,

odnosno za $i_1 = i_2 = \cdots = i_N = i$ iz (14) proizlazi

$$\sigma_{i^N} = q_{ii}^{N(N-1)}, \tag{15}$$

gdje je $i^N := \underbrace{ii...i}_{\text{N puta}}$.

✐ Definiramo iterirane komutatore

$$Y_{i_1} := e_{i_1},$$

$$Y_{i_1 i_2} := \left[Y_{i_1}, e_{i_2} \right]_{q_{i_2 i_1}} = \left[e_{i_1}, e_{i_2} \right]_{q_{i_2 i_1}} = e_{i_1} e_{i_2} - q_{i_2 i_1} e_{i_2} e_{i_1},$$

$$Y_{i_1 i_2 i_3} := \left[Y_{i_1 i_2}, e_{i_3} \right]_{q_{i_3 i_1} q_{i_3 i_2}} = \left[\left[e_{i_1}, e_{i_2} \right]_{q_{i_2 i_1}}, e_{i_3} \right]_{q_{i_3 i_1} q_{i_3 i_2}}$$

$$= \left(e_{i_1} e_{i_2} - q_{i_2 i_1} e_{i_2} e_{i_1} \right) e_{i_3} - q_{i_3 i_1} q_{i_3 i_2} e_{i_3} \left(e_{i_1} e_{i_2} - q_{i_2 i_1} e_{i_2} e_{i_1} \right)$$

$$= e_{i_1} e_{i_2} e_{i_3} - q_{i_2 i_1} e_{i_2} e_{i_1} e_{i_3} - q_{i_3 i_1} q_{i_3 i_2} e_{i_3} e_{i_1} e_{i_2} + q_{i_3 i_1} q_{i_3 i_2} q_{i_2 i_1} e_{i_3} e_{i_2} e_{i_1},$$

$$\vdots$$

$$Y_{i_1 i_2 \dots i_p} := \left[Y_{i_1 i_2 \dots i_{p-1}}, e_{i_p} \right]_{q_{i_p i_1} q_{i_p i_2} \cdots q_{i_p i_{p-1}}}$$

$$= Y_{i_1 i_2 \dots i_{p-1}} e_{i_p} - q_{i_p i_1} q_{i_p i_2} \dots q_{i_p i_{p-1}} e_{i_p} Y_{i_1 i_2 \dots i_{p-1}} \qquad (16)$$

$2 \le p \le N$.

Specijalno za $p = n$, $i_1 = 1$, $i_2 = 2, \dots, i_n = n$ imamo

$$Y_{12\dots n} = \left[Y_{12\dots(n-1)}, e_n \right]_{q_{n1} q_{n2} \cdots q_{n(n-1)}}$$

$$= \left[\left[Y_{12\dots(n-2)}, e_{n-1} \right]_{q_{(n-1)1} q_{(n-1)2} \cdots q_{(n-1)(n-2)}}, e_n \right]_{q_{n1} q_{n2} \cdots q_{n(n-1)}}$$

$$\vdots$$

$$= \left[\left[\dots \left[\left[Y_{12}, e_3 \right]_{q_{31} q_{32}}, e_4 \right]_{q_{41} q_{42} q_{43}}, \dots, e_{n-1} \right]_{q_{(n-1)1} q_{(n-1)2} \cdots q_{(n-1)(n-2)}}, e_n \right]_{q_{n1} q_{n2} \cdots q_{n(n-1)}}$$

Ako je $i_1 = i_2 = i$, onda je iterirani komutator $Y_{i_1 i_2}$ oblika

$$Y_{ii} = \left(1 - q_{ii} \right) e_i^2. \qquad (17)$$

Analogno, ako je $i_1 = i_2 = i_3 = i$, onda je iterirani komutator $Y_{i_1 i_2 i_3}$ oblika

$$Y_{iii} = \left(1 - q_{ii} \right) \left(1 - q_{ii}^2 \right) e_i^3, \qquad (18)$$

odnosno općenito za $i_1 = i_2 = \cdots = i_n = i$ imamo

$$Y_{i^n} = \prod_{k=1}^{n-1} \left(1 - q_{ii}^k \right) e_i^n, \qquad (19)$$

pri čemu je $i^n = \underbrace{ii\dots i}_{\text{n puta}}$.

✎ Definiramo jednostavne komutatore

$$X^{i_1 i_2} := \left[e_{i_1}, e_{i_2} \right]_{q_{i_2 i_1}} = e_{i_1} e_{i_2} - q_{i_2 i_1} e_{i_2} e_{i_1}$$

$$X^{i_1 i_2 i_3} := \left[e_{i_1} e_{i_2}, e_{i_3} \right]_{q_{i_3 i_1} q_{i_3 i_2}} = e_{i_1} e_{i_2} e_{i_3} - q_{i_3 i_1} q_{i_3 i_2} e_{i_3} e_{i_1} e_{i_2}$$

$$\vdots$$

$$X^{i_1 i_2 \dots i_p} := \left[e_{i_1} e_{i_2} \cdots e_{i_{p-1}}, e_{i_p} \right]_{q_{i_p i_1} q_{i_p i_2} \dots q_{i_p i_{p-1}}} \tag{20}$$

$$= e_{i_1} e_{i_2} \cdots e_{i_{p-1}} e_{i_p} - q_{i_p i_1} q_{i_p i_2} \dots q_{i_p i_{p-1}} e_{i_p} e_{i_1} e_{i_2} \cdots e_{i_{p-1}}$$

$2 \le p \le N$, gdje je $X^{i_1} := e_{i_1}$.

Specijalno za $i_1 = i_2 = \cdots = i_N = i$ iz (20) proizlazi

$$X^{i^N} = \left[e_i^{N-1}, e_i \right]_{q_{ii}^{N-1}}. \tag{21}$$

Primjedba 2.1.10

Formula (19) daje direktnu vezu između n-te potencije $e_{i_s}^n = \left(e_{i_s} \right)^n$ bilo kojeg generatora e_{i_s}, $i_s \in \mathcal{N}$ algebre $\boldsymbol{\mathcal{B}}$ i pripadnog iteriranog komutatora $Y_{i_s^n}$

$$e_{i_s}^n = \frac{Y_{i_s^n}}{\prod_{k=1}^{n-1} \left(1 - q_{i_s i_s}^k \right)},$$

pri čemu je $1 - q_{i_s i_s}^k \neq 0$, $1 \le s \le N$, $n \ge 2$.

Propozicija 2.1.11

Za svaki $i, j \in \mathcal{N}$ vrijede sljedeće tvrdnje.

(*i*) Ako je $Y_{ij} = 0,\ Y_{ji} = 0$, onda je $\sigma_{ij} = 1$.

(*ii*) Ako je $\sigma_{ij} = 1$, onda je $Y_{ij} = -q_{ji} Y_{ji}$, odnosno $Y_{ji} = -q_{ij} Y_{ij}$.

Dokaz:

Pretpostavimo da je $i \neq j$, $i, j \in \mathcal{N} = \{ i_1, i_2, \dots, i_N \}$. Tada

(*i*) $$Y_{ij} = 0 \quad \Rightarrow \quad \left[e_i, e_j \right]_{q_{ji}} = 0, \quad \text{odnosno} \quad e_i e_j - q_{ji} e_j e_i = 0, \tag{22}$$

$$Y_{ji} = 0 \quad \Rightarrow \quad \left[e_j, e_i \right]_{q_{ij}} = 0, \quad \text{odnosno} \quad e_j e_i - q_{ij} e_i e_j = 0. \tag{23}$$

Rješavanjem sustava jednadžbi (22) i (23) dobivamo $e_j e_i \left(1 - q_{ij} q_{ji}\right) = 0$, što povlači da je $q_{ij} q_{ji} = 1$, odnosno $\sigma_{ij} = 1$.

(*ii*) Iz $\sigma_{ij} = 1$ proizlazi $q_{ij} = \dfrac{1}{q_{ji}}$, odnosno $q_{ji} = \dfrac{1}{q_{ij}}$. Time je

$$Y_{ij} = \left[e_i, e_j\right]_{q_{ji}} = e_i e_j - q_{ji}\, e_j e_i = -q_{ji}\left(e_j e_i - \frac{1}{q_{ji}} e_i e_j\right) = -q_{ji}\left(e_j e_i - q_{ij}\, e_i e_j\right) = -q_{ji} Y_{ji}$$

Analogno se pokazuje da je $Y_{ji} = -q_{ij} Y_{ij}$ ako je $\sigma_{ij} = 1$.

Pretpostavimo sada da je $i = j$, $i, j \in \mathcal{N} = \{i_1, i_2, ..., i_N\}$.

Tada je $Y_{ii} = [e_i, e_i]_{q_{ii}} = (1 - q_{ii})\, e_i^2$.

Dakle, ako je $Y_{ii} = 0$, onda je $q_{ii} = 1$, što povlači da je $\sigma_{ii} = q_{ii}^2 = 1$.

Time smo potvrdili da tvrdnja (*i*) vrijedi također za $i = j$.

Provjerimo tvrdnju (*ii*) uz pretpostavku da je $i = j$.

U ovom slučaju treba provjeriti da li uvjet $\sigma_{ii} = 1$ povlači da je $Y_{ii} = -q_{ii} Y_{ii}$ ili $(1 + q_{ii}) Y_{ii} = 0$, gdje je: $Y_{ii} = [e_i, e_i]_{q_{ii}} = (1 - q_{ii})\, e_i^2$.

Neka je $\sigma_{ii} = 1$.

Tada iz $\sigma_{ii} = q_{ii}^2 = 1$ proizlazi $q_{ii} = 1$ ili $q_{ii} = -1$.

Ako je $q_{ii} = 1$, onda je $Y_{ii} = 0$, stoga je $(1 + q_{ii}) Y_{ii} = 0$, a za $q_{ii} = -1$ direktno proizlazi da je $(1 + q_{ii}) Y_{ii} = 0$.

Time zaključujemo da tvrdnje (*i*), (*ii*) vrijede za svaki $i, j \in \mathcal{N}$.

Primjedba 2.1.12

Na osnovu rečenog lako se dokazuje da vrijedi:

(*i*) ako je $Y_{j_1 j_2 j_3 \cdots j_n} = 0,\ Y_{j_2 j_1 j_3 \cdots j_n} = 0$, onda je $\sigma_{j_1 j_2} = 1$,

(*ii*) ako je $\sigma_{j_1 j_2} = 1$, onda je $Y_{j_1 j_2 j_3 \cdots j_n} = -q_{j_2 j_1} Y_{j_2 j_1 j_3 \cdots j_n}$,

(tj. $Y_{j_1 j_2 j_3 \cdots j_n} + q_{j_2 j_1} Y_{j_2 j_1 j_3 \cdots j_n} = 0$), gdje je $j_1, j_2, ..., j_n \in \mathcal{N}$.

2. 2. Računanje prostora konstanti u algebri $\mathcal{B}$

Osnovni problem koji se nameće, a koji će se detaljno proučavati u ovoj radnji, je određivanje prostora konstanti u slobodnoj asocijativnoj $\mathbb{C}$-algebri s jedinicom $\mathcal{B} = \mathbb{C}\langle e_{i_1}, e_{i_2}, \ldots, e_{i_N} \rangle$ u kojoj je uvedena **q**-diferencijalna struktura nizom linearnih (diferencijalnih) operatora $\partial_{i_s} : \mathcal{B} \to \mathcal{B}$, $1 \le s \le N$ na algebri $\mathcal{B}$ zadanih relacijom (6).

Podsjetimo se, element $C \in \mathcal{B}$ naziva se konstanta u algebri $\mathcal{B}$ ako je $\partial_{i_s} C = 0$ za svaki $1 \le s \le N$.

Uvedimo sada operator stupnja $\partial : \mathcal{B} \to \mathcal{B}$ na algebri $\mathcal{B}$ formulom

$$\partial := \sum_{s=1}^{N} e_{i_s} \partial_{i_s} ,$$

gdje su $e_{i_s} \partial_{i_s} : \mathcal{B} \to \mathcal{B}$, $1 \le s \le N$ linearni operatori na algebri $\mathcal{B}$.

Tada imamo

$$\partial\, C = \sum_{s=1}^{N} e_{i_s} \partial_{i_s} C = 0 \quad \text{ako i samo ako je} \quad \partial_{i_s} C = 0 \quad \text{za svaki } 1 \le s \le N ,$$

na osnovu čega zaključujemo da se konstanta u algebri $\mathcal{B}$ može definirati kao bilo koji element $C \in \mathcal{B}$ za koji je $\partial\, C = 0$ ili $C \in Ker\, \partial$, gdje $Ker\, \partial$ označava jezgru operatora stupnja $\partial : \mathcal{B} \to \mathcal{B}$.

Dakle, problem nalaženja prostora konstanti u algebri $\mathcal{B}$ može se promatrati kao problem određivanja jezgre operatora stupnja ∂.

S druge strane, koristeći činjenicu da operator stupnja čuva rastav algebre $\mathcal{B}$ na težinske potprostore, tj. da je $\partial(\mathcal{B}_Q) \subset \mathcal{B}_Q$ za svaki multiskup $Q = \{ l_1 \le l_2 \le \ldots \le l_n \}$ nad $\mathcal{N} = \{ i_1, i_2, \ldots, i_N \}$, zaključujemo da se problem određivanja konstanti u algebri $\mathcal{B}$ može svesti na problem određivanja konstanti u pojedinačnim težinskim potprostorima $\mathcal{B}_Q$ algebre $\mathcal{B}$, što je ekvivalentno traženju jezgre operatora $\partial |_{\mathcal{B}_Q} : \mathcal{B}_Q \to \mathcal{B}_Q$.

Pogledajmo sada kako djeluje operator $\partial|_{\mathcal{B}_Q} : \mathcal{B}_Q \to \mathcal{B}_Q$ na elementima baze težinskog potprostora

$$\mathcal{B}_Q = span_{\mathbb{C}} \left\{ e_{j_1 j_2 \cdots j_n} = e_{j_1} e_{j_2} \cdots e_{j_n} \mid j_1 j_2 ... j_n = \text{permutacija multiskupa Q} \right\}$$

pridruženog multiskupu $Q = \{ l_1 \le l_2 \le ... \le l_n \}$ nad $\mathcal{N} = \{ i_1, i_2, ..., i_N \}$.

Imamo

$$\partial|_{\mathcal{B}_Q} \left(e_{j_1 j_2 \cdots j_n} \right) = \partial|_{\mathcal{B}_Q} \left(e_{j_1 j_2 \cdots j_{p-1} j_p j_{p+1} \cdots j_n} \right)$$

$$= \sum_{s=1}^{N} e_{i_s} \partial_{i_s} \left(e_{j_1 j_2 \cdots j_{p-1} j_p j_{p+1} \cdots j_n} \right)$$

odnosno primjenom formule (7) dobivamo

$$\partial|_{\mathcal{B}_Q} \left(e_{j_1 j_2 \cdots j_n} \right) = \sum_{s=1}^{N} \sum_{\substack{1 \le p \le n \\ j_p = i_s}} q_{i_s j_1} q_{i_s j_2} \cdots q_{i_s j_{p-1}} \, e_{i_s j_1 j_2 \cdots j_{p-1} j_{p+1} \cdots j_n}$$

$$= \sum_{s=1}^{N} \sum_{\substack{1 \le p \le n \\ j_p = i_s}} \left(\prod_{r=1}^{p-1} q_{i_s j_r} \right) e_{i_s j_1 j_2 \cdots j_{p-1} j_{p+1} \cdots j_n} \tag{24}$$

$j_1 j_2 ... j_n \in \widehat{Q} = S_n Q$.

U monomijalnoj bazi težinskog potprostora $\mathcal{B}_Q$ operatoru $\partial|_{\mathcal{B}_Q}$ je pridružena kvadratna matrica, koju ćemo označiti sa B_Q.

Drugim rječima retci, tj. stupci matrice B_Q su indeksirani elementima baze težinskog potprostora $\mathcal{B}_Q$ pa je red matrice B_Q jednak $\dim \mathcal{B}_Q = \text{Card } \widehat{Q}$.

Pritom je $\text{Card } \widehat{Q} = \#$ permutacija multiskupa Q.

Specijalno, ako je Q podskup od $\mathcal{N} = \{ i_1, i_2, ..., i_N \}$, onda je relacija (24) oblika

$$\partial|_{\mathcal{B}_Q} \left(e_{j_1 j_2 \cdots j_n} \right) = \sum_{p=1}^{n} \left(\prod_{r=1}^{p-1} q_{j_p j_r} \right) e_{j_p j_1 j_2 \cdots j_{p-1} j_{p+1} \cdots j_n} . \tag{25}$$

Relaciju (25) možemo pisati ovako

$$\mathbf{B}_Q \, e_{j_1 j_2 \cdots j_n} = \sum_{p=1}^{n} \left(\prod_{r=1}^{p-1} q_{j_p j_r} \right) e_{j_p j_1 j_2 \cdots j_{p-1} j_{p+1} \cdots j_n} \tag{26}$$

gdje je $\mathbf{B}_Q = \partial|_{\mathcal{B}_Q}$ ili u raspisanom obliku

$$\mathbf{B}_Q\, e_{j_1 j_2 \cdots j_n} = 1\, e_{j_1 j_2 \cdots j_n} + q_{j_2 j_1}\, e_{j_2 j_1 j_3 \cdots j_n} + q_{j_3 j_1} q_{j_3 j_2}\, e_{j_3 j_1 j_2 j_4 \cdots j_n} + \cdots$$

$$+ q_{j_p j_1} q_{j_p j_2} \cdots q_{j_p j_{p-1}}\, e_{j_p j_1 j_2 \cdots j_{p-1} j_{p+1} \cdots j_n} + \cdots + q_{j_n j_1} q_{j_n j_2} \cdots q_{j_n j_{n-1}}\, e_{j_n j_1 j_2 \cdots j_{n-1}} \qquad (27)$$

U ovom slučaju je red matrice B_Q jednak $\dim \mathcal{B}_Q = \mathrm{Card}\, \widehat{Q} = n!$.

Nužan i dovoljan uvjet egzistencije netrivijalne (bazične) konstante u težinskom potprostoru $\mathcal{B}_Q$ pridruženog multiskupu $Q = \{l_1 \le l_2 \le \ldots \le l_n\}$ nad $\mathcal{N}$ je

$$\det B_Q = 0$$

ili B_Q je singularna matrica.

Drugim rječima, ako je $\det B_Q \neq 0$, onda u težinskom potprostoru $\mathcal{B}_Q$ nema netrivijalne konstante. Netrivijalna konstanta u $\mathcal{B}_Q$ će postojati ako i samo ako je determinanta matrice B_Q jednaka nuli.

Time se nameće prvi problem:

faktorizacija determinante matrice B_Q, a samim time i faktorizacija matrice B_Q.

Taj će se problem detaljnije proučavati u četvrtom poglavlju po uzoru na radove Svrtana-Meljanca, Kroba i drugih.

Nadalje, iz jednadžbe $\det B_Q = 0$ proizaći će određeni uvjeti na parametre q_{ij}, $i, j \in \mathcal{N}$ za koje će postojati netrivijalna konstanta u $\mathcal{B}_Q$, ali isto tako i u $\mathcal{B}_{\overline{Q}}$ za neki pravi podskup $\overline{Q} \subset Q$, stoga se nameće drugi problem:

određivanje vrijednosti parametra q_{ij}, $i, j \in \mathcal{N}$, tj. tzv. *uvjeta kocikličnosti* za koji postoji netrivijalna (bazična) konstanta u $\mathcal{B}_Q$, ali ne postoji u $\mathcal{B}_{\overline{Q}}$ za svaki pravi podskup $\overline{Q} \subset Q$.

Pretpostavimo da je uvjet kocikličnosti zadovoljen. Tada u $\mathcal{B}_Q$ postoji do na faktor netrivijalna (bazična) konstanta, koju ćemo uz podesnu normalizaciju označavati sa C_Q. Pritom za sve preostale vrijednosti parametara q_{ij}, $i, j \in \mathcal{N}$ za koje je $\det B_Q = 0$, a koje ne zadovoljavaju uvjet kocikličnosti, dobivamo tzv. bazične konstante, koje imaju svojstvo da proizlaze iz C_Q, na osnovu čega zaključujemo da netrivijalna bazična konstanta C_Q generira obostrani ideal $\boldsymbol{I}_Q$ u $\mathcal{B}_Q$, tj. $\boldsymbol{I}_Q = \langle C_Q \rangle$.

Prirodno se postavlja sljedeći problem:

kolika je dimenzija prostora konstanti u $\mathcal{B}_{\mathrm{Q}}$ ako je uvjet kocikličnosti zadovoljen i kakav je zapis konstante C_{Q}?

Taj će se problem detaljno proučavati u sljedećim pododjeljcima ovog odjeljka s obzirom na bilo koji težinski potprostor (generički ili degenerirani) $\mathcal{B}_{\mathrm{Q}}$ algebre $\mathcal{B}$ takav da je $2 \le \mathrm{Card}\,\mathrm{Q} \le 4$ (ili pravi podskup $\mathcal{B}_{\mathrm{Q}} \subset \mathcal{B}(\mathrm{n})$, $2 \le \mathrm{n} \le 4$). Pritom će se pokazati da u $\mathcal{B}(1)$ nema netrivijalnih (baznih) konstanti.

Također će se pokazati da konstante u bilo kojem degeneriranom težinskom potprostoru možemo konstruirati iz konstanti nekog generičkog težinskog potprostora u određenoj identifikaciji elemenata u skupu Q, što dovodi do zaključka da je pri izračunavanju netrivijalnih konstanti u $\mathcal{B}(\mathrm{n})$, $\mathrm{n} \ge 2$ dovoljno izračunati konstante u generičkim težinskim potprostorima.

2. 2. 1. Računanje konstanti u $\mathcal{B}(1)$

Potprostor $\mathcal{B}(1)$ se sastoji od svih homogenih nekomutativnih polinoma prvog stupnja u N varijabli $\{\mathrm{e}_{\mathrm{i}_s}\}_{1 \le s \le \mathrm{N}}$. Pretpostavimo da je

$$C = \sum_{s=1}^{\mathrm{N}} \alpha_s \, \mathrm{e}_{\mathrm{i}_s}$$

konstanta u potprostoru $\mathcal{B}(1)$. Tada primjenom definicije 2.1.6 imamo:

$$\partial_{\mathrm{i}} C = 0 \quad \text{za svaki } \mathrm{i} \in \mathcal{N} = \{\mathrm{i}_1, \mathrm{i}_2, ..., \mathrm{i}_{\mathrm{N}}\}.$$

Dobivamo:

$$0 = \partial_{\mathrm{i}} \left(\sum_{s=1}^{\mathrm{N}} \alpha_s \, \mathrm{e}_{\mathrm{i}_s} \right) = \sum_{s=1}^{\mathrm{N}} \alpha_s \left(\partial_{\mathrm{i}} \, \mathrm{e}_{\mathrm{i}_s} \right) = \sum_{s=1}^{\mathrm{N}} \alpha_s \, \delta_{\mathrm{i}\,\mathrm{i}_s}$$

što povlači da je:

$$\alpha_s = 0 \quad \text{za svaki} \quad 1 \le s \le \mathrm{N},$$

jer je: $\delta_{\mathrm{i}\,\mathrm{i}_s} = \begin{cases} 1 & \text{za } \ \mathrm{i} = \mathrm{i}_s \\ 0 & \text{za } \ \mathrm{i} \ne \mathrm{i}_s \end{cases}.$

Odatle slijedi $\qquad C = 0,$

odnosno u $\mathcal{B}(1)$ nema netrivijalnih konstanti.

2. 2. 2. Računanje prostora konstanti u $\mathcal{B}(2)$

Potprostor $\mathcal{B}(2)$ se sastoji od svih homogenih nekomutativnih polinoma drugog stupnja generiranih varijablama $\mathrm{e}_{\mathrm{i}_1},...,\mathrm{e}_{\mathrm{i}_\mathrm{N}}$. Imamo:

$$\sum_{r,s=1}^{\mathrm{N}} \alpha_{rs}\, \mathrm{e}_{\mathrm{i}_r} \mathrm{e}_{\mathrm{i}_s} = \sum_{1 \le r < s \le \mathrm{N}} \left(\alpha_{rs}\, \mathrm{e}_{\mathrm{i}_r} \mathrm{e}_{\mathrm{i}_s} + \alpha_{sr}\, \mathrm{e}_{\mathrm{i}_s} \mathrm{e}_{\mathrm{i}_r} \right) + \sum_{r=1}^{\mathrm{N}} \alpha_{rr}\, \mathrm{e}_{\mathrm{i}_r}^2 ,$$

$\alpha_{rs}, \alpha_{rs} \in \mathbb{C}$.

Neka je $\mathrm{Q} = \{ l_1 \le l_2 \}$ bilo koji dvočlani multiskup nad $\mathcal{N} = \{ \mathrm{i}_1, \mathrm{i}_2, ..., \mathrm{i}_\mathrm{N} \}$ s pripadnim težinskim potprostorom $\mathcal{B}_\mathrm{Q} \subset \mathcal{B}(2)$.

Pritom je $\mathcal{B}_\mathrm{Q} = span_{\mathbb{C}} \left\{ \mathrm{e}_{\mathrm{j}_1\mathrm{j}_2} = \mathrm{e}_{\mathrm{j}_1} \mathrm{e}_{\mathrm{j}_2} \mid \mathrm{j}_1\mathrm{j}_2 = \text{permutacija multiskupa Q} \right\}$.

Djelovanjem operatora stupnja na težinskom potprostoru $\mathcal{B}_\mathrm{Q} \subset \mathcal{B}(2)$

$$\partial|_{\mathcal{B}_\mathrm{Q}} : \mathcal{B}_\mathrm{Q} \to \mathcal{B}_\mathrm{Q}$$

iz formule (24) proizlazi:

$$\partial|_{\mathcal{B}_\mathrm{Q}} \left(\mathrm{e}_{\mathrm{j}_1\mathrm{j}_2} \right) = 1\, \mathrm{e}_{\mathrm{j}_1\mathrm{j}_2} + \mathrm{q}_{\mathrm{j}_2\mathrm{j}_1}\, \mathrm{e}_{\mathrm{j}_2\mathrm{j}_1} , \tag{28}$$

gdje je $\mathrm{j}_1\mathrm{j}_2 \in \widehat{\mathrm{Q}} = \mathrm{S}_2\mathrm{Q}$.

✐ Neka je $\mathrm{Q} = \{ l_1, l_2 \}$ skup. Tada je njemu pridruženi generički težinski potprostor $\mathcal{B}_{l_1 l_2}$, $l_1 < l_2$ dan sa $\mathcal{B}_{l_1 l_2} = span_{\mathbb{C}} \left\{ \mathrm{e}_{l_1 l_2}, \mathrm{e}_{l_2 l_1} \right\}$.

Imamo: $\widehat{\mathrm{Q}} = \mathrm{S}_2\mathrm{Q} = \{ l_1 l_2, l_2 l_1 \}$, stoga iz (28) proizlazi:

$$\partial|_{\mathcal{B}_{l_1 l_2}} \left(\mathrm{e}_{l_1 l_2} \right) = 1\, \mathrm{e}_{l_1 l_2} + \mathrm{q}_{l_2 l_1} \mathrm{e}_{l_2 l_1} ,$$

$$\partial|_{\mathcal{B}_{l_1 l_2}} \left(\mathrm{e}_{l_2 l_1} \right) = 1\, \mathrm{e}_{l_2 l_1} + \mathrm{q}_{l_1 l_2} \mathrm{e}_{l_1 l_2} = \mathrm{q}_{l_1 l_2} \mathrm{e}_{l_1 l_2} + 1\, \mathrm{e}_{l_2 l_1} ,$$

pa je matrica $\mathrm{B}_{l_1 l_2}$, pridružena operatoru $\partial|_{\mathcal{B}_{l_1 l_2}} : \mathcal{B}_{l_1 l_2} \to \mathcal{B}_{l_1 l_2}$ u monomijalnoj bazi generičkog potprostora $\mathcal{B}_{l_1 l_2}$, dana sa:

$$\mathrm{B}_{l_1 l_2} = \begin{array}{c c} & \begin{array}{cc} l_1 l_2 & l_2 l_1 \end{array} \\ \begin{array}{c} l_1 l_2 \\ l_2 l_1 \end{array} & \begin{bmatrix} 1 & \mathrm{q}_{l_1 l_2} \\ \mathrm{q}_{l_2 l_1} & 1 \end{bmatrix} \end{array}$$

Njena determinanta je $\det \mathrm{B}_{l_1 l_2} = 1 - \sigma_{l_1 l_2}$, gdje je $\sigma_{l_1 l_2} := \mathrm{q}_{l_1 l_2} \mathrm{q}_{l_2 l_1}$.

Matrica $B_{l_1 l_2}$ je singularna ako i samo ako je $\det B_{l_1 l_2} = 0$, tj. $\sigma_{l_1 l_2} = 1$.

Ako je $\sigma_{l_1 l_2} \neq 1$, onda u potprostoru $\mathcal{B}_{l_1 l_2}$ nema netrivijalnih konstanti. Netrivijalna konstanta u $\mathcal{B}_{l_1 l_2}$ postoji jedino ako je $\sigma_{l_1 l_2} = 1$.

Ako je $\sigma_{l_1 l_2} = 1$, onda je $\left\{ -q_{l_1 l_2} e_{l_1 l_2} + e_{l_2 l_1} \right\}$ baza prostora konstanti u potprostoru $\mathcal{B}_{l_1 l_2}$. Jednostavnim izračunavanjem dobiva se da je netrivijalna konstanta u potprostoru $\mathcal{B}_{l_1 l_2}$ oblika:

$$C_{l_1 l_2} = \alpha \, Y_{l_2 l_1}, \quad \alpha \in \mathbb{C} \setminus \{0\}.$$

Pritom je: $Y_{l_2 l_1} = \left[e_{l_2}, e_{l_1} \right]_{q_{l_1 l_2}} = e_{l_2 l_1} - q_{l_1 l_2} e_{l_1 l_2}$, gdje je: $e_{ij} := e_i e_j$.

Dobili smo da je prostor konstanti u generičkom potprostoru $\mathcal{B}_{l_1 l_2}$ jednodimenzionalan te da je iterirani komutator $Y_{l_2 l_1}$ moguća bazična konstanta u $\mathcal{B}_{l_1 l_2}$ ako je $\sigma_{l_1 l_2} = 1$.

Primjedba 2.2.2.1

Ako je $\sigma_{l_1 l_2} = 1$, onda su iterirani komutatori $Y_{l_1 l_2}$ i $Y_{l_2 l_1}$ linearno zavisni (propozicija 2.1.11), stoga možemo uzeti da je ideal $\boldsymbol{I}_{l_1 l_2}$ u potprostoru $\mathcal{B}_{l_1 l_2}$ generiran komutatorom $Y_{l_1 l_2}$, tj. $\boldsymbol{I}_{l_1 l_2} = \left\langle Y_{l_1 l_2} \right\rangle$, ako je $l_1 < l_2$.

✎ Neka je $l_1 = l_2 = l$. Tada je multiskupu $Q = \{l^2\}$, $l \in \mathcal{N}$ pridružen degenerirani težinski potprostor $\mathcal{B}_{l^2} = span_{\mathbb{C}} \{e_{ll}\}$, gdje je $e_{ll} = e_l e_l = e_{l^2}$.

Iz formule (24) slijedi da u monomijalnoj bazi potprostora $\mathcal{B}_{l^2}$ operatoru $\partial|_{\mathcal{B}_{l^2}} : \mathcal{B}_{l^2} \to \mathcal{B}_{l^2}$ je pridružena 1×1 matrica B_{l^2}, kojoj je jedini element jednak $1 + q_{ll}$, što povlači da je: $\det B_{l^2} = 1 + q_{ll}$.

Napomena:

Primijetimo da za $l_1 = l_2 = l$ iz (24) proizlazi: $\partial|_{\mathcal{B}_{l^2}} (e_{ll}) = (1 + q_{ll}) e_{ll}$,

što povlači da je

$$B_{l^2} = \begin{array}{cc} & l^2 \\ l^2 & \left[1 + q_{ll} \right] \end{array}$$

odnosno: $\det B_{l^2} = 1 + q_{ll}$.

Pritom je: $\widehat{Q} = S_2 Q = \{ll\} = \{l^2\}$.

Matrica B_{l^2} je singularna ako i samo ako je $\mathrm{q}_{ll} = -1$. Tada je bilo koja netrivijalna konstanta u $\mathcal{B}_{l^2}$ oblika $C_{ll} = \alpha\, \mathrm{e}_{l^2}$, $\alpha \in \mathbb{C} \setminus \{0\}$, gdje je $\mathrm{e}_{l^2} := \mathrm{e}_l^2$.

Ako je $\mathrm{q}_{ll} \neq -1$, onda u $\mathcal{B}_{l^2}$ nema netrivijalnih konstanti.

Specijalno za $\alpha = 0$ imamo da je $C_{ll} = 0$ trivijalna konstanta u potprostoru $\mathcal{B}_{l^2}$.

Dobili smo da je prostor konstanti u degeneriranom potprostoru $\mathcal{B}_{l^2}$ jednodimenzionalan i da je $C_{ll} = \mathrm{e}_{l^2}$ bazična konstanta u $\mathcal{B}_{l^2}$ ako je $[2]_{\mathrm{q}_{ll}} = 0$.

Pritom je $[2]_{\mathrm{q}_{ll}} = 1 + \mathrm{q}_{ll}$.

Primjedba 2.2.2.2

Uzimajući u obzir primjedbu 2.1.10 za $\mathrm{q}_{ll} \neq 1$ imamo:

$$\mathrm{e}_{l^2} = \frac{\mathrm{Y}_{l^2}}{1 - \mathrm{q}_{ll}} = \frac{\mathrm{Y}_{ll}}{1 - \mathrm{q}_{ll}}. \tag{29}$$

S druge strane, primijetimo da iz $\sigma_{l_1 l_2} = 1$ za $l_1 = l_2 = l$ proizlazi: $\mathrm{q}_{ll}^2 = 1$,

odnosno: $\mathrm{q}_{ll} = 1$ ili $\mathrm{q}_{ll} = -1$.

Dakle, zbog uvjeta $\mathrm{q}_{ll} \neq 1$ u (29) imamo da će bazična konstanta e_{l^2} biti linearno zavisna s iteriranim komutatorom Y_{ll} ako je $\mathrm{q}_{ll} = -1$.

Pritom za $\mathrm{q}_{ll} = -1$ iz (29) dobivamo: $\mathrm{e}_{l^2} = \frac{1}{2} \mathrm{Y}_{ll}$

ili
$$\mathrm{Y}_{ll} = 2\, \mathrm{e}_{l^2}. \tag{30}$$

Na osnovu rečenog zaključujemo:

Netrivijalna konstanta $C_{ll} = \beta\, \mathrm{e}_{l^2}$, $\beta \in \mathbb{C} \setminus \{0\}$ u degeneriranom potprostoru $\mathcal{B}_{l^2}$ proizlazi iz netrivijalne konstante $C_{l_1 l_2} = \alpha\, \mathrm{Y}_{l_2 l_1}$, $\alpha \in \mathbb{C} \setminus \{0\}$ odgovarajućeg generičkog težinskog potprostoru $\mathcal{B}_{l_1 l_2}$ ako stavimo: $l_1 = l_2 = l$.

Pritom se uvjet kocikličnosti $\sigma_{l_1 l_2} = 1$ degenerira na uvjet $\mathrm{q}_{ll} = -1$.

Dimenzija prostora konstanti u $\mathcal{B}(2)$ je manja ili jednaka $\binom{\mathrm{N}+1}{2}$.

Dimenzija je maksimalna ako je $\sigma_{l_1 l_2} = 1$ za svaki $l_1, l_2 \in \mathcal{N}$, $l_1 < l_2$ i $\mathrm{q}_{ll} = -1$ za svaki $l \in \mathcal{N}$.

Konstante u potprostoru $\mathcal{B}(2)$ generiraju obostrani ideal $\boldsymbol{I}_2 = \left\langle \mathrm{Y}_{l_1 l_2} \right\rangle$ ako je $\sigma_{l_1 l_2} = 1$, $l_1 \le l_2$, $l_1, l_2 \in \mathcal{N}$. Pritom je $\mathrm{Y}_{l_1 l_2} = \left[\mathrm{e}_{l_1}, \mathrm{e}_{l_2} \right]_{\mathrm{q}_{l_2 l_1}} = \mathrm{e}_{l_1 l_2} - \mathrm{q}_{l_2 l_1} \mathrm{e}_{l_2 l_1}$.

2. 2. 3. Računanje prostora konstanti u $\mathcal{B}(3)$

Neka je $\mathrm{Q} = \{ l_1 \le l_2 \le l_3 \}$ bilo koji tročlani multiskup nad $\mathcal{N} = \{ \mathrm{i}_1, \mathrm{i}_2, ..., \mathrm{i}_\mathrm{N} \}$ i neka je $\mathcal{B}_\mathrm{Q} \subset \mathcal{B}(3)$ težinski potprostor pridružen multiskupu Q, gdje je

$$\mathcal{B}_\mathrm{Q} = span_{\mathbb{C}} \left\{ \mathrm{e}_{\mathrm{j}_1 \mathrm{j}_2 \mathrm{j}_3} = \mathrm{e}_{\mathrm{j}_1} \mathrm{e}_{\mathrm{j}_2} \mathrm{e}_{\mathrm{j}_3} \mid \mathrm{j}_1 \mathrm{j}_2 \mathrm{j}_3 = \text{permutacija multiskupa Q} \right\} \subset \mathcal{B}(3) .$$

Tada djelovanjem operatora stupnja $\partial |_{\mathcal{B}_\mathrm{Q}} : \mathcal{B}_\mathrm{Q} \to \mathcal{B}_\mathrm{Q}$ na potprostoru $\mathcal{B}_\mathrm{Q}$ primjenom formule (24) dobivamo

$$\partial |_{\mathcal{B}_\mathrm{Q}} \left(\mathrm{e}_{\mathrm{j}_1 \mathrm{j}_2 \mathrm{j}_3} \right) = 1\, \mathrm{e}_{\mathrm{j}_1 \mathrm{j}_2 \mathrm{j}_3} + \mathrm{q}_{\mathrm{j}_2 \mathrm{j}_1} \mathrm{e}_{\mathrm{j}_2 \mathrm{j}_1 \mathrm{j}_3} + \mathrm{q}_{\mathrm{j}_3 \mathrm{j}_1} \mathrm{q}_{\mathrm{j}_3 \mathrm{j}_2} \mathrm{e}_{\mathrm{j}_3 \mathrm{j}_1 \mathrm{j}_2} , \tag{31}$$

gdje je $\mathrm{j}_1 \mathrm{j}_2 \mathrm{j}_3 \in \widehat{\mathrm{Q}} = \mathrm{S}_3 \mathrm{Q}$.

1. <u>Generički slučaj:</u>

Neka je $\mathcal{B}_{l_1 l_2 l_3}$, $l_1 < l_2 < l_3$ generički težinski potprostor pridružen skupu $\mathrm{Q} = \{ l_1, l_2, l_3 \}$. Pritom je $\widehat{\mathrm{Q}} = \mathrm{S}_3 \mathrm{Q} = \{ l_1 l_2 l_3, l_1 l_3 l_2, l_3 l_1 l_2, l_3 l_2 l_1, l_2 l_3 l_1, l_2 l_1 l_3 \}$, stoga iz (31) proizlazi

$$\partial |_{\mathcal{B}_{l_1 l_2 l_3}} \left(\mathrm{e}_{l_1 l_2 l_3} \right) = 1\, \mathrm{e}_{l_1 l_2 l_3} + \mathrm{q}_{l_2 l_1} \mathrm{e}_{l_2 l_1 l_3} + \mathrm{q}_{l_3 l_1} \mathrm{q}_{l_3 l_2} \mathrm{e}_{l_3 l_1 l_2} ,$$

$$\partial |_{\mathcal{B}_{l_1 l_2 l_3}} \left(\mathrm{e}_{l_1 l_3 l_2} \right) = 1\, \mathrm{e}_{l_1 l_3 l_2} + \mathrm{q}_{l_3 l_1} \mathrm{e}_{l_3 l_1 l_2} + \mathrm{q}_{l_2 l_1} \mathrm{q}_{l_2 l_3} \mathrm{e}_{l_2 l_1 l_3} ,$$

$$\partial |_{\mathcal{B}_{l_1 l_2 l_3}} \left(\mathrm{e}_{l_3 l_1 l_2} \right) = 1\, \mathrm{e}_{l_3 l_1 l_2} + \mathrm{q}_{l_1 l_3} \mathrm{e}_{l_1 l_3 l_2} + \mathrm{q}_{l_2 l_3} \mathrm{q}_{l_2 l_1} \mathrm{e}_{l_2 l_3 l_1} ,$$

$$\partial |_{\mathcal{B}_{l_1 l_2 l_3}} \left(\mathrm{e}_{l_3 l_2 l_1} \right) = 1\, \mathrm{e}_{l_3 l_2 l_1} + \mathrm{q}_{l_2 l_3} \mathrm{e}_{l_2 l_3 l_1} + \mathrm{q}_{l_1 l_3} \mathrm{q}_{l_1 l_2} \mathrm{e}_{l_1 l_3 l_2} ,$$

$$\partial |_{\mathcal{B}_{l_1 l_2 l_3}} \left(\mathrm{e}_{l_2 l_3 l_1} \right) = 1\, \mathrm{e}_{l_2 l_3 l_1} + \mathrm{q}_{l_3 l_2} \mathrm{e}_{l_3 l_2 l_1} + \mathrm{q}_{l_1 l_2} \mathrm{q}_{l_1 l_3} \mathrm{e}_{l_1 l_2 l_3} ,$$

$$\partial |_{\mathcal{B}_{l_1 l_2 l_3}} \left(\mathrm{e}_{l_2 l_1 l_3} \right) = 1\, \mathrm{e}_{l_2 l_1 l_3} + \mathrm{q}_{l_1 l_2} \mathrm{e}_{l_1 l_2 l_3} + \mathrm{q}_{l_3 l_2} \mathrm{q}_{l_3 l_1} \mathrm{e}_{l_3 l_2 l_1} .$$

U monomijalnoj bazi generičkog potprostora $\mathcal{B}_{l_1 l_2 l_3}$ operatoru $\partial |_{\mathcal{B}_{l_1 l_2 l_3}} : \mathcal{B}_{l_1 l_2 l_3} \to \mathcal{B}_{l_1 l_2 l_3}$ je pridružena matrica $\mathrm{B}_{l_1 l_2 l_3}$

$$
\mathrm{B}_{l_1l_2l_3}=\begin{array}{cc}
 & \begin{array}{cccccc} l_1l_2l_3 & l_1l_3l_2 & l_3l_1l_2 & l_3l_2l_1 & l_2l_3l_1 & l_2l_1l_3 \end{array} \\
\begin{array}{c} l_1l_2l_3 \\ l_1l_3l_2 \\ l_3l_1l_2 \\ l_3l_2l_1 \\ l_2l_3l_1 \\ l_2l_1l_3 \end{array} &
\begin{bmatrix}
1 & 0 & 0 & 0 & \mathrm{q}_{l_1l_2}\mathrm{q}_{l_1l_3} & \mathrm{q}_{l_1l_2} \\
0 & 1 & \mathrm{q}_{l_1l_3} & \mathrm{q}_{l_1l_3}\mathrm{q}_{l_1l_2} & 0 & 0 \\
\mathrm{q}_{l_3l_1}\mathrm{q}_{l_3l_2} & \mathrm{q}_{l_3l_1} & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & \mathrm{q}_{l_3l_2} & \mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_1} \\
0 & 0 & \mathrm{q}_{l_2l_3}\mathrm{q}_{l_2l_1} & \mathrm{q}_{l_2l_3} & 1 & 0 \\
\mathrm{q}_{l_2l_1} & \mathrm{q}_{l_2l_1}\mathrm{q}_{l_2l_3} & 0 & 0 & 0 & 1
\end{bmatrix}
\end{array}
$$

Izračunavanjem determinante matrice $\mathrm{B}_{l_1l_2l_3}$ dobivamo da je

$$
\begin{aligned}
\det \mathrm{B}_{l_1l_2l_3} &= \left(1-\sigma_{l_1l_2}\right)\left(1-\sigma_{l_1l_3}\right)\left(1-\sigma_{l_2l_3}\right)\left(1-\sigma_{l_1l_2l_3}\right) \\
&= \prod_{1\le i<j\le 3}\left(1-\sigma_{l_il_j}\right)\left(1-\sigma_{l_1l_2l_3}\right).
\end{aligned}
$$

Pritom je $\sigma_{\mathrm{ij}}=\mathrm{q}_{\mathrm{ij}}\mathrm{q}_{\mathrm{ji}}$, $\sigma_{l_1l_2l_3}=\sigma_{l_1l_2}\sigma_{l_1l_3}\sigma_{l_2l_3}$.

Ako je $\sigma_{l_1l_2}\neq 1$, $\sigma_{l_1l_3}\neq 1$, $\sigma_{l_2l_3}\neq 1$, $\sigma_{l_1l_2l_3}\neq 1$, onda u $\mathcal{B}_{l_1l_2l_3}$ nema netrivijalnih konstanti.

U generičkom težinskom potprostoru $\mathcal{B}_{l_1l_2l_3}$ postojati će netrivijalna bazična konstanta ako i samo ako je $\det \mathrm{B}_{l_1l_2l_3}=0$, tj. $\sigma_{l_1l_2}=1 \ \vee \ \sigma_{l_1l_3}=1 \ \vee \ \sigma_{l_2l_3}=1 \ \vee \ \sigma_{l_1l_2l_3}=1$.

U nastavku će se dokazati da je prostor konstanti u $\mathcal{B}_{l_1l_2l_3}$ jednodimenzionalan te da se za $\sigma_{l_1l_2l_3}=1$ dobiva konstanta, koju ćemo uz podesnu normalizaciju označiti sa $C_{l_1l_2l_3}$. Konstanta $C_{l_1l_2l_3}$ generirati će ideal $\boldsymbol{I}_{l_1l_2l_3}$ u potprostoru $\mathcal{B}_{l_1l_2l_3}$, ako je $l_1\le l_2\le l_3$, $l_1,l_2,l_3\in\mathcal{N}$.

Radi jednostavnosti zapisa, pretpostavimo da je $l_1=1$, $l_2=2$, $l_3=3$.

Tada je skupu $\mathrm{Q}=\{1,2,3\}$ pridružen generički težinski potprostor $\mathcal{B}_{123}$ te je u skladu s gore navedenim u bazi potprostora $\mathcal{B}_{123}$ operatoru $\partial|_{\mathcal{B}_{123}}:\mathcal{B}_{123}\to\mathcal{B}_{123}$ pridružena matrica B_{123}, kojoj je determinanta $\det \mathrm{B}_{123}=\left(1-\sigma_{12}\right)\left(1-\sigma_{13}\right)\left(1-\sigma_{23}\right)\left(1-\sigma_{123}\right)$.

Slijedi izračunavanje prostora konstanti u generičkom težinskom potprostoru $\mathcal{B}_{123}$ u ovisnosti o izborima vrijednosti parametara q_{ij} za koje je $\det \mathrm{B}_{123}=0$.

Ako je $\sigma_{12}\neq 1$, $\sigma_{13}\neq 1$, $\sigma_{23}\neq 1$, $\sigma_{123}\neq 1$, onda u $\mathcal{B}_{123}$ nema konstanti.

♦ Ako je $\sigma_{123}=1$, $\sigma_{12}\neq 1$, $\sigma_{13}\neq 1$, $\sigma_{23}\neq 1$, onda vektor

$$\left(1,\ -\frac{\sigma_{13}\sigma_{23}-1}{q_{23}\left(\sigma_{13}-1\right)},\ \frac{q_{31}\left(\sigma_{23}-1\right)}{q_{23}\left(\sigma_{13}-1\right)},\ q_{21}q_{31}q_{32},\ -\frac{q_{21}q_{31}\left(\sigma_{13}\sigma_{23}-1\right)}{\sigma_{13}-1},\ \frac{q_{21}\sigma_{13}\left(\sigma_{23}-1\right)}{\sigma_{13}-1}\right)$$

pripada jezgri operatora $\partial|_{\mathcal{B}_{123}}$, što znači da bazična konstanta može biti oblika

$$C_{123}=\frac{1-\sigma_{13}}{q_{31}}\left(e_{123}+q_{21}q_{31}q_{32}\,e_{321}\right)+\frac{1-\sigma_{12}}{q_{12}}\left(e_{231}+q_{32}q_{12}q_{13}\,e_{132}\right)$$
$$+\frac{1-\sigma_{23}}{q_{23}}\left(e_{312}+q_{13}q_{23}q_{21}\,e_{213}\right). \tag{32}$$

Koristeći svojstvo $\sigma_{ji}=\sigma_{ij}$ možemo pisati

$$C_{123}=\frac{1-\sigma_{12}}{q_{12}}\left(e_{231}+q_{32}q_{12}q_{13}\,e_{132}\right)+\frac{1-\sigma_{31}}{q_{31}}\left(e_{123}+q_{21}q_{31}q_{32}\,e_{321}\right)$$
$$+\frac{1-\sigma_{23}}{q_{23}}\left(e_{312}+q_{13}q_{23}q_{21}\,e_{213}\right)$$

ili kraće

$$C_{123}=\sum_{cyc}\frac{1-\sigma_{12}}{q_{12}}\left(e_{231}+q_{32}q_{12}q_{13}\,e_{132}\right) \tag{33}$$

gdje $\sum\limits_{cyc}$ označuje cikličku sumu.

Uvodimo oznaku

$$\widetilde{X}_{p_1p_2p_3}:=e_{p_1p_2p_3}+\prod_{1\le j<k\le 3}q_{p_kp_j}\,w_3\,e_{p_1p_2p_3}\ , \tag{34}$$

gdje w_3 označava najdulju permutaciju u S_3 i $w_3\,e_{p_1p_2p_3}$ je zapravo $e_{p_3p_2p_1}$ pa je

$$\widetilde{X}_{p_1p_2p_3}=e_{p_1p_2p_3}+q_{p_2p_1}q_{p_3p_1}q_{p_3p_2}\,e_{p_3p_2p_1}\ .$$

Uzimajući u obzir uvedenu oznaku (33) u nastavku će se podrazumijevati da je

$$C_{123}=\frac{1-\sigma_{12}}{q_{12}}\widetilde{X}_{231}+\frac{1-\sigma_{13}}{q_{31}}\widetilde{X}_{123}+\frac{1-\sigma_{23}}{q_{23}}\widetilde{X}_{312}\ . \tag{35}$$

ili

$$C_{123}=\sum_{cyc}\frac{1-\sigma_{12}}{q_{12}}\,\widetilde{X}_{231}$$

bazična konstanta u $\mathcal{B}_{123}$ ako je $\sigma_{123}=1$, $\sigma_{12}\neq 1$, $\sigma_{13}\neq 1$, $\sigma_{23}\neq 1$.

- Ako je $\sigma_{12}=1$, $\sigma_{13}\neq 1$, $\sigma_{23}\neq 1$, onda je vektor $\left(1,\,0,\,-q_{31}q_{32},\,q_{31}q_{32}q_{21},\,0,\,-q_{21}\right)$ u jezgri operatora $\partial|_{\mathcal{B}_{123}}$, tj. $\left\{e_{123}-q_{31}q_{32}\,e_{312}+q_{31}q_{32}q_{21}\,e_{321}-q_{21}\,e_{213}\right\}$ je baza prostora konstanti u težinskom potprostoru $\mathcal{B}_{123}$.

  Jednostavnim izračunavanjem dobiva se da je iterirani komutator

  $$Y_{123}=\Big[\big[e_1,\,e_2\big]_{q_{21}},\,e_3\Big]_{q_{31}q_{32}}$$

  moguća bazična konstanta u $\mathcal{B}_{123}$ ako je $\sigma_{12}=1$, $\sigma_{13}\neq 1$, $\sigma_{23}\neq 1$.

- Ako je $\sigma_{13}=1$, $\sigma_{12}\neq 1$, $\sigma_{23}\neq 1$, onda vektor $\left(0,\,1,\,-q_{31},\,0,\,q_{23}q_{21}q_{31},\,-q_{23}q_{21}\right)$ pripada jezgri operatora $\partial|_{\mathcal{B}_{123}}$, tj. $\left\{e_{132}-q_{31}\,e_{312}+q_{23}q_{21}q_{31}\,e_{231}-q_{23}q_{21}\,e_{213}\right\}$ je baza prostora konstanti u težinskom potprostoru $\mathcal{B}_{123}$ pa je u ovom slučaju iterirani komutator $Y_{132}=\Big[\big[e_1,\,e_3\big]_{q_{31}},\,e_2\Big]_{q_{21}q_{23}}$ moguća bazična konstanta u $\mathcal{B}_{123}$.

- Ako je $\sigma_{23}=1$, $\sigma_{12}\neq 1$, $\sigma_{13}\neq 1$, onda je vektor $\left(-q_{12}q_{13},\,q_{12}q_{13}q_{32},\,0,\,-q_{32},\,1,\,0\right)$ u jezgri operatora $\partial|_{\mathcal{B}_{123}}$, tj. $\left\{-q_{12}q_{13}\,e_{123}+q_{12}q_{13}q_{32}\,e_{132}-q_{32}\,e_{321}+e_{231}\right\}$ je baza prostora konstanti u težinskom potprostoru $\mathcal{B}_{123}$, što povlači da je moguća bazična konstanta iterirani komutator oblika $Y_{231}=\Big[\big[e_2,\,e_3\big]_{q_{32}},\,e_1\Big]_{q_{12}q_{13}}$.

Pritom vrijedi $Y_{i_1i_2i_3}=-q_{i_2i_1}Y_{i_2i_1i_3}$ ako je $\sigma_{i_1i_2}=1$.

Napomena:

Neka je $\sigma_{12}=1$, $\sigma_{13}=1$, $\sigma_{23}\neq 1$. Tada dobivamo da oba vektora

$$\left(1,\,0,\,-q_{31}q_{32},\,q_{31}q_{32}q_{21},\,0,\,-q_{21}\right),\ \left(0,\,1,\,-q_{31},\,0,\,q_{23}q_{21}q_{31},\,-q_{23}q_{21}\right)$$

pripadaju jezgri operatora $\partial|_{\mathcal{B}_{123}}$, što znači da je bazična konstanta prikaziva pomoću iteriranih komutatora Y_{123} i Y_{132}.

Analogno se dobiva da je bazična konstanta u $\mathcal{B}_{123}$ prikaziva pomoću iteriranih komutatora Y_{213} i Y_{231} ako je $\sigma_{21}=1$, $\sigma_{23}=1$, $\sigma_{13}\neq 1$, odnosno iteriranih komutatora Y_{312} i Y_{321} ako je $\sigma_{31}=1$, $\sigma_{32}=1$, $\sigma_{12}\neq 1$.

Pritom je $\sigma_{ji}=\sigma_{ij}$ za svaki $i\neq j$.

Linearna kombinacija iteriranih komutatora $Y_{i_1 i_2 i_3}$ i $Y_{i_1 i_3 i_2}$ neće se posebno isticati kao bazična konstanta u $\mathcal{B}_{123}$, jer iz pretpostavke da su komutatori $Y_{i_1 i_2 i_3}$ i $Y_{i_1 i_3 i_2}$ bazične konstante proizlazi da je njihova linearna kombinacija također bazična konstanta.

Tvrdnja 2.2.3.1

Ako je $\sigma_{12} = \sigma_{13} = \sigma_{23} = 1$, onda su iterirani komutatori Y_{123}, Y_{132} i Y_{231} linearno zavisni.

Dokažimo danu tvrdnju.

Neka je $\sigma_{12} = \sigma_{13} = \sigma_{23} = 1$ tako da vrijedi

$$\lambda_1 Y_{123} + \lambda_2 Y_{132} + \lambda_3 Y_{231} = 0 \tag{36}$$

Želimo pokazati da postoji barem jedan $\lambda_i \neq 0$, $i = 1,2,3$.

Po definiciji je

$$\begin{aligned} Y_{123} = \Big[[e_1, e_2]_{q_{21}}, e_3 \Big]_{q_{31}q_{32}} &= e_1e_2e_3 - q_{21}\, e_2e_1e_3 - q_{31}q_{32}\, e_3e_1e_2 + q_{31}q_{32}q_{21}\, e_3e_2e_1 \\ &= e_{123} - q_{21}\, e_{213} - q_{31}q_{32}\, e_{312} + q_{31}q_{32}q_{21}\, e_{321} \end{aligned}$$

$$\begin{aligned} Y_{132} = \Big[[e_1, e_3]_{q_{31}}, e_2 \Big]_{q_{21}q_{23}} &= e_1e_3e_2 - q_{31}\, e_3e_1e_2 - q_{21}q_{23}\, e_2e_1e_3 + q_{21}q_{23}q_{31}\, e_2e_3e_1 \\ &= e_{132} - q_{31}\, e_{312} - q_{21}q_{23}\, e_{213} + q_{21}q_{23}q_{31}\, e_{231} \end{aligned}$$

$$\begin{aligned} Y_{231} = \Big[[e_2, e_3]_{q_{32}}, e_1 \Big]_{q_{12}q_{13}} &= e_2e_3e_1 - q_{32}\, e_3e_2e_1 - q_{12}q_{13}\, e_1e_2e_3 + q_{12}q_{13}q_{32}\, e_1e_3e_2 \\ &= e_{231} - q_{32}\, e_{321} - q_{12}q_{13}\, e_{123} + q_{12}q_{13}q_{32}\, e_{132} \end{aligned}$$

Uvrštavanjem komutatora Y_{123}, Y_{132} i Y_{231} u (36) dobivamo

$$\begin{aligned} &\lambda_1 e_{123} - \lambda_1 q_{21}\, e_{213} - \lambda_1 q_{31}q_{32}\, e_{312} + \lambda_1 q_{31}q_{32}q_{21}\, e_{321} \\ &+ \lambda_2 e_{132} - \lambda_2 q_{31}\, e_{312} - \lambda_2 q_{21}q_{23}\, e_{213} + \lambda_2 q_{21}q_{23}q_{31}\, e_{231} \\ &+ \lambda_3 e_{231} - \lambda_3 q_{32}\, e_{321} - \lambda_3 q_{12}q_{13}\, e_{123} + \lambda_3 q_{12}q_{13}q_{32}\, e_{132} = 0 \end{aligned}$$

odnosno

$$\begin{aligned} &(\lambda_1 - \lambda_3 q_{12}q_{13})e_{123} + (\lambda_2 + \lambda_3 q_{12}q_{13}q_{32})e_{132} \\ &+ (-\lambda_1 q_{31}q_{32} - \lambda_2 q_{31})e_{312} + (\lambda_1 q_{31}q_{32}q_{21} - \lambda_3 q_{32})e_{321} \\ &+ (\lambda_2 q_{21}q_{23}q_{31} + \lambda_3)e_{231} + (-\lambda_1 q_{21} - \lambda_2 q_{21}q_{23})e_{213} = 0 \end{aligned}$$

na osnovu čega proizlazi

$$\left.\begin{aligned} &\lambda_1\left(1-\sigma_{12}\sigma_{13}\right)=0\,, \\ &\lambda_2\left(1-\sigma_{23}\right)=0\,, \\ &\lambda_3\left(1-\sigma_{12}\sigma_{13}\sigma_{23}\right)=0\,. \end{aligned}\right\} \tag{37}$$

Koristeći pretpostavku $\sigma_{12}=1$, $\sigma_{13}=1$, $\sigma_{23}=1$ iz (37) proizlazi da koeficijenti λ_i, $i=1,2,3$ ne moraju nužno svi biti jednaki nuli, što povlači da su iterirani komutatori Y_{123}, Y_{132} i Y_{231} linearno zavisni.

Primjer 2.2.3.2

Želimo pokazati da bazična konstanta Y_{123} proizlazi iz bazične konstante C_{123} ako je $\sigma_{123}=1$, $\sigma_{12}=1$, $\sigma_{13}\neq 1$, $\sigma_{23}\neq 1$, gdje je $\sigma_{123}=\sigma_{12}\sigma_{13}\sigma_{23}$.

Neka je $\sigma_{123}=1$, $\sigma_{12}=1$, $\sigma_{13}\neq 1$, $\sigma_{23}\neq 1$. Tada imamo

$$\sigma_{12}=1\,,\ \sigma_{13}\sigma_{23}=1\,,\ \sigma_{13}\neq 1\,,\ \sigma_{23}\neq 1. \tag{38}$$

Uvrštavanjem $\sigma_{12}=1$ u bazičnu konstantu C_{123} danu relacijom (32) dobiva se nova konstanta, koju ćemo označiti sa C'

$$\begin{aligned} C' &= \frac{1-\sigma_{13}}{\mathrm{q}_{31}}\ \left(\mathrm{e}_{123}+\mathrm{q}_{21}\mathrm{q}_{31}\mathrm{q}_{32}\,\mathrm{e}_{321}\right)+\mathrm{q}_{32}\left(\sigma_{13}-1\right)\left(\mathrm{e}_{312}+\mathrm{q}_{13}\mathrm{q}_{23}\mathrm{q}_{21}\,\mathrm{e}_{213}\right) \\ &= \frac{1-\sigma_{13}}{\mathrm{q}_{31}}\left(\mathrm{e}_{123}+\mathrm{q}_{21}\mathrm{q}_{31}\mathrm{q}_{32}\,\mathrm{e}_{321}-\mathrm{q}_{31}\mathrm{q}_{32}\,\mathrm{e}_{312}-\mathrm{q}_{31}\mathrm{q}_{13}\mathrm{q}_{23}\mathrm{q}_{32}\mathrm{q}_{21}\,\mathrm{e}_{213}\right) \\ &= \frac{1-\sigma_{13}}{\mathrm{q}_{31}}\mathrm{Y}_{123} \end{aligned} \tag{39}$$

Pritom smo koristili sljedeće

- uvjeti $\sigma_{123}=1$, $\sigma_{12}=1$ povlače $\mathrm{q}_{32}\left(\sigma_{13}-1\right)=\dfrac{1-\sigma_{23}}{\mathrm{q}_{23}}$,
- uvjeti (38) povlače $\mathrm{q}_{31}\mathrm{q}_{13}\mathrm{q}_{23}\mathrm{q}_{32}\mathrm{q}_{21}=\mathrm{q}_{21}$ i
- definiciju iteriranog komutatora $\mathrm{Y}_{123}=\mathrm{e}_{123}-\mathrm{q}_{21}\,\mathrm{e}_{213}-\mathrm{q}_{31}\mathrm{q}_{32}\,\mathrm{e}_{312}+\mathrm{q}_{31}\mathrm{q}_{32}\mathrm{q}_{21}\,\mathrm{e}_{321}$.

Lako se može pokazati da je bilo koji iterirani komutator $\mathrm{Y}_{i_1 i_2 i_3}$, $i_1,i_2,i_3\in\{1,2,3\}$, $i_1\neq i_2\neq i_3$ linearno zavisan s bazičnom konstantom C_{123} (tj. da proizlazi iz C_{123}) ako je $\sigma_{123}=1$, $\sigma_{i_1 i_2}=1$, $\sigma_{i_1 i_3}\neq 1$, $\sigma_{i_2 i_3}\neq 1$.

Primijetimo da vrijedi $C_{j_1j_2j_3} = \lambda\, C_{123}, \quad \lambda \in \mathbb{C}\setminus\{0\},$

gdje je $j_1j_2j_3$ bilo koja permutacija skupa $\{1,2,3\}$. Pritom je $C_{j_1j_2j_3}$ oblika

$$C_{j_1j_2j_3} = \sum_{cyc} \frac{1-\sigma_{j_1j_2}}{q_{j_1j_2}} \widetilde{X}_{j_2j_3j_1} \,. \tag{40}$$

gdje $\sum\limits_{cyc}$ označuje cikličku sumu. $\widetilde{X}_{p_1p_2p_3}$ je definirano relacijom (34).

Uzimajući u obzir gore navedeno zaključujemo da je prostor konstanti u generičkom težinskom potprostoru $\mathcal{B}_{123}$ jednodimenzionalan te da je ideal $\boldsymbol{I}_{123} = \langle\, C_{123} \rangle$ u $\mathcal{B}_{123}$ generiran bazičnom konstantom C_{123} ako je $\sigma_{123} = 1$.

Zaključujemo:

Prostor konstanti u bilo kojem generičkom težinskom potprostoru $\mathcal{B}_{l_1l_2l_3}$, $l_1 < l_2 < l_3$, $l_1, l_2, l_3 \in \mathcal{N} = \{i_1, i_2, ..., i_N\}$ algebre $\mathcal{B}$ je jednodimenzionalan.

Ideal $\boldsymbol{I}_{l_1l_2l_3} = \langle\, C_{l_1l_2l_3} \rangle$ u $\mathcal{B}_{l_1l_2l_3}$ biti će generiran bazičnom konstantom $C_{l_1l_2l_3}$ ako je $\sigma_{l_1l_2l_3} = 1$, $l_1 < l_2 < l_3$.

Razlikujemo sljedeće slučajeve

(1) Bazična konstanta $C_{l_1l_2l_3}$ može biti oblika

$$C_{l_1l_2l_3} = \sum_{cyc} \frac{1-\sigma_{l_1l_2}}{q_{l_1l_2}} \left(e_{l_2l_3l_1} + q_{l_3l_2} q_{l_1l_2} q_{l_1l_3}\, e_{l_1l_3l_2} \right)$$

ili kraće $$C_{l_1l_2l_3} = \sum_{cyc} \frac{1-\sigma_{l_1l_2}}{q_{l_1l_2}} \widetilde{X}_{l_2l_3l_1} \,.$$

ako je $\sigma_{l_1l_2l_3} = 1$, $\sigma_{l_1l_2} \neq 1$, $\sigma_{l_1l_3} \neq 1$, $\sigma_{l_2l_3} \neq 1$.

Pritom je $\widetilde{X}_{p_1p_2p_3} = e_{p_1p_2p_3} + q_{p_2p_1} q_{p_3p_1} q_{p_3p_2}\, e_{p_3p_2p_1}$ te $\sum\limits_{cyc}$ označuje cikličku sumu.

(2) Ako je $\sigma_{l_1l_2l_3} = 1$, $\sigma_{i_1i_2} = 1$, $\sigma_{i_1i_3} \neq 1$, $\sigma_{i_2i_3} \neq 1$, $i_1, i_2, i_3 \in \{l_1, l_2, l_3\}$, $i_1 \neq i_2 \neq i_3$, onda je bazična konstanta $C_{l_1l_2l_3}$ proporcionalna iteriranom komutatoru $Y_{i_1i_2i_3}$ i pišemo

$$C_{l_1l_2l_3} = \alpha\, Y_{i_1i_2i_3}, \quad \alpha \in \mathbb{C}\setminus\{0\}.$$

(3) Ako je $\sigma_{l_1l_2l_3} \neq 1$, $\sigma_{l_1l_2} \neq 1$, $\sigma_{l_1l_3} \neq 1$, $\sigma_{l_2l_3} \neq 1$, onda u generičkom težinskom potprostoru $\mathcal{B}_{l_1l_2l_3}$ nema konstanti.

2. <u>Degenerirani slučaj:</u>

Pretpostavimo da je $l_3 = l_1$.

Tada je multiskupu $\mathrm{Q} = \{l_1, l_1, l_2\} = \{l_1^2, l_2\}$ nad $\mathcal{N} = \{\mathrm{i}_1, \mathrm{i}_2, \ldots, \mathrm{i}_\mathrm{N}\}$ pridruženi degenerirani težinski potprostor dan sa $\boldsymbol{\mathcal{B}}_{l_1^2 l_2} = span_{\mathbb{C}} \{\mathrm{e}_{l_1 l_1 l_2}, \mathrm{e}_{l_1 l_2 l_1}, \mathrm{e}_{l_2 l_1 l_1}\}$.

U ovom slučaju je $\widehat{\mathrm{Q}} = \mathrm{S}_3\mathrm{Q} = \{l_1 l_1 l_2, l_1 l_2 l_1, l_2 l_1 l_1\}$ pa iz formule (24) proizlazi

$$\partial|_{\boldsymbol{\mathcal{B}}_{l_1^2 l_2}} \left(\mathrm{e}_{l_1 l_1 l_2}\right) = 1\,\mathrm{e}_{l_1 l_1 l_2} + \mathrm{q}_{l_1 l_1}\mathrm{e}_{l_1 l_1 l_2} + \mathrm{q}_{l_2 l_1}\mathrm{q}_{l_2 l_1}\mathrm{e}_{l_2 l_1 l_1} = \left(1 + \mathrm{q}_{l_1 l_1}\right)\mathrm{e}_{l_1 l_1 l_2} + \mathrm{q}^2_{l_2 l_1}\mathrm{e}_{l_2 l_1 l_1},$$

$$\partial|_{\boldsymbol{\mathcal{B}}_{l_1^2 l_2}} \left(\mathrm{e}_{l_1 l_2 l_1}\right) = 1\,\mathrm{e}_{l_1 l_2 l_1} + \mathrm{q}_{l_2 l_1}\mathrm{e}_{l_2 l_1 l_1} + \mathrm{q}_{l_1 l_1}\mathrm{q}_{l_1 l_2}\mathrm{e}_{l_1 l_1 l_2} = \mathrm{q}_{l_1 l_1}\mathrm{q}_{l_1 l_2}\mathrm{e}_{l_1 l_1 l_2} + 1\,\mathrm{e}_{l_1 l_2 l_1} + \mathrm{q}_{l_2 l_1}\mathrm{e}_{l_2 l_1 l_1},$$

$$\partial|_{\boldsymbol{\mathcal{B}}_{l_1^2 l_2}} \left(\mathrm{e}_{l_2 l_1 l_1}\right) = 1\,\mathrm{e}_{l_2 l_1 l_1} + \mathrm{q}_{l_1 l_2}\mathrm{e}_{l_1 l_2 l_1} + \mathrm{q}_{l_1 l_2}\mathrm{q}_{l_1 l_1}\mathrm{e}_{l_1 l_1 l_2} = \mathrm{q}_{l_1 l_2}\left(1 + \mathrm{q}_{l_1 l_1}\right)\mathrm{e}_{l_1 l_2 l_1} + 1\,\mathrm{e}_{l_2 l_1 l_1},$$

što povlači da u monomijalnoj bazi potprostora $\boldsymbol{\mathcal{B}}_{l_1^2 l_2}$ operatoru $\partial|_{\boldsymbol{\mathcal{B}}_{l_1^2 l_2}} : \boldsymbol{\mathcal{B}}_{l_1^2 l_2} \to \boldsymbol{\mathcal{B}}_{l_1^2 l_2}$ je pridružena matrica $\mathrm{B}_{l_1^2 l_2}$ oblika

$$\mathrm{B}_{l_1^2 l_2} = \begin{array}{c} \\ l_1 l_1 l_2 \\ l_1 l_2 l_1 \\ l_2 l_1 l_1 \end{array} \!\!\begin{array}{c} \begin{array}{ccc} l_1 l_1 l_2 & \quad l_1 l_2 l_1 & \quad\quad l_2 l_1 l_1 \end{array} \\ \begin{bmatrix} 1 + \mathrm{q}_{l_1 l_1} & \mathrm{q}_{l_1 l_1}\mathrm{q}_{l_1 l_2} & 0 \\ 0 & 1 & \mathrm{q}_{12}\left(1 + \mathrm{q}_{l_1 l_1}\right) \\ \mathrm{q}^2_{l_2 l_1} & \mathrm{q}_{l_2 l_1} & 1 \end{bmatrix} \end{array}$$

kojoj je determinanta

$$\det \mathrm{B}_{l_1^2 l_2} = \left(1 + \mathrm{q}_{l_1 l_1}\right)\left(1 - \sigma_{l_1 l_2}\right)\left(1 - \mathrm{q}_{l_1 l_1}\sigma_{l_1 l_2}\right)$$

$$= \left(1 + \mathrm{q}_{l_1 l_1}\right)\prod_{\mathrm{k}=0}^{1}\left(1 - \mathrm{q}^{\mathrm{k}}_{l_1 l_1}\sigma_{l_1 l_2}\right) = [2]_{\mathrm{q}_{l_1 l_1}} \prod_{\mathrm{k}=0}^{1}\left(1 - \mathrm{q}^{\mathrm{k}}_{l_1 l_1}\sigma_{l_1 l_2}\right) = [2]_{\mathrm{q}_{l_1 l_1}}! \prod_{\mathrm{k}=0}^{1}\left(1 - \mathrm{q}^{\mathrm{k}}_{l_1 l_1}\sigma_{l_1 l_2}\right).$$

Pritom je $[2]_{\mathrm{q}_{l_1 l_1}}! := [2]_{\mathrm{q}_{l_1 l_1}} \cdot [1]_{\mathrm{q}_{l_1 l_1}} = [2]_{\mathrm{q}_{l_1 l_1}} = 1 + \mathrm{q}_{l_1 l_1}$, $[1]_{\mathrm{q}_{l_1 l_1}} := 1$.

Ako je $\mathrm{q}_{l_1 l_1}\sigma_{l_1 l_2} \neq 1$, $\sigma_{l_1 l_2} \neq 1$, $[2]_{\mathrm{q}_{l_1 l_1}} \neq 0$, onda u $\boldsymbol{\mathcal{B}}_{l_1^2 l_2}$ nema konstanti.

U degeneriranom težinskom potprostoru $\boldsymbol{\mathcal{B}}_{l_1^2 l_2}$ postojati će netrivijalna bazična konstanta ako i samo ako je $\det \mathrm{B}_{l_1^2 l_2} = 0$, tj. $\mathrm{q}_{l_1 l_1}\sigma_{l_1 l_2} = 1$ ili $\sigma_{l_1 l_2} = 1$ ili $[2]_{\mathrm{q}_{l_1 l_1}} = 0$.

U nastavku će se dokazati da je tada prostor konstanti u $\boldsymbol{\mathcal{B}}_{l_1^2 l_2}$ jednodimenzionalan.

Ako je $\mathrm{q}_{l_1 l_1}\sigma_{l_1 l_2} = 1$, onda ćemo konstruirati konstantu $C_{l_1 l_1 l_2}$, koja će generirati ideal $\boldsymbol{I}_{l_1 l_1 l_2}$ u potprostoru $\boldsymbol{\mathcal{B}}_{l_1^2 l_2}$.

Radi jednostavnosti zapisa pretpostavimo da je $l_1 = 1$, $l_2 = 2$.

Tada je multiskupu $Q = \{1^2, 2\}$ pridružen degenerirani težinski potprostor $\mathcal{B}_{1^2 2}$ pa je u monomijalnoj bazi potprostora $\mathcal{B}_{1^2 2}$ operatoru $\partial|_{\mathcal{B}_{1^2 2}} : \mathcal{B}_{1^2 2} \to \mathcal{B}_{1^2 2}$ pridružena matrica $B_{1^2 2}$. Pritom je $\det B_{1^2 2} = (1 + q_{11})(1 - \sigma_{12})(1 - q_{11}\sigma_{12})$.

Slijedi izračunavanje prostora konstanti u degeneriranom težinskom potprostoru $\mathcal{B}_{1^2 2}$ u ovisnosti o izborima vrijednosti parametara q_{ij} za koje je $\det B_{1^2 2} = 0$.

Ako je $q_{11}\sigma_{12} \neq 1$, $\sigma_{12} \neq 1$, $[2]_{q_{11}} \neq 0$, onda u $\mathcal{B}_{1^2 2}$ nema konstanti.

Pritom je $[2]_{q_{11}} = 1 + q_{11}$.

- ♦ Ako je $q_{11}\sigma_{12} = 1$, $\sigma_{12} \neq 1$, $[2]_{q_{11}} \neq 0$ ili $q_{11}\sigma_{12} = 1$, $\sigma_{12} = 1$, $\left([2]_{q_{11}} \neq 0\right)$, onda vektor $\left(q_{11}q_{12}^2,\ -q_{12}(1 + q_{11}),\ 1\right)$ pripada jezgri operatora $\partial|_{\mathcal{B}_{1^2 2}}$, tj. $\{q_{11}q_{12}^2 e_{112} - q_{12}(1 + q_{11})e_{121} + e_{211}\}$ je baza prostora konstanti u $\mathcal{B}_{1^2 2}$ pa za baznu konstantu možemo uzeti

$$C_{112} = Y_{211}, \tag{41}$$

gdje je

$$Y_{211} = [Y_{21}, e_1]_{q_{11}q_{12}} = \left[[e_2, e_1]_{q_{12}}, e_1\right]_{q_{11}q_{12}} = e_{211} - q_{12}(1 + q_{11})e_{121} + q_{11}q_{12}^2 e_{112},$$

(uz uobičajene pokrate: $e_{j_1 j_2 j_3} = e_{j_1} e_{j_2} e_{j_3}$).

- ➢ Ako je $[2]_{q_{11}} = 0$, $\sigma_{12} \neq 1$, onda je $\{e_{112} - q_{21}^2 e_{211}\}$ baza prostora konstanti u $\mathcal{B}_{1^2 2}$ pa se bazična konstanta može pisati u obliku jednostavnog komutatora

$$X^{112} := \left[e_1^2, e_2\right]_{q_{21}^2} = e_{112} - q_{21}^2 e_{211}. \tag{42}$$

Prirodno se nameće sljedeće pitanje:

da li su komutatori Y_{211} i X^{112} linearno zavisni, odnosno da li bazična konstanta X^{112} može proizaći iz bazične konstante Y_{211} ako je $q_{11}\sigma_{12} = 1$, $[2]_{q_{11}} = 0$?

Pretpostavimo da je $q_{11}\sigma_{12} = 1$, $[2]_{q_{11}} = 0$, gdje je $[2]_{q_{11}} = q_{11} + 1$.

Tada primjenom uvjeta $q_{11}\sigma_{12} = 1$ dobivamo

$$\begin{aligned} Y_{211} = \left[[e_2, e_1]_{q_{12}}, e_1\right]_{q_{11}q_{12}} &= e_{211} - q_{12}(1+q_{11})e_{121} + q_{11}q_{12}^2 e_{112} \\ &= q_{11}q_{12}^2\left(e_{112} - \frac{1}{q_{11}q_{12}}(1+q_{11})e_{121} + \frac{1}{q_{11}q_{12}^2}e_{211}\right) \\ &= q_{11}q_{12}^2\left(e_{112} - q_{21}(1+q_{11})e_{121} + q_{11}q_{21}^2 e_{211}\right). \end{aligned} \tag{43}$$

Uvrštavanjem $q_{11} = -1$ u (43) dobiva se nova konstanta, koju ćemo označiti sa Y'_{211}. Dobivamo

$$Y'_{211} = -q_{12}^2 X^{112}, \tag{44}$$

pri čemu smo koristili (42).

Komutatori Y_{211} i X^{112} su proporcionalni, tj. jednostavni komutator X^{112} može se dobiti iz iteriranog komutatora Y_{211} ako je $q_{11}\sigma_{12} = 1$, $[2]_{q_{11}} = 0$, stoga možemo uzeti da je ideal $\boldsymbol{I}_{112} = \langle C_{112} \rangle$ u potprostoru $\mathcal{B}_{1^2 2}\ (= \mathcal{B}_{112})$ generiran baznim konstantom $C_{112} = Y_{211}$ ako je $q_{11}\sigma_{12} = 1$.

Primjedba 2.2.3.3

Za iterirane komutatore vrijede sljedeća svojstva.

(*i*) Ako je $\sigma_{12} = 1$, $[2]_{q_{11}} \neq 0$, onda je $Y_{211} = -q_{12} Y_{121}$.

(*ii*) Ako je $[2]_{q_{11}} = 0$, onda je $Y_{211} = -\frac{q_{12}^2}{2} Y_{112}$.

Zaključujemo:

Prostor konstanti u bilo kojem degeneriranom težinskom potprostoru $\mathcal{B}_{l_1^2 l_2}$, $l_1 \neq l_2$, $l_1, l_2 \in \mathcal{N} = \{i_1, i_2, ..., i_N\}$ algebre $\mathcal{B}$ je jednodimenzionalan.

Ideal $\boldsymbol{I}_{l_1 l_1 l_2} = \langle C_{l_1 l_1 l_2} \rangle$ u $\mathcal{B}_{l_1^2 l_2}$ biti će generiran baznim konstantom $C_{l_1 l_1 l_2}$ ako je $q_{l_1 l_1}\sigma_{l_1 l_2} = 1$, $l_1 \neq l_2$.

Razlikujemo sljedeće slučajeve

(1) Bazična konstanta $C_{l_1l_1l_2}$ može biti oblika $C_{l_1l_1l_2} = \mathrm{Y}_{l_2l_1l_1}$

ako je $\mathrm{q}_{l_1l_1}\sigma_{l_1l_2} = 1$, $\sigma_{l_1l_2} \neq 1$, $[2]_{\mathrm{q}_{l_1l_1}} \neq 0$ ili ako je $\mathrm{q}_{l_1l_1}\sigma_{l_1l_2} = 1$, $\sigma_{l_1l_2} = 1$, $[2]_{\mathrm{q}_{l_1l_1}} \neq 0$.

Pritom je $\mathrm{Y}_{l_2l_1l_1} = \left[\mathrm{Y}_{l_2l_1}, \mathrm{e}_{l_1}\right]_{\mathrm{q}_{l_1l_1}\mathrm{q}_{l_1l_2}} = \mathrm{e}_{l_2l_1l_1} - \mathrm{q}_{l_1l_2}\left(1+\mathrm{q}_{l_1l_1}\right)\mathrm{e}_{l_1l_2l_1} + \mathrm{q}_{l_1l_1}\mathrm{q}^2_{l_1l_2}\mathrm{e}_{l_1l_1l_2}$.

(2) Ako je $\mathrm{q}_{l_1l_1}\sigma_{l_1l_2} = 1$, $[2]_{\mathrm{q}_{l_1l_1}} = 0$, onda je bazična konstanta $C_{l_1l_1l_2}$ linearno zavisna s jednostavnim komutatorom $\mathrm{X}^{l_1l_1l_2}$ i pišemo $C_{l_1l_1l_2} = \alpha\,\mathrm{X}^{l_1l_1l_2}$, $\alpha \in \mathbb{C}\setminus\{0\}$.

Pritom je $\mathrm{X}^{l_1l_1l_2} = \left[\mathrm{e}^2_{l_1}, \mathrm{e}_{l_2}\right]_{\mathrm{q}^2_{l_2l_1}} = \mathrm{e}_{l_1l_1l_2} - \mathrm{q}^2_{l_2l_1}\,\mathrm{e}_{l_2l_1l_1}$.

(3) Ako je $\mathrm{q}_{l_1l_1}\sigma_{l_1l_2} \neq 1$, $\sigma_{l_1l_2} \neq 1$, $[2]_{\mathrm{q}_{l_1l_1}} \neq 0$, onda u $\mathcal{B}_{l_1^2l_2}$ nema konstanti.

Pretpostavimo sada da je $l_1 = l_2 = l_3 = l$.

Tada je multiskupu $\mathrm{Q} = \{l^3\}$ nad $\mathcal{N} = \{\mathrm{i}_1, \mathrm{i}_2, ..., \mathrm{i}_\mathrm{N}\}$ pridruženi degenerirani težinski potprostor $\mathcal{B}_{l^3} = span_{\mathbb{C}}\{\mathrm{e}_{lll}\}$, gdje je $\mathrm{e}_{lll} = \mathrm{e}_l\mathrm{e}_l\mathrm{e}_l = \mathrm{e}_{l^3}$.

Imamo da je $\widehat{\mathrm{Q}} = \mathrm{S}_3\mathrm{Q} = \{lll\} = \{l^3\}$, stoga primjenom formule (24) proizlazi

$$\partial|_{\mathcal{B}_{l^3}}(\mathrm{e}_{lll}) = \left(1+\mathrm{q}_{ll}+\mathrm{q}^2_{ll}\right)\mathrm{e}_{lll},$$

što povlači da u monomijalnoj bazi potprostora $\mathcal{B}_{l^3}$ operatoru $\partial|_{\mathcal{B}_{l^3}} : \mathcal{B}_{l^3} \to \mathcal{B}_{l^3}$ je pridružena 1×1 matrica B_{l^3}, kojoj je jedini element jednak $1+\mathrm{q}_{ll}+\mathrm{q}^2_{ll}$ pa je

$$\det \mathrm{B}_{l^3} = 1+\mathrm{q}_{ll}+\mathrm{q}^2_{ll}.$$

Matrica B_{l^3} je singularna ako i samo ako je $1+\mathrm{q}_{ll}+\mathrm{q}^2_{ll} = 0$, tj. $\mathrm{q}_{ll} = \dfrac{-1\pm i\sqrt{3}}{2}$.

Tada je bilo koja netrivijalna konstanta u potprostoru $\mathcal{B}_{l^3}$ oblika $C_{lll} = \alpha\,\mathrm{e}_{l^3}$, $\alpha \in \mathbb{C}\setminus\{0\}$, gdje je $\mathrm{e}_{l^3} = \mathrm{e}^3_l$. Ako je $1+\mathrm{q}_{ll}+\mathrm{q}^2_{ll} \neq 0$, onda u $\mathcal{B}_{l^3}$ nema konstanti.

Dobili smo da je prostor konstanti u degeneriranom potprostoru $\mathcal{B}_{l^3}$ jednodimenzionalan i da je $C_{lll} = \mathrm{e}_{l^3}$ bazična konstanta u $\mathcal{B}_{l^3}$ ako je $[3]_{\mathrm{q}_{ll}} = 0$.

Pritom je $[3]_{\mathrm{q}_{ll}} = 1+\mathrm{q}_{ll}+\mathrm{q}^2_{ll}$.

U nastavku se želi pokazati da se ideal $\boldsymbol{I}_3$ u potprostoru $\mathcal{B}(3)$ može generirati konstantama oblika $C_{l_1l_2l_3}$, ako je $\sigma_{l_1l_2l_3}=1$, $l_1\le l_2\le l_3$, $l_1,l_2,l_3\in\mathcal{N}$.

Da bi se to pokazalo potrebno je dokazati tvrdnju:

> bazična konstanta u bilo kojem degeneriranom težinskom potprostoru $\mathcal{B}_{\mathrm{Q}}\subset\mathcal{B}(3)$ (Q je multiskup koji nije skup) može se konstruirati iz bazične konstante nekog generičkog težinskog potprostora $\mathcal{B}_{\mathrm{Q}}\subset\mathcal{B}(3)$ (Q je skup) određenom specijalizacijom koja je inducirana poistovjećivanjem nekih elemenata u skupu Q.

Tvrdnja 2.2.3.4

Iz bazične konstante $C_{l_1l_2l_3}$ generičkog težinskog potprostora $\mathcal{B}_{l_1l_2l_3}$ dobiva se bazična konstanta $C'_{l_1l_1l_2}$ u odgovarajućem degeneriranom težinskom potprostoru $\mathcal{B}_{l_1^2l_2}$ ako se pritom npr. poistovijete elementi l_3 i l_1.

Neka je $$\sigma_{l_1l_2l_3}=1,\ \sigma_{l_1l_2}\neq 1,\ \sigma_{l_1l_3}\neq 1,\ \sigma_{l_2l_3}\neq 1. \tag{45}$$

Tada je u generičkom težinskom potprostoru $\mathcal{B}_{l_1l_2l_3}$ bazična konstanta $C_{l_1l_2l_3}$ dana sa

$$C_{l_1l_2l_3}=\frac{1-\sigma_{l_1l_2}}{\mathrm{q}_{l_1l_2}}\left(\mathrm{e}_{l_2l_3l_1}+\mathrm{q}_{l_3l_2}\mathrm{q}_{l_1l_2}\mathrm{q}_{l_1l_3}\,\mathrm{e}_{l_1l_3l_2}\right)+\frac{1-\sigma_{l_1l_3}}{\mathrm{q}_{l_3l_1}}\left(\mathrm{e}_{l_1l_2l_3}+\mathrm{q}_{l_2l_1}\mathrm{q}_{l_3l_1}\mathrm{q}_{l_3l_2}\,\mathrm{e}_{l_3l_2l_1}\right)$$
$$+\frac{1-\sigma_{l_2l_3}}{\mathrm{q}_{l_2l_3}}\left(\mathrm{e}_{l_3l_1l_2}+\mathrm{q}_{l_1l_3}\mathrm{q}_{l_2l_3}\mathrm{q}_{l_2l_1}\,\mathrm{e}_{l_2l_1l_3}\right).$$

Pretpostavimo da je $l_3=l_1$. Tada iz (45) proizlazi

$$\mathrm{q}_{l_1l_1}^2\sigma_{l_1l_2}^2=1,\ \mathrm{q}_{l_1l_1}^2\neq 1,\ \sigma_{l_1l_2}\neq 1$$

ili

$$\left(1-\mathrm{q}_{l_1l_1}\sigma_{l_1l_2}\right)\left(1+\mathrm{q}_{l_1l_1}\sigma_{l_1l_2}\right)=0,\ \ \mathrm{q}_{l_1l_1}\neq 1,\ \ \mathrm{q}_{l_1l_1}\neq -1,\ \ \sigma_{l_1l_2}\neq 1, \tag{46}$$

gdje je $l_1<l_2$, $l_1,l_2\in\mathcal{N}$.

Označimo sa $C'_{l_1l_1l_2}$ bazičnu konstantu u odgovarajućem degeneriranom težinskom potprostoru $\mathcal{B}_{l_1^2l_2}$, koja će se dobiti iz bazične konstante $C_{k_1k_2k_3}$ za $l_3=l_1$. Tada je

$$C'_{l_1l_1l_2}=\frac{1-\mathrm{q}_{l_1l_1}^2}{\mathrm{q}_{l_1l_1}}\left(1+\mathrm{q}_{l_1l_1}\sigma_{l_1l_2}\right)\mathrm{e}_{l_1l_2l_1}+\frac{1-\sigma_{l_1l_2}}{\sigma_{l_1l_2}}\left(\mathrm{q}_{l_1l_2}\left(1+\mathrm{q}_{l_1l_1}\sigma_{l_1l_2}\right)\mathrm{e}_{l_1l_1l_2}+\mathrm{q}_{l_2l_1}\left(1+\mathrm{q}_{l_1l_1}\sigma_{l_1l_2}\right)\mathrm{e}_{l_2l_1l_1}\right).$$

Uzimajući u obzir (46) zaključujemo sljedeće

(i) Ako je $q_{l_1l_1}\sigma_{l_1l_2} = -1$, $q_{l_1l_1} \neq 1$, $q_{l_1l_1} \neq -1$, $\sigma_{l_1l_2} \neq 1$, onda je $C'_{l_1l_1l_2} = 0$, tj. $C'_{l_1l_1l_2}$ je trivijalna konstanta.

(ii) Ako je $q_{l_1l_1}\sigma_{l_1l_2} = 1$, $q_{l_1l_1} \neq 1$, $q_{l_1l_1} \neq -1$, $\sigma_{l_1l_2} \neq 1$, onda je

$$
\begin{aligned}
C'_{l_1l_1l_2} &= 2\left(1-q_{l_1l_1}\right)\left(\frac{1+q_{l_1l_1}}{q_{l_1l_1}}e_{l_1l_2l_1} - q_{l_1l_2}e_{l_1l_1l_2} - q_{l_2l_1}e_{l_2l_1l_1}\right) \\
&= 2\frac{1-\sigma_{l_1l_2}}{q_{l_1l_2}}\left(e_{l_2l_1l_1} - q_{l_1l_2}\left(1+q_{l_1l_1}\right)e_{l_1l_2l_1} + q_{l_1l_1}q^2_{l_1l_2}e_{l_1l_1l_2}\right) \\
&= 2\frac{1-\sigma_{l_1l_2}}{q_{l_1l_2}}Y_{l_2l_1l_1} \\
&= 2\frac{1-\sigma_{l_1l_2}}{q_{l_1l_2}}C_{l_1l_1l_2}.
\end{aligned}
$$

Dobili smo da se bazična konstanta $C_{l_1l_1l_2}$ u degeneriranom težinskom potprostoru $\mathcal{B}_{l_1^2l_2}$ može konstruirati iz bazične konstante $C_{l_1l_2l_3}$ generičkog težinskog potprostora $\mathcal{B}_{l_1l_2l_3}$ ako pritom u skupu Q element l_3 poistovijetimo s elementom l_1.
Pritom se uvjet kocikličnosti $\sigma_{l_1l_2l_3} = 1$ degenerira na uvjet $q_{l_1l_1}\sigma_{l_1l_2} = 1$.

Tvrdnja 2.2.3.5

Iz bazične konstante $C_{l_1l_2l_3}$ generičkog težinskog potprostora $\mathcal{B}_{l_1l_2l_3}$ može se dobiti bazična konstanta C'_{lll} u degeneriranom težinskom potprostoru $\mathcal{B}_{l^3}$ ako stavimo $l_1 = l_2 = l_3 = l$.

Dakle, pretpostavimo da je $l_1 = l_2 = l_3 = l$. Tada iz (45) proizlazi

$$q_{ll}^6 = 1,\ q_{ll}^2 \neq 1$$

ili

$$\left(1-q_{ll}+q_{ll}^2\right)\left(1+q_{ll}+q_{ll}^2\right) = 0,\ \ q_{ll} \neq 1,\ \ q_{ll} \neq -1. \tag{47}$$

Označimo sa C'_{lll} bazičnu konstantu u degeneriranom težinskom potprostoru $\mathcal{B}_{l^3}$, koja će se dobiti iz bazične konstante $C_{k_1k_2k_3}$ za $l_1 = l_2 = l_3 = l$.

Tada je

$$C'_{lll} = 3\frac{1-q_{ll}^2}{q_{ll}}\left(1+q_{ll}^3\right)e_{lll} = \frac{3\left(1-q_{ll}^2\right)\left(1+q_{ll}\right)\left(1-q_{ll}+q_{ll}^2\right)}{q_{ll}}e_{l^3}\,.$$

Uzimajući u obzir (47) zaključujemo sljedeće

(*i*) Ako je $1-q_{ll}+q_{ll}^2=0$, onda je $C'_{lll}=0$, tj. C'_{lll} je trivijalna konstanta.

(*ii*) Ako je $1+q_{ll}+q_{ll}^2=0$, onda je $C'_{lll}=\alpha\, C_{lll}$, $\alpha\in\mathbb{C}\setminus\{0\}$.

Pritom se koristi svojstvo da

$$1-q_{ll}+q_{ll}^2=0,\ \text{tj. } 1+q_{ll}+q_{ll}^2=0 \quad \text{povlači} \quad q_{ll}\neq 1,\ q_{ll}\neq -1.$$

Dobili smo da se bazična konstanta C_{lll} u degeneriranom težinskom potprostoru $\mathcal{B}_{l^3}$ može konstruirati iz bazične konstante $C_{l_1l_2l_3}$ generičkog težinskog potprostora $\mathcal{B}_{l_1l_2l_3}$ ako stavimo $l_1=l_2=l_3=l$.

Pritom se uvjet kocikličnosti $\sigma_{l_1l_2l_3}=1$ degenerira na uvjet $1+q_{ll}+q_{ll}^2=0$.

Zaključujemo:

Konstante u potprostoru $\mathcal{B}(3)$ generiraju obostrani ideal $\boldsymbol{I}_3=\left\langle C_{l_1l_2l_3}\right\rangle$ ako je $\sigma_{l_1l_2l_3}=1$, $l_1\le l_2\le l_3$, $l_1,l_2,l_3\in\mathcal{N}$.

Pritom je

$$C_{l_1l_2l_3}=\sum_{cyc}\frac{1-\sigma_{l_1l_2}}{q_{l_1l_2}}\widetilde{X}_{l_2l_3l_1}\,,$$

gdje je $\widetilde{X}_{p_1p_2p_3}=e_{p_1p_2p_3}+q_{p_2p_1}q_{p_3p_1}q_{p_3p_2}\,e_{p_3p_2p_1}$ te $\sum\limits_{cyc}$ označuje cikličku sumu.

Dimenzija prostora konstanti u $\mathcal{B}(3)$ je manja ili jednaka $\binom{N+2}{3}$.

Dimenzija je maksimalna ako je

$\sigma_{l_1l_2l_3}=1$ za svaki $l_1,l_2,l_3\in\mathcal{N}$, $l_1<l_2<l_3$,

$q_{l_1l_1}\sigma_{l_1l_2}=1$ za svaki $l_1,l_2\in\mathcal{N}$, $l_1\neq l_2$ i

$1+q_{ll}+q_{ll}^2=0$ za svaki $l\in\mathcal{N}$.

2. 2. 4. Računanje prostora konstanti u $\mathcal{B}(4)$

U ovom odjeljku će predmet razmatranja biti nekomutirajući polinomi četvrtog stupnja u varijablama $e_{i_1},...,e_{i_N}$.

Odredimo prostor konstanti u potprostoru $\mathcal{B}(4)$.

Neka je $Q=\{l_1 \le l_2 \le l_3 \le l_4\}$ multiskup nad $\mathcal{N}=\{i_1,i_2,...,i_N\}$. Tada je multiskupu Q pridružen težinski potprostor $\mathcal{B}_Q \subset \mathcal{B}(4)$ takav da je

$$\mathcal{B}_Q = span_{\mathbb{C}}\left\{e_{j_1j_2j_3j_4}=e_{j_1}e_{j_2}e_{j_3}e_{j_4} \mid j_1j_2j_3j_4 = \text{permutacija multiskupa Q}\right\}.$$

Iz formule (24) slijedi da je operator $\partial|_{\mathcal{B}_Q}:\mathcal{B}_Q \to \mathcal{B}_Q$ na potprostoru $\mathcal{B}_Q$ definiran sa

$$\partial|_{\mathcal{B}_Q}\left(e_{j_1j_2j_3j_4}\right)=1\,e_{j_1j_2j_3j_4}+q_{j_2j_1}\,e_{j_2j_1j_3j_4}+q_{j_3j_1}q_{j_3j_2}\,e_{j_3j_1j_2j_4}+q_{j_4j_1}q_{j_4j_2}q_{j_4j_3}\,e_{j_4j_1j_2j_3}, \tag{48}$$

$j_1j_2j_3j_4 \in \widehat{Q}=S_4Q$.

U daljnjim razmatranjima promatrati će se najprije generički težinski potprostor $\mathcal{B}_{l_1l_2l_3l_4}$, $l_1<l_2<l_3<l_4$ pridružen skupu $Q=\{l_1,l_2,l_3,l_4\}$, a potom će se u skupu Q poistovijetiti neki elementi. Time će se dobiti odgovarajući multiskupovi (koji nisu skupovi) i njima pridruženi degenerirani potprostori.

U potprostoru $\mathcal{B}(4)$ razlikujemo četiri tipa degeneriranih potprostora

1) $\mathcal{B}_{i_1^2 i_2 i_3}$, $i_1 \neq i_2 < i_3$, $i_1, i_2, i_3 \in \mathcal{N}=\{i_1,i_2,...,i_N\}$,

2) $\mathcal{B}_{i_1^2 i_2^2}$, $i_1 < i_2$, $i_1, i_2 \in \mathcal{N}$,

3) $\mathcal{B}_{i_1^3 i_2}$, $i_1 \neq i_2$, $i_1, i_2 \in \mathcal{N}$,

4) $\mathcal{B}_{i^4}$ $i \in \mathcal{N}$.

S druge strane navedene degenerirane potprostore možemo konstruirati tako da ih pridružimo odgovarajućem multiskupu Q, koji će se dobiti iz skupa $Q=\{l_1,l_2,l_3,l_4\}$ podskupa od $\mathcal{N}=\{i_1,i_2,...,i_N\}$ tako da

1) poistovijetimo bilo koja dva elementa skupa Q (a preostala dva ostavimo nepromijenjena),

2) poistovijetimo po dva elementa skupa Q,

3) poistovijetimo bilo koja tri elementa skupa Q,

4) poistovijetimo sva četiri elementa skupa Q.

Konkretno

1) Pretpostavimo da je $l_4 = l_1$.

Tada će iz skupa $\mathrm{Q} = \{l_1, l_2, l_3, l_4\}$ nastati multiskup $\mathrm{Q} = \{l_1^2, l_2, l_3\}$ te će njemu pridruženi degenerirani težinski potprostor biti

$$\mathcal{B}_{l_1^2 l_2 l_3} = span_{\mathbb{C}}\left\{\mathrm{e}_{l_1 l_1 l_2 l_3}, \mathrm{e}_{l_1 l_1 l_3 l_2}, \mathrm{e}_{l_1 l_2 l_1 l_3}, \mathrm{e}_{l_1 l_2 l_3 l_1}, \mathrm{e}_{l_1 l_3 l_1 l_2}, \mathrm{e}_{l_1 l_3 l_2 l_1}, \mathrm{e}_{l_2 l_1 l_1 l_3}, \mathrm{e}_{l_2 l_1 l_3 l_1}, \mathrm{e}_{l_2 l_3 l_1 l_1}, \mathrm{e}_{l_3 l_1 l_1 l_2}, \mathrm{e}_{l_3 l_1 l_2 l_1}, \mathrm{e}_{l_3 l_2 l_1 l_1}\right\},$$

2) Pretpostavimo da je $l_4 = l_1$, $l_3 = l_2$.

Tada će iz skupa $\mathrm{Q} = \{l_1, l_2, l_3, l_4\}$ nastati multiskup $\mathrm{Q} = \{l_1^2, l_2^2\}$, kojemu će pridruženi degenerirani težinski potprostor biti

$$\mathcal{B}_{l_1^2 l_2^2} = span_{\mathbb{C}}\left\{\mathrm{e}_{l_1 l_1 l_2 l_2}, \mathrm{e}_{l_1 l_2 l_1 l_2}, \mathrm{e}_{l_1 l_2 l_2 l_1}, \mathrm{e}_{l_2 l_1 l_1 l_2}, \mathrm{e}_{l_2 l_1 l_2 l_1}, \mathrm{e}_{l_2 l_2 l_1 l_1}\right\},$$

3) Pretpostavimo da je $l_4 = l_3 = l_1$.

Tada će iz skupa $\mathrm{Q} = \{l_1, l_2, l_3, l_4\}$ nastati multiskup $\mathrm{Q} = \{l_1^3, l_2\}$, kojemu će pridruženi degenerirani težinski potprostor biti

$$\mathcal{B}_{l_1^3 l_2} = span_{\mathbb{C}}\left\{\mathrm{e}_{l_1 l_1 l_1 l_2}, \mathrm{e}_{l_1 l_1 l_2 l_1}, \mathrm{e}_{l_1 l_2 l_1 l_1}, \mathrm{e}_{l_2 l_1 l_1 l_1}\right\},$$

4) Pretpostavimo da je $l_4 = l_3 = l_2 = l_1 = l$.

Tada će iz skupa $\mathrm{Q} = \{l_1, l_2, l_3, l_4\}$ nastati multiskup $\mathrm{Q} = \{l_1^4\}$ te će njemu pridruženi degenerirani težinski potprostor biti $\mathcal{B}_{l^4} = span_{\mathbb{C}}\{\mathrm{e}_{llll}\}$, gdje je $\mathrm{e}_{llll} = \mathrm{e}_l \mathrm{e}_l \mathrm{e}_l \mathrm{e}_l = \mathrm{e}_{l^4}$.

Izračunajmo najprije prostor konstanti u generičkom težinskom potprostoru

$$\begin{aligned}\mathcal{B}_{l_1 l_2 l_3 l_4} = span_{\mathbb{C}}\{&\mathrm{e}_{l_1 l_2 l_3 l_4}, \mathrm{e}_{l_1 l_2 l_4 l_3}, \mathrm{e}_{l_1 l_4 l_2 l_3}, \mathrm{e}_{l_4 l_1 l_2 l_3}, \mathrm{e}_{l_4 l_1 l_3 l_2}, \mathrm{e}_{l_1 l_4 l_3 l_2}, \mathrm{e}_{l_1 l_3 l_4 l_2}, \mathrm{e}_{l_1 l_3 l_2 l_4}, \mathrm{e}_{l_3 l_1 l_2 l_4},\\ &\mathrm{e}_{l_3 l_1 l_4 l_2}, \mathrm{e}_{l_3 l_4 l_1 l_2}, \mathrm{e}_{l_4 l_3 l_1 l_2}, \mathrm{e}_{l_4 l_3 l_2 l_1}, \mathrm{e}_{l_3 l_4 l_2 l_1}, \mathrm{e}_{l_3 l_2 l_4 l_1}, \mathrm{e}_{l_3 l_2 l_1 l_4}, \mathrm{e}_{l_2 l_3 l_1 l_4}, \mathrm{e}_{l_2 l_3 l_4 l_1},\\ &\mathrm{e}_{l_2 l_4 l_3 l_1}, \mathrm{e}_{l_4 l_2 l_3 l_1}, \mathrm{e}_{l_4 l_2 l_1 l_3}, \mathrm{e}_{l_2 l_4 l_1 l_3}, \mathrm{e}_{l_2 l_1 l_4 l_3}, \mathrm{e}_{l_2 l_1 l_3 l_4}\},\end{aligned}$$

pridruženom skupu $\mathrm{Q} = \{l_1, l_2, l_3, l_4\} \subset \mathcal{N} = \{\mathrm{i}_1, \mathrm{i}_2, ..., \mathrm{i}_\mathrm{N}\}$ uz uobičajenu pokratu $\mathrm{e}_{\mathrm{ijkm}} := \mathrm{e}_\mathrm{i} \mathrm{e}_\mathrm{j} \mathrm{e}_\mathrm{k} \mathrm{e}_\mathrm{m}$.

Pritom koristimo Johnson-Trotterov uređaj na skupu permutacija.

Radi jednostavnosti zapisa pretpostavimo da je $l_1 = 1$, $l_2 = 2$, $l_3 = 3$, $l_4 = 4$. Tada djelovanjem operatora stupnja $\partial|_{\mathcal{B}_{1234}} : \mathcal{B}_{1234} \to \mathcal{B}_{1234}$ na potprostoru $\mathcal{B}_{1234}$ primjenom formule (24) dobivamo

$$\partial|_{\mathcal{B}_{1234}}\left(e_{j_1j_2j_3j_4}\right) = 1\,e_{j_1j_2j_3j_4} + q_{j_2j_1}\,e_{j_2j_1j_3j_4} + q_{j_3j_1}q_{j_3j_2}\,e_{j_3j_1j_2j_4} + q_{j_4j_1}q_{j_4j_2}q_{j_4j_3}\,e_{j_4j_1j_2j_3} \tag{49}$$

gdje je $j_1j_2j_3j_4 \in \{1234, 1243, 1423, 4123, 4132, 1432, 1342, 1324, 3124, 3142, 3412, 4312,$
$4321, 3421, 3241, 3214, 2314, 2341, 2431, 4231, 4213, 2413, 2143, 2134\}$.

U monomijalnoj bazi generičkog težinskog potprostora $\mathcal{B}_{1234}$ operatoru $\partial|_{\mathcal{B}_{1234}}$ je pridružena kvadratna matrica B_{1234} reda $4! = 24$.

Izračunavanjem determinante matrice B_{1234} dobivamo

$$\det B_{1234} = (1-\sigma_{12})^2(1-\sigma_{13})^2(1-\sigma_{14})^2(1-\sigma_{23})^2(1-\sigma_{24})^2(1-\sigma_{34})^2$$
$$(1-\sigma_{123})(1-\sigma_{124})(1-\sigma_{134})(1-\sigma_{234})(1-\sigma_{1234})^2$$

ili kraće

$$\det B_{1234} = \prod_{1\le i<j\le 4}(1-\sigma_{ij})^2 \cdot \prod_{1\le i<j<k\le 4}(1-\sigma_{ijk}) \cdot (1-\sigma_{1234})^2. \tag{50}$$

Ako je $\sigma_{ij} \neq 1$, $1 \le i < j \le 4$, $\sigma_{ijk} \neq 1$, $1 \le i < j < k \le 4$, $\sigma_{1234} \neq 1$, onda u $\mathcal{B}_{1234}$ nema konstanti. U $\mathcal{B}_{1234}$ postoji netrivijalna bazična konstanta ako i samo ako je

$$\sigma_{12} = 1 \;\vee\; \sigma_{13} = 1 \;\vee\; \sigma_{14} = 1 \;\vee\; \sigma_{23} = 1 \;\vee\; \sigma_{24} = 1 \;\vee\; \sigma_{34} = 1$$
$$\vee\; \sigma_{123} = 1 \;\vee\; \sigma_{124} = 1 \;\vee\; \sigma_{134} = 1 \;\vee\; \sigma_{234} = 1 \;\vee\; \sigma_{1234} = 1, \tag{51}$$

gdje je $\sigma_{1234} = \sigma_{12}\sigma_{13}\sigma_{14}\sigma_{23}\sigma_{24}\sigma_{34}$, $\sigma_{ijk} = \sigma_{ij}\sigma_{ik}\sigma_{jk}$, $\sigma_{ab} = q_{ab}q_{ba}$.

Ako je $\sigma_{1234} = 1$, $\sigma_{ij} \neq 1$, $1 \le i < j \le 4$, $\sigma_{ijk} \neq 1$, $1 \le i < j < k \le 4$, onda je prostor konstanti u $\mathcal{B}_{1234}$ dvodimenzionalan.

Dobivamo da se bazične konstante, koje ćemo označiti s C'_{1234}, C''_{1234}, mogu pisati u obliku

$$\begin{aligned}C'_{1234} = {} & q_{31}\left(1-\sigma_{14}\right)\left(1-\sigma_{124}\right)\left(1-\sigma_{134}\right)\left[\frac{1-\sigma_{13}}{q_{31}}\left(\widetilde{X}_{1234}-q_{41}q_{42}q_{43}\sigma_{123}\widetilde{X}_{4123}\right)\right.\\
&\left.+\frac{1-\sigma_{12}}{q_{12}}\left(\widetilde{X}_{2314}-q_{41}q_{42}q_{43}\sigma_{123}\widetilde{X}_{4231}\right)+q_{32}\left(\sigma_{12}\sigma_{13}-1\right)\left(\widetilde{X}_{3124}-q_{41}q_{42}q_{43}\widetilde{X}_{4312}\right)\right]\\
&+q_{21}\,q_{14}q_{41}q_{42}q_{43}\left(1-\sigma_{13}\right)\left(1-\sigma_{123}\right)\left(1-\sigma_{134}\right)\left[\frac{1-\sigma_{12}}{q_{21}}\left(\widetilde{X}_{1423}-q_{31}q_{32}q_{34}\sigma_{124}\widetilde{X}_{3142}\right)\right.\\
&\left.+\frac{1-\sigma_{14}}{q_{14}}\left(\widetilde{X}_{4213}-q_{31}q_{32}q_{34}\sigma_{124}\widetilde{X}_{3421}\right)+q_{24}\left(\sigma_{12}\sigma_{14}-1\right)\left(\widetilde{X}_{2143}-q_{31}q_{32}q_{34}\widetilde{X}_{3214}\right)\right]\\
&+q_{41}\,q_{13}q_{31}q_{32}q_{34}\,q_{14}q_{41}q_{42}q_{43}\left(1-\sigma_{12}\right)\left(1-\sigma_{123}\right)\left(1-\sigma_{124}\right)\left[\frac{1-\sigma_{14}}{q_{41}}\left(\widetilde{X}_{1342}-q_{21}q_{23}q_{24}\sigma_{134}\widetilde{X}_{2134}\right)\right.\\
&\left.+\frac{1-\sigma_{13}}{q_{13}}\left(\widetilde{X}_{3412}-q_{21}q_{23}q_{24}\sigma_{134}\widetilde{X}_{2341}\right)+q_{43}\left(\sigma_{13}\sigma_{14}-1\right)\left(\widetilde{X}_{4132}-q_{21}q_{23}q_{24}\widetilde{X}_{2413}\right)\right],\end{aligned}$$

$$\begin{aligned}C''_{1234} = {} & \frac{q_{31}\left(1-\sigma_{14}\right)\left(1-\sigma_{124}\right)\left(1-\sigma_{134}\right)}{\sigma_{14}\sigma_{24}}\left[\frac{1-\sigma_{13}}{q_{31}}\left(\widetilde{X}_{1234}-q_{41}q_{42}q_{43}\widetilde{X}_{4123}\right)\right.\\
&\left.+\frac{1-\sigma_{12}}{q_{12}}\left(\widetilde{X}_{2314}-q_{41}q_{42}q_{43}\widetilde{X}_{4231}\right)+\frac{\sigma_{12}\sigma_{13}-1}{q_{23}\sigma_{12}\sigma_{13}}\left(\widetilde{X}_{3124}-q_{41}q_{42}q_{43}\sigma_{123}\widetilde{X}_{4312}\right)\right]\\
&+q_{21}\,q_{14}q_{41}q_{42}q_{43}\left(1-\sigma_{13}\right)\left(1-\sigma_{123}\right)\left(1-\sigma_{134}\right)\left[\frac{1-\sigma_{12}}{q_{21}}\left(\widetilde{X}_{1423}-q_{31}q_{32}q_{34}\widetilde{X}_{3142}\right)\right.\\
&\left.+\frac{1-\sigma_{14}}{q_{14}}\left(\widetilde{X}_{4213}-q_{31}q_{32}q_{34}\widetilde{X}_{3421}\right)+\frac{\sigma_{12}\sigma_{14}-1}{q_{42}\sigma_{12}\sigma_{14}}\left(\widetilde{X}_{2143}-q_{31}q_{32}q_{34}\sigma_{124}\widetilde{X}_{3214}\right)\right]\\
&+\frac{q_{41}\,q_{13}q_{31}q_{32}q_{34}\,q_{14}q_{41}q_{42}q_{43}\left(1-\sigma_{12}\right)\left(1-\sigma_{123}\right)\left(1-\sigma_{124}\right)}{\sigma_{12}\sigma_{14}\sigma_{23}\sigma_{24}}\left[\frac{1-\sigma_{14}}{q_{41}}\left(\widetilde{X}_{1342}-q_{21}q_{23}q_{24}\widetilde{X}_{2134}\right)\right.\\
&\left.+\frac{1-\sigma_{13}}{q_{13}}\left(\widetilde{X}_{3412}-q_{21}q_{23}q_{24}\widetilde{X}_{2341}\right)+\frac{\sigma_{13}\sigma_{14}-1}{q_{34}\sigma_{13}\sigma_{14}}\left(\widetilde{X}_{4132}-q_{21}q_{23}q_{24}\sigma_{134}\widetilde{X}_{2413}\right)\right].\end{aligned}$$

Koristili smo oznaku

$$\widetilde{X}_{p_1p_2p_3p_4} := e_{p_1p_2p_3p_4} - \prod_{1\le j<k\le 4} q_{p_kp_j}\, w_4\, e_{p_1p_2p_3p_4}\,, \tag{52}$$

gdje w_4 označava najdulju permutaciju u S_4, stoga je (52) dano sa

$$\widetilde{X}_{p_1p_2p_3p_4} = e_{p_1p_2p_3p_4} - q_{p_2p_1}q_{p_3p_1}q_{p_3p_2}q_{p_4p_1}q_{p_4p_2}q_{p_4p_3}\,e_{p_4p_3p_2p_1}\,.$$

Primjedba 2.2.4.1

Uvodimo oznake

$$Z_{p_1p_2p_3p_4} := \left(1-\sigma_{p_1p_4}\right)\left(1-\sigma_{p_1p_2p_4}\right)\left(1-\sigma_{p_1p_3p_4}\right)\left[\frac{1-\sigma_{p_1p_3}}{q_{p_3p_1}} V_{p_1p_2p_3,\,p_4} + \frac{1-\sigma_{p_1p_2}}{q_{p_1p_2}} V_{p_2p_3p_1,\,p_4} + q_{p_3p_2}\left(\sigma_{p_1p_2}\sigma_{p_1p_3}-1\right) W_{p_3p_1p_2,\,p_4}\right] \tag{53}$$

$$Z'_{p_1p_2p_3p_4} := \left(1-\sigma_{p_1p_4}\right)\left(1-\sigma_{p_1p_2p_4}\right)\left(1-\sigma_{p_1p_3p_4}\right)\left[\frac{1-\sigma_{p_1p_3}}{q_{p_3p_1}} W_{p_1p_2p_3,\,p_4} + \frac{1-\sigma_{p_1p_2}}{q_{p_1p_2}} W_{p_2p_3p_1,\,p_4} + \frac{\sigma_{p_1p_2}\sigma_{p_1p_3}-1}{q_{p_2p_3}\sigma_{p_1p_2}\sigma_{p_1p_3}} V_{p_3p_1p_2,\,p_4}\right] \tag{54}$$

gdje je

$$V_{p_1p_2p_3,\,p_4} := \widetilde{X}_{p_1p_2p_3p_4} - \prod_{1\le j<4} q_{p_4p_j}\sigma_{p_1p_2p_3}\widetilde{X}_{p_4p_1p_2p_3}\,, \tag{55}$$

$$W_{p_1p_2p_3,\,p_4} := \widetilde{X}_{p_1p_2p_3p_4} - \prod_{1\le j<4} q_{p_4p_j}\widetilde{X}_{p_4p_1p_2p_3}\,, \tag{56}$$

$\widetilde{X}_{p_1p_2p_3p_4}$ je definirano sa (52).

Pritom je

$$\prod_{1\le j<4} q_{p_4p_j}\sigma_{p_1p_2p_3} = \frac{1}{\prod_{1\le j<4} q_{p_jp_4}}\,, \tag{57}$$

što direktno proizlazi iz pretpostavke $\sigma_{p_1p_2p_3p_4}=1$.

S obzirom na uvedene oznake u primjedbi 2.2.4.1 imamo da bazične konstante C'_{1234}, C''_{1234} u $\mathcal{B}_{1234}$ možemo napisati u obliku

$$\left.\begin{aligned} C'_{1234} &= q_{31}\,Z_{1234} + q_{21}\,q_{14}q_{41}q_{42}q_{43}\,Z_{1423} + q_{41}\,q_{13}q_{31}q_{32}q_{34}\,q_{14}q_{41}q_{42}q_{43}\,Z_{1342}\,,\\ C''_{1234} &= \frac{q_{31}}{\sigma_{14}\sigma_{24}}Z'_{1234} + q_{21}\,q_{14}q_{41}q_{42}q_{43}\,Z'_{1423} + \frac{q_{41}\,q_{13}q_{31}q_{32}q_{34}\,q_{14}q_{41}q_{42}q_{43}}{\sigma_{12}\sigma_{14}\sigma_{23}\sigma_{24}}Z'_{1342} \end{aligned}\right\} \tag{58}$$

ako vrijedi uvjet kocikličnosti $\sigma_{1234}=1$, $\sigma_{ij}\neq 1$, $1\le i<j\le 4$, $\sigma_{ijk}\neq 1$, $1\le i<j<k\le 4$.

Detaljnim izračunavanjem bazičnih konstanti u generičkom težinskom potprostoru $\mathcal{B}_{1234}$ došlo se je do sljedećeg zaključka:

Pretpostavimo da je uvjet kocikličnosti zadovoljen, tj. $\sigma_{1234}=1$, $\sigma_{ij}\neq 1$, $1\le i<j\le 4$, $\sigma_{ijk}\neq 1$, $1\le i<j<k\le 4$. Tada za bazične konstante uzimamo C'_{1234}, C''_{1234} u $\mathcal{B}_{1234}$. Za sve preostale vrijednosti parametara q_{ij} za koje je $\det B_{1234}=0$, a koje ne zadovoljavaju uvjet kocikličnosti dobivaju se bazične konstante proporcionalne s C'_{1234}, C''_{1234}. Konkretno

(1) Ako je $\sigma_{1234}=1$, $\sigma_{123}=1$, $\sigma_{ij}\neq 1$, $1\le i<j\le 4$, onda se dobiva da je prostor konstanti u $\mathcal{B}_{1234}$ jednodimenzionalan te da je moguća bazična konstanta iterirani komutator $[C_{123}, e_4]_{q_{41}q_{42}q_{43}}$ proporcionalan s C'_{1234}, ali isto tako i s C''_{1234}.

Dakle, iz bazične konstante C'_{1234}, odnosno C''_{1234} može se konstruirati moguća bazična konstanta C_{1234} takva da je $C_{1234}=\alpha\,[C_{123}, e_4]_{q_{41}q_{42}q_{43}}$, $\alpha\in\mathbb{C}\setminus\{0\}$.

Pritom je $$[C_{123}, e_4]_{q_{41}q_{42}q_{43}} = C_{123}\,e_4 - q_{41}q_{42}q_{43}\,e_4\,C_{123},$$

$$C_{123}=\sum_{cyc}\frac{1-\sigma_{12}}{q_{12}}\widetilde{X}_{231} \quad \text{(ciklička suma)},$$

$$\widetilde{X}_{p_1p_2p_3}=e_{p_1p_2p_3}+q_{p_2p_1}q_{p_3p_1}q_{p_3p_2}\,e_{p_3p_2p_1}.$$

(2) Ako je $\sigma_{1234}=1$, $\sigma_{12}=1$, $\sigma_{123}\neq 1$, $\sigma_{13}\neq 1$, $\sigma_{14}\neq 1$, $\sigma_{23}\neq 1$, $\sigma_{24}\neq 1$, $\sigma_{34}\neq 1$, onda se dobiva da je prostor konstanti u $\mathcal{B}_{1234}$ dvodimenzionalan. U ovom slučaju imamo da su bazične konstante C'_{1234}, C''_{1234} proporcionalne iteriranim komutatorima Y_{2134}, Y_{2143}. Pritom je

$$Y_{2134}=\Big[[Y_{21}, e_3]_{q_{31}q_{32}}, e_4\Big]_{q_{41}q_{42}q_{43}}, \quad Y_{2143}=\Big[[Y_{21}, e_4]_{q_{41}q_{42}}, e_3\Big]_{q_{31}q_{32}q_{34}}.$$

Na osnovu rečenog zaključujemo da je prostor konstanti u generičkom težinskom potprostoru $\mathcal{B}_{1234}$ najviše dvodimenzionalan.

Ideal $\boldsymbol{I}_{1234}=\langle C'_{1234}, C''_{1234}\rangle\subset\mathcal{B}_{1234}$ je generiran bazičnim konstantama C'_{1234}, C''_{1234} ako je $\sigma_{1234}=1$.

Slijedi izračunavanje prostora konstanti, tj. bazičnih netrivijalnih konstanti u degeneriranim težinskim potprostorima $\mathcal{B}_{l_1^2 l_2 l_3}$, $\mathcal{B}_{l_1^2 l_2^2}$, $\mathcal{B}_{l_1^3 l_2}$ i $\mathcal{B}_{l^4}$.

✐ Izračunavanje prostora konstanti u $\mathcal{B}_{l_1^2 l_2 l_3}$.

Pretpostavimo da je $l_1 = 1$, $l_2 = 2$, $l_3 = 3$. Tada primjenom formule (24) dobivamo

$$\partial|_{\mathcal{B}_{1^2 23}}\left(e_{j_1 j_2 j_3 j_4}\right) = 1\, e_{j_1 j_2 j_3 j_4} + q_{j_2 j_1}\, e_{j_2 j_1 j_3 j_4} + q_{j_3 j_1} q_{j_3 j_2}\, e_{j_3 j_1 j_2 j_4} + q_{j_4 j_1} q_{j_4 j_2} q_{j_4 j_3}\, e_{j_4 j_1 j_2 j_3} \qquad (59)$$

gdje je $j_1 j_2 j_3 j_4 \in \{1123, 1132, 1213, 1231, 1312, 1321, 2113, 2131, 2311, 3112, 3121, 3211\}$.

Pritom je u monomijalnoj bazi degeneriranog težinskog potprostora $\mathcal{B}_{1^2 23}$ operatoru $\partial|_{\mathcal{B}_{1^2 23}} : \mathcal{B}_{1^2 23} \to \mathcal{B}_{1^2 23}$ pridružena kvadratna matrica $B_{1^2 23} = B_{1123}$ reda $\frac{4!}{2!} = 12$.

Izračunavanjem determinante matrice B_{1123} dobivamo

$$\det B_{1^2 23} = (1+q_{11})^2 (1-\sigma_{12})^2 (1-\sigma_{13})^2 (1-\sigma_{23})$$
$$(1-q_{11}\sigma_{12})(1-q_{11}\sigma_{13})(1-\sigma_{123})\left(1-q_{11}^2\sigma_{12}^2\sigma_{13}^2\sigma_{23}\right)$$

ili

$$\det B_{1^2 23} = \prod_{k=1}^{2}\left([k]_{q_{11}}!\right)^2 \prod_{m=2}^{3}\left(1-q_{11}^{2-k}\sigma_{1m}\right)^k \prod_{r=0}^{2}\left(1-q_{11}^{2\cdot\binom{r}{2}}\sigma_{12}^r\sigma_{13}^r\sigma_{23}\right).$$

U težinskom potprostoru $\mathcal{B}_{1^2 23}$ nema konstanti ako je $[2]_{q_{11}} \neq 0$, $\sigma_{12} \neq 1$, $\sigma_{13} \neq 1$, $\sigma_{23} \neq 1$, $\sigma_{123} \neq 1$, $q_{11}\sigma_{12} \neq 1$, $q_{11}\sigma_{13} \neq 1$, $q_{11}^2\sigma_{12}^2\sigma_{13}^2\sigma_{23} \neq 1$.

Netrivijalne bazične konstante u potprostoru $\mathcal{B}_{1^2 23}$ postoje ako i samo ako je $q_{11} = -1$ $\vee\ \sigma_{12} = 1 \vee \sigma_{13} = 1 \vee \sigma_{23} = 1 \vee \sigma_{123} = 1 \vee q_{11}\sigma_{12} = 1 \vee q_{11}\sigma_{13} = 1 \vee q_{11}^2\sigma_{12}^2\sigma_{13}^2\sigma_{23} = 1$.

- Ako je $q_{11}^2\sigma_{12}^2\sigma_{13}^2\sigma_{23} = 1$, $q_{11}\sigma_{12} \neq 1$, $q_{11}\sigma_{13} \neq 1$, $\sigma_{123} \neq 1$, $\sigma_{12} \neq 1$, $\sigma_{13} \neq 1$, $\sigma_{23} \neq 1$, $[2]_{q_{11}} \neq 0$, onda je prostor konstanti u $\mathcal{B}_{1^2 23}$ jednodimenzionalan, a bazična konstanta može se pisati u obliku

$$C_{1123} = \left(\frac{1-q_{11}\sigma_{13}}{q_{31}}\right)\cdot\left(\frac{1-\sigma_{13}}{q_{31}}\right)\widetilde{X}_{1123} + \left(\frac{1-q_{11}\sigma_{12}}{q_{12}}\right)\cdot\left(\frac{1-\sigma_{12}}{q_{12}}\right)\widetilde{X}_{2311}$$
$$+\left(\frac{1-q_{11}\sigma_{12}\sigma_{13}}{q_{12}q_{31}}\right)\cdot\left(\frac{1-q_{11}^2\sigma_{12}\sigma_{13}}{q_{11}q_{12}q_{31}}\right)\widetilde{X}_{2113} + \left(\frac{1+q_{11}}{q_{11}}\right)\cdot\left(\frac{1-q_{11}\sigma_{13}}{q_{31}}\right)\cdot\left(\frac{1-q_{11}\sigma_{12}}{q_{12}}\right)\widetilde{X}_{1231}$$
$$-\left(\frac{1+q_{11}}{q_{11}}\right)\cdot\left(\frac{1-q_{11}\sigma_{13}}{q_{31}}\right)\cdot\left(\frac{1-q_{11}\sigma_{12}\sigma_{13}}{q_{12}q_{31}}\right)\widetilde{X}_{1213} - \left(\frac{1+q_{11}}{q_{11}}\right)\cdot\left(\frac{1-q_{11}\sigma_{12}}{q_{12}}\right)\cdot\left(\frac{1-q_{11}\sigma_{12}\sigma_{13}}{q_{12}q_{31}}\right)\widetilde{X}_{2131}$$
$$(60)$$

Pritom je $\widetilde{X}_{p_1p_2p_3p_4}$ definirano relacijom (52).

Napomena:

Za bilo koji izbor vrijednosti parametara q_{ij} za koje je $\det B_{1^223} = 0$ dobiva se konstanta, koja je linearno zavisna s bazičnom konstantom C_{1123} danom sa (60). Ideal $\boldsymbol{I}_{1123} = \langle C_{1123} \rangle$ u $\boldsymbol{\mathcal{B}}_{1^223}$ je generiran bazičnom konstantom C_{1123} ako je $q_{11}^2\sigma_{12}^2\sigma_{13}^2\sigma_{23} = 1$.

Konkretno imamo

- Ako je $\sigma_{123} = 1$, $q_{11}\sigma_{12} \neq 1$, $q_{11}\sigma_{13} \neq 1$, $\sigma_{12} \neq 1$, $\sigma_{13} \neq 1$, $\sigma_{23} \neq 1$, $[2]_{q_{11}} \neq 0$,

  onda je

$$C_{1123} = \lambda [C_{123}\ , e_1]_{q_{11}q_{12}q_{13}}, \tag{61}$$

$\lambda \in \mathbb{C} \setminus \{0\}$ (vidi primjedbu 2.2.4.3).

- Ako je $q_{11}\sigma_{12} = 1$, $q_{11}\sigma_{13} \neq 1$, $\sigma_{12} \neq 1$, $\sigma_{13} \neq 1$, $\sigma_{23} \neq 1$, $[2]_{q_{11}} \neq 0$, onda je

$$C_{1123} = \lambda Y_{2113}, \tag{62}$$

$\lambda \in \mathbb{C} \setminus \{0\}$. Analogno, za $q_{11}\sigma_{13} = 1$, $q_{11}\sigma_{12} \neq 1$, $\sigma_{12} \neq 1$, $\sigma_{13} \neq 1$, $\sigma_{23} \neq 1$, $[2]_{q_{11}} \neq 0$ dobivamo

$$C_{1123} = \lambda Y_{3112}, \tag{63}$$

$\lambda \in \mathbb{C} \setminus \{0\}$.

- Ako je $\sigma_{12} = 1$, $q_{11}\sigma_{13} \neq 1$, $\sigma_{13} \neq 1$, $\sigma_{23} \neq 1$, $[2]_{q_{11}} \neq 0$, onda je prostor konstanti dvodimenzionalan te se bazična konstanta C_{1123} može zapisati u obliku linearne kombinacije komutatora Y_{2113} i Y_{2131}

$$C_{1123} = \alpha_1 Y_{2113} + \alpha_2 Y_{2131}, \tag{64}$$

$\alpha_1, \alpha_2 \in \mathbb{C} \setminus \{0\}$.

Uzimajući u obzir da je $Y_{j_1j_2j_3j_4} = -q_{j_2j_1} Y_{j_2j_1j_3j_4}$, ako je $\sigma_{j_1j_2} = 1$ zaključujemo da se konstanta (64) može pisati u obliku

$$C_{1123} = \alpha_1 Y_{j_1j_2 13} + \alpha_2 Y_{j_1j_2 31}, \tag{65}$$

gdje je $j_1j_2 \in S_2$, tj. j_1j_2 je bilo koja permutacija skupa $\{1, 2\}$.

Analogno se dobiva da je $\quad C_{1123} = \alpha_1 \mathrm{Y}_{j_1 j_2 12} + \alpha_2 \mathrm{Y}_{j_1 j_2 21}$, (66)

ako je $\sigma_{13} = 1$, $q_{11}\sigma_{12} \neq 1$, $\sigma_{12} \neq 1$, $\sigma_{23} \neq 1$, $[2]_{q_{11}} \neq 0$.

Pritom je $j_1 j_2$ bilo koja permutacija skupa $\{1,3\}$.

- Ako je $\sigma_{23} = 1$, $q_{11}\sigma_{12} \neq 1$, $q_{11}\sigma_{13} \neq 1$, $\sigma_{12} \neq 1$, $\sigma_{13} \neq 1$, $[2]_{q_{11}} \neq 0$, onda je prostor konstanti jednodimenzionalan, pri čemu se dobiva da je bazična konstanta C_{1123} linearno zavisna s iteriranim komutatorom $\mathrm{Y}_{2311} = -q_{32}\mathrm{Y}_{3211}$ i pišemo

$$C_{1123} = \lambda \mathrm{Y}_{j_1 j_2 11}, \tag{67}$$

$\lambda \in \mathbb{C} \setminus \{0\}$, gdje je $j_1 j_2$ bilo koja permutacija skupa $\{2,3\}$.

- Ako je $[2]_{q_{11}} = 0$, $\sigma_{123} \neq 1$, $\sigma_{12} \neq 1$, $\sigma_{13} \neq 1$, $\sigma_{23} \neq 1$, onda je prostor konstanti dvodimenzionalan. U ovom slučaju imamo da se bazična konstanta C_{1123} može pisati kao linearna kombinacija iteriranih komutatora Y_{1123} i Y_{1132}

$$C_{1123} = \alpha_1 \mathrm{Y}_{1123} + \alpha_2 \mathrm{Y}_{1132}, \tag{68}$$

$\alpha_1, \alpha_2 \in \mathbb{C} \setminus \{0\}$. Pritom je

$$\mathrm{Y}_{11 j_1 j_2} = 2\left[\mathrm{X}^{11 j_1}, e_{j_2}\right]_{q_{j_2 1}^2 q_{j_2 j_1}}, \quad j_1 \neq j_2, \; j_1, j_2 \in \{2,3\},$$

$$\mathrm{X}^{11 j_1} = \left[e_1^2, e_{j_1}\right]_{q_{j_1 1}^2}.$$

Primjedba 2.2.4.2

Iz navedenog proizlazi da se iz bazične konstante C_{1123} može konstruirati bilo koji iterirani komutator Y_η, $\eta =$ permutacija multiskupa $\{1^2,2,3\}$ (ili linearna kombinacija nekih od njih) u ovisnosti o odgovarajućim izborima vrijednosti parametara q_{ij} za koje je $\det \mathrm{B}_{1^2 23} = 0$.

Navedimo još nekoliko primjera.

- Ako je $\sigma_{12} = 1$, $\sigma_{23} = 1$, $[2]_{q_{11}} \neq 0$, $\sigma_{13} \neq 1$, onda je

$$C_{1123} = \alpha_1 \mathrm{Y}_{j_1 j_2 13} + \alpha_2 \mathrm{Y}_{j_1 j_2 31} + \alpha_3 \mathrm{Y}_{j_3 j_4 11},$$

gdje je $\quad j_1 j_2$ je bilo koja permutacija skupa $\{1,2\}$,

$j_3 j_4$ je bilo koja permutacija skupa $\{2,3\}$.

Analogno, ako je $\sigma_{13}=1,\ \sigma_{23}=1,\ [2]_{q_{11}}\neq 0,\ \sigma_{12}\neq 1$, onda je

$$C_{1123}=\alpha_1 Y_{j_1 j_2 12}+\alpha_2 Y_{j_1 j_2 21}+\alpha_3 Y_{j_3 j_4 11},$$

gdje je $j_1 j_2$ je bilo koja permutacija skupa $\{1,3\}$,

$j_3 j_4$ je bilo koja permutacija skupa $\{2,3\}$.

- Ako je $\sigma_{12}=1,\ \sigma_{13}=1,\ [2]_{q_{11}}\neq 0,\ \sigma_{23}\neq 1$, onda je

$$\begin{aligned} C_{1123}&=\alpha_1 Y_{j_1 j_2} Y_{j_3 j_4}+\alpha_2 Y_{j_3 j_4} Y_{j_1 j_2} \\ &+\gamma\left(Y_{211}\, e_3+q_{12}q_{13}q_{31}^2 q_{32}\left(1+q_{11}\right)e_1 e_3 Y_{21}+q_{11}^* q_{31}^2 q_{32} q_{12}^2\, e_3 Y_{112}\right) \\ &+\delta\left(Y_{311}\, e_2+q_{12}q_{13}q_{21}^2 q_{23}\left(1+q_{11}\right)e_1 e_2 Y_{31}+q_{11}^* q_{21}^2 q_{23} q_{13}^2\, e_2 Y_{113}\right) \end{aligned}$$

$j_1 j_2$ je bilo koja permutacija skupa $\{1,2\}$, $j_3 j_4$ je bilo koja permutacija skupa $\{1,3\}$.

Pritom je: $q_{ii}^*=\dfrac{1}{1-q_{ii}}$.

- Ako je $\sigma_{12}=1,\ \sigma_{13}=1,\ [2]_{q_{11}}=0,\ \sigma_{23}\neq 1$, onda je

$$C_{1123}=\alpha_1 Y_{j_1 j_2} Y_{j_3 j_4}+\alpha_2 Y_{j_3 j_4} Y_{j_1 j_2}+\alpha_3 Y_{1123}+\alpha_4 Y_{1132}$$

$j_1 j_2$ je bilo koja permutacija skupa $\{1,2\}$, a $j_3 j_4$ je bilo koja permutacija skupa $\{1,3\}$.

- Ako je $\sigma_{12}=1,\ \sigma_{13}=1,\ \sigma_{23}=1,\ [2]_{q_{11}}=0$, onda je

$$C_{1123}=\alpha_1 Y_{j_1 j_2} Y_{j_3 j_4}+\alpha_2 Y_{j_3 j_4} Y_{j_1 j_2}+\alpha_3 Y_{11} Y_{j_5 j_6}+\alpha_4 Y_{k_1 k_2 k_3 2}+\varepsilon\left(e_3 Y_{112}-q_{13}^2\left[Y_{11} e_3,\, e_2\right]_{q_{21}^2 q_{23}}\right)$$

gdje je $j_1 j_2=$ permutacija skupa $\{1,2\}$, $j_3 j_4=$ permutacija skupa $\{1,3\}$,

$j_5 j_6=$ permutacija skupa $\{2,3\}$, $k_1 k_2 k_3=$ permutacija multiskupa $\{1^2,3\}$.

Izračunavanje prostora konstanti u $\mathcal{B}_{l_1^2 l_2^2}$.

Pretpostavimo da je $l_1=1,\ l_2=2$.

Tada je multiskupu $Q=\{1^2,2^2\}$ pridružen degenerirani težinski potprostor

$$\mathcal{B}_{1^2 2^2}=span_{\mathbb{C}}\left\{e_{j_1 j_2 j_3 j_4}\mid j_1 j_2 j_3 j_4=\text{permutacija multiskupa } Q=\{1^2,2^2\}\right\}$$

$$=span_{\mathbb{C}}\left\{e_{1122},e_{1212},e_{1221},e_{2112},e_{2121},e_{2211}\right\}.$$

Primjenom formule (24) dobivamo

$$\partial|_{\mathcal{B}_{1^2 2^2}}\left(e_{j_1 j_2 j_3 j_4}\right)=1\,e_{j_1 j_2 j_3 j_4}+q_{j_2 j_1}\,e_{j_2 j_1 j_3 j_4}+q_{j_3 j_1}q_{j_3 j_2}\,e_{j_3 j_1 j_2 j_4}+q_{j_4 j_1}q_{j_4 j_2}q_{j_4 j_3}\,e_{j_4 j_1 j_2 j_3}$$

gdje je $j_1 j_2 j_3 j_4 \in S_4 Q=\{1122,1212,1221,2112,2121,2211\}$.

Operatoru $\partial|_{\mathcal{B}_{1^2 2^2}}:\mathcal{B}_{1^2 2^2}\to\mathcal{B}_{1^2 2^2}$ u monomijalnoj bazi degeneriranog potprostora $\mathcal{B}_{1^2 2^2}$ pridružena je kvadratna matrica $B_{1^2 2^2}\ (=B_{1122})$ reda $\dfrac{4!}{2!2!}=6$ takva da je

$$B_{1^2 2^2}=\begin{array}{c}\\ 1122\\ 1212\\ 1221\\ 2112\\ 2121\\ 2211\end{array}\!\!\begin{array}{c}\begin{array}{cccccc}1122 & 1212 & 1221 & 2112 & 2121 & 2211\end{array}\\ \begin{bmatrix}1+q_{11} & q_{11}q_{12} & q_{11}q_{12}^2 & 0 & 0 & 0\\ 0 & 1 & 0 & q_{12}(1+q_{11}) & q_{11}q_{12}^2 & 0\\ 0 & 0 & 1 & 0 & q_{12} & q_{12}^2(1+q_{11})\\ q_{21}^2(1+q_{22}) & q_{21} & 0 & 1 & 0 & 0\\ 0 & q_{22}q_{21}^2 & q_{21}(1+q_{22}) & 0 & 1 & 0\\ 0 & 0 & 0 & q_{22}q_{21}^2 & q_{22}q_{21} & 1+q_{22}\end{bmatrix}\end{array}.$$

Izračunavanjem determinante matrice $B_{1^2 2^2}$ dobivamo

$$\det B_{1^2 2^2}=(1+q_{11})(1+q_{22})(1-\sigma_{12})^2(1-q_{11}\sigma_{12})(1-q_{22}\sigma_{12})(1+q_{11}q_{22}\sigma_{12}^2)$$

ili

$$\det B_{1^2 2^2}=\prod_{k=1}^{2}[k]_{q_{11}}!\ \left(1-q_{11}^{2-k}\sigma_{12}\right)^k\prod_{r=0}^{2}\left(1+(-1)^r q_{11}^{\binom{r}{2}}q_{22}\sigma_{12}^r\right).$$

U potprostoru $\mathcal{B}_{1^2 2^2}$ nema konstanti ako $[2]_{q_{11}}\neq 0$, $[2]_{q_{22}}\neq 0$, $\sigma_{12}\neq 1$, $q_{11}\sigma_{12}\neq 1$, $q_{22}\sigma_{12}\neq 1$, $q_{11}q_{22}\sigma_{12}^2\neq -1$.

Netrivijalne konstante postoje ako i samo ako je

$$[2]_{q_{11}}=0\ \vee\ [2]_{q_{22}}=0\ \vee\ \sigma_{12}=1\ \vee\ q_{11}\sigma_{12}=1\ \vee\ q_{22}\sigma_{12}=1\ \vee\ q_{11}q_{22}\sigma_{12}^2=-1.$$

Napomena:

Ideal $\boldsymbol{I}_{1122}=\langle C_{1122}\rangle$ u $\mathcal{B}_{1^2 2^2}$ moguće je generirati baričnom konstantom C_{1122} oblika (69) ako je $q_{11}q_{22}\sigma_{12}^2=-1$. Za sve preostale izbore vrijednosti parametara q_{ij} za koje je $\det B_{1^2 2^2}=0$ dobiti će se bazične konstante koje će biti linearno zavisne s bazičnom konstantom C_{1122}.

♦ Ako je $q_{11}q_{22}\sigma_{12}^2 = -1$, $q_{11}\sigma_{12} \neq 1$, $q_{22}\sigma_{12} \neq 1$, $\sigma_{12} \neq 1$, $[2]_{q_{11}} \neq 0$, $[2]_{q_{22}} \neq 0$, onda je prostor konstanti u $\mathcal{B}_{1^2 2^2}$ jednodimenzionalan.

Pritom se dobiva bazična konstanta, koju ćemo označiti s C_{1122}, a koju je moguće zapisati u obliku

$$C_{1122} = \frac{1-\sigma_{12}}{q_{12}} \widetilde{X}_{2211} - \frac{1}{q_{21}^2} \cdot \left(\frac{1+q_{11}}{q_{11}} \right) \cdot \left(\frac{1+q_{22}}{q_{22}} \right) \widetilde{X}_{1212}$$

$$+ \left(\frac{1+q_{11}}{q_{11}} \right) \cdot \left(\frac{1-q_{11}\sigma_{12}}{q_{21}} \right) e_1 e_2^2 e_1 + \left(\frac{1+q_{22}}{q_{22}} \right) \cdot \left(\frac{1-q_{22}\sigma_{12}}{q_{21}} \right) e_2 e_1^2 e_2 , \qquad (69)$$

gdje je $\widetilde{X}_{p_1p_2p_3p_4}$ definirano relacijom (52).

➢ Ako je $q_{11}\sigma_{12} = 1$, $\sigma_{12} \neq 1$, $[2]_{q_{11}} \neq 0$, $[2]_{q_{22}} \neq 0$, onda je bazična konstanta C_{1122} proporcionalna iteriranom komutatoru Y_{2112} i pišemo

$$C_{1122} = \lambda Y_{2112} , \qquad (70)$$

$\lambda \in \mathbb{C} \setminus \{0\}$, gdje je $Y_{2112} = [Y_{211}, e_2]_{q_{21}^2 q_{22}}$.

Analogno se za $q_{22}\sigma_{12} = 1$, $\sigma_{12} \neq 1$, $[2]_{q_{11}} \neq 0$, $[2]_{q_{22}} \neq 0$ dobiva $C_{1122} = \lambda Y_{1221}$.

➢ Ako je $\sigma_{12} = 1$, $[2]_{q_{11}} \neq 0$, $[2]_{q_{22}} \neq 0$, onda imamo

$$C_{1122} = \lambda_1 Y_{j_1j_2} e_1 e_2 + \lambda_2 e_1 Y_{j_1j_2} e_2 + \lambda_3 e_2 Y_{j_1j_2} e_1 + \lambda_4 e_2 e_1 Y_{j_1j_2} + \alpha Y_{j_1j_2} Y_{j_1j_2} , \qquad (71)$$

gdje je $j_1 j_2 =$ permutacija skupa $\{1,2\}$, $\alpha, \lambda_k \in \mathbb{C} \setminus \{0\}$, $1 \leq k \leq 4$.

Specijalno iz (71) proizlazi $C'_{1122} = \alpha_1 Y_{1212} + \alpha_2 Y_{j_1j_2} Y_{j_1j_2}$,

$$C''_{1122} = \lambda_1 Y_{j_1j_21} e_2 + \lambda_2 e_2 Y_{j_1j_21} + \alpha Y_{j_1j_2} Y_{j_1j_2} ,$$

$j_1 j_2 =$ permutacija skupa $\{1,2\}$

Pritom $Y_{j_1j_2} Y_{j_1j_2}$ može biti oblika Y_{12}^2, $Y_{12}Y_{21}$, $Y_{21}Y_{12}$, Y_{21}^2.

➢ Ako je $[2]_{q_{11}} = 0$, $[2]_{q_{22}} \neq 0$, $\sigma_{12} \neq 1$, onda je bazična konstanta proporcionalna iteriranom komutatoru Y_{1122} i pišemo

$$C_{1122} = \lambda Y_{1122} , \qquad (72)$$

$\lambda \in \mathbb{C} \setminus \{0\}$, gdje je $Y_{1122} = \left[[Y_{11}, e_2]_{q_{21}^2}, e_2 \right]_{q_{21}^2 q_{22}}$.

Analogno se za $[2]_{q_{22}} = 0$, $[2]_{q_{11}} \neq 0$, $\sigma_{12} \neq 1$ dobiva $C_{1122} = \lambda Y_{2211}$, $\lambda \in \mathbb{C} \setminus \{0\}$.

✎ Izračunavanje prostora konstanti u $\mathcal{B}_{l_1^3 l_2} = span_{\mathbb{C}} \{ e_{l_1 l_1 l_1 l_2}, e_{l_1 l_1 l_2 l_1}, e_{l_1 l_2 l_1 l_1}, e_{l_2 l_1 l_1 l_1} \}$.

Pretpostavimo da je $l_1 = 1$, $l_2 = 2$.

Tada je multiskupu $Q = \{1^3, 2\}$ pridružen degenerirani težinski potprostor $\mathcal{B}_{1^3 2}$. Primjenom formule (24) dobivamo

$$\partial|_{\mathcal{B}_{1^3 2}} \left(e_{j_1 j_2 j_3 j_4} \right) = 1\, e_{j_1 j_2 j_3 j_4} + q_{j_2 j_1}\, e_{j_2 j_1 j_3 j_4} + q_{j_3 j_1} q_{j_3 j_2}\, e_{j_3 j_1 j_2 j_4} + q_{j_4 j_1} q_{j_4 j_2} q_{j_4 j_3}\, e_{j_4 j_1 j_2 j_3}$$

gdje je $j_1 j_2 j_3 j_4 \in S_4 Q = \{1112, 1121, 1211, 2111\}$.

Operatoru $\partial|_{\mathcal{B}_{1^3 2}} : \mathcal{B}_{1^3 2} \to \mathcal{B}_{1^3 2}$ u monomijalnoj bazi potprostora $\mathcal{B}_{1^3 2}$ pridružena je kvadratna matrica $B_{1^3 2}\ (= B_{1112})$ reda $\frac{4!}{3!} = 4$ takva da je

$$B_{1^3 2} = \begin{array}{c} \\ 1112 \\ 1121 \\ 1211 \\ 2111 \end{array} \begin{array}{c} \begin{array}{cccc} 1112 & 1121 & 1211 & 2111 \end{array} \\ \begin{bmatrix} 1 + q_{11} + q_{11}^2 & q_{11}^2 q_{12} & 0 & 0 \\ 0 & 1 + q_{11} & q_{11} q_{12} (1 + q_{11}) & 0 \\ 0 & 0 & 1 & q_{12} \left(1 + q_{11} + q_{11}^2\right) \\ q_{21}^3 & q_{21}^2 & q_{21} & 1 \end{bmatrix} \end{array}.$$

Njena determinanta je

$$\det B_{1^3 2} = (1 + q_{11})\left(1 + q_{11} + q_{11}^2\right)(1 - \sigma_{12})(1 - q_{11}\sigma_{12})\left(1 - q_{11}^2 \sigma_{12}\right)$$

$$= [2]_{q_{11}} [3]_{q_{11}} \prod_{k=0}^{2} \left(1 - q_{11}^k \sigma_{12}\right),$$

odnosno

$$\det B_{1^3 2} = [3]_{q_{11}}! \prod_{k=0}^{2} \left(1 - q_{11}^k \sigma_{12}\right).$$

Ako je $[2]_{q_{11}} \neq 0$, $[3]_{q_{11}} \neq 0$, $\sigma_{12} \neq 1$, $q_{11}\sigma_{12} \neq 1$, $q_{11}^2 \sigma_{12} \neq 1$, onda u potprostoru $\mathcal{B}_{1^3 2}$ nema konstanti. Netrivijalne konstante postoje ako i samo ako je

$$[2]_{q_{11}} = 0 \ \vee\ [3]_{q_{11}} = 0 \ \vee\ \sigma_{12} = 1 \ \vee\ q_{11}\sigma_{12} = 1 \ \vee\ q_{11}^2 \sigma_{12} = 1.$$

Ideal $\boldsymbol{I}_{1112} = \langle C_{1112} \rangle$ u $\boldsymbol{\mathcal{B}}_{1^3 2}$ moguće je generirati bazičnom konstantom C_{1112} oblika (73) ako je $q_{11}^2 \sigma_{12} = 1$. Za sve preostale izbore vrijednosti parametara q_{ij} za koje je $\det B_{1^3 2} = 0$ dobiti će se bazične konstante koje će biti linearno zavisne s bazičnom konstantom C_{1112}.

- Ako je $q_{11}^2 \sigma_{12} = 1$, $q_{11} \sigma_{12} \neq 1$, $\sigma_{12} \neq 1$, $[3]_{q_{11}} \neq 0$, $[2]_{q_{11}} \neq 0$ ili $q_{11}^2 \sigma_{12} = 1$, $q_{11} \sigma_{12} = 1$, $\sigma_{12} \neq 1$, $[3]_{q_{11}} \neq 0$, $[2]_{q_{11}} \neq 0$ ili $q_{11}^2 \sigma_{12} = 1$, $\sigma_{12} = 1$, $[3]_{q_{11}} \neq 0$, $[2]_{q_{11}} \neq 0$, onda se dobiva da je

$$\left\{ -q_{11}^3 q_{12}^3 e_{1112} + q_{11} q_{12}^2 \left(1 + q_{11} + q_{11}^2\right) e_{1121} - q_{12} \left(1 + q_{11} + q_{11}^2\right) e_{1211} + e_{2111} \right\}$$

baza prostora konstanti u $\boldsymbol{\mathcal{B}}_{1^3 2}$, što znači da će bazična konstanta C_{1112} biti proporcionalna s iteriranim komutatorom Y_{2111} i pišemo

$$C_{1112} = \lambda Y_{2111}, \quad \lambda \in \mathbb{C} \setminus \{0\}, \tag{73}$$

gdje je $Y_{2111} = \left[\left[Y_{21}, e_1 \right]_{q_{11} q_{12}}, e_1 \right]_{q_{11}^2 q_{12}}$

$$= e_{2111} - q_{12} \left(1 + q_{11} + q_{11}^2\right) e_{1211} + q_{11} q_{12}^2 \left(1 + q_{11} + q_{11}^2\right) e_{1121} - q_{11}^3 q_{12}^3 e_{1112}.$$

- Ako je $q_{11}^2 \sigma_{12} = 1$, $[3]_{q_{11}} = 0$, $\sigma_{12} \neq 1$, onda je prostor konstanti jednodimenzionalan te je baza prostora konstanti u $\boldsymbol{\mathcal{B}}_{1^3 2}$ dana sa $\left\{ e_{1112} - q_{21}^3 e_{2111} \right\}$, što znači da će bazična konstanta C_{1112} biti proporcionalna s jednostavnim komutatorom X^{1112} i pišemo

$$C_{1112} = \lambda X^{1112}, \quad \lambda \in \mathbb{C} \setminus \{0\}. \tag{74}$$

Pritom je $X^{1112} = \left[e_1^3, e_2 \right]_{q_{21}^3} = e_{1112} - q_{21}^3 e_{2111}$.

- Ako je $q_{11}^2 \sigma_{12} = 1, [2]_{q_{11}} = 0$, onda se dobiva da je prostor konstanti dvodimenzionalan. U ovom slučaju imamo da se bazična konstanta $C_{1112} \left(= Y_{2111}\right)$ može zapisati kao linearna kombinacija

$$C_{1112} = \alpha_1 Y_{11} Y_{j_1 j_2} + \alpha_2 Y_{j_1 j_2} Y_{11}. \tag{75}$$

Tu je $j_1 j_2 =$ permutacija skupa $\{1,2\}$.

✎ Izračunavanje prostora konstanti u $\mathcal{B}_{\iota^4} = span_{\mathbb{C}}\left\{ e_{\iota\iota\iota\iota} \mid e_{\iota\iota\iota\iota} = e_{\iota^4} \right\}$.

U ovom slučaju imamo da je degeneriran potprostor $\mathcal{B}_{\iota^4}$ pridružen multiskupu $Q = \left\{ \iota^4 \right\}$ nad $\mathcal{N} = \left\{ i_1, i_2, ..., i_N \right\}$.

Imamo $\widehat{Q} = S_3 Q = \left\{ \iota\iota\iota\iota \right\} = \left\{ \iota^4 \right\}$.

Iz formule (24) slijedi da u monomijalnoj bazi potprostora $\mathcal{B}_{\iota^4}$ operatoru $\partial|_{\mathcal{B}_{\iota^4}} : \mathcal{B}_{\iota^4} \to \mathcal{B}_{\iota^4}$ je pridružena 1×1 matrica $B_{\iota^4} = \left[1 + q_{\iota} + q_{\iota}^2 + q_{\iota}^3 \right]$, što povlači da je

$$\det B_{\iota^3} = 1 + q_{\iota} + q_{\iota}^2 + q_{\iota}^3 . \tag{76}$$

Matrica B_{ι^4} je singularna ako i samo ako je $1 + q_{\iota} + q_{\iota}^2 + q_{\iota}^3 = 0$. Tada je bilo koja netrivijalna konstanta u $\mathcal{B}_{\iota^4}$ oblika $C_{\iota\iota\iota\iota} = \alpha\, e_{\iota^4}$, $\alpha \in \mathbb{C} \setminus \left\{ 0 \right\}$, gdje je $e_{\iota^4} = e_{\iota}^4$.

Ako je $1 + q_{\iota} + q_{\iota}^2 + q_{\iota}^3 \neq 0$, onda u $\mathcal{B}_{\iota^4}$ nema konstanti.

Prostor konstanti u degeneriranom potprostoru $\mathcal{B}_{\iota^4}$ je jednodimenzionalan te je $C_{\iota\iota\iota\iota} = e_{\iota^4}$ bazična konstanta u $\mathcal{B}_{\iota^4}$ ako je $\left[4 \right]_{q_{\iota}} = 0$. Pritom je $\left[4 \right]_{q_{\iota}} = 1 + q_{\iota} + q_{\iota}^2 + q_{\iota}^3$.

Primjedba 2.2.4.3

Koristeći primjedbu 2.1.10 imamo da su e_{ι^4} i $Y_{\iota\iota\iota\iota}$ linearno zavisni, tj. bazična konstanta $C_{\iota\iota\iota\iota}\ (= e_{\iota^4})$ je proporcionalna iteriranom komutatoru $Y_{\iota\iota\iota\iota}$, gdje je

$$Y_{\iota\iota\iota\iota} = \left(1 - q_{\iota} \right)\left(1 - q_{\iota}^2 \right)\left(1 - q_{\iota}^3 \right) e_{\iota^4}$$

ili:

$$Y_{\iota\iota\iota\iota} = \left(1 - q_{\iota} \right)^3 \left(1 + q_{\iota} \right)\left(1 + q_{\iota} + q_{\iota}^2 \right) e_{\iota^4} . \tag{77}$$

Pritom je $q_{\iota} \neq 1$, $q_{\iota} \neq -1$, $1 + q_{\iota} + q_{\iota}^2 \neq 0$.

Napomenimo da se pod uvjetom $\left[4 \right]_{q_{\iota}} = 0$ podrazumijeva da je $1 + q_{\iota}^2 = 0$, $q_{\iota} \neq -1$ (ako se želi dobiti da je $Y_{\iota\iota\iota\iota}$ netrivijalna konstanta), jer je po definiciji

$$\left[4 \right]_{q_{\iota}} = 1 + q_{\iota} + q_{\iota}^2 + q_{\iota}^3 = \left(1 + q_{\iota} \right)\left(1 + q_{\iota}^2 \right).$$

Primijetimo da se uvrštavanjem $q_{\iota} = -1$ u (77) dobiva $Y_{\iota\iota\iota\iota} = 0$, što znači da je $Y_{\iota\iota\iota\iota}$, a samim time i $C_{\iota\iota\iota\iota}$ trivijalna konstanta.

Resumirajmo gore provedena razmatranja.

✐ Prostor konstanti u bilo kojem generičkom težinskom potprostoru $\mathcal{B}_{l_1 l_2 l_3 l_4}$, $l_1 < l_2 < l_3 < l_4$, $l_1, l_2, l_3, l_4 \in \mathcal{N} = \{\mathrm{i}_1, \mathrm{i}_2, ..., \mathrm{i}_\mathrm{N}\}$ algebre $\mathcal{B}$ je najviše dvodimenzionalan. Ideal $\boldsymbol{I}_{l_1 l_2 l_3 l_4} = \langle C'_{l_1 l_2 l_3 l_4}, C''_{l_1 l_2 l_3 l_4} \rangle \subset \mathcal{B}_{l_1 l_2 l_3 l_4}$ je generiran baz̆ičnim konstantama $C'_{l_1 l_2 l_3 l_4}$, $C''_{l_1 l_2 l_3 l_4}$ u $\mathcal{B}_{l_1 l_2 l_3 l_4}$ ako je $\sigma_{l_1 l_2 l_3 l_4} = 1$.

Razlikujemo sljedeće slučajeve

(1) Ako je $\sigma_{l_1 l_2 l_3 l_4} = 1$, $\sigma_{l_i l_j} \neq 1$, $1 \leq \mathrm{i} < \mathrm{j} \leq 4$, $\sigma_{l_a l_b l_c} \neq 1$, $1 \leq \mathrm{a} < \mathrm{b} < \mathrm{c} \leq 4$,

onda se bazične konstante $C'_{l_1 l_2 l_3 l_4}$, $C''_{l_1 l_2 l_3 l_4}$ mogu zapisati u obliku

$$C'_{l_1 l_2 l_3 l_4} = \mathrm{q}_{l_3 l_1} Z_{l_1 l_2 l_3 l_4} + \mathrm{q}_{l_2 l_1}\, \mathrm{q}_{l_1 l_4} \mathrm{q}_{l_4 l_1} \mathrm{q}_{l_4 l_2} \mathrm{q}_{l_4 l_3} Z_{l_1 l_4 l_2 l_3}$$
$$+ \mathrm{q}_{l_4 l_1}\, \mathrm{q}_{l_1 l_3} \mathrm{q}_{l_3 l_1} \mathrm{q}_{l_3 l_2} \mathrm{q}_{l_3 l_4}\, \mathrm{q}_{l_1 l_4} \mathrm{q}_{l_4 l_1} \mathrm{q}_{l_4 l_2} \mathrm{q}_{l_4 l_3} Z_{l_1 l_3 l_4 l_2},$$

$$C''_{l_1 l_2 l_3 l_4} = \frac{\mathrm{q}_{l_3 l_1}}{\sigma_{l_1 l_4} \sigma_{l_2 l_4}} Z'_{l_1 l_2 l_3 l_4} + \mathrm{q}_{l_2 l_1}\, \mathrm{q}_{l_1 l_4} \mathrm{q}_{l_4 l_1} \mathrm{q}_{l_4 l_2} \mathrm{q}_{l_4 l_3} Z'_{l_1 l_4 l_2 l_3}$$
$$+ \frac{\mathrm{q}_{l_4 l_1}\, \mathrm{q}_{l_1 l_3} \mathrm{q}_{l_3 l_1} \mathrm{q}_{l_3 l_2} \mathrm{q}_{l_3 l_4}\, \mathrm{q}_{l_1 l_4} \mathrm{q}_{l_4 l_1} \mathrm{q}_{l_4 l_2} \mathrm{q}_{l_4 l_3}}{\sigma_{l_1 l_2} \sigma_{l_1 l_4} \sigma_{l_2 l_3} \sigma_{l_2 l_4}} Z'_{l_1 l_3 l_4 l_2},$$

pri čemu je

$$Z_{\mathrm{p_1 p_2 p_3 p_4}} = \left(1 - \sigma_{\mathrm{p_1 p_4}}\right)\left(1 - \sigma_{\mathrm{p_1 p_2 p_4}}\right)\left(1 - \sigma_{\mathrm{p_1 p_3 p_4}}\right)\left[\frac{1 - \sigma_{\mathrm{p_1 p_3}}}{\mathrm{q_{p_3 p_1}}} V_{\mathrm{p_1 p_2 p_3},\, \mathrm{p_4}}\right.$$
$$\left. + \frac{1 - \sigma_{\mathrm{p_1 p_2}}}{\mathrm{q_{p_1 p_2}}} V_{\mathrm{p_2 p_3 p_1},\, \mathrm{p_4}} + \mathrm{q_{p_3 p_2}} \left(\sigma_{\mathrm{p_1 p_2}} \sigma_{\mathrm{p_1 p_3}} - 1\right) W_{\mathrm{p_3 p_1 p_2},\, \mathrm{p_4}}\right],$$

$$Z'_{\mathrm{p_1 p_2 p_3 p_4}} = \left(1 - \sigma_{\mathrm{p_1 p_4}}\right)\left(1 - \sigma_{\mathrm{p_1 p_2 p_4}}\right)\left(1 - \sigma_{\mathrm{p_1 p_3 p_4}}\right)\left[\frac{1 - \sigma_{\mathrm{p_1 p_3}}}{\mathrm{q_{p_3 p_1}}} W_{\mathrm{p_1 p_2 p_3},\, \mathrm{p_4}}\right.$$
$$\left. + \frac{1 - \sigma_{\mathrm{p_1 p_2}}}{\mathrm{q_{p_1 p_2}}} W_{\mathrm{p_2 p_3 p_1},\, \mathrm{p_4}} + \frac{\sigma_{\mathrm{p_1 p_2}} \sigma_{\mathrm{p_1 p_3}} - 1}{\mathrm{q_{p_2 p_3}} \sigma_{\mathrm{p_1 p_2}} \sigma_{\mathrm{p_1 p_3}}} V_{\mathrm{p_3 p_1 p_2},\, \mathrm{p_4}}\right],$$

$$V_{\mathrm{p_1 p_2 p_3},\, \mathrm{p_4}} = \widetilde{\mathrm{X}}_{\mathrm{p_1 p_2 p_3 p_4}} - \prod_{1 \leq \mathrm{j} < 4} \mathrm{q}_{\mathrm{p_4 p_j}} \sigma_{\mathrm{p_1 p_2 p_3}} \widetilde{\mathrm{X}}_{\mathrm{p_4 p_1 p_2 p_3}},$$

$$W_{\mathrm{p_1 p_2 p_3},\, \mathrm{p_4}} = \widetilde{\mathrm{X}}_{\mathrm{p_1 p_2 p_3 p_4}} - \prod_{1 \leq \mathrm{j} < 4} \mathrm{q}_{\mathrm{p_4 p_j}} \widetilde{\mathrm{X}}_{\mathrm{p_4 p_1 p_2 p_3}},$$

$$\widetilde{\mathrm{X}}_{\mathrm{p_1 p_2 p_3 p_4}} = \mathrm{e}_{\mathrm{p_1 p_2 p_3 p_4}} - \prod_{1 \leq \mathrm{j} < \mathrm{k} \leq 4} \mathrm{q}_{\mathrm{p_k p_j}}\, \mathrm{e}_{\mathrm{p_4 p_3 p_2 p_1}}.$$

(2) Ako je $\sigma_{l_1l_2l_3l_4}=1$, $\sigma_{l_il_jl_k}=1$, $i,j,k\in\{1,2,3,4\}$, $i\neq j\neq k$, $\sigma_{l_il_j}\neq 1$, $1\le i<j\le 4$, onda su obe bazične konstante $C'_{l_1l_2l_3l_4}$, $C''_{l_1l_2l_3l_4}$ proporcionalne iteriranom komutatoru $\left[C_{l_il_jl_k}, e_{l_m}\right]_{q_{l_ml_i}q_{l_ml_j}q_{l_ml_k}}$, $m\in\{1,2,3,4\}\setminus\{i,j,k\}$.

Pritom je $\left[C_{l_il_jl_k}, e_{l_m}\right]_{q_{l_ml_i}q_{l_ml_j}q_{l_ml_k}} = C_{l_il_jl_k}\, e_{l_m} - q_{l_ml_i}q_{l_ml_j}q_{l_ml_k}\, e_{l_m}\, C_{l_il_jl_k}$, gdje je

$C_{l_il_jl_k} = \sum\limits_{cyc} \dfrac{1-\sigma_{l_il_j}}{q_{l_il_j}}\widetilde{X}_{l_jl_kl_i}$ (ciklička suma), $\widetilde{X}_{p_1p_2p_3} = e_{p_1p_2p_3} + q_{p_2p_1}q_{p_3p_1}q_{p_3p_2}\, e_{p_3p_2p_1}$.

(3) Ako je $\sigma_{l_1l_2l_3l_4}=1$, $\sigma_{l_il_j}=1$, $i,j\in\{1,2,3,4\}$, $i\neq j$, $\sigma_{l_al_b}\neq 1$, $a,b\in\{1,2,3,4\}$, $a\neq b$ (a, b nisu istovremeno jednaki i, j), onda su bazične konstante $C'_{l_1l_2l_3l_4}$, $C''_{l_1l_2l_3l_4}$ proporcionalne iteriranim komutatorima $Y_{l_il_jl_kl_m}$, $Y_{l_il_jl_ml_k}$, $k,m\in\{1,2,3,4\}\setminus\{i,j\}$, $k\neq m$.

Pritom je $Y_{l_jl_il_kl_m} = -q_{l_il_j}Y_{l_il_jl_kl_m}$, $Y_{l_jl_il_ml_k} = -q_{l_il_j}Y_{l_il_jl_ml_k}$, jer je $\sigma_{l_il_j}=1$.

(4) Ako je $\sigma_{l_1l_2l_3l_4}\neq 1$, $\sigma_{l_il_j}\neq 1$, $1\le i<j\le 4$, $\sigma_{l_al_bl_c}\neq 1$, $1\le a<b<c\le 4$, onda u generičkom težinskom potprostoru $\mathcal{B}_{l_1l_2l_3l_4}$ nema konstanti.

✎ Prostor konstanti u bilo kojem degeneriranom težinskom potprostoru $\mathcal{B}_{l_1^2l_2l_3}$, $l_1\neq l_2<l_3$, $l_1,l_2,l_3\in\mathcal{N}=\{i_1,i_2,...,i_N\}$ algebre $\mathcal{B}$ je jednodimenzionalan.

Ideal $\boldsymbol{I}_{l_1l_1l_2l_3} = \langle C_{l_1l_1l_2l_3}\rangle \subset \mathcal{B}_{l_1^2l_2l_3}$ biti će generiran bazičnom konstantom $C_{l_1l_1l_2l_3}$ ako je $q^2_{l_1l_1}\sigma^2_{l_1l_2}\sigma^2_{l_1l_3}\sigma_{l_2l_3}=1$, $l_1\neq l_2<l_3$.

Razlikujemo sljedeće slučajeve

(1) Ako je $q^2_{l_1l_1}\sigma^2_{l_1l_2}\sigma^2_{l_1l_3}\sigma_{l_2l_3}=1$, $\sigma_{l_1l_2l_3}\neq 1$, $q_{l_1l_1}\sigma_{l_1l_j}\neq 1$, $j=2,3$, $\sigma_{l_il_j}\neq 1$, $1\le i<j\le 3$, $[2]_{q_{l_1l_1}}\neq 0$, onda se bazična konstanta $C_{l_1l_1l_2l_3}$ može zapisati u obliku

$$
\begin{aligned}
C_{l_1l_1l_2l_3} = {} & \frac{1-q_{l_1l_1}\sigma_{l_1l_3}}{q_{l_3l_1}}\cdot\frac{1-\sigma_{l_1l_3}}{q_{l_3l_1}}\widetilde{X}_{l_1l_1l_2l_3} + \frac{1-q_{l_1l_1}\sigma_{l_1l_2}}{q_{l_1l_2}}\cdot\frac{1-\sigma_{l_1l_2}}{q_{l_1l_2}}\widetilde{X}_{l_2l_3l_1l_1} \\
& + \frac{1-q_{l_1l_1}\sigma_{l_1l_2}\sigma_{l_1l_3}}{q_{l_1l_2}q_{l_3l_1}}\cdot\frac{1-q^2_{l_1l_1}\sigma_{l_1l_2}\sigma_{l_1l_3}}{q_{l_1l_1}q_{l_1l_2}q_{l_3l_1}}\widetilde{X}_{l_2l_1l_1l_3} + \frac{1+q_{l_1l_1}}{q_{l_1l_1}}\cdot\frac{1-q_{l_1l_1}\sigma_{l_1l_3}}{q_{l_3l_1}}\cdot\frac{1-q_{l_1l_1}\sigma_{l_1l_2}}{q_{l_1l_2}}\widetilde{X}_{l_1l_2l_3l_1} \\
& - \frac{1+q_{l_1l_1}}{q_{l_1l_1}}\cdot\frac{1-q_{l_1l_1}\sigma_{l_1l_3}}{q_{l_3l_1}}\cdot\frac{1-q_{l_1l_1}\sigma_{l_1l_2}\sigma_{l_1l_3}}{q_{l_1l_2}q_{l_3l_1}}\widetilde{X}_{l_1l_2l_1l_3} - \frac{1+q_{l_1l_1}}{q_{l_1l_1}}\cdot\frac{1-q_{l_1l_1}\sigma_{l_1l_2}}{q_{l_1l_2}}\cdot\frac{1-q_{l_1l_1}\sigma_{l_1l_2}\sigma_{l_1l_3}}{q_{l_1l_2}q_{l_3l_1}}\widetilde{X}_{l_2l_1l_3l_1}
\end{aligned}
$$

$\widetilde{X}_{p_1p_2p_3p_4}$ je definiran relacijom (52).

(2) Ako je $q_{l_1l_1}^2\sigma_{l_1l_2}^2\sigma_{l_1l_3}^2\sigma_{l_2l_3}=1$, $\sigma_{l_1l_2l_3}=1$, $q_{l_1l_1}\sigma_{l_1l_j}\neq 1$, $j=2,3$, $\sigma_{l_il_j}\neq 1$, $1\leq i<j\leq 3$, $[2]_{q_{l_1l_1}}\neq 0$, onda je $$C_{l_1l_1l_2l_3}=\lambda\left[C_{l_1l_2l_3}, e_{l_1}\right]_{q_{l_1l_1}q_{l_1l_2}q_{l_1l_3}},\ \lambda\in\mathbb{C}\setminus\{0\}.$$

(3) Ako je $q_{l_1l_1}^2\sigma_{l_1l_2}^2\sigma_{l_1l_3}^2\sigma_{l_2l_3}=1$, $\sigma_{l_1l_2l_3}\neq 1$, $q_{l_1l_1}\sigma_{l_1l_j}=1$, $j\in\{2,3\}$, $q_{l_1l_1}\sigma_{l_1l_m}\neq 1$, $m\in\{2,3\}\setminus\{j\}$, $\sigma_{l_il_j}\neq 1$, $1\leq i<j\leq 3$, $[2]_{q_{l_1l_1}}\neq 0$, onda je

$$C_{l_1l_1l_2l_3}=\lambda\, Y_{l_jl_1l_1l_m},\ \lambda\in\mathbb{C}\setminus\{0\}.$$

(4) Ako je $q_{l_1l_1}^2\sigma_{l_1l_2}^2\sigma_{l_1l_3}^2\sigma_{l_2l_3}=1$, $\sigma_{l_1l_2l_3}\neq 1$, $q_{l_1l_1}\sigma_{l_1l_j}\neq 1$, $j=2,3$, $\sigma_{l_1l_j}=1$, $j\in\{2,3\}$, $\sigma_{l_1l_m}\neq 1$, $m\in\{2,3\}\setminus\{j\}$, $\sigma_{l_2l_3}\neq 1$, $[2]_{q_{l_1l_1}}\neq 0$, onda je

$$C_{l_1l_1l_2l_3}=\alpha_1 Y_{l_1l_jl_1l_m}+\alpha_2 Y_{l_1l_jl_ml_1},\ \alpha_1,\alpha_2\in\mathbb{C}\setminus\{0\}.$$

Pritom je $Y_{l_jl_1l_1l_m}=-q_{l_1l_j}Y_{l_1l_jl_1l_m}$, $Y_{l_jl_1l_ml_1}=-q_{l_1l_j}Y_{l_1l_jl_ml_1}$.

(5) Ako je $q_{l_1l_1}^2\sigma_{l_1l_2}^2\sigma_{l_1l_3}^2\sigma_{l_2l_3}=1$, $\sigma_{l_1l_2l_3}\neq 1$, $q_{l_1l_1}\sigma_{l_1l_j}\neq 1$, $j=2,3$, $\sigma_{l_1l_j}\neq 1$, $j\in\{2,3\}$, $\sigma_{l_2l_3}=1$, $[2]_{q_{l_1l_1}}\neq 0$, onda je $C_{l_1l_1l_2l_3}=\alpha\, Y_{l_2l_3l_1l_1}$, $\alpha\in\mathbb{C}\setminus\{0\}$.

Pritom je $Y_{l_3l_2l_1l_1}=-q_{l_2l_3}Y_{l_2l_3l_1l_1}$.

(6) Ako je $q_{l_1l_1}^2\sigma_{l_1l_2}^2\sigma_{l_1l_3}^2\sigma_{l_2l_3}=1$, $\sigma_{l_1l_2l_3}=1$, $q_{l_1l_1}\sigma_{l_1l_j}\neq 1$, $j=2,3$, $\sigma_{l_il_j}\neq 1$, $1\leq i<j\leq 3$, $[2]_{q_{l_1l_1}}=0$, onda je $C_{l_1l_1l_2l_3}=\alpha_1 Y_{l_1l_1l_2l_3}+\alpha_2 Y_{l_1l_1l_3l_2}$, $\alpha_1,\alpha_2\in\mathbb{C}\setminus\{0\}$.

(7) Ako je $q_{l_1l_1}^2\sigma_{l_1l_2}^2\sigma_{l_1l_3}^2\sigma_{l_2l_3}\neq 1$, $\sigma_{l_1l_2l_3}\neq 1$, $q_{l_1l_1}\sigma_{l_1l_j}\neq 1$, $j=2,3$, $\sigma_{l_il_j}\neq 1$, $1\leq i<j\leq 3$, $[2]_{q_{l_1l_1}}\neq 0$, onda u $\mathcal{B}_{l_1^2l_2l_3}$ nema konstanti.

✎ Prostor konstanti u bilo kojem degeneriranom težinskom potprostoru $\mathcal{B}_{l_1^2l_2^2}$, $l_1<l_2$, $l_1,l_2\in\mathcal{N}=\{i_1,i_2,...,i_N\}$ algebre $\mathcal{B}$ je jednodimenzionalan.

Ideal $\boldsymbol{I}_{l_1l_1l_2l_2}=\langle C_{l_1l_1l_2l_2}\rangle\subset\mathcal{B}_{l_1^2l_2^2}$ biti će generiran bazičnom konstantom $C_{l_1l_1l_2l_2}$ ako je $q_{l_1l_1}q_{l_2l_2}\sigma_{l_1l_2}^2=-1$, $l_1<l_2$.

Razlikujemo slučajeve

(1) Ako je $q_{l_1l_1}q_{l_2l_2}\sigma_{l_1l_2}^2=-1$, $q_{l_jl_j}\sigma_{l_1l_2}\neq 1$, $j=1,2$, $\sigma_{l_1l_2}\neq 1$, $[2]_{q_{l_jl_j}}\neq 0$, $j=1,2$, onda se bazična konstanta $C_{l_1l_1l_2l_2}$ može zapisati u obliku

$$C_{l_1l_1l_2l_2}=\frac{1-\sigma_{l_1l_2}}{q_{l_1l_2}}\widetilde{X}_{l_2l_2l_1l_1}-\frac{1}{q_{l_2l_1}^2}\cdot\frac{1+q_{l_1l_1}}{q_{l_1l_1}}\cdot\frac{1+q_{l_2l_2}}{q_{l_2l_2}}\widetilde{X}_{l_1l_2l_1l_2}$$
$$+\frac{1+q_{l_1l_1}}{q_{l_1l_1}}\cdot\frac{1-q_{l_1l_1}\sigma_{l_1l_2}}{q_{l_2l_1}}e_{l_1l_2l_2l_1}+\frac{1+q_{l_2l_2}}{q_{l_2l_2}}\cdot\frac{1-q_{l_2l_2}\sigma_{l_1l_2}}{q_{l_2l_1}}e_{l_2l_1l_1l_2},$$

gdje je $\widetilde{X}_{p_1p_2p_3p_4}$ definirano relacijom (52).

(2) Ako je $q_{l_1l_1}q_{l_2l_2}\sigma_{l_1l_2}^2=-1$, $q_{l_jl_j}\sigma_{l_1l_2}=1$, $j\in\{1,2\}$, $q_{l_kl_k}\sigma_{l_1l_2}\neq 1$, $k\in\{1,2\}\setminus\{j\}$, $\sigma_{l_1l_2}\neq 1$, $[2]_{q_{l_jl_j}}\neq 0$, $j=1,2$, onda je $C_{l_1l_1l_2l_2}=\lambda Y_{l_kl_jl_jl_k}$, $\lambda\in\mathbb{C}\setminus\{0\}$.

(3) Ako je $q_{l_1l_1}q_{l_2l_2}\sigma_{l_1l_2}^2=-1$, $q_{l_jl_j}\sigma_{l_1l_2}\neq 1$, $j=1,2$, $\sigma_{l_1l_2}=1$, $[2]_{q_{l_jl_j}}\neq 0$, $j=1,2$, onda je $C_{l_1l_1l_2l_2}=\lambda_1 Y_{j_1j_2}\,e_{l_1l_2}+\lambda_2\,e_{l_1}Y_{j_1j_2}\,e_{l_2}+\lambda_3\,e_{l_2}Y_{j_1j_2}\,e_{l_1}+\lambda_4\,e_{l_2l_1}Y_{j_1j_2}+\alpha\,Y_{j_1j_2}Y_{j_1j_2}$,
gdje je $j_1j_2=$ permutacija skupa $\{l_1,l_2\}$, $\alpha,\lambda_k\in\mathbb{C}\setminus\{0\}$, $1\le k\le 4$.

(4) Ako je $q_{l_1l_1}q_{l_2l_2}\sigma_{l_1l_2}^2=-1$, $q_{l_jl_j}\sigma_{l_1l_2}\neq 1$, $j=1,2$, $\sigma_{l_1l_2}\neq 1$, $[2]_{q_{l_jl_j}}=0$, $j\in\{1,2\}$, $[2]_{q_{l_kl_k}}\neq 0$, $k\in\{1,2\}\setminus\{j\}$, onda je $C_{l_1l_1l_2l_2}=\lambda Y_{l_jl_jl_kl_k}$, $\lambda\in\mathbb{C}\setminus\{0\}$.

(5) Ako je $q_{l_1l_1}q_{l_2l_2}\sigma_{l_1l_2}^2\neq -1$, $q_{l_jl_j}\sigma_{l_1l_2}\neq 1$, $j=1,2$, $\sigma_{l_1l_2}\neq 1$, $[2]_{q_{l_jl_j}}\neq 0$, $j=1,2$, onda u $\mathcal{B}_{l_1^2l_2^2}$ nema konstanti.

✐ Prostor konstanti u bilo kojem degeneriranom težinskom potprostoru $\mathcal{B}_{l_1^3l_2}$, $l_1\neq l_2$, $l_1,l_2\in\mathcal{N}=\{i_1,i_2,...,i_N\}$ algebre $\mathcal{B}$ je jednodimenzionalan.
Ideal $\boldsymbol{I}_{l_1l_1l_1l_2}=\langle C_{l_1l_1l_1l_2}\rangle\subset\mathcal{B}_{l_1^3l_2}$ biti će generiran bazičnom konstantom $C_{l_1l_1l_1l_2}$ ako je $q_{l_1l_1}^2\sigma_{l_1l_2}=1$, $l_1\neq l_2$.

Razlikujemo slučajeve

(1) Ako je $q_{l_1l_1}^2\sigma_{l_1l_2}=1$, $q_{l_1l_1}\sigma_{l_1l_2}\neq 1$, $\sigma_{l_1l_2}\neq 1$, $[k]_{q_{l_1l_1}}\neq 0$, $k=2,3$, onda je

$$C_{l_1l_1l_1l_2}=\lambda\,Y_{l_2l_1l_1l_1},\;\;\lambda\in\mathbb{C}\setminus\{0\}.$$

Pritom se (85) također dobiva i za $q_{l_1l_1}^2\sigma_{l_1l_2}=1$, $q_{l_1l_1}\sigma_{l_1l_2}=1$, $\sigma_{l_1l_2}\neq 1$, $[k]_{q_{l_1l_1}}\neq 0$, $k=2,3$, ali i za $q_{l_1l_1}^2\sigma_{l_1l_2}=1$, $q_{l_1l_1}\sigma_{l_1l_2}\neq 1$, $\sigma_{l_1l_2}=1$, $[k]_{q_{l_1l_1}}\neq 0$, $k=2,3$.

(2) Ako je $q_{l_1l_1}^2\sigma_{l_1l_2}=1$, $q_{l_1l_1}\sigma_{l_1l_2}\neq 1$, $\sigma_{l_1l_2}\neq 1$, $[3]_{q_{l_1l_1}}=0$, $[2]_{q_{l_1l_1}}\neq 0$, onda je

$$C_{l_1l_1l_1l_2}=\lambda\,Y_{l_1l_1l_1l_2},\;\;\lambda\in\mathbb{C}\setminus\{0\}.$$

(3) Ako je $q_{l_1l_1}^2\sigma_{l_1l_2}=1$, $[2]_{q_{l_1l_1}}=0$, $\left(\text{tj. } [3]_{q_{l_1l_1}}\neq 0\right)$, onda je

$$C_{l_1l_1l_1l_2}=\alpha_1\,Y_{l_1l_1}Y_{j_1j_2}+\alpha_2\,Y_{j_1j_2}Y_{l_1l_1},$$

$\alpha_1,\alpha_2\in\mathbb{C}\setminus\{0\}$, gdje je $j_1j_2=$ permutacija skupa $\{l_1,l_2\}$.

Pritom iz $q_{l_1l_1}^2\sigma_{l_1l_2}=1$, $[2]_{q_{l_1l_1}}=0$ proizlazi $\sigma_{l_1l_2}=1$.

(4) Ako je $q_{l_1l_1}^2\sigma_{l_1l_2}\neq 1$, $q_{l_1l_1}\sigma_{l_1l_2}\neq 1$, $\sigma_{l_1l_2}\neq 1$, $[k]_{q_{l_1l_1}}\neq 0$, $k=2,3$, onda u $\mathcal{B}_{l_1^3l_2}$ nema konstanti.

✐ Prostor konstanti u bilo kojem degeneriranom težinskom potprostoru $\mathcal{B}_{l^4}$, $l\in\mathcal{N}=\{i_1,i_2,\ldots,i_N\}$ algebre $\mathcal{B}$ je jednodimenzionalan. Ideal $\boldsymbol{I}_{llll}=\langle C_{llll}\rangle\subset\mathcal{B}_{l^4}$ biti će generiran baznom konstantom $C_{llll}=\alpha\,e_{l^4}$ ako je $[4]_{q_{ll}}=0$.

Pritom je $[4]_{q_{ll}}=1+q_{ll}+q_{ll}^2+q_{ll}^3$. Ako je $[4]_{q_{ll}}\neq 0$, onda u $\mathcal{B}_{l^4}$ nema konstanti.

Primjedba 2.2.4.4

Koristeći činjenicu da se bazična konstanta u bilo kojem degeneriranom težinskom potprostoru $\mathcal{B}_Q\subset\mathcal{B}(4)$ (Q je multiskup koji nije skup) može konstruirati iz bazične konstante nekog generičkog težinskog potprostora $\mathcal{B}_Q\subset\mathcal{B}(3)$ (Q je skup) određenom specijalizacijom koja je inducirana poistovjećivanjem nekih elemenata u skupu Q zaključujemo da će konstante u potprostoru $\mathcal{B}(4)$ generirati ideal $\boldsymbol{I}_4=\langle C_{l_1l_2l_3l_4}\rangle$ ako je $\sigma_{l_1l_2l_3l_4}=1$, $l_1\leq l_2\leq l_3\leq l_4$, $l_1,l_2,l_3,l_4\in\mathcal{N}$.

Pritom je dimenzija prostora konstanti u $\mathcal{B}(4)$ manja ili jednaka $2\cdot\binom{N+3}{4}$.

3. Zakrenuta grupovna algebra $\mathcal{A}_n$

U ovom poglavlju provesti ćemo razmatranja analogno kao i u [MS1] time da će se koristiti notacije zakrenute grupovne algebre (twisted group algebra).

Neka je $R_n := \mathbb{C}[X_{km} \mid 1 \le k, m \le n]$ prsten polinoma u n^2 komutirajućih varijabli X_{km}, a S_n simetrična grupa skupa $\{1, 2, ..., n\}$. Elementi skupa S_n su bijekcije n-članog skupa $\{1, 2, ..., n\}$, a množenje je kompozicija.

Neka je $X := \{X_{km} \mid 1 \le k, m \le n\}$. Pogledajmo najprije prirodno djelovanje $S_n \times X \to X$ simetrične grupe S_n na skupu X definirano sa

$$g.X_{km} := X_{g(k)\,g(m)}, \tag{1}$$

$g \in S_n$. Tada to djelovanje inducira djelovanje $S_n \times R_n \to R_n$ simetrične grupe S_n na prstenu polinoma $R_n = \mathbb{C}[X_{km} \mid 1 \le k, m \le n]$ takvo da je:

$$g.p(..., X_{km}, ...) = p(..., X_{g(k)\,g(m)}, ...), \tag{2}$$

$g \in S_n$. Primijetimo, (obična) grupovna algebra simetrične grupe S_n, jednaka $\mathbb{C}[S_n] = \left\{ \sum_{\sigma \in S_n} c_\sigma \, \sigma \mid c_\sigma \in \mathbb{C} \right\}$, je slobodni vektorski prostor generiran skupom S_n na kojemu je množenje vektora baze dano s množenjem u grupi S_n i pišemo:

$$\left(\sum_{\sigma \in S_n} c'_\sigma \, \sigma \right) \cdot \left(\sum_{\tau \in S_n} c''_\tau \, \tau \right) = \sum_{\sigma, \tau \in S_n} c'_\sigma c''_\tau \, \sigma \circ \tau .$$

Definirajmo sada općenitiju grupovnu algebru simetrične grupe S_n s koeficijentima u prstenu polinoma R_n, pri čemu se koristi gore spomenuto djelovanje S_n na R_n. Tako definirana općenitija grupovna algebra naziva se zakrenuta grupovna algebra simetrične grupe S_n s koeficijentima u prstenu polinoma R_n i označava se sa

$$\mathcal{A}_n := R_n \rtimes \mathbb{C}[S_n]. \tag{3}$$

U zakrenutoj grupovnoj algebri $\mathcal{A}_n = \left\{ \sum_{g_i \in S_n} p_i \, g_i \mid p_i \in R_n \right\}$ je množenje definirano sa

$$(p_1 \, g_1) \cdot (p_2 \, g_2) := (p_1 \cdot g_1.p_2) \, g_1 g_2, \tag{4}$$

gdje je $g_1.p_2$ definirano pravilom (2), a $g_1 g_2$ je produkt od g_1 sa g_2 u S_n.

Neka je $g \in S_n$ bilo koja permutacija iz skupa S_n.

Uvodimo oznaku

$$I(g) = \{(a,b) \mid 1 \le a < b \le n,\ g(a) > g(b)\}$$

za skup inverzija permutacije $g \in S_n$.

Definicija 3.1

Bilo kojoj permutaciji $g \in S_n$ pridružimo monom $X_g \in R_n$ u prstenu polinoma $R_n = \mathbb{C}[X_{km} \mid 1 \le k, m \le n]$ definiran sa

$$\begin{aligned} X_g &:= \prod_{a<b,\ g^{-1}(a)>g^{-1}(b)} X_{ab} \\ &= \prod_{(a,b)\in I(g^{-1})} X_{ab}\,. \end{aligned} \tag{5}$$

Definicija 3.2

Bilo kojem nepraznom skupu $A \subseteq \{1,2,...,n\}$ pridružimo monom $X_A \in R_n$ definiran sa

$$\begin{aligned} X_A &:= \prod_{(a,b)\in A\times A,\ a\neq b} X_{ab} = \prod_{(a,b)\in A\times A,\ a<b} X_{ab}\cdot X_{ba} \\ &= \prod_{(a,b)\in A\times A,\ a<b} X_{\{a,b\}}\,. \end{aligned} \tag{6}$$

Uočimo da je $X_{\{a,b\}} = X_{ab} \cdot X_{ba}$ (slučaj za $A = \{a,b\}$).

Definicija 3.3

Bilo kojoj permutaciji $g \in S_n$ pridružimo element $\tilde{g}$ u zakrenutoj grupovnoj algebri $\mathcal{A}_n$ definiran sa

$$\tilde{g} := X_g\, g\,. \tag{7}$$

Uočimo da koeficijent X_g kodira sve inverzije od g^{-1}, a samim time i od g.

Napomena:

U nastavku će se često koristiti elementi iz zakrenute grupovne algebre $\mathcal{A}_n$, definirani sa 3.3.

Propozicija 3.4 (Multiparametarski identitet)

Neka su $g_1, g_2 \in S_n$ bilo koje dvije permutacije iz skupa S_n.

Tada za odgovarajuće elemente $\widetilde{g_1}, \widetilde{g_2} \in \mathcal{A}_n$ algebre $\mathcal{A}_n$ vrijedi

$$\widetilde{g_1} \cdot \widetilde{g_2} = X(g_1, g_2) \cdot \widetilde{g_1 \circ g_2} \tag{8}$$

pri čemu je

$$X(g_1, g_2) = \prod_{(a,b) \in I(g_1^{-1}) - I(g_2^{-1} g_1^{-1})} X_{\{a,b\}} = \prod_{(a,b) \in I(g_1) \cap I(g_2^{-1})} X_{\{g_1(a), g_1(b)\}}, \tag{9}$$

gdje je $X_{\{a,b\}} = X_{ab} \cdot X_{ba}$.

Dokaz:

Za $g_1, g_2 \in S_n$ imamo

$$\widetilde{g_1} = X_{g_1} g_1 = \left(\prod_{(a,b) \in I(g_1^{-1})} X_{ab} \right) g_1,$$

$$\widetilde{g_2} = X_{g_2} g_2 = \left(\prod_{(a',b') \in I(g_2^{-1})} X_{a'b'} \right) g_2,$$

stoga je

$$\begin{aligned}
\widetilde{g_1} \cdot \widetilde{g_2} &= \left(X_{g_1} g_1 \right) \cdot \left(X_{g_2} g_2 \right) \\
&= \left(X_{g_1} \cdot g_1 . X_{g_2} \right) g_1 \circ g_2 \\
&= \left(X_{g_1} \cdot g_1 . \prod_{(a',b') \in I(g_2^{-1})} X_{a'b'} \right) g_1 \circ g_2 \\
&= \left(X_{g_1} \cdot g_1 . \prod_{a'<b',\ g_2^{-1}(a') > g_2^{-1}(b')} X_{a'b'} \right) g_1 \circ g_2 \\
&= \left(X_{g_1} \cdot \prod_{a'<b',\ g_2^{-1}(a') > g_2^{-1}(b')} X_{g_1(a') g_1(b')} \right) g_1 \circ g_2 .
\end{aligned}$$

Označimo $a = g_1(a'),\ b = g_1(b'),$

tada je

$$a' = g_1^{-1}(a),\ b' = g_1^{-1}(b),$$

na osnovu čega dobivamo

$$
=\left(X_{g_1}\cdot \prod_{g_1^{-1}(a)<g_1^{-1}(b),\; g_2^{-1}\left(g_1^{-1}(a)\right)>g_2^{-1}\left(g_1^{-1}(b)\right)} X_{ab} \right) g_1\circ g_2
$$

$$
=\left(\prod_{(a,b)\in I\left(g_1^{-1}\right)} X_{ab}\cdot \prod_{\left(g_1^{-1}(a),g_1^{-1}(b)\right)\in I\left(g_2^{-1}\right)} X_{ab} \right) g_1\circ g_2
$$

$$
=\left(\prod_{(a,b)\in I\left(g_1^{-1}\right)} X_{ab}\cdot \prod_{(a,b)\in I\left(g_2^{-1}g_1^{-1}\right)-I\left(g_1^{-1}\right)} X_{ab}\cdot \prod_{(b,a)\in I\left(g_1^{-1}\right)-I\left(g_2^{-1}g_1^{-1}\right)} X_{ab} \right) g_1\circ g_2
$$

$$
\overset{(10)}{=}\left(\prod_{(a,b)\in I\left(g_1^{-1}\right)} X_{ab}\cdot \left(\prod_{(a,b)\in I\left(g_1^{-1}\right)\cap I\left(g_2^{-1}g_1^{-1}\right)} X_{ab}^{-1}\cdot \prod_{(a,b)\in I\left(g_2^{-1}g_1^{-1}\right)} X_{ab} \right)\cdot \prod_{(b,a)\in I\left(g_1^{-1}\right)-I\left(g_2^{-1}g_1^{-1}\right)} X_{ab} \right) g_1\circ g_2
$$

$$
=\left(\left(\prod_{(a,b)\in I\left(g_1^{-1}\right)} X_{ab}\cdot \prod_{(a,b)\in I\left(g_1^{-1}\right)\cap I\left(g_2^{-1}g_1^{-1}\right)} X_{ab}^{-1} \right)\cdot \prod_{(b,a)\in I\left(g_1^{-1}\right)-I\left(g_2^{-1}g_1^{-1}\right)} X_{ab}\cdot \prod_{(a,b)\in I\left(g_2^{-1}g_1^{-1}\right)} X_{ab} \right) g_1\circ g_2
$$

$$
\overset{(11)}{=}\left(\prod_{(a,b)\in I\left(g_1^{-1}\right)-I\left(g_2^{-1}g_1^{-1}\right)} X_{ab}\cdot \prod_{(b,a)\in I\left(g_1^{-1}\right)-I\left(g_2^{-1}g_1^{-1}\right)} X_{ab}\cdot \prod_{(a,b)\in I\left(g_2^{-1}g_1^{-1}\right)} X_{ab} \right) g_1\circ g_2
$$

$$
=\left(\prod_{(a,b)\in I\left(g_1^{-1}\right)-I\left(g_2^{-1}g_1^{-1}\right)} X_{ab}\,X_{ba}\cdot \prod_{(a,b)\in I\left(g_2^{-1}g_1^{-1}\right)} X_{ab} \right) g_1\circ g_2
$$

$$
=\prod_{(a,b)\in I\left(g_1^{-1}\right)-I\left(g_2^{-1}g_1^{-1}\right)} X_{\{a,b\}}\cdot \left(\left(\prod_{(a,b)\in I\left(g_2^{-1}g_1^{-1}\right)} X_{ab} \right) g_1\circ g_2 \right)
$$

$$
=X\left(g_1,g_2\right)\cdot \widetilde{g_1\circ g_2}\,.
$$

U gornjem izvodu korišćene su sljedeće relacije

$$
\prod_{(a,b)\in I\left(g_2^{-1}g_1^{-1}\right)-I\left(g_1^{-1}\right)} X_{ab} = \prod_{(a,b)\in I\left(g_1^{-1}\right)\cap I\left(g_2^{-1}g_1^{-1}\right)} X_{ab}^{-1}\cdot \prod_{(a,b)\in I\left(g_2^{-1}g_1^{-1}\right)} X_{ab}\,, \tag{10}
$$

$$
\prod_{(a,b)\in I\left(g_1^{-1}\right)} X_{ab}\cdot \prod_{(a,b)\in I\left(g_1^{-1}\right)\cap I\left(g_2^{-1}g_1^{-1}\right)} X_{ab}^{-1} = \prod_{(a,b)\in I\left(g_1^{-1}\right)-I\left(g_2^{-1}g_1^{-1}\right)} X_{ab}\,. \tag{11}
$$

Pritom relacija (10) proizlazi iz

$$
\prod_{(a,b)\in I\left(g_2^{-1}g_1^{-1}\right)} X_{ab} \coloneqq \prod_{(a,b)\in I\left(g_1^{-1}\right)\cap I\left(g_2^{-1}g_1^{-1}\right)} X_{ab}\cdot \prod_{(a,b)\in I\left(g_2^{-1}g_1^{-1}\right)-I\left(g_1^{-1}\right)} X_{ab}\,,
$$

a relacija (11) iz

$$
\prod_{(a,b)\in I\left(g_1^{-1}\right)} X_{ab} \coloneqq \prod_{(a,b)\in I\left(g_1^{-1}\right)-I\left(g_2^{-1}g_1^{-1}\right)} X_{ab}\cdot \prod_{(a,b)\in I\left(g_1^{-1}\right)\cap I\left(g_2^{-1}g_1^{-1}\right)} X_{ab}\,.
$$

Napomena:

U nastavku ćemo kompoziciju permutacija $g_1 \circ g_2$ označavati sa $g_1 g_2$.

Primjedba 3.5

Faktor $X(g_1, g_2)$ vodi brigu o smanjivanju broja inverzija kod množenja permutacija g_1 sa g_2.

Ako su skupovi inverzija $I(g_1)$ i $I(g_2)$ tzv. relativno prosti, tj. ako je

$$\mathrm{Card}(I(g_1 g_2)) = \mathrm{Card}(I(g_1)) + \mathrm{Card}(I(g_2)),$$

onda je $X(g_1, g_2) = 1$ pa iz formule (8) dobivamo $\widetilde{g_1} \cdot \widetilde{g_2} = \widetilde{g_1 g_2}$.

Primjeri 3.6

(1) Uzmimo $g_1 = 132 \in S_3$, $g_2 = 231 \in S_3$.

Tada je $g_1^{-1} = 132$, $g_2^{-1} = 312$.

Imamo $\quad I(g_1^{-1}) = \{(2,3)\}$, $I(g_2^{-1}) = \{(1,2),(1,3)\}$,

odnosno $\quad I(g_1) = \{(2,3)\}$, $I(g_2) = \{(1,3),(2,3)\}$,

U algebri $\mathcal{A}_3$ za elemente

$$\widetilde{g_1} = \prod_{(a,b)\in I(g_1^{-1})} X_{ab}\, g_1 = X_{23}\, g_1,$$

$$\widetilde{g_2} = \prod_{(a,b)\in I(g_2^{-1})} X_{ab}\, g_2 = X_{12}\, X_{13}\, g_2,$$

koristeći propoziciju 3.4. dobivamo $\widetilde{g_1} \cdot \widetilde{g_2} = X(g_1, g_2)\widetilde{g_1 g_2}$, odnosno $\widetilde{g_1} \cdot \widetilde{g_2} = \widetilde{g_1 g_2}$, jer je primjenom formule (9)

$$X(g_1, g_2) = \prod_{(a,b)\in I(g_1)\cap I(g_2^{-1})} X_{\{g_1(a),\, g_1(b)\}} = \prod_{(a,b)\in\varnothing} X_{\{g_1(a),\, g_1(b)\}} = 1,$$

što povlači da su skupovi inverzija $I(g_1)$ i $I(g_2)$ relativno prosti.

S druge strane iz

$$g_1 g_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

proizlazi $I(g_1 g_2) = \{(1,2),(1,3),(2,3)\}$, tj. $\mathrm{Card}(I(g_1 g_2)) = 3$.

Uzimajući u obzir da je $\mathrm{Card}(I(g_1)) = 1$, $\mathrm{Card}(I(g_2)) = 2$, dobivamo da je

$$\mathrm{Card}\left(I\left(g_1 g_2\right)\right)=\mathrm{Card}\left(I\left(g_1\right)\right)+\mathrm{Card}\left(I\left(g_2\right)\right),$$

na osnovu čega možemo zaključiti da je $X\left(g_1, g_2\right)=1$.

Izračunajmo sada $\widetilde{g_1} \cdot \widetilde{g_2}$.

Koristeći definiciju množenja u zakrenutoj grupovnoj algebri dobivamo:

$$\begin{aligned}\widetilde{g_1} \cdot \widetilde{g_2} &= \left(X_{23}\, g_1\right)\left(X_{12}\, X_{13}\, g_2\right)=X_{23}\, X_{g_1(1)\, g_1(2)}\, X_{g_1(1)\, g_1(3)}\, g_1 g_2=X_{23}\, X_{13}\, X_{12}\, g_1 g_2 \\ &= X_{12}\, X_{13}\, X_{23}\, g_1 g_2=\widetilde{g_1 g_2}.\end{aligned}$$

Uočimo da je $\left(g_1 g_2\right)^{-1}=g_1 g_2$, tj. $I\left(\left(g_1 g_2\right)^{-1}\right)=I\left(g_1 g_2\right)=\{(1,2),(1,3),(2,3)\}$ te da je

$$\widetilde{g_1 g_2}=\prod_{(a,b) \in I\left(\left(g_1 g_2\right)^{-1}\right)} X_{ab}\, g_1 g_2=X_{12}\, X_{13}\, X_{23}\, g_1 g_2.$$

(2) Uzmimo sada $g_1=132 \in S_3$, $g_2=312 \in S_3$.

Tada je $g_1^{-1}=132$, $g_2^{-1}=231$.

Imamo $I\left(g_1^{-1}\right)=\{(2,3)\}$, $I\left(g_2^{-1}\right)=\{(1,3),(2,3)\}$

$I\left(g_1\right)=\{(2,3)\}$, $I\left(g_2\right)=\{(1,2),(1,3)\}$

U ovom slučaju imamo

$$\widetilde{g_1}=\prod_{(a,b) \in I\left(g_1^{-1}\right)} X_{ab}\, g_1=X_{23}\, g_1,$$

$$\widetilde{g_2}=\prod_{(a,b) \in I\left(g_2^{-1}\right)} X_{ab}\, g_2=X_{13}\, X_{23}\, g_2.$$

Primjenom formule (9) dobivamo

$$X\left(g_1, g_2\right)=\prod_{(a,b) \in I\left(g_1\right) \cap I\left(g_2^{-1}\right)} X_{\{g_1(a),\, g_1(b)\}}=\prod_{(a,b) \in\{(2,3)\}} X_{\{g_1(a),\, g_1(b)\}}=X_{\{3,2\}}=X_{\{2,3\}},$$

(g_1 i g_2 nisu "relativno prosti").

S druge strane iz

$$g_1 g_2=\begin{pmatrix}1 & 2 & 3 \\ 1 & 3 & 2\end{pmatrix} \cdot \begin{pmatrix}1 & 2 & 3 \\ 3 & 1 & 2\end{pmatrix}=\begin{pmatrix}1 & 2 & 3 \\ 2 & 1 & 3\end{pmatrix}$$

proizlazi $I\left(g_1 g_2\right)=\{(1,2)\}$, tj. $\mathrm{Card}\left(I\left(g_1 g_2\right)\right)=1$.

Uzimajući u obzir da je $\mathrm{Card}\left(I\left(g_1\right)\right)=1$, $\mathrm{Card}\left(I\left(g_2\right)\right)=2$, dobivamo da je

$$\mathrm{Card}\left(I\left(g_1 g_2\right)\right) \neq \mathrm{Card}\left(I\left(g_1\right)\right)+\mathrm{Card}\left(I\left(g_2\right)\right),$$

tj. $X\left(g_1, g_2\right) \neq 1$.

Lako se pokaže da je

$$\widetilde{g_1}\cdot\widetilde{g_2}=\left(X_{23}\,g_1\right)\left(X_{13}\,X_{23}\,g_2\right)=X_{23}\,X_{g_1(1)\,g_1(3)}\,X_{g_1(2)\,g_1(3)}\,g_1g_2=X_{23}\,X_{12}\,X_{32}\,g_1g_2$$

$$=X_{\{2,3\}}X_{12}\,g_1g_2\,,$$

$$\widetilde{g_1g_2}=\prod_{(a,b)\in I\left(\left(g_1g_2\right)^{-1}\right)}X_{ab}\,g_1g_2=X_{12}\,g_1g_2\,,$$

odnosno $\widetilde{g_1}\cdot\widetilde{g_2}=X_{\{2,3\}}\widetilde{g_1g_2}$

Pritom smo koristili $\left(g_1g_2\right)^{-1}=213=g_1g_2$, odnosno $I\left(\left(g_1g_2\right)^{-1}\right)=I\left(g_1g_2\right)=\left\{(1,2)\right\}$.

Definicija 3.7

Za $1\le a\le b\le n$ označimo sa $t_{a,b}$ cikličku permutaciju, koja preslikava

$$(a)\mapsto(b)\mapsto(b-1)\mapsto(b-2)\mapsto\cdots\mapsto(a+1)\mapsto(a)$$

te sve preostale brojeve k, $1\le k<a$ ili $b<k\le n$ fiksira, tj.

$$t_{a,b}(k)=\begin{cases}k & 1\le k<a \text{ ili } b<k\le n,\\ b & k=a,\\ k-1 & a<k\le b.\end{cases}$$

U dvorednoj notaciji imamo

$$t_{a,b}=\begin{pmatrix}1 & 2 & \cdots & a-1 & a & a+1 & a+2 & \cdots & b & b+1 & \cdots & n-1 & n\\ 1 & 2 & \cdots & a-1 & b & a & a+1 & \cdots & b-1 & b+1 & \cdots & n-1 & n\end{pmatrix},$$

ili u skraćenom obliku

$$t_{a,b}=\begin{pmatrix}a & a+1 & a+2 & \cdots & b\\ b & a & a+1 & \cdots & b-1\end{pmatrix}. \tag{12}$$

Pritom je skup inverzija od $t_{a,b}$ dan sa

$$I\left(t_{a,b}\right)=\left\{(a,j)|\ a<j\le b,\ t_{a,b}(a)>t_{a,b}(j)\right\}=\left\{(a,a+1),(a,a+2),...,(a,b-1),(a,b)\right\}$$

i vrijedi $\operatorname{Card}\left(I\left(t_{a,b}\right)\right)=\left|I\left(t_{a,b}\right)\right|=b-a$.

Za duljinu $\left|t_{a,b}\right|\ \left(=\ell\left(t_{a,b}\right)\right)$ cikličke permutacije $t_{a,b}$ imamo $\left|t_{a,b}\right|=b-a+1$.

Primjedba 3.8

Inverz cikličke permutacije $t_{a,b}$ je dan sa

$$t_{a,b}^{-1}(k)=\begin{cases} k & 1\le k<a \text{ ili } b<k\le n, \\ k+1 & a\le k<b, \\ a & k=b, \end{cases} \tag{13}$$

odnosno

$$t_{a,b}^{-1}=\begin{pmatrix} a & a+1 & \cdots & b-1 & b \\ a+1 & a+2 & \cdots & b & a \end{pmatrix}.$$

Pritom je skup inverzija od $t_{a,b}^{-1}$ dan sa

$$I\left(t_{a,b}^{-1}\right)=\left\{(i,b) \mid a\le i<b,\ t_{a,b}^{-1}(i)>t_{a,b}^{-1}(b)\right\}=\left\{(a,b),(a+1,b),\dots,(b-2,b),(b-1,b)\right\},$$

te je $\left|I\left(t_{a,b}^{-1}\right)\right|=b-a$. Jasno, $\left|t_{a,b}^{-1}\right|=\left|t_{a,b}\right|=b-a+1$.

Notacija 3.9

Za bilo koji $1\le a\le b\le n$ definira se

$$t_{b,a}\coloneqq t_{a,b}^{-1}. \tag{14}$$

Pritom je $I\left(t_{b,a}\right)=I\left(t_{a,b}^{-1}\right)$.

Promatrajmo sada specijalne slučajeve od $t_{b,a}$, $1\le a\le b\le n$.

- Ako je $b=a$, onda je

$$t_{a,a}=\mathrm{id}, \tag{15}$$

što povlači da je $I\left(t_{a,a}\right)=\varnothing$.

- Ako je $b=a+1$, onda primjenom relacije (13) dobivamo

$$t_{a+1,a}(k)=\begin{cases} k & 1\le k<a \text{ ili } a+1<k\le n, \\ a+1 & k=a, \\ a & k=a+1, \end{cases} \tag{16}$$

ili

$$t_{a,a+1}=\begin{pmatrix} a & a+1 \\ a+1 & a \end{pmatrix}$$

pa je $I\left(t_{a+1,a}\right)=\left\{(a,a+1)\right\}$.

Napomena:

Očito je $t_{a,a+1}^{-1} = t_{a,a+1}$ za svaki $1 \le a \le n-1$, tj. vrijedi $t_{a+1,a} = t_{a,a+1}$.

Time je $I\left(t_{a+1,a}\right) = I\left(t_{a,a+1}\right) = \left\{(a, a+1)\right\}$.

Za transpoziciju susjednih elemenata uvodimo pokratu

$$t_a := t_{a+1,a} . \tag{17}$$

za $1 \le a \le n-1$.

Pritom je $I(t_a) = \left\{(a, a+1)\right\}$.

Primjenom definicije 3.1 dobivamo

$$\left.\begin{aligned}
X_{t_{a,b}} &= \prod_{(i',j') \in I\left(t_{a,b}^{-1}\right)} X_{i'j'} = \prod_{i=a}^{b-1} X_{ib} , \\
X_{t_{b,a}} &= X_{t_{a,b}^{-1}} = \prod_{(i',j') \in I\left(t_{a,b}\right)} X_{i'j'} = \prod_{j=a+1}^{b} X_{aj} , \\
X_{t_{a,a}} &= X_{t_{a,a}^{-1}} = \prod_{(i',j') \in \emptyset} X_{i'j'} = 1 , \\
X_{t_a} &= \prod_{(i,j) \in \{(a,a+1)\}} X_{ij} = X_{a\,a+1} .
\end{aligned}\right\} \tag{18}$$

i analogno primjenom definicije 3.3 dobivamo sljedeće elemente u zakrenutoj grupovnoj algebri $\mathcal{A}_n$

$$\left.\begin{aligned}
\widetilde{t_{a,b}} &= \left(\prod_{i=a}^{b-1} X_{ib}\right) t_{a,b} , \\
\widetilde{t_{b,a}} &= \left(\prod_{j=a+1}^{b} X_{aj}\right) t_{b,a} , \\
\widetilde{t_{a,a}} &= id , \\
\widetilde{t_a} &= X_{a\,a+1}\, t_a .
\end{aligned}\right\} \tag{19}$$

Lema 3.10

Za $1 \le a \le n-1$ vrijedi

$$\widetilde{t_a}^{\,2} = X_{\{a,a+1\}}\, id , \tag{20}$$

gdje je $X_{\{a,a+1\}} = X_{a\,a+1} \cdot X_{a+1\,a}$.

Dokaz:

Primjenom propozicije 3.4 imamo da je

$$\widetilde{t_a}^2 = \widetilde{t_a}\widetilde{t_a} = X(t_a, t_a)\cdot \underbrace{\widetilde{t_a t_a}}_{=id} = \prod_{(i,j)\in I(t_a)\cap I(t_a^{-1})} X_{\{t_a(i), t_a(j)\}}\cdot id$$

$$= \prod_{(i,j)\in\{(a,a+1)\}} X_{\{t_a(i), t_a(j)\}}\, id = X_{\{t_a(a), t_a(a+1)\}}\, id = X_{\{a+1, a\}}\, id = X_{\{a, a+1\}}\, id\,.$$

Time je lema dokazana.

Dokažimo da se svaka ciklička permutacija $t_{a,b}$ $(1 \le a < b \le n)$ rastavlja u produkt jednostavnih transpozicija ovako:

$$t_{a,b} = t_{b-1}t_{b-2}\cdots t_{a+1}t_a = \overleftarrow{\prod_{a\le j\le b-1}} t_j \tag{21}$$

i analogno njen inverz $t_{b,a}$:

$$t_{b,a} = t_a t_{a+1}\cdots t_{b-2}t_{b-1} = \prod_{j=a}^{b-1} t_j\,. \tag{22}$$

Pritom formula (22) proizlazi direktno iz formule (21), pri čemu se koristi svojstvo da je $t_k^{-1} = t_k$ za svaki $1 \le k \le n-1$.

Dokaz formule (21) provodimo indukcijom.

Ako je $b = a+1$, onda je $t_{a,a+1} = t_a$,

što je u suglasnosti s gore navedenim. Pretpostavimo da je $t_{a,b} = t_{b-1}t_{b-2}\cdots t_{a+1}t_a$ za svaki $1 \le a < b \le n$. Tada treba dokazati da vrijedi $t_{a,b+1} = t_b t_{b-1}t_{b-2}\cdots t_{a+1}t_a$.

Koristeći definiciju 3.7 dobivamo

$$t_{a,b+1}(k) = \begin{cases} k & 1\le k < a, \\ b+1 & k = a, \\ k-1 & a < k \le b, \\ b & k = b+1, \\ k & b+1 < k \le n. \end{cases}$$

S druge strane imamo $$t_b t_{a,b}(k) = \begin{cases} t_b(k) = k & 1\le k < a, \\ t_b(b) = b+1 & k = a, \\ t_b(k-1) = k-1 & a < k \le b, \\ t_b(b+1) = b & k = b+1, \\ t_b(k) = k & b+1 < k \le n, \end{cases}$$

stoga dobivamo $t_{a,b+1}(k) = t_b t_{a,b}(k)$ za svaki $1 \le k \le n$.

Uzimajući u obzir pretpostavku $t_{a,b} = t_{b-1}t_{b-2}\cdots t_{a+1}t_a$ zaključujemo da je

$$t_{a,b+1}(k) = t_b\left(t_{b-1}t_{b-2}\cdots t_{a+1}t_a\right)(k) = t_b t_{b-1}t_{b-2}\cdots t_{a+1}t_a(k)$$

za svaki $1 \le k \le n$.

Lema 3.11

Za svaki $1 \le a < b \le n$ vrijede sljedeće tvrdnje

(i) $\widetilde{t_{b-1}} \cdot \widetilde{t_{b-2}} \cdots \widetilde{t_{a+1}} \cdot \widetilde{t_a} = \widetilde{t_{a,b}} = \left(\prod_{i=a}^{b-1} X_{ib}\right) t_{a,b}$,

(ii) $\widetilde{t_a} \cdot \widetilde{t_{a+1}} \cdots \widetilde{t_{b-2}} \cdot \widetilde{t_{b-1}} = \widetilde{t_{b,a}} = \left(\prod_{j=a+1}^{b} X_{aj}\right) t_{b,a}$.

Dokaz:

Za dokaz tvrdnje *(i)* koristiti ćemo matematičku indukciju. Dokažimo najprije prvi dio tvrdnje *(i)*, tj. $\widetilde{t_{b-1}} \cdot \widetilde{t_{b-2}} \cdots \widetilde{t_{a+1}} \cdot \widetilde{t_a} = \widetilde{t_{a,b}}$ za svaki $1 \le a < b \le n$.

Ako je $b = a+1$, onda je $\widetilde{t_a} = \widetilde{t_{a,a+1}}$, što je u suglasnosti s gore navedenim.

Pretpostavimo da je $\widetilde{t_{b-1}} \cdot \widetilde{t_{b-2}} \cdots \widetilde{t_{a+1}} \cdot \widetilde{t_a} = \widetilde{t_{a,b}}$ za svaki $1 \le a < b \le n$.

Tada treba dokazati da vrijedi $\widetilde{t_b} \cdot \widetilde{t_{b-1}} \cdot \widetilde{t_{b-2}} \cdots \widetilde{t_{a+1}} \cdot \widetilde{t_a} = \widetilde{t_{a,b+1}}$.

Koristeći pretpostavku indukcije dobivamo

$$\begin{aligned}\widetilde{t_b} \cdot \widetilde{t_{b-1}} \cdot \widetilde{t_{b-2}} \cdots \widetilde{t_{a+1}} \cdot \widetilde{t_a} &= \widetilde{t_b} \cdot \left(\widetilde{t_{b-1}} \cdot \widetilde{t_{b-2}} \cdots \widetilde{t_{a+1}} \cdot \widetilde{t_a}\right) = \widetilde{t_b} \cdot \widetilde{t_{a,b}} \overset{\text{prop. } 3.4}{=} X\left(t_b, t_{a,b}\right) \cdot \widetilde{t_b t_{a,b}} \\ &= \widetilde{t_b t_{a,b}},\end{aligned}$$

pri čemu se primjenom formule (9) dobiva $X\left(t_b, t_{a,b}\right) = \prod_{\underbrace{(i,j)\in I(t_b)\cap I\left(t_{a,b}^{-1}\right)}_{=\varnothing}} X_{\{t_b(i),\, t_b(j)\}} = 1$,

jer je $\quad I\left(t_b\right) = \left\{(b, b+1)\right\}$,

$$I\left(t_{a,b}\right) = \left\{(a,b), (a+1,b), \ldots, (b-1,b)\right\}.$$

S druge strane, koristeći dokazano svojstvo $t_b t_{a,b} = t_{a,b+1}$ imamo da je $\widetilde{t_b t_{a,b}} = \widetilde{t_{a,b+1}}$, što povlači da je $\widetilde{t_b} \cdot \widetilde{t_{b-1}} \cdot \widetilde{t_{b-2}} \cdots \widetilde{t_{a+1}} \cdot \widetilde{t_a} = \widetilde{t_{a,b+1}}$ za svaki $1 \le a < b \le n$.

Time je dokazan prvi dio tvrdnje *(i)*.

Drugi dio tvrdnje *(i)*, tj. $\widetilde{t_{b-1}} \cdot \widetilde{t_{b-2}} \cdots \widetilde{t_{a+1}} \cdot \widetilde{t_a} = \left(\prod_{i=a}^{b-1} X_{ib}\right) t_{a,b}$, slijedi iz (19).

Analogno se matematičkom indukcijom dokazuje tvrdnja *(ii)*.

Lema 3.12

U algebri $\mathcal{A}_n$ vrijede tzv. pletenične relacije (braid relations)

$$\widetilde{t_a}\cdot\widetilde{t_{a+1}}\cdot\widetilde{t_a}=\widetilde{t_{a+1}}\cdot\widetilde{t_a}\cdot\widetilde{t_{a+1}} \qquad \text{za svaki } 1\le a\le n-2, \tag{23}$$

$$\widetilde{t_a}\cdot\widetilde{t_b}=\widetilde{t_b}\cdot\widetilde{t_a} \qquad \text{za svaki } 1\le a,b\le n-1,\ |a-b|\ge 2. \tag{24}$$

Dokažimo te relacije.

Dokaz relacije (23). Po definiciji je

$$t_a(k)=\begin{cases} k & 1\le k<a \ \text{ ili } \ a+1<k\le n,\\ a+1 & k=a,\\ a & k=a+1,\end{cases}$$

odnosno

$$t_{a+1}(k)=\begin{cases} k & 1\le k<a+1 \ \text{ ili } \ a+2<k\le n,\\ a+2 & k=a+1,\\ a+1 & k=a+2,\end{cases}$$

stoga dobivamo

$$t_a t_{a+1} t_a(k)=\begin{cases} t_a t_{a+1}(k)=t_a(k)=k & 1\le k<a \ \text{ ili } \ a+2<k\le n,\\ t_a t_{a+1}(a+1)=t_a(a+2)=a+2 & k=a,\\ t_a t_{a+1}(a)=t_a(a)=a+1 & k=a+1,\\ t_a t_{a+1}(a+2)=t_a(a+1)=a & k=a+2,\end{cases}$$

odnosno

$$t_{a+1} t_a t_{a+1}(k)=\begin{cases} t_{a+1} t_a(k)=t_{a+1}(k)=k & 1\le k<a \ \text{ ili } \ a+2<k\le n,\\ t_{a+1} t_a(a)=t_{a+1}(a+1)=a+2 & k=a,\\ t_{a+1} t_a(a+2)=t_{a+1}(a+2)=a+1 & k=a+1,\\ t_{a+1} t_a(a+1)=t_{a+1}(a)=a & k=a+2.\end{cases}$$

Dakle, $t_a t_{a+1} t_a(k)=t_{a+1} t_a t_{a+1}(k)$ za svaki $1\le k\le n$.

Izračunajmo sada

$$\begin{aligned}\widetilde{t_a}\cdot\widetilde{t_{a+1}}\cdot\widetilde{t_a}&=(X_{a\,a+1}\,t_a)\cdot(X_{a+1\,a+2}\,t_{a+1})\cdot(X_{a\,a+1}\,t_a)\\ &=\left(X_{a\,a+1}\,X_{t_a(a+1)\,t_a(a+2)}\right)\cdot(t_a t_{a+1})\cdot(X_{a\,a+1}\,t_a)\\ &=\left(X_{a\,a+1}\,X_{a\,a+2}\,X_{t_a t_{a+1}(a)\,t_a t_{a+1}(a+1)}\right)t_a t_{a+1} t_a\\ &=(X_{a\,a+1}\,X_{a\,a+2}\,X_{a+1\,a+2})\,t_a t_{a+1} t_a,\end{aligned} \tag{25}$$

$$\widetilde{t_{a+1}}\cdot\widetilde{t_a}\cdot\widetilde{t_{a+1}}=\left(X_{a+1\,a+2}\,t_{a+1}\right)\cdot\left(X_{a\,a+1}\,t_a\right)\cdot\left(X_{a+1\,a+2}\,t_{a+1}\right)$$

$$=\left(X_{a+1\,a+2}\,X_{t_{a+1}(a)\,t_{a+1}(a+1)}\right)\cdot\left(t_{a+1}\,t_a\right)\cdot\left(X_{a+1\,a+2}\,t_{a+1}\right)$$

$$=\left(X_{a+1\,a+2}\,X_{a\,a+2}\,X_{t_{a+1}t_a(a+1)\,t_{a+1}t_a(a+2)}\right)t_{a+1}\,t_a\,t_{a+1}$$

$$=\left(X_{a+1\,a+2}\,X_{a\,a+2}\,X_{a\,a+1}\right)t_{a+1}\,t_a\,t_{a+1}$$

$$=\left(X_{a\,a+1}\,X_{a\,a+2}\,X_{a+1\,a+2}\right)t_{a+1}\,t_a\,t_{a+1}\,. \tag{26}$$

Uzimajući u obzir da je $t_a t_{a+1} t_a(k)=t_{a+1} t_a t_{a+1}(k)$ za svaki $1\le k\le n$ iz (25) i (26) proizlazi tražena relacija (23).

Slijedi dokaz relacije (24). Po definiciji je

$$t_a(k)=\begin{cases} k & 1\le k<a \text{ ili } a+1<k\le n,\\ a+1 & k=a,\\ a & k=a+1,\end{cases}$$

$$t_b(k)=\begin{cases} k & 1\le k<b \text{ ili } b+1<k\le n,\\ b+1 & k=b,\\ b & k=b+1,\end{cases}$$

stoga je za svaki $1\le a,b\le n-1$, $|a-b|\ge 2$ dobivamo

$$t_a t_b(k)=\begin{cases} t_a(k)=k & 1\le k<a,b \text{ ili } a+1,b+1<k\le n,\\ t_a(a)=a+1 & k=a,\\ t_a(a+1)=a & k=a+1,\\ t_a(b+1)=b+1 & k=b,\\ t_a(b)=b & k=b+1,\end{cases}$$

odnosno

$$t_b t_a(k)=\begin{cases} t_b(k)=k & 1\le k<a,b \text{ ili } a+1,b+1<k\le n,\\ t_b(a+1)=a+1 & k=a,\\ t_b(a)=a & k=a+1,\\ t_b(b)=b+1 & k=b,\\ t_b(b+1)=b & k=b+1,\end{cases}$$

što povlači da je $t_a t_b(k)=t_b t_a(k)$ za svaki $1\le k\le n$, $1\le a,b\le n-1$, $|a-b|\ge 2$.

Izračunajmo

$$\widetilde{t_a}\cdot\widetilde{t_b}=\left(X_{a\,a+1}\,t_a\right)\left(X_{b\,b+1}\,t_b\right)$$

$$=\left(X_{a\,a+1}\,X_{t_a(b)\,t_a(b+1)}\right)t_a\,t_b=\left(X_{a\,a+1}\,X_{b\,b+1}\right)t_a\,t_b\,, \tag{27}$$

$$\widetilde{t_b}\cdot\widetilde{t_a}=\left(X_{b\,b+1}\,t_b\right)\left(X_{a\,a+1}\,t_a\right)$$

$$=\left(X_{b\,b+1}\,X_{t_b(a)\,t_b(a+1)}\right)t_b\,t_a=\left(X_{b\,b+1}\,X_{a\,a+1}\right)t_b\,t_a\,. \tag{28}$$

Tada iz (27) i (28) uz prethodno navedeno proizlazi relacija (24).

Korolar 3.13

Neka je $g\in S_n$ bilo koja permutacija u skupu S_n i neka je $t_{b,a}$, $1\le a<b\le n$ inverz cikličke permutacije $t_{a,b}$.

Tada vrijedi

$$\widetilde{g}\cdot\widetilde{t_{b,a}}=\left(\prod_{a<j\le b,\ g(a)>g(j)}X_{\{g(j),\,g(a)\}}\right)\cdot\widetilde{gt_{b,a}}\,, \tag{29}$$

$\widetilde{g},\widetilde{t_{b,a}}\in\mathcal{A}_n$.

Dokaz:

Primjenom definicije 3.3 imamo

$$\widetilde{g}=X_g\,g=\left(\prod_{(i,j)\in I\left(g^{-1}\right)}X_{ij}\right)g=\left(\prod_{i<j,\ g^{-1}(i)>g^{-1}(j)}X_{ij}\right)g\,,$$

$$\widetilde{t_{b,a}}=X_{t_{b,a}}\,t_{b,a}=\left(\prod_{(i',j')\in I\left(t_{a,b}\right)}X_{i'j'}\right)t_{b,a}=\left(\prod_{a<j\le b,\ t_{a,b}(a)>t_{a,b}(j)}X_{aj}\right)t_{b,a}\,,\ \ 1\le a<b\le n\,.$$

Koristeći propoziciju 3.4 dobivamo

$$\widetilde{g}\cdot\widetilde{t_{b,a}}=X\left(g,t_{b,a}\right)\widetilde{gt_{b,a}}=\left(\prod_{(i,j)\in I\left(g^{-1}\right)-I\left(t_{a,b}g^{-1}\right)}X_{\{i,j\}}\right)\cdot\widetilde{gt_{b,a}}=\left(\prod_{(i,j)\in I(g)\cap I\left(t_{a,b}\right)}X_{\{g(i),\,g(j)\}}\right)\cdot\widetilde{gt_{b,a}}$$

$$=\left(\prod_{a<j\le b,\ g(a)>g(j)}X_{\{g(a),\,g(j)\}}\right)\cdot\widetilde{gt_{b,a}}$$

gdje je $X_{\{a,b\}}=X_{ab}\,X_{ba}$.

Pritom se koristilo svojstvo

$$I(g)\cap I\left(t_{a,b}\right)=\left\{(a,j)\,|\ 1\le a<j\le b\le n,\ \ g(a)>g(j)\right\},$$

pri čemu je

$$I(g)=\left\{(a',b')\,|\,1\le a'<b'\le n,\ \ g(a')>g(b')\right\},$$

$$I\left(t_{a,b}\right)=\left\{(a,j)\,|\ 1\le a<j\le b\le n,\ \ t_{a,b}(a)>t_{a,b}(j)\right\}=\left\{(a,a+1),(a,a+2),\ldots,(a,b-1),(a,b)\right\}.$$

Primjedba 3.14

Ako je $g \in S_j \times S_{n-j}$, $1 \le j \le k \le n$, onda za $\widetilde{g}, \widetilde{t_{k,j}} \in \boldsymbol{\mathcal{A}}_n$ vrijedi

$$\widetilde{g} \cdot \widetilde{t_{k,j}} = \widetilde{g t_{k,j}} . \tag{30}$$

Relacija (30) proizlazi iz primjedbe 3.5, pri čemu je $\left|I\left(g t_{k,j}\right)\right| = \left|I(g)\right| + \left|I\left(t_{k,j}\right)\right|$,
gdje je: $\left|I(g)\right| = \mathrm{Card}\left(I(g)\right)$.

Teorem 3.15 (Rastav permutacije u produkt cikličkih permutacija)

Za bilo koju permutaciju $g \in S_n$ postoje $t_{k_j,j}$, $j \le k_j \le n$ za svaki $j \in \{1, 2, ..., n\}$
tako da je

$$g = t_{k_n,n} \cdot t_{k_{n-1},n-1} \cdots t_{k_j,j} \cdots t_{k_2,2} \cdot t_{k_1,1}, \tag{31}$$

Dokaz:

Neka je $g \in S_n$ bilo koja permutacija skupu S_n.

Tražimo cikličku permutaciju $t_{k_1,1}$, $1 \le k_1 \le n$, takvu da permutacija $g t_{k_1,1}^{-1} := g_1$ fiksira 1, tj. $g_1 \in S_1 \times S_{n-1}$. Tada mora vrijediti:

$$g(k_1) = g_1\left(t_{k_1,1}(k_1)\right) = g_1(1) = 1.$$

Zaključujemo $k_1 = g^{-1}(1)$, stoga je $t_{g^{-1}(1),1}$ tražena ciklička permutacija.

Nadalje, za permutaciju $g_1 \in S_1 \times S_{n-1}$ tražimo cikličku permutaciju $t_{k_2,2}$, ($2 \le k_2 \le n$), takvu da permutacija $g_1\, t_{k_2,2}^{-1} := g_2$ fiksira 1 i 2, tj. $g_2 \in S_1 \times S_1 \times S_{n-2} = S_1^2 \times S_{n-2}$.
Tada mora vrijediti:

$$g_1(k_2) = g_2\left(t_{k_2,2}(k_2)\right) = g_2(2) = 2,$$

stoga se iz $g_1(k_2) = 2$ dobiva $k_2 = g_1^{-1}(2)$ pa je tražena ciklička permutacija dana sa $t_{k_2,2} = t_{g_1^{-1}(2),2}$.

Ponavljanjem navedenog postupka za svaki $1 \le j \le n$ zaključujemo da za permutaciju $g_{j-1} \in S_1^{j-1} \times S_{n-j+1}$ postoji ciklička permutacija $t_{k_j,j}$, ($j \le k_j \le n$) takva da za permutaciju g_j definiranu sa $g_{j-1}\, t_{k_j,j}^{-1} = g_j$ vrijedi $g_j(j) = j$. Pritom je $k_j = g_{j-1}^{-1}(j)$, stoga je tražena ciklička permutacija $t_{k_j,j}$ dana sa $t_{g_{j-1}^{-1}(j),j}$ i nužno je $g_j \in S_1^j \times S_{n-j}$.

Uzimajući u obzir navedeno dobivamo da se bilo koja permutacija $g \in S_n$ može rastaviti u produkt (31).

Komentar 3.16

Osim navedenog rastava (31), permutacija $g \in S_n$ može se rastaviti i u obliku sljedećih produkata cikličkih permutacija:

(i) $$g = t_{1,k_1} \cdot t_{2,k_2} \cdots t_{j,k_j} \cdots t_{n-1,k_{n-1}} \cdot t_{n,k_n} ,$$

gdje je $j \le k_j \le n$ za svaki $j \in \{1,2,...,n\}$;

(ii) $$g = t_{l_n,1} \cdot t_{l_{n-1},2} \cdots t_{l_j,n-j+1} \cdots t_{l_2,n-1} \cdot t_{l_1,n} ,$$

gdje je $1 \le l_j \le n-j+1$ za svaki $j \in \{1,2,...,n\}$;

(iii) $$g = t_{n,l_1} \cdot t_{n-1,l_2} \cdots t_{n-j+1,l_j} \cdots t_{2,l_{n-1}} \cdot t_{1,l_n} ,$$

gdje je $1 \le l_j \le n-j+1$ za svaki $j \in \{1,2,...,n\}$.

Obrazložimo ukratko navedene rastave:

Neka je $g \in S_n$ bilo koja permutacija.

(i) Tražimo cikličku permutaciju t_{1,k_1} ($1 \le k_1 \le n$) takvu da permutacija $t_{1,k_1}^{-1}\, g := g_1$ fiksira 1, tj. $g_1 \in S_1 \times S_{n-1}$. Tada mora vrijediti: $g(1) = t_{1,k_1}(g_1(1)) = t_{1,k_1}(1) = k_1$, stoga je tražena ciklička permutacija dana sa $t_{1,k_1} = t_{1,g(1)}$.

Ponavljanjem navedenog postupka za svaki $1 \le j \le n$ zaključujemo da za permutaciju $g_{j-1} \in S_1^{j-1} \times S_{n-j+1}$ postoji ciklička permutacija t_{j,k_j}, ($j \le k_j \le n$) takva da za permutaciju g_j definiranu sa $t_{j,k_j}^{-1}\, g_{j-1} = g_j$ vrijedi $g_j(j) = j$, tj. $g_j \in S_1^j \times S_{n-j}$. Pritom je $k_j = g_{j-1}(j)$, stoga je $t_{j,g_{j-1}(j)}$ tražena ciklička permutacija.

(ii) Tražimo cikličku permutaciju $t_{l_1,n}$, $1 \le l_1 \le n$, takvu da permutacija $g t_{l_1,n}^{-1} := g_1$ fiksira n, tj. $g_1 \in S_{n-1} \times S_1$. Tada mora vrijediti: $g(l_1) = g_1(t_{l_1,n}(l_1)) = g_1(n) = n$, iz čega se dobiva $l_1 = g^{-1}(n)$, stoga je tražena ciklička permutacija dana sa $t_{l_1,n} = t_{g^{-1}(n),n}$.

Analogno, za permutaciju $g_1 \in S_{n-1} \times S_1$ tražimo cikličku permutaciju $t_{l_2,n-1}$, ($1 \le l_2 \le n-1$), takvu da permutacija $g_1\, t_{l_2,n-1}^{-1} := g_2$ fiksira n i $(n-1)$, tj. $g_2 \in S_{n-2} \times S_1^2$. Tada mora vrijediti: $g_1(l_2) = g_2(t_{l_2,n-1}(l_2)) = g_2(n-1) = n-1$.

Zaključujemo $l_2 = g_1^{-1}(n-1)$, stoga je $t_{g_1^{-1}(n-1),n-1}$ tražena ciklička permutacija.

Ponavljanjem navedenog postupka za svaki $1 \leq j \leq n$ zaključujemo da za permutaciju $g_{j-1} \in S_{n-j+1} \times S_1^{j-1}$ postoji ciklička permutacija $t_{l_j,n-j+1}$, $(1 \leq l_j \leq n-j+1)$ takva da za permutaciju $g_{j-1}\, t_{l_j,n-j+1}^{-1} = g_j$ vrijedi $g_j(n-j+1) = n-j+1$ i nužno je $g_j \in S_{n-j} \times S_1^j$. Pritom dobivamo $l_j = g_{j-1}^{-1}(n-j+1)$ pa je $t_{l_j,n-j+1} = t_{g_{j-1}^{-1}(n-j+1),n-j+1}$ tražena ciklička permutacija.

(iii) Tražimo cikličku permutaciju t_{n,l_1}, $1 \leq l_1 \leq n$, takvu da permutacija $t_{n,l_1}^{-1}\, g := g_1$ fiksira n, tj. $g_1 \in S_{n-1} \times S_1$. Tada mora vrijediti: $g(n) = t_{n,l_1}(g_1(n)) = t_{n,l_1}(n) = l_1$ pa je tražena ciklička permutacija dana sa $t_{n,l_1} = t_{n,g(n)}$.

Ponavljanjem navedenog postupka za svaki $1 \leq j \leq n$ zaključujemo da za permutaciju $g_{j-1} \in S_{n-j+1} \times S_1^{j-1}$ postoji ciklička permutacija t_{n-j+1,l_j}, $(1 \leq l_j \leq n-j+1)$ takva da za permutaciju $t_{n-j+1,l_j}^{-1}\, g_{j-1} = g_j$ vrijedi $g_j(n-j+1) = n-j+1$ i nužno je $g_j \in S_{n-j} \times S_1^j$. Pritom dobivamo $l_j = g_{j-1}(n-j+1)$ pa je $t_{n-j+1,l_j} = t_{n-j+1,g_{j-1}(n-j+1)}$ tražena ciklička permutacija.

✐ U nastavku će se za rastav bilo koje permutacije $g \in S_n$, $n \geq 2$ u produkt cikličkih permutacija koristiti formula (31) iz teorema 3.15.

Primjer 3.17

Promatrajmo $S_3 = \{123, 132, 312, 321, 231, 213\}$ skup svih permutacija tročlanog skupa $\{1,2,3\}$. Primjenom teorema 3.15 dobivamo

$$123 = t_{3,3}t_{2,2}t_{1,1}, \qquad 321 = t_{3,3}t_{3,2}t_{3,1},$$

$$132 = t_{3,3}t_{3,2}t_{1,1}, \qquad 231 = t_{3,3}t_{2,2}t_{3,1},$$

$$312 = t_{3,3}t_{3,2}t_{2,1}, \qquad 213 = t_{3,3}t_{2,2}t_{2,1}.$$

Tada su odgovarajući elementi u $\mathcal{A}_3$ dani sa

$$\widetilde{123} = \widetilde{t_{3,3}t_{2,2}t_{1,1}} = \widetilde{t_{3,3}} \cdot \widetilde{t_{2,2}} \cdot \widetilde{t_{1,1}}, \qquad \widetilde{321} = \widetilde{t_{3,3}t_{3,2}t_{3,1}} = \widetilde{t_{3,3}} \cdot \widetilde{t_{3,2}} \cdot \widetilde{t_{3,1}},$$

$$\widetilde{132} = \widetilde{t_{3,3}t_{3,2}t_{1,1}} = \widetilde{t_{3,3}} \cdot \widetilde{t_{3,2}} \cdot \widetilde{t_{1,1}}, \qquad \widetilde{231} = \widetilde{t_{3,3}t_{2,2}t_{3,1}} = \widetilde{t_{3,3}} \cdot \widetilde{t_{2,2}} \cdot \widetilde{t_{3,1}},$$

$$\widetilde{312} = \widetilde{t_{3,3}t_{3,2}t_{2,1}} = \widetilde{t_{3,3}} \cdot \widetilde{t_{3,2}} \cdot \widetilde{t_{2,1}}, \qquad \widetilde{213} = \widetilde{t_{3,3}t_{2,2}t_{2,1}} = \widetilde{t_{3,3}} \cdot \widetilde{t_{2,2}} \cdot \widetilde{t_{2,1}}.$$

Slijedeće računanje ilustrira opću situaciju, koju ćemo kasnije koristiti u mnogim računanjima.

Promatrajmo u algebri $\mathcal{A}_3$ element $\widetilde{\alpha_3}$ takav da je $\widetilde{\alpha_3} = \sum\limits_{g \in S_3} \widetilde{g}$, odnosno

$$
\begin{aligned}
\widetilde{\alpha_3} &= \sum_{g \in S_3} \widetilde{g} = \widetilde{123} + \widetilde{132} + \widetilde{312} + \widetilde{321} + \widetilde{231} + \widetilde{213} \\
&= \widetilde{t_{3,3}} \cdot \widetilde{t_{2,2}} \cdot \widetilde{t_{1,1}} + \widetilde{t_{3,3}} \cdot \widetilde{t_{3,2}} \cdot \widetilde{t_{1,1}} + \widetilde{t_{3,3}} \cdot \widetilde{t_{3,2}} \cdot \widetilde{t_{2,1}} + \widetilde{t_{3,3}} \cdot \widetilde{t_{3,2}} \cdot \widetilde{t_{3,1}} + \widetilde{t_{3,3}} \cdot \widetilde{t_{2,2}} \cdot \widetilde{t_{3,1}} + \widetilde{t_{3,3}} \cdot \widetilde{t_{2,2}} \cdot \widetilde{t_{2,1}} \\
&= \left(\widetilde{t_{3,3}}\right) \cdot \left(\widetilde{t_{2,2}} \cdot \widetilde{t_{1,1}} + \widetilde{t_{3,2}} \cdot \widetilde{t_{1,1}} + \widetilde{t_{3,2}} \cdot \widetilde{t_{2,1}} + \widetilde{t_{3,2}} \cdot \widetilde{t_{3,1}} + \widetilde{t_{2,2}} \cdot \widetilde{t_{3,1}} + \widetilde{t_{2,2}} \cdot \widetilde{t_{2,1}}\right) \\
&= \left(\widetilde{t_{3,3}}\right) \cdot \left(\widetilde{t_{3,2}} \cdot \left(\widetilde{t_{3,1}} + \widetilde{t_{2,1}} + \widetilde{t_{1,1}}\right) + \widetilde{t_{2,2}} \cdot \left(\widetilde{t_{3,1}} + \widetilde{t_{2,1}} + \widetilde{t_{1,1}}\right)\right) \\
&= \left(\underbrace{\widetilde{t_{3,3}}}_{=\mathrm{id}}\right) \cdot \left(\widetilde{t_{3,2}} + \underbrace{\widetilde{t_{2,2}}}_{=\mathrm{id}}\right) \cdot \left(\widetilde{t_{3,1}} + \widetilde{t_{2,1}} + \underbrace{\widetilde{t_{1,1}}}_{=\mathrm{id}}\right) \\
&= \left(\widetilde{t_{3,2}} + \mathrm{id}\right) \cdot \left(\widetilde{t_{3,1}} + \widetilde{t_{2,1}} + \mathrm{id}\right) \qquad (32)
\end{aligned}
$$

i uvedimo oznake

$$
\begin{aligned}
\widetilde{\beta_1} &= \widetilde{t_{3,3}} = \mathrm{id}\,, \\
\widetilde{\beta_2} &= \widetilde{t_{3,2}} + \widetilde{t_{2,2}} = \widetilde{t_{3,2}} + \mathrm{id}\,, \\
\widetilde{\beta_3} &= \widetilde{t_{3,1}} + \widetilde{t_{2,1}} + \widetilde{t_{1,1}} = \widetilde{t_{3,1}} + \widetilde{t_{2,1}} + \mathrm{id}\,.
\end{aligned}
$$

Tada iz relacije (32) proizlazi

$$\widetilde{\alpha_3} = \widetilde{\beta_2} \cdot \widetilde{\beta_3}\,.$$

Zaključujemo sljedeće

ako u algebri $\mathcal{A}_3$ odaberemo element $\widetilde{\alpha_3}$ takav da je $\widetilde{\alpha_3} = \sum\limits_{g \in S_3} \widetilde{g}$, onda se $\widetilde{\alpha_3}$ može faktorizirati po formuli $\widetilde{\alpha_3} = \widetilde{\beta_1} \cdot \widetilde{\beta_2} \cdot \widetilde{\beta_3} = \widetilde{\beta_2} \cdot \widetilde{\beta_3}$.

Pritom su elementi $\widetilde{\beta_1}, \widetilde{\beta_2}, \widetilde{\beta_3} \in \mathcal{A}_3$ dani formulama

$$\widetilde{\beta_1} = \mathrm{id}\,, \quad \widetilde{\beta_2} = \widetilde{t_{3,2}} + \mathrm{id}\,, \quad \widetilde{\beta_3} = \widetilde{t_{3,1}} + \widetilde{t_{2,1}} + \mathrm{id}\,.$$

U nastavku će se dokazati da se $\widetilde{\alpha_n} \in \mathcal{A}_n$, $\widetilde{\alpha_n} = \sum\limits_{g \in S_n} \widetilde{g}$ može faktorizirati po formuli $\widetilde{\alpha_n} = \widetilde{\beta_2} \cdots \widetilde{\beta_{n-1}} \cdot \widetilde{\beta_n}$.

Definicija 3.18

U algebri $\mathcal{A}_n$ definiraju se elementi $\widetilde{\beta_{n-k+1}} \in \mathcal{A}_n$, $1 \le k \le n$ na sljedeći način

$$\widetilde{\beta_{n-k+1}} := \widetilde{t_{n,k}} + \widetilde{t_{n-1,k}} + \cdots + \widetilde{t_{k+1,k}} + \widetilde{t_{k,k}} \tag{33}$$

ili kraće

$$\widetilde{\beta_{n-k+1}} = \sum_{k \le s \le n}^{\leftarrow} \widetilde{t_{s,k}} = \sum_{k+1 \le s \le n}^{\leftarrow} \widetilde{t_{s,k}} + \mathrm{id}, \tag{34}$$

gdje je $\widetilde{t_{s,s}} = \mathrm{id}$ (oznaka $\overset{\leftarrow}{\sum}$ sugerira da sumande zapišemo u obrnutom redoslijedu).

U raspisanom obliku imamo

$$\widetilde{\beta_n} = \widetilde{t_{n,1}} + \widetilde{t_{n-1,1}} + \cdots + \widetilde{t_{2,1}} + \widetilde{t_{1,1}} = \widetilde{t_{n,1}} + \widetilde{t_{n-1,1}} + \cdots + \widetilde{t_{2,1}} + \mathrm{id}$$

$$\widetilde{\beta_{n-1}} = \widetilde{t_{n,2}} + \widetilde{t_{n-1,2}} + \cdots + \widetilde{t_{3,2}} + \widetilde{t_{2,2}} = \widetilde{t_{n,2}} + \widetilde{t_{n-1,2}} + \cdots + \widetilde{t_{3,2}} + \mathrm{id}$$

$$\vdots$$

$$\widetilde{\beta_{n-k+1}} = \widetilde{t_{n,k}} + \widetilde{t_{n-1,k}} + \cdots + \widetilde{t_{k+1,k}} + \widetilde{t_{k,k}} = \widetilde{t_{n,k}} + \widetilde{t_{n-1,k}} + \cdots + \widetilde{t_{k+1,k}} + \mathrm{id}$$

$$\widetilde{\beta_{n-k}} = \widetilde{t_{n,k+1}} + \widetilde{t_{n-1,k+1}} + \cdots + \widetilde{t_{k+2,k+1}} + \widetilde{t_{k+1,k+1}} = \widetilde{t_{n,k+1}} + \widetilde{t_{n-1,k+1}} + \cdots + \widetilde{t_{k+2,k+1}} + \mathrm{id}$$

$$\vdots$$

$$\widetilde{\beta_3} = \widetilde{t_{n,n-2}} + \widetilde{t_{n-1,n-2}} + \widetilde{t_{n-2,n-2}} = \widetilde{t_{n,n-2}} + \widetilde{t_{n-1,n-2}} + \mathrm{id},$$

$$\widetilde{\beta_2} = \widetilde{t_{n,n-1}} + \widetilde{t_{n-1,n-1}} = \widetilde{t_{n,n-1}} + \mathrm{id},$$

$$\widetilde{\beta_1} = \widetilde{t_{n,n}} = \mathrm{id}.$$

Propozicija 3.19

Neka je
$$\widetilde{\alpha_n} = \sum_{g \in S_n} \widetilde{g}. \tag{35}$$

Tada je

$$\widetilde{\alpha_n} = \widetilde{\beta_2} \cdots \widetilde{\beta_{n-1}} \cdot \widetilde{\beta_n} = \prod_{1 \le k \le n-1}^{\leftarrow} \widetilde{\beta_{n-k+1}}. \tag{36}$$

Pritom su elementi $\widetilde{\beta_{n-k+1}} \in \mathcal{A}_n$ definirani sa (33), tj. (34).

Dokaz:

Relacija (36) je poopćenje prethodno navedene formule (32) za $n = 3$.

Prema teoremu 3.15 o rastavu bilo koje permutacije $g \in S_n$ na produkt cikličkih permutacija imamo:

$$
\begin{aligned}
\widetilde{\alpha_n} &= \sum_{g\in S_n}\widetilde{g} = \sum_{\substack{g_1\in S_1\times S_{n-1}\\ 1\le k_1\le n}} \widetilde{g_1\, t_{k_1,1}} = \sum_{\substack{g_1\in S_1\times S_{n-1}\\ 1\le k_1\le n}} \widetilde{g_1}\,\widetilde{t_{k_1,1}} \\
&= \left(\sum_{g_1\in S_1\times S_{n-1}} \widetilde{g_1}\right)\cdot\left(\sum_{k_1=1}^{n}\widetilde{t_{k_1,1}}\right) \\
&= \left(\sum_{\substack{g_2\in S_1^2\times S_{n-2}\\ 2\le k_2\le n}} \widetilde{g_2\, t_{k_2,2}}\right)\cdot\left(\sum_{k_1=1}^{n}\widetilde{t_{k_1,1}}\right) \\
&= \left(\sum_{g_2\in S_1^2\times S_{n-2}} \widetilde{g_2}\right)\cdot\left(\sum_{k_2=2}^{n}\widetilde{t_{k_2,2}}\right)\cdot\left(\sum_{k_1=1}^{n}\widetilde{t_{k_1,1}}\right) \\
&= \cdots \\
&= \left(\sum_{g_j\in S_1^j\times S_{n-j}} \widetilde{g_j}\right)\cdot\left(\sum_{k_j=j}^{n}\widetilde{t_{k_j,j}}\right)\cdots\left(\sum_{k_2=2}^{n}\widetilde{t_{k_2,2}}\right)\cdot\left(\sum_{k_1=1}^{n}\widetilde{t_{k_1,1}}\right) \\
&= \cdots \\
&= \underbrace{\left(\underbrace{\widetilde{t_{k_n,n}}}_{=\widetilde{t_{n,n}}}\right)}_{=id}\cdot\left(\sum_{k_{n-1}=n-1}^{n}\widetilde{t_{k_{n-1},n-1}}\right)\cdots\left(\sum_{k_{n-j+1}=n-j+1}^{n}\widetilde{t_{k_{n-j+1},n-j+1}}\right)\cdots\left(\sum_{k_j=j}^{n}\widetilde{t_{k_j,j}}\right)\cdots\left(\sum_{k_2=2}^{n}\widetilde{t_{k_2,2}}\right)\cdot\left(\sum_{k_1=1}^{n}\widetilde{t_{k_1,1}}\right) \\
&= \left(\underbrace{\sum_{k_{n-1}=n-1}^{n}\widetilde{t_{k_{n-1},n-1}}}_{=\widetilde{\beta_2}}\right)\cdots\left(\underbrace{\sum_{k_{n-j+1}=n-j+1}^{n}\widetilde{t_{k_{n-j+1},n-j+1}}}_{=\widetilde{\beta_j}}\right)\cdots\left(\underbrace{\sum_{k_j=j}^{n}\widetilde{t_{k_j,j}}}_{=\widetilde{\beta_{n-j+1}}}\right)\cdots\left(\underbrace{\sum_{k_2=2}^{n}\widetilde{t_{k_2,2}}}_{=\widetilde{\beta_{n-1}}}\right)\cdot\left(\underbrace{\sum_{k_1=1}^{n}\widetilde{t_{k_1,1}}}_{=\widetilde{\beta_n}}\right) \\
&= \widetilde{\beta_2}\cdots\widetilde{\beta_{n-1}}\cdot\widetilde{\beta_n} = \overleftarrow{\prod_{1\le k\le n-1}}\widetilde{\beta_{n-k+1}}\,.
\end{aligned}
$$

U nastavku će se pokazati da se svaki element $\widetilde{\beta_{n-k+1}} \in \mathcal{A}_n$, $1 \le k \le n-1$ može dalje dekomponirati po formuli $\widetilde{\beta_{n-k+1}} = \widetilde{\delta_{n-k+1}}\cdot\widetilde{\gamma_{n-k+1}}^{-1}$ pomoću elemenata $\widetilde{\gamma_{n-k+1}}$ i $\widetilde{\delta_{n-k+1}}$ algebre $\mathcal{A}_n$, koje ćemo definirati u 3.21.

Kao pripremu navedimo neke rezultate koji će nam trebati u dokazu takve dekompozicije.

Lema 3.20

Pretpostavimo da je $1 \le k \le p < s \le n$. Tada u algebri $\mathcal{A}_n$ vrijedi relacija

$$\widetilde{t_{p,k}} \cdot \widetilde{t_{s,k}} = \widetilde{t_k}^2 \cdot \widetilde{t_{s,k+1}} \cdot \widetilde{t_{p-1,k}}. \tag{37}$$

Pritom je $\widetilde{t_k}^2 = \widetilde{t_{k+1,k}}^2 = X_{\{k,k+1\}}\,\mathrm{id}$ (vidi lemu 3.10).

Dokaz:

Primjenom propozicije 3.4 dobivamo

$$\begin{aligned}
\widetilde{t_{p,k}} \cdot \widetilde{t_{s,k}} &= \left(\prod_{(a,b) \in I(t_{p,k}) \cap I(t_{s,k}^{-1})} X_{\{t_{p,k}(a),\, t_{p,k}(b)\}} \right) \cdot \widetilde{t_{p,k} t_{s,k}} \\
&= \left(X_{\{t_{p,k}(k),\, t_{p,k}(p)\}} \right) \cdot \widetilde{t_{p,k} t_{s,k}} \\
&= \left(X_{\{k+1,k\}} \right) \cdot \widetilde{t_{p,k} t_{s,k}} \\
&= \widetilde{t_k}^2 \cdot \widetilde{t_{p,k} t_{s,k}}.
\end{aligned} \tag{38}$$

Pritom se koristilo da je

$$I(t_{p,k}) \cap I(t_{k,s}) = \{(k,p)\},$$

gdje za $1 \le k \le p < s \le n$ je

$$I(t_{p,k}) = \{(k,p),(k+1,p),\ldots,(p-2,p),(p-1,p)\},$$

$$I(t_{k,s}) = \{(k,k+1),(k,k+2),\ldots,(k,p),\ldots,(k,s-1),(k,s)\}.$$

Uzimajući u obzir da je

$$\widetilde{t_{p,k} t_{s,k}} = \left(\prod_{(a,b) \in I\left((t_{p,k} t_{s,k})^{-1}\right)} X_{ab} \right) \cdot t_{p,k} t_{s,k} = \left(\prod_{(a,b) \in I(t_{k,s} t_{k,p})} X_{ab} \right) \cdot t_{p,k} t_{s,k},$$

imamo da se izraz (38) može pisati u obliku

$$\widetilde{t_{p,k} t_{s,k}} = \widetilde{t_k}^2 \cdot \left(\prod_{(a,b) \in I(t_{k,s} t_{k,p})} X_{ab} \right) \cdot t_{p,k} t_{s,k}. \tag{39}$$

Raspišimo sada desnu stranu jednakosti (37). Primjenom propozicije 3.4 dobivamo

$$
\begin{aligned}
\widetilde{t_{s,k+1}}\cdot\widetilde{t_{p-1,k}} &= \left(\prod_{(a,b)\in I\left(t_{s,k+1}\right)\cap I\left(t_{p-1,k}^{-1}\right)} X_{\left\{t_{s,k+1}(a),\,t_{s,k+1}(b)\right\}}\right)\cdot\widetilde{t_{s,k+1}t_{p-1,k}}\\
&= \left(\prod_{(a,b)\in I\left(t_{s,k+1}\right)\cap I\left(t_{k,p-1}\right)} X_{\left\{t_{s,k+1}(a),\,t_{s,k+1}(b)\right\}}\right)\cdot\widetilde{t_{s,k+1}t_{p-1,k}}\\
&= \widetilde{t_{s,k+1}t_{p-1,k}}
\end{aligned}
$$

pri čemu se koristilo da je

$$\prod_{(a,b)\in I\left(t_{s,k+1}\right)\cap I\left(t_{k,p-1}\right)} X_{\left\{t_{s,k+1}(a),\,t_{s,k+1}(b)\right\}} = 1,$$

jer je

$$I\left(t_{s,k+1}\right)\cap I\left(t_{k,p-1}\right)=\varnothing,$$

što proizlazi iz

$$I\left(t_{s,k+1}\right)=\left\{(k+1,s),(k+2,s),\ldots,(p-1,s),(p,s),\ldots,(s-2,s),(s-1,s)\right\},$$

$$I\left(t_{k,p-1}\right)=\left\{(k,k+1),(k,k+2),\ldots,(k,p-2),(k,p-1)\right\},$$

gdje je $1\le k\le p<s\le n$.

Time dobivamo

$$
\begin{aligned}
\widetilde{t_{s,k+1}}\cdot\widetilde{t_{p-1,k}} = \widetilde{t_{s,k+1}t_{p-1,k}} &= \left(\prod_{(a,b)\in I\left(\left(t_{s,k+1}t_{p-1,k}\right)^{-1}\right)} X_{ab}\right)\cdot t_{s,k+1}t_{p-1,k}\\
&= \left(\prod_{(a,b)\in I\left(t_{k,p-1}t_{k+1,s}\right)} X_{ab}\right)\cdot t_{s,k+1}t_{p-1,k}\,. \qquad (40)
\end{aligned}
$$

Sada ćemo usporediti permutacije $t_{p,k}t_{s,k}$ i $t_{s,k+1}t_{p-1,k}$.

Imamo da za $1\le k\le p<s\le n$ vrijedi

$$t_{p,k}=\begin{pmatrix} k & k+1 & \cdots & p-2 & p-1 & p & p+1 & \cdots & s-1 & s\\ k+1 & k+2 & \cdots & p-1 & p & k & p+1 & \cdots & s-1 & s\end{pmatrix},$$

$$t_{s,k}=\begin{pmatrix} k & k+1 & \cdots & p-2 & p-1 & p & p+1 & \cdots & s-1 & s\\ k+1 & k+2 & \cdots & p-1 & p & p+1 & p+2 & \cdots & s & k\end{pmatrix},$$

$$t_{s,k+1}=\begin{pmatrix} k & k+1 & \cdots & p-2 & p-1 & p & p+1 & \cdots & s-1 & s\\ k & k+2 & \cdots & p-1 & p & p+1 & p+2 & \cdots & s & k+1\end{pmatrix},$$

$$t_{p-1,k}=\begin{pmatrix} k & k+1 & \cdots & p-2 & p-1 & p & p+1 & \cdots & s-1 & s\\ k+1 & k+2 & \cdots & p-1 & k & p & p+1 & \cdots & s-1 & s\end{pmatrix},$$

stoga je

$$t_{p,k}t_{s,k}=\begin{pmatrix} k & k+1 & \cdots & p-2 & p-1 & p & p+1 & \cdots & s-1 & s \\ k+2 & k+3 & \cdots & p & k & p+1 & p+2 & \cdots & s & k+1 \end{pmatrix},$$

odnosno

$$t_{s,k+1}t_{p-1,k}=\begin{pmatrix} k & k+1 & \cdots & p-2 & p-1 & p & p+1 & \cdots & s-1 & s \\ k+2 & k+3 & \cdots & p & k & p+1 & p+2 & \cdots & s & k+1 \end{pmatrix},$$

što povlači da je

$$t_{p,k}t_{s,k}=t_{s,k+1}t_{p-1,k}.$$

Iz jednakosti cikličkih permutacija proizlazi jednakost njihovih skupova inverzija, tj.

$$I\left(t_{p,k}t_{s,k}\right)=I\left(t_{s,k+1}t_{p-1,k}\right),$$

a samim time vrijedi da je

$$I\left(\left(t_{p,k}t_{s,k}\right)^{-1}\right)=I\left(\left(t_{s,k+1}t_{p-1,k}\right)^{-1}\right)$$

odnosno

$$I\left(t_{k,s}t_{k,p}\right)=I\left(t_{k,p-1}t_{k+1,s}\right)$$

gdje je $t_{a,b}=t_{b,a}^{-1}$, $1\le k\le p<s\le n$.

Na osnovu rečenog iz (40) proizlazi

$$\widetilde{t_{s,k+1}}\cdot\widetilde{t_{p-1,k}}=\left(\prod_{(a,b)\in I\left(t_{k,s}t_{k,p}\right)}X_{ab}\right)\cdot t_{p,k}t_{s,k},$$

što uvrštavanjem u (39) dokazuje tvrdnju (37).

Definicija 3.21

Za svaki $1\le k\le n-1$ u algebri $\mathcal{A}_n$ definiramo elemente $\widetilde{\gamma_{n-k+1}}$ i $\widetilde{\delta_{n-k+1}}$ na sljedeći način

$$\widetilde{\gamma_{n-k+1}}:=\left(id-\widetilde{t_{n,k}}\right)\cdot\left(id-\widetilde{t_{n-1,k}}\right)\cdots\left(id-\widetilde{t_{k+2,k}}\right)\cdot\left(id-\widetilde{t_{k+1,k}}\right), \tag{41}$$

$$\widetilde{\delta_{n-k+1}}:=\left(id-\widetilde{t_k}^{2}\,\widetilde{t_{n,k+1}}\right)\cdot\left(id-\widetilde{t_k}^{2}\,\widetilde{t_{n-1,k+1}}\right)\cdots\left(id-\widetilde{t_k}^{2}\,\widetilde{t_{k+2,k+1}}\right)\cdot\left(id-\widetilde{t_k}^{2}\,\widetilde{t_{k+1,k+1}}\right) \tag{42}$$

Pritom je $\widetilde{t_k}^{2}=\widetilde{t_{k+1,k}}^{2}=X_{\{k,k+1\}}\,id$.

Propozicija 3.22

Za elemente $\widetilde{\beta_{n-k+1}} = \widetilde{t_{n,k}} + \widetilde{t_{n-1,k}} + \cdots + \widetilde{t_{k+1,k}} + \mathrm{id} \qquad (1 \le k \le n-1)$

definirane u 3.18, vrijedi faktorizacija

$$\widetilde{\beta_{n-k+1}} = \widetilde{\delta_{n-k+1}} \cdot \widetilde{\gamma_{n-k+1}}^{-1} . \tag{43}$$

Dokaz:

Primijetimo da se elementi $\widetilde{\gamma_{n-k+1}}$ i $\widetilde{\delta_{n-k+1}}$, $1 \le k \le n-1$ definirani u 3.21 mogu kraće zapisati u obliku

$$\widetilde{\gamma_{n-k+1}} = \overleftarrow{\prod_{k+1 \le s \le n}} \left(\mathrm{id} - \widetilde{t_{s,k}}\right),$$

$$\widetilde{\delta_{n-k+1}} = \overleftarrow{\prod_{k+1 \le s \le n}} \left(\mathrm{id} - \widetilde{t_k}^{\,2}\, \widetilde{t_{s,k+1}}\right).$$

Element $\widetilde{\beta_{n-k+1}}$ iz definicije 3.18 možemo kraće zapisati

$$\widetilde{\beta_{n-k+1}} = \overleftarrow{\sum_{k \le p \le n}} \widetilde{t_{p,k}} .$$

Neka je $k \le s \le n$. Uvodimo oznaku

$$\widetilde{\beta_{n-k+1,\,s}} := \overleftarrow{\sum_{k \le p \le s}} \widetilde{t_{p,k}} . \tag{44}$$

Specijalno iz (44) proizlazi $\widetilde{\beta_{n-k+1,\,k}} = \overleftarrow{\sum_{k \le p \le k}} \widetilde{t_{p,k}} = \widetilde{t_{k,k}} = \mathrm{id}$ ako je $s = k$,

$$\widetilde{\beta_{n-k+1,\,n}} = \overleftarrow{\sum_{k \le p \le n}} \widetilde{t_{p,k}} = \widetilde{\beta_{n-k+1}} \qquad \text{ako je } s = n .$$

Pretpostavimo da je $s \ge k+1$ fiksiran. Tada primjenom (44) dobivamo

$$\begin{aligned}
\widetilde{\beta_{n-k+1,\,s}} \cdot \left(\mathrm{id} - \widetilde{t_{s,k}}\right) &= \widetilde{\beta_{n-k+1,\,s}} - \widetilde{\beta_{n-k+1,\,s}} \cdot \widetilde{t_{s,k}} \\
&= \overleftarrow{\sum_{k \le p \le s}} \widetilde{t_{p,k}} - \overleftarrow{\sum_{k \le p \le s}} \widetilde{t_{p,k}} \cdot \widetilde{t_{s,k}} \\
&= \widetilde{t_{s,k}} + \overleftarrow{\sum_{k \le p \le s-1}} \widetilde{t_{p,k}} - \overleftarrow{\sum_{k+1 \le p \le s}} \widetilde{t_{p,k}} \cdot \widetilde{t_{s,k}} - \underbrace{\widetilde{t_{k,k}}}_{=\mathrm{id}} \cdot \widetilde{t_{s,k}} \\
&= \overleftarrow{\sum_{k \le p \le s-1}} \widetilde{t_{p,k}} - \overleftarrow{\sum_{k+1 \le p \le s}} \widetilde{t_{p,k}} \cdot \widetilde{t_{s,k}}
\end{aligned}$$

$$\overset{\text{lema 3.20}}{=} \overleftarrow{\sum_{k\le p\le s-1}} \widetilde{t_{p,k}} - \overleftarrow{\sum_{k+1\le p\le s}} \widetilde{t_k}^{\,2} \cdot \widetilde{t_{s,k+1}} \cdot \widetilde{t_{p-1,k}}$$

$$= \overleftarrow{\sum_{k\le p\le s-1}} \widetilde{t_{p,k}} - \overleftarrow{\sum_{k\le p\le s-1}} \widetilde{t_k}^{\,2} \cdot \widetilde{t_{s,k+1}} \cdot \widetilde{t_{p,k}}$$

$$= \overleftarrow{\sum_{k\le p\le s-1}} \left(\mathrm{id} - \widetilde{t_k}^{\,2}\widetilde{t_{s,k+1}}\right)\cdot \widetilde{t_{p,k}}$$

$$= \left(\mathrm{id} - \widetilde{t_k}^{\,2}\widetilde{t_{s,k+1}}\right)\cdot \overleftarrow{\sum_{k\le p\le s-1}} \widetilde{t_{p,k}}$$

$$= \left(\mathrm{id} - \widetilde{t_k}^{\,2}\widetilde{t_{s,k+1}}\right)\cdot \widetilde{\beta_{n-k+1,\, s-1}}\,.$$

Time imamo

$$\widetilde{\beta_{n-k+1,\, s}}\cdot\left(\mathrm{id} - \widetilde{t_{s,k}}\right) = \left(\mathrm{id} - \widetilde{t_k}^{\,2}\widetilde{t_{s,k+1}}\right)\cdot \widetilde{\beta_{n-k+1,\, s-1}}\,. \tag{45}$$

Indukcijom po $k+1\le s\le n$ iz dobivene jednakosti (45) proizlazi

$$\left.\begin{array}{l}
\widetilde{\beta_{n-k+1,\, k+1}}\cdot\left(\mathrm{id} - \widetilde{t_{k+1,k}}\right) = \left(\mathrm{id} - \widetilde{t_k}^{\,2}\widetilde{t_{k+1,k+1}}\right)\cdot \underbrace{\widetilde{\beta_{n-k+1,\, k}}}_{=\mathrm{id}}\,, \\[2ex]
\widetilde{\beta_{n-k+1,\, k+2}}\cdot\left(\mathrm{id} - \widetilde{t_{k+2,k}}\right) = \left(\mathrm{id} - \widetilde{t_k}^{\,2}\widetilde{t_{k+2,k+1}}\right)\cdot \widetilde{\beta_{n-k+1,\, k+1}}\,, \\
\qquad\vdots \\
\widetilde{\beta_{n-k+1,\, n-1}}\cdot\left(\mathrm{id} - \widetilde{t_{n-1,k}}\right) = \left(\mathrm{id} - \widetilde{t_k}^{\,2}\widetilde{t_{n-1,k+1}}\right)\cdot \widetilde{\beta_{n-k+1,\, n-2}}\,,
\end{array}\right\} \tag{46}$$

$$\widetilde{\beta_{n-k+1,\, n}}\cdot\left(\mathrm{id} - \widetilde{t_{n,k}}\right) = \left(\mathrm{id} - \widetilde{t_k}^{\,2}\widetilde{t_{n,k+1}}\right)\cdot \widetilde{\beta_{n-k+1,\, n-1}}\,. \tag{47}$$

Pomnožimo li sdesna izraz (47) sa $\left(\mathrm{id} - \widetilde{t_{n-1,k}}\right)\cdots\left(\mathrm{id} - \widetilde{t_{k+2,k}}\right)\cdot\left(\mathrm{id} - \widetilde{t_{k+1,k}}\right)$ dobivamo

$$\widetilde{\beta_{n-k+1,\, n}}\cdot\left(\mathrm{id} - \widetilde{t_{n,k}}\right)\cdot\left(\mathrm{id} - \widetilde{t_{n-1,k}}\right)\cdots\left(\mathrm{id} - \widetilde{t_{k+2,k}}\right)\cdot\left(\mathrm{id} - \widetilde{t_{k+1,k}}\right)$$

$$= \left(\mathrm{id} - \widetilde{t_k}^{\,2}\widetilde{t_{n,k+1}}\right)\cdot \widetilde{\beta_{n-k+1,\, n-1}}\cdot\left(\mathrm{id} - \widetilde{t_{n-1,k}}\right)\cdots\left(\mathrm{id} - \widetilde{t_{k+2,k}}\right)\cdot\left(\mathrm{id} - \widetilde{t_{k+1,k}}\right)$$

odakle primjenom jednakosti (46) proizlazi da je

$$\widetilde{\beta_{n-k+1,\, n}}\cdot\left(\mathrm{id} - \widetilde{t_{n,k}}\right)\cdot\left(\mathrm{id} - \widetilde{t_{n-1,k}}\right)\cdots\left(\mathrm{id} - \widetilde{t_{k+2,k}}\right)\cdot\left(\mathrm{id} - \widetilde{t_{k+1,k}}\right)$$

$$= \left(\mathrm{id} - \widetilde{t_k}^{\,2}\;\widetilde{t_{n,k+1}}\right)\cdot\left(\mathrm{id} - \widetilde{t_k}^{\,2}\;\widetilde{t_{n-1,k+1}}\right)\cdots\left(\mathrm{id} - \widetilde{t_k}^{\,2}\;\widetilde{t_{k+2,k+1}}\right)\cdot\left(\mathrm{id} - \widetilde{t_k}^{\,2}\;\widetilde{t_{k+1,k+1}}\right)$$

ili

$$\widetilde{\beta_{n-k+1}} \cdot \prod_{k+1\le s\le n}^{\leftarrow}\left(\mathrm{id}-\widetilde{t_{s,k}}\right)=\prod_{k+1\le s\le n}^{\leftarrow}\left(\mathrm{id}-\widetilde{t_k}^{\,2}\,\widetilde{t_{s,k+1}}\right)$$

(gdje je $\widetilde{\beta_{n-k+1,\,n}}=\widetilde{\beta_{n-k+1}}$), što povlači da je

$$\widetilde{\beta_{n-k+1}}\cdot\widetilde{\gamma_{n-k+1}}=\widetilde{\delta_{n-k+1}}\,,$$

odnosno

$$\widetilde{\beta_{n-k+1}}=\widetilde{\delta_{n-k+1}}\cdot\widetilde{\gamma_{n-k+1}}^{\,-1},\quad 1\le k\le n-1.$$

Time je dokazana propozicija 3.22.

Lema 3.23

Neka je $w_n = n(n-1)(n-2)\cdots 21$ najdulja permutacija u S_n i neka je $g\in S_n$ bilo koja permutacija u S_n.

Tada u algebri $\mathcal{A}_n$ imamo elemente $\widetilde{w_n}$ i $\widetilde{g}$ za koje vrijedi

$$\widetilde{gw_n}\cdot\widetilde{w_n}=\widetilde{w_n}\cdot\widetilde{w_ng}=\left(\prod_{i<j,\ g^{-1}(i)<g^{-1}(j)} X_{\{i,j\}}\right)\widetilde{g} \tag{48}$$

Dokaz:

Primijetimo da je najdulja permutacija $w_n\in S_n$ definirana sa

$$w_n=\begin{pmatrix}1 & 2 & 3 & \cdots & n-2 & n-1 & n\\ n & n-1 & n-2 & \cdots & 3 & 2 & 1\end{pmatrix},$$

odnosno

$$w_n(k)=n-k+1,\quad 1\le k\le n.$$

Pritom vrijede svojstva

(*i*) $\quad w_n^{-1}=w_n$,

(*ii*) $\quad w_n w_n = w_n^{-1}w_n^{-1}=\mathrm{id}$,

(*iii*) $\quad I(w_n)=I(w_n^{-1})=\{(a,b)\mid 1\le a<b\le n\}$,

(*iv*) $\quad I(w_n w_n)=I(w_n^{-1}w_n^{-1})=\varnothing$,

(*v*) $\quad w_n = t_{1,n}\,t_{2,n}\,t_{3,n}\cdots t_{n-2,n}\,t_{n-1,n}$.

Neka je $g\in S_n$ bilo koja permutacija iz skupa S_n.

Tada primjenom propozicije 3.4 dobivamo

$$\begin{aligned}
\widetilde{gw_n}\cdot\widetilde{w_n} &= X(gw_n, w_n)\cdot\widetilde{gw_nw_n} \\
&= X(gw_n, w_n)\cdot\widetilde{g} \\
&= \left(\prod_{(a,b)\in I(w_n^{-1}g^{-1})-I(w_n^{-1}w_n^{-1}g^{-1})} X_{\{a,b\}}\right)\widetilde{g} \\
&= \left(\prod_{(a,b)\in I(w_n^{-1}g^{-1})-I(g^{-1})} X_{\{a,b\}}\right)\widetilde{g} \\
&= \left(\underbrace{\prod_{(a,b)\in I(g^{-1})\cap I(w_n^{-1}g^{-1})} X_{\{a,b\}}^{-1}}_{=1}\prod_{(a,b)\in I(w_n^{-1}g^{-1})} X_{\{a,b\}}\right)\widetilde{g} \\
&= \left(\prod_{(a,b)\in I(w_n^{-1}g^{-1})} X_{\{a,b\}}\right)\widetilde{g}\,.
\end{aligned}$$

Uzimajući u obzir da je

$$(w_n^{-1}g^{-1})(k) = w_n^{-1}(g^{-1}(k)) = n - g^{-1}(k) + 1\,,$$

imamo da je

$$\begin{aligned}
I(w_n^{-1}g^{-1}) &= \{(a,b)\,|\,1\le a<b\le n,\ \ (n-g^{-1}(a)+1) > (n-g^{-1}(b)+1)\} \\
&= \{(a,b)\,|\,1\le a<b\le n,\ \ g^{-1}(a) < g^{-1}(b)\},
\end{aligned}$$

što povlači
$$\widetilde{gw_n}\cdot\widetilde{w_n} = \left(\prod_{a<b,\ g^{-1}(a)<g^{-1}(b)} X_{\{a,b\}}\right)\widetilde{g}\,. \tag{49}$$

Napomena:

Primijetimo da vrijedi $\displaystyle\prod_{(a,b)\in I(g^{-1})\cap I(w_n^{-1}g^{-1})} X_{\{a,b\}}^{-1} = 1$,

jer je

$$I(g^{-1}) = \{(a,b)\,|\,1\le a<b\le n,\ \ g^{-1}(a) > g^{-1}(b)\},$$

$$\begin{aligned}
I(w_n^{-1}g^{-1}) &= \{(a,b)\,|\,1\le a<b\le n,\ \ (n-g^{-1}(a)+1) > (n-g^{-1}(b)+1)\} \\
&= \{(a,b)\,|\,1\le a<b\le n,\ \ g^{-1}(a) < g^{-1}(b)\},
\end{aligned}$$

odnosno

$$I(g^{-1})\cap I(w_n^{-1}g^{-1}) = \varnothing\,.$$

S druge strane je

$$\begin{aligned}\widetilde{w_n}\cdot\widetilde{w_n g} &= X(w_n, w_n g)\cdot\widetilde{w_n w_n g}\\ &= X(w_n, w_n g)\cdot\widetilde{g}\\ &= \left(\prod_{(a,b)\in I(w_n^{-1})-I(g^{-1}w_n^{-1}w_n^{-1})} X_{\{a,b\}}\right)\widetilde{g}\\ &= \left(\prod_{(a,b)\in I(w_n^{-1})-I(g^{-1})} X_{\{a,b\}}\right)\widetilde{g}\,.\end{aligned}$$

Pritom je

$$\begin{aligned}I(w_n^{-1})-I(g^{-1}) &= \{(a,b)\,|\,1\le a<b\le n\}\setminus\{(a,b)\,|\,1\le a<b\le n,\ g^{-1}(a)>g^{-1}(b)\}\\ &= \{(a,b)\,|\,1\le a<b\le n,\ g^{-1}(a)<g^{-1}(b)\}\end{aligned}$$

pa je
$$\widetilde{w_n}\cdot\widetilde{w_n g} = \left(\prod_{a<b,\ g^{-1}(a)<g^{-1}(b)} X_{\{a,b\}}\right)\widetilde{g}\,. \tag{50}$$

Iz dobivenih izraza (49) i (50) proizlazi jednakost (48) za bilo koji $g\in S_n$ i najdulju permutaciju $w_n = n(n-1)(n-2)\cdots 21$ u skupu S_n.

Napomena:

Primjenom svojstva (*i*) na svojsvo (*v*) najdulje permutacije $w_n\in S_n$ dobivamo

$$w_n = t_{1,n}\,t_{2,n}\,t_{3,n}\cdots t_{n-2,n}\,t_{n-1,n} = t_{n-1,n}^{-1}\,t_{n-2,n}^{-1}\cdots t_{3,n}^{-1}\,t_{2,n}^{-1}\,t_{1,n}^{-1}\,,$$

odnosno
$$w_n = t_{n,n-1}\,t_{n,n-2}\cdots t_{n,3}\,t_{n,2}\,t_{n,1}\,. \tag{51}$$

Pritom se koristila definicija 3.9.

Rezimirajmo prethodno izloženo.

U algebri $\mathcal{A}_n$ imamo element $\widetilde{\alpha_n} = \sum_{g\in S_n}\widetilde{g}$. Propozicijom 3.19 dokazali smo da se element $\widetilde{\alpha_n}$ može faktorizirati elementima $\widetilde{\beta_{n-k+1}}\in\mathcal{A}_n$, $1\le k\le n-1$ formulom $\widetilde{\alpha_n} = \overleftarrow{\prod_{1\le k\le n-1}}\widetilde{\beta_{n-k+1}}$, gdje su elementi $\widetilde{\beta_{n-k+1}}$ definirani u 3.18 na sljedeći način $\widetilde{\beta_{n-k+1}} = \overleftarrow{\sum_{k\le s\le n}}\widetilde{t_{s,k}} = \overleftarrow{\sum_{k+1\le s\le n}}\widetilde{t_{s,k}} + \mathrm{id}$. Pritom je $\widetilde{\beta_1} = \mathrm{id}$.

Nadalje, primjenom propozicije 3.22 imamo da se svaki element $\widetilde{\beta_{n-k+1}}$ može faktorizirati po formuli $\widetilde{\beta_{n-k+1}} = \widetilde{\delta_{n-k+1}} \cdot \widetilde{\gamma_{n-k+1}}^{-1}$, gdje je (prema definiciji 3.21)

$$\widetilde{\gamma_{n-k+1}} = \prod_{k+1\le s\le n}^{\leftarrow} \left(\mathrm{id} - \widetilde{t_{s,k}}\right),$$

$$\widetilde{\delta_{n-k+1}} = \prod_{k+1\le s\le n}^{\leftarrow} \left(\mathrm{id} - \widetilde{t_{k+1,k}}^{2}\ \widetilde{t_{s,k+1}}\right).$$

Imamo da se element $\widetilde{\beta_{n-k+1}}$, $1 \le k \le n-1$ može faktorizirati primjenom formule

$$\widetilde{\beta_{n-k+1}} = \prod_{k+1\le s\le n}^{\leftarrow} \left(\mathrm{id} - \widetilde{t_{k+1,k}}^{2}\ \widetilde{t_{s,k+1}}\right) \cdot \left[\prod_{k+1\le s\le n}^{\leftarrow} \left(\mathrm{id} - \widetilde{t_{s,k}}\right)\right]^{-1}$$

ili

$$\widetilde{\beta_{n-k+1}} = \prod_{k+1\le s\le n}^{\leftarrow} \left(\mathrm{id} - \widetilde{t_{k+1,k}}^{2}\ \widetilde{t_{s,k+1}}\right) \cdot \prod_{k+1\le s\le n}^{\rightarrow} \left(\mathrm{id} - \widetilde{t_{s,k}}\right)^{-1}, \tag{52}$$

stoga se element $\widetilde{\alpha_n} \in \mathcal{A}_n$, $\widetilde{\alpha_n} = \sum_{g\in S_n} \widetilde{g}$ može faktorizirati na sljedeći način

$$\widetilde{\alpha_n} = \prod_{1\le k\le n-1}^{\leftarrow} \widetilde{\beta_{n-k+1}} = \prod_{1\le k\le n-1}^{\leftarrow} \left(\prod_{k+1\le s\le n}^{\leftarrow} \left(\mathrm{id} - \widetilde{t_{k+1,k}}^{2}\ \widetilde{t_{s,k+1}}\right) \cdot \prod_{k+1\le s\le n}^{\rightarrow} \left(\mathrm{id} - \widetilde{t_{s,k}}\right)^{-1} \right). \tag{53}$$

Slijedi detaljniji raspis elemenata $\widetilde{\beta_{n-k+1}}$, $\widetilde{\gamma_{n-k+1}}$ i $\widetilde{\delta_{n-k+1}}$, $1 \le k \le n-1$ s ciljem da se bolje razumije formula (52), a samim time i formula (53).

Neka je n fiksiran.

Tada za $k = 1$ imamo:

$$\widetilde{\beta_n} = \widetilde{t_{n,1}} + \widetilde{t_{n-1,1}} + \cdots + \widetilde{t_{3,1}} + \widetilde{t_{2,1}} + \mathrm{id},$$

$$\widetilde{\gamma_n} = \left(\mathrm{id} - \widetilde{t_{n,1}}\right)\cdot\left(\mathrm{id} - \widetilde{t_{n-1,1}}\right)\cdots\left(\mathrm{id} - \widetilde{t_{3,1}}\right)\cdot\left(\mathrm{id} - \widetilde{t_{2,1}}\right),$$

$$\widetilde{\delta_n} = \left(\mathrm{id} - \widetilde{t_{2,1}}^{2}\ \widetilde{t_{n,2}}\right)\cdot\left(\mathrm{id} - \widetilde{t_{2,1}}^{2}\ \widetilde{t_{n-1,2}}\right)\cdots\left(\mathrm{id} - \widetilde{t_{2,1}}^{2}\ \widetilde{t_{3,2}}\right)\cdot\left(\mathrm{id} - \widetilde{t_{2,1}}^{2}\ \widetilde{t_{2,2}}\right),$$

stoga je

$$\boxed{\begin{aligned} \widetilde{\beta_n} = {} & \left(\mathrm{id} - \widetilde{t_{2,1}}^{2}\ \widetilde{t_{n,2}}\right)\cdot\left(\mathrm{id} - \widetilde{t_{2,1}}^{2}\ \widetilde{t_{n-1,2}}\right)\cdots\left(\mathrm{id} - \widetilde{t_{2,1}}^{2}\ \widetilde{t_{3,2}}\right)\cdot\left(\mathrm{id} - \widetilde{t_{2,1}}^{2}\ \widetilde{t_{2,2}}\right)\cdot \\ & \cdot\left(\mathrm{id} - \widetilde{t_{2,1}}\right)^{-1}\cdot\left(\mathrm{id} - \widetilde{t_{3,1}}\right)^{-1}\cdots\left(\mathrm{id} - \widetilde{t_{n-1,1}}\right)^{-1}\cdot\left(\mathrm{id} - \widetilde{t_{n,1}}\right)^{-1}. \end{aligned}} \tag{54}$$

Ako je $k=2$, onda je $n-k+1=n-1$ i vrijedi:

$$\widetilde{\beta_{n-1}}=\widetilde{t_{n,2}}+\widetilde{t_{n-1,2}}+\cdots+\widetilde{t_{4,2}}+\widetilde{t_{3,2}}+\mathrm{id}\,,$$

$$\widetilde{\gamma_{n-1}}=\left(\mathrm{id}-\widetilde{t_{n,2}}\right)\cdot\left(\mathrm{id}-\widetilde{t_{n-1,2}}\right)\cdots\left(\mathrm{id}-\widetilde{t_{4,2}}\right)\cdot\left(\mathrm{id}-\widetilde{t_{3,2}}\right),$$

$$\widetilde{\delta_{n-1}}=\left(\mathrm{id}-\widetilde{t_{3,2}}^{\,2}\,\widetilde{t_{n,3}}\right)\cdot\left(\mathrm{id}-\widetilde{t_{3,2}}^{\,2}\,\widetilde{t_{n-1,3}}\right)\cdots\left(\mathrm{id}-\widetilde{t_{3,2}}^{\,2}\,\widetilde{t_{4,3}}\right)\cdot\left(\mathrm{id}-\widetilde{t_{3,2}}^{\,2}\,\widetilde{t_{3,3}}\right)$$

pa je

$$\boxed{\begin{aligned}\widetilde{\beta_{n-1}}=&\left(\mathrm{id}-\widetilde{t_{3,2}}^{\,2}\,\widetilde{t_{n,3}}\right)\cdot\left(\mathrm{id}-\widetilde{t_{3,2}}^{\,2}\,\widetilde{t_{n-1,3}}\right)\cdots\left(\mathrm{id}-\widetilde{t_{3,2}}^{\,2}\,\widetilde{t_{4,3}}\right)\cdot\left(\mathrm{id}-\widetilde{t_{3,2}}^{\,2}\,\widetilde{t_{3,3}}\right)\cdot\left(\mathrm{id}-\widetilde{t_{3,2}}\right)^{-1}\cdot\\&\cdot\left(\mathrm{id}-\widetilde{t_{4,2}}\right)^{-1}\cdots\left(\mathrm{id}-\widetilde{t_{n-1,2}}\right)^{-1}\cdot\left(\mathrm{id}-\widetilde{t_{n,2}}\right)^{-1}.\end{aligned}}$$

Ako je $k=p-1$, onda je $n-k+1=n-p+2$ i vrijedi:

$$\widetilde{\beta_{n-p+2}}=\widetilde{t_{n,p-1}}+\widetilde{t_{n-1,p-1}}+\cdots+\widetilde{t_{p+1,p-1}}+\widetilde{t_{p,p-1}}+\mathrm{id}\,,$$

$$\widetilde{\gamma_{n-p+2}}=\left(\mathrm{id}-\widetilde{t_{n,p-1}}\right)\cdot\left(\mathrm{id}-\widetilde{t_{n-1,p-1}}\right)\cdots\left(\mathrm{id}-\widetilde{t_{p+1,p-1}}\right)\cdot\left(\mathrm{id}-\widetilde{t_{p,p-1}}\right),$$

$$\widetilde{\delta_{n-p+2}}=\left(\mathrm{id}-\widetilde{t_{p,p-1}}^{\,2}\,\widetilde{t_{n,p}}\right)\cdot\left(\mathrm{id}-\widetilde{t_{p,p-1}}^{\,2}\,\widetilde{t_{n-1,p}}\right)\cdots\left(\mathrm{id}-\widetilde{t_{p,p-1}}^{\,2}\,\widetilde{t_{p+1,p}}\right)\cdot\left(\mathrm{id}-\widetilde{t_{p,p-1}}^{\,2}\,\widetilde{t_{p,p}}\right),$$

stoga je

$$\boxed{\begin{aligned}\widetilde{\beta_{n-p+2}}=&\left(\mathrm{id}-\widetilde{t_{p,p-1}}^{\,2}\,\widetilde{t_{n,p}}\right)\cdot\left(\mathrm{id}-\widetilde{t_{p,p-1}}^{\,2}\,\widetilde{t_{n-1,p}}\right)\cdots\left(\mathrm{id}-\widetilde{t_{p,p-1}}^{\,2}\,\widetilde{t_{p+1,p}}\right)\cdot\left(\mathrm{id}-\widetilde{t_{p,p-1}}^{\,2}\,\widetilde{t_{p,p}}\right)\cdot\\&\cdot\left(\mathrm{id}-\widetilde{t_{p,p-1}}\right)^{-1}\cdot\left(\mathrm{id}-\widetilde{t_{p+1,p-1}}\right)^{-1}\cdots\left(\mathrm{id}-\widetilde{t_{n-1,p-1}}\right)^{-1}\cdot\left(\mathrm{id}-\widetilde{t_{n,p-1}}\right)^{-1}.\end{aligned}}$$

Analogno ako je $k=p$, onda je $n-k+1=n-p+1$ i vrijedi:

$$\widetilde{\beta_{n-p+1}}=\widetilde{t_{n,p}}+\widetilde{t_{n-1,p}}+\cdots+\widetilde{t_{p+2,p}}+\widetilde{t_{p+1,p}}+\mathrm{id}\,,$$

$$\widetilde{\gamma_{n-p+1}}=\left(\mathrm{id}-\widetilde{t_{n,p}}\right)\cdot\left(\mathrm{id}-\widetilde{t_{n-1,p}}\right)\cdots\left(\mathrm{id}-\widetilde{t_{p+2,p}}\right)\cdot\left(\mathrm{id}-\widetilde{t_{p+1,p}}\right),$$

$$\widetilde{\delta_{n-p+1}}=\left(\mathrm{id}-\widetilde{t_{p+1,p}}^{\,2}\,\widetilde{t_{n,p+1}}\right)\cdot\left(\mathrm{id}-\widetilde{t_{p+1,p}}^{\,2}\,\widetilde{t_{n-1,p+1}}\right)\cdots\left(\mathrm{id}-\widetilde{t_{p+1,p}}^{\,2}\,\widetilde{t_{p+2,p+1}}\right)\cdot\left(\mathrm{id}-\widetilde{t_{p+1,p}}^{\,2}\,\widetilde{t_{p+1,p+1}}\right)$$

pa je

$$\boxed{\begin{aligned}\widetilde{\beta_{n-p+1}}=&\left(\mathrm{id}-\widetilde{t_{p+1,p}}^{\,2}\,\widetilde{t_{n,p+1}}\right)\cdot\left(\mathrm{id}-\widetilde{t_{p+1,p}}^{\,2}\,\widetilde{t_{n-1,p+1}}\right)\cdots\left(\mathrm{id}-\widetilde{t_{p+1,p}}^{\,2}\,\widetilde{t_{p+2,p+1}}\right)\cdot\\&\cdot\left(\mathrm{id}-\widetilde{t_{p+1,p}}^{\,2}\,\widetilde{t_{p+1,p+1}}\right)\cdot\left(\mathrm{id}-\widetilde{t_{p+1,p}}\right)^{-1}\cdot\left(\mathrm{id}-\widetilde{t_{p+2,p}}\right)^{-1}\cdots\left(\mathrm{id}-\widetilde{t_{n-1,p}}\right)^{-1}\cdot\left(\mathrm{id}-\widetilde{t_{n,p}}\right)^{-1}.\end{aligned}}$$

Navedimo još, ako je $k = n - 2$, onda je $n - k + 1 = 3$ te je:

$$\widetilde{\beta_3} = \widetilde{t_{n,n-2}} + \widetilde{t_{n-1,n-2}} + \mathrm{id},$$

$$\widetilde{\gamma_3} = \left(\mathrm{id} - \widetilde{t_{n,n-2}}\right) \cdot \left(\mathrm{id} - \widetilde{t_{n-1,n-2}}\right),$$

$$\widetilde{\delta_3} = \left(\mathrm{id} - \widetilde{t_{n-1,n-2}}^{\,2}\ \widetilde{t_{n,n-1}}\right) \cdot \left(\mathrm{id} - \widetilde{t_{n-1,n-2}}^{\,2}\ \widetilde{t_{n-1,n-1}}\right)$$

pa je

$$\boxed{\widetilde{\beta_3} = \left(\mathrm{id} - \widetilde{t_{n-1,n-2}}^{\,2}\ \widetilde{t_{n,n-1}}\right) \cdot \left(\mathrm{id} - \widetilde{t_{n-1,n-2}}^{\,2}\ \widetilde{t_{n-1,n-1}}\right) \cdot \left(\mathrm{id} - \widetilde{t_{n-1,n-2}}\right)^{-1} \cdot \left(\mathrm{id} - \widetilde{t_{n,n-2}}\right)^{-1},}$$

odnosno ako je $k = n - 1$, onda je $n - k + 1 = 2$ i vrijedi:

$$\widetilde{\beta_2} = \widetilde{t_{n,n-1}} + \mathrm{id},$$

$$\widetilde{\gamma_2} = \mathrm{id} - \widetilde{t_{n,n-1}},$$

$$\widetilde{\delta_2} = \mathrm{id} - \widetilde{t_{n,n-1}}^{\,2}\ \widetilde{t_{n,n}},$$

tj.

$$\boxed{\widetilde{\beta_2} = \left(\mathrm{id} - \widetilde{t_{n,n-1}}^{\,2}\ \widetilde{t_{n,n}}\right) \cdot \left(\mathrm{id} - \widetilde{t_{n,n-1}}\right)^{-1}.}$$

Napomena:

Primjenom tvrdnje *(ii)* iz leme 3.11 imamo da se svaki element $\widetilde{t_{b,a}} \in \boldsymbol{\mathcal{A}}_n$, $1 \le a < b \le n$ može dekomponirati pomoću elementarnih transpozicija $\widetilde{t_j}$, $a \le j \le b-1$ na sljedeći način $\widetilde{t_{b,a}} = \widetilde{t_a} \cdot \widetilde{t_{a+1}} \cdots \widetilde{t_{b-2}} \cdot \widetilde{t_{b-1}} = \prod_{j=a}^{b-1} \widetilde{t_j}$, stoga možemo pisati

$$\widetilde{\beta_{n-k+1}} = \overleftarrow{\sum_{k+1 \le s \le n}} \left(\prod_{j=k}^{s-1} \widetilde{t_j} \right) + \mathrm{id}, \tag{55}$$

$$\widetilde{\gamma_{n-k+1}} = \overleftarrow{\prod_{k+1 \le s \le n}} \left(\mathrm{id} - \prod_{j=k}^{s-1} \widetilde{t_j} \right), \tag{56}$$

$$\widetilde{\delta_{n-k+1}} = \overleftarrow{\prod_{k+1 \le s \le n}} \left(\mathrm{id} - \widetilde{t_{k+1,k}}^{\,2} \cdot \prod_{j=k+1}^{s-1} \widetilde{t_j} \right) = \overleftarrow{\prod_{k+1 \le s \le n}} \left(\mathrm{id} - \widetilde{t_k}^{\,2} \cdot \prod_{j=k+1}^{s-1} \widetilde{t_j} \right). \tag{57}$$

ili u raspisanom obliku

$$\widetilde{\beta_{n-k+1}} = \left(\widetilde{t_k} \cdot \widetilde{t_{k+1}} \cdots \widetilde{t_{n-1}}\right) + \left(\widetilde{t_k} \cdot \widetilde{t_{k+1}} \cdots \widetilde{t_{n-2}}\right) + \cdots + \left(\widetilde{t_k} \cdot \widetilde{t_{k+1}}\right) + \widetilde{t_k} + \mathrm{id}$$

$$\widetilde{\gamma_{n-k+1}} = \left(\mathrm{id} - \widetilde{t_k} \cdot \widetilde{t_{k+1}} \cdots \widetilde{t_{n-1}}\right) \cdot \left(\mathrm{id} - \widetilde{t_k} \cdot \widetilde{t_{k+1}} \cdots \widetilde{t_{n-2}}\right) \cdots \left(\mathrm{id} - \widetilde{t_k} \cdot \widetilde{t_{k+1}}\right) \cdot \left(\mathrm{id} - \widetilde{t_k}\right)$$

$$\widetilde{\delta_{n-k+1}} = \left(\mathrm{id} - \widetilde{t_k}^{\,2} \cdot \widetilde{t_{k+1}} \cdot \widetilde{t_{k+2}} \cdots \widetilde{t_{n-1}}\right) \cdot \left(\mathrm{id} - \widetilde{t_k}^{\,2} \cdot \widetilde{t_{k+1}} \cdot \widetilde{t_{k+2}} \cdots \widetilde{t_{n-2}}\right) \cdots \left(\mathrm{id} - \widetilde{t_k}^{\,2} \cdot \widetilde{t_{k+1}}\right) \cdot \left(\mathrm{id} - \widetilde{t_k}^{\,2}\right).$$

Pritom za $s = k+1$ je $\prod_{j=k+1}^{k} \widetilde{t_j} := id$.

U ovom slučaju imamo da se izraz (52) može pisati u obliku

$$\widetilde{\beta_{n-k+1}} = \overleftarrow{\prod_{k+1 \le s \le n}} \left(id - \widetilde{t_k}^2 \cdot \prod_{j=k+1}^{s-1} \widetilde{t_j} \right) \cdot \overrightarrow{\prod_{k+1 \le s \le n}} \left(id - \prod_{j=k}^{s-1} \widetilde{t_j} \right)^{-1} \tag{58}$$

ili raspisano

$$\widetilde{\beta_{n-k+1}} = \left(id - \widetilde{t_k}^2 \cdot \widetilde{t_{k+1}} \cdot \widetilde{t_{k+2}} \cdots \widetilde{t_{n-1}} \right) \cdot \left(id - \widetilde{t_k}^2 \cdot \widetilde{t_{k+1}} \cdot \widetilde{t_{k+2}} \cdots \widetilde{t_{n-2}} \right) \cdots \left(id - \widetilde{t_k}^2 \cdot \widetilde{t_{k+1}} \right) \cdot \left(id - \widetilde{t_k}^2 \right)$$
$$\cdot \left(id - \widetilde{t_k} \right)^{-1} \cdot \left(id - \widetilde{t_k} \cdot \widetilde{t_{k+1}} \right)^{-1} \cdots \left(id - \widetilde{t_k} \cdot \widetilde{t_{k+1}} \cdots \widetilde{t_{n-2}} \right)^{-1} \cdot \left(id - \widetilde{t_k} \cdot \widetilde{t_{k+1}} \cdots \widetilde{t_{n-1}} \right)^{-1}.$$

Koristeći (58) imamo da se element $\widetilde{\alpha_n} \in \mathcal{A}_n$ dan izrazom (53) može pisati u obliku

$$\widetilde{\alpha_n} = \overleftarrow{\prod_{1 \le k \le n-1}} \widetilde{\beta_{n-k+1}} = \overleftarrow{\prod_{1 \le k \le n-1}} \left(\overleftarrow{\prod_{k+1 \le s \le n}} \left(id - \widetilde{t_k}^2 \cdot \prod_{j=k+1}^{s-1} \widetilde{t_j} \right) \cdot \overrightarrow{\prod_{k+1 \le s \le n}} \left(id - \prod_{j=k}^{s-1} \widetilde{t_j} \right)^{-1} \right). \tag{59}$$

Primjer 3.24

Slijedi prikaz faktorizacije elementa $\widetilde{\alpha_n} \in \mathcal{A}_n$, $\widetilde{\alpha_n} = \sum_{g \in S_n} \widetilde{g}$ za $n = 2,3,4$ primjenom formule (53), odnosno formule (59).

Neka je $n = 2$. Tada specijalno u algebri $\mathcal{A}_2$ imamo da je element $\widetilde{\alpha_2} = \sum_{g \in S_2} \widetilde{g}$ jednak elementu $\widetilde{\beta_2} = \widetilde{t_{2,1}} + id$ (definicija 3.18), tj. $\widetilde{\alpha_2} = \widetilde{\beta_2}$.

Primjenom definicije 3.21 proizlazi $\widetilde{\gamma_2} = id - \widetilde{t_{2,1}}$, $\widetilde{\delta_2} = id - \widetilde{t_{2,1}}^2\, \widetilde{t_{2,2}} = id - \widetilde{t_{2,1}}^2$, stoga u ovom slučaju imamo da je formula (52), tj. (53) oblika

$$\widetilde{\alpha_2} = \widetilde{\beta_2} = \left(id - \widetilde{t_{2,1}}^2 \right) \cdot \left(id - \widetilde{t_{2,1}} \right)^{-1}. \tag{60}$$

S druge strane, uzimajući u obzir da je $\widetilde{t_{2,1}} = \widetilde{t_1}$, tj. $\widetilde{t_{2,1}}^2 = \widetilde{t_1}^2$ imamo da se izraz (60) može pisati u obliku

$$\widetilde{\alpha_2} = \widetilde{\beta_2} = \left(id - \widetilde{t_1}^2 \right) \cdot \left(id - \widetilde{t_1} \right)^{-1},$$

što ujedno direktno proizlazi iz formule (58), tj. (59) za $n = 2$, $k = 1$.

Neka je $n=3$. Tada je $1 \le k \le 2$. U algebri $\mathcal{A}_3$ imamo element $\widetilde{\alpha_3} = \sum_{g \in S_3} \widetilde{g}$, koji se faktorizira elementima $\widetilde{\beta_2}, \widetilde{\beta_3}$ formulom $\widetilde{\alpha_3} = \widetilde{\beta_2} \cdot \widetilde{\beta_3}$ (propozicija 3.19). Primjenom formule (52) dobivamo

$$\widetilde{\beta_3} = \left(\mathrm{id} - \widetilde{t_{2,1}}^2 \widetilde{t_{3,2}}\right) \cdot \left(\mathrm{id} - \widetilde{t_{2,1}}^2\right) \cdot \left(\mathrm{id} - \widetilde{t_{2,1}}\right)^{-1} \cdot \left(\mathrm{id} - \widetilde{t_{3,1}}\right)^{-1} \qquad \text{za } k=1,$$

$$\widetilde{\beta_2} = \left(\mathrm{id} - \widetilde{t_{3,2}}^2\right) \cdot \left(\mathrm{id} - \widetilde{t_{3,2}}\right)^{-1} \qquad \text{za } k=2,$$

pri čemu se koristilo da je $\widetilde{t_{3,2}}^2\, \widetilde{t_{3,3}} = \widetilde{t_{3,2}}^2$, $\widetilde{t_{2,1}}^2\, \widetilde{t_{2,2}} = \widetilde{t_{2,1}}^2$, jer je $\widetilde{t_{3,3}} = \widetilde{t_{2,2}} = \mathrm{id}$.

Time dobivamo

$$\widetilde{\alpha_3} = \widetilde{\beta_2} \cdot \widetilde{\beta_3} = \left(\mathrm{id} - \widetilde{t_{3,2}}^2\right) \cdot \left(\mathrm{id} - \widetilde{t_{3,2}}\right)^{-1} \cdot \left(\mathrm{id} - \widetilde{t_{2,1}}^2 \widetilde{t_{3,2}}\right) \cdot \left(\mathrm{id} - \widetilde{t_{2,1}}^2\right) \cdot \left(\mathrm{id} - \widetilde{t_{2,1}}\right)^{-1} \cdot \left(\mathrm{id} - \widetilde{t_{3,1}}\right)^{-1},$$

što direktno proizlazi iz formule (53).

Nadalje, u ovom slučaju imamo da je formula (59) oblika

$$\widetilde{\alpha_3} = \left(\mathrm{id} - \widetilde{t}_2^{\,2}\right) \cdot \left(\mathrm{id} - \widetilde{t}_2\right)^{-1} \cdot \left(\mathrm{id} - \widetilde{t}_1^{\,2} \widetilde{t}_2\right) \cdot \left(\mathrm{id} - \widetilde{t}_1^{\,2}\right) \cdot \left(\mathrm{id} - \widetilde{t}_1\right)^{-1} \cdot \left(\mathrm{id} - \widetilde{t}_1 \widetilde{t}_2\right)^{-1}.$$

Neka je $n=4$. Tada je $1 \le k \le 3$. U algebri $\mathcal{A}_4$ imamo element $\widetilde{\alpha_4} = \sum_{g \in S_4} \widetilde{g}$, koji se primjenom formule (53) može faktorizirati na sljedeći način

$$\begin{aligned}
\widetilde{\alpha_4} &= \widetilde{\beta_2} \cdot \widetilde{\beta_3} \cdot \widetilde{\beta_4} \\
&= \left(\mathrm{id} - \widetilde{t_{4,3}}^2\right) \cdot \left(\mathrm{id} - \widetilde{t_{4,3}}\right)^{-1} \cdot \left(\mathrm{id} - \widetilde{t_{3,2}}^2 \widetilde{t_{4,3}}\right) \cdot \left(\mathrm{id} - \widetilde{t_{3,2}}^2\right) \cdot \left(\mathrm{id} - \widetilde{t_{3,2}}\right)^{-1} \cdot \left(\mathrm{id} - \widetilde{t_{4,2}}\right)^{-1} \cdot \\
&\quad \cdot \left(\mathrm{id} - \widetilde{t_{2,1}}^2 \widetilde{t_{4,2}}\right) \cdot \left(\mathrm{id} - \widetilde{t_{2,1}}^2 \widetilde{t_{3,2}}\right) \cdot \left(\mathrm{id} - \widetilde{t_{2,1}}^2\right) \cdot \left(\mathrm{id} - \widetilde{t_{2,1}}\right)^{-1} \cdot \left(\mathrm{id} - \widetilde{t_{3,1}}\right)^{-1} \cdot \left(\mathrm{id} - \widetilde{t_{4,1}}\right)^{-1}
\end{aligned}$$

ili ekvivalentno primjenom formule (59)

$$\begin{aligned}
\widetilde{\alpha_4} &= \left(\mathrm{id} - \widetilde{t}_3^{\,2}\right) \cdot \left(\mathrm{id} - \widetilde{t}_3\right)^{-1} \cdot \left(\mathrm{id} - \widetilde{t}_2^{\,2} \widetilde{t}_3\right) \cdot \left(\mathrm{id} - \widetilde{t}_2^{\,2}\right) \cdot \left(\mathrm{id} - \widetilde{t}_2\right)^{-1} \cdot \left(\mathrm{id} - \widetilde{t}_2 \widetilde{t}_3\right)^{-1} \cdot \\
&\quad \cdot \left(\mathrm{id} - \widetilde{t}_1^{\,2} \widetilde{t}_2 \widetilde{t}_3\right) \cdot \left(\mathrm{id} - \widetilde{t}_1^{\,2} \widetilde{t}_2\right) \cdot \left(\mathrm{id} - \widetilde{t}_1^{\,2}\right) \cdot \left(\mathrm{id} - \widetilde{t}_1\right)^{-1} \cdot \left(\mathrm{id} - \widetilde{t}_1 \widetilde{t}_2\right)^{-1} \cdot \left(\mathrm{id} - \widetilde{t}_1 \widetilde{t}_2 \widetilde{t}_3\right)^{-1}.
\end{aligned}$$

Pritom se elementi

$$\widetilde{\beta_4} = \widetilde{t_{4,1}} + \widetilde{t_{3,1}} + \widetilde{t_{2,1}} + \mathrm{id} = \widetilde{t}_1 \widetilde{t}_2 \widetilde{t}_3 + \widetilde{t}_1 \widetilde{t}_2 + \widetilde{t}_1 + \mathrm{id},$$

$$\widetilde{\beta_3} = \widetilde{t_{4,2}} + \widetilde{t_{3,2}} + \mathrm{id} = \widetilde{t}_2 \widetilde{t}_3 + \widetilde{t}_2 + \mathrm{id},$$

$$\widetilde{\beta_2} = \widetilde{t_{4,3}} + \mathrm{id} = \widetilde{t}_3 + \mathrm{id},$$

primjenom formule (52), tj. (58) faktoriziraju ovako

$$\widetilde{\beta_4} = \left(\mathrm{id} - \widetilde{t_{2,1}}^2 \widetilde{t_{4,2}}\right)\cdot\left(\mathrm{id} - \widetilde{t_{2,1}}^2 \widetilde{t_{3,2}}\right)\cdot\left(\mathrm{id} - \widetilde{t_{2,1}}^2\right)\cdot\left(\mathrm{id} - \widetilde{t_{2,1}}\right)^{-1}\cdot\left(\mathrm{id} - \widetilde{t_{3,1}}\right)^{-1}\cdot\left(\mathrm{id} - \widetilde{t_{4,1}}\right)^{-1}$$
$$= \left(\mathrm{id} - \widetilde{t_1}^2 \widetilde{t_2}\widetilde{t_3}\right)\cdot\left(\mathrm{id} - \widetilde{t_1}^2 \widetilde{t_2}\right)\cdot\left(\mathrm{id} - \widetilde{t_1}^2\right)\cdot\left(\mathrm{id} - \widetilde{t_1}\right)^{-1}\cdot\left(\mathrm{id} - \widetilde{t_1}\widetilde{t_2}\right)^{-1}\cdot\left(\mathrm{id} - \widetilde{t_1}\widetilde{t_2}\widetilde{t_3}\right)^{-1},$$

$$\widetilde{\beta_3} = \left(\mathrm{id} - \widetilde{t_{3,2}}^2 \widetilde{t_{4,3}}\right)\cdot\left(\mathrm{id} - \widetilde{t_{3,2}}^2\right)\cdot\left(\mathrm{id} - \widetilde{t_{3,2}}\right)^{-1}\cdot\left(\mathrm{id} - \widetilde{t_{4,2}}\right)^{-1}$$
$$= \left(\mathrm{id} - \widetilde{t_2}^2 \widetilde{t_3}\right)\cdot\left(\mathrm{id} - \widetilde{t_2}^2\right)\cdot\left(\mathrm{id} - \widetilde{t_2}\right)^{-1}\cdot\left(\mathrm{id} - \widetilde{t_2}\widetilde{t_3}\right)^{-1},$$

$$\widetilde{\beta_2} = \left(\mathrm{id} - \widetilde{t_{4,3}}^2\right)\cdot\left(\mathrm{id} - \widetilde{t_{4,3}}\right)^{-1} = \left(\mathrm{id} - \widetilde{t_3}^2\right)\cdot\left(\mathrm{id} - \widetilde{t_3}\right)^{-1}.$$

Uočimo da je

$$\widetilde{\gamma_4} = \left(\mathrm{id} - \widetilde{t_{4,1}}\right)\cdot\left(\mathrm{id} - \widetilde{t_{3,1}}\right)\cdot\left(\mathrm{id} - \widetilde{t_{2,1}}\right) = \left(\mathrm{id} - \widetilde{t_1}\widetilde{t_2}\widetilde{t_3}\right)\cdot\left(\mathrm{id} - \widetilde{t_1}\widetilde{t_2}\right)\cdot\left(\mathrm{id} - \widetilde{t_1}\right),$$

$$\widetilde{\delta_4} = \left(\mathrm{id} - \widetilde{t_{2,1}}^2 \widetilde{t_{4,2}}\right)\cdot\left(\mathrm{id} - \widetilde{t_{2,1}}^2 \widetilde{t_{3,2}}\right)\cdot\left(\mathrm{id} - \widetilde{t_{2,1}}^2\right) = \left(\mathrm{id} - \widetilde{t_1}^2 \widetilde{t_2}\widetilde{t_3}\right)\cdot\left(\mathrm{id} - \widetilde{t_1}^2 \widetilde{t_2}\right)\cdot\left(\mathrm{id} - \widetilde{t_1}^2\right),$$

$$\widetilde{\gamma_3} = \left(\mathrm{id} - \widetilde{t_{4,2}}\right)\cdot\left(\mathrm{id} - \widetilde{t_{3,2}}\right) = \left(\mathrm{id} - \widetilde{t_2}\widetilde{t_3}\right)\cdot\left(\mathrm{id} - \widetilde{t_2}\right),$$

$$\widetilde{\delta_3} = \left(\mathrm{id} - \widetilde{t_{3,2}}^2 \widetilde{t_{4,3}}\right)\cdot\left(\mathrm{id} - \widetilde{t_{3,2}}^2\right) = \left(\mathrm{id} - \widetilde{t_2}^2 \widetilde{t_3}\right)\cdot\left(\mathrm{id} - \widetilde{t_2}^2\right),$$

$$\widetilde{\gamma_2} = \mathrm{id} - \widetilde{t_{4,3}} = \mathrm{id} - \widetilde{t_3}, \qquad \widetilde{\delta_2} = \mathrm{id} - \widetilde{t_{4,3}}^2 = \mathrm{id} - \widetilde{t_3}^2.$$

Vratimo se na proučavanje cikličkih permutacija $t_{a,b}$, $1 \le a \le b \le n$ i njihovih inverza $t_{b,a} = t_{a,b}^{-1}$ u skupu S_n. Pritom je

$$t_{a,b}(p) = \begin{cases} p & 1 \le p < a \text{ ili } b < p \le n \\ b & p = a \\ p-1 & a < p \le b \end{cases}$$

odnosno

$$t_{b,a}(p) = \begin{cases} p & 1 \le p < a \text{ ili } b < p \le n \\ p+1 & a \le p < b \\ a & p = b \end{cases}$$

stoga je

$$t_{n,1}(p) = \begin{cases} p+1 & 1 \le p < n \\ 1 & p = n, \end{cases}$$

$$t_{n,1}^{-1}(p) = t_{1,n}(p) = \begin{cases} n & p = 1 \\ p-1 & 1 < p \le n. \end{cases}$$

Lako se pokaže da vrijedi $t_{n,1} \cdot t_{b,a} = t_{b+1,a+1} \cdot t_{n,1}$, $1 \le a \le b \le n$, što je analogno zapisu

$$t_{b,a} = t_{n,1}^{-1} \cdot t_{b+1,a+1} \cdot t_{n,1}. \tag{61}$$

Lema 3.25

Vrijede sljedeće jednakosti

$$\text{(i)}\qquad \widetilde{t_{b,a}} = t_{n,1}^{-1}\cdot \widetilde{t_{b+1,a+1}}\cdot t_{n,1} \qquad 1\le a\le b\le n, \tag{62}$$

$$\text{(ii)}\qquad X_{\{a,\,a+1\}}\,\mathrm{id} = t_{n,1}^{-1}\cdot X_{\{a+1,a+2\}}\cdot t_{n,1} \qquad 1\le a\le n-1. \tag{63}$$

Dokaz:

(i) Neka je $1\le a\le b\le n$. Uzimajući u obzir formulu (19) imamo da je

$\widetilde{t_{b,a}} = \left(\prod_{j=a+1}^{b} X_{aj}\right)t_{b,a}$ i analogno $\widetilde{t_{b+1,a+1}} = \left(\prod_{j=a+2}^{b+1} X_{a+1\,j}\right)t_{b+1,a+1}$. Dobivamo

$$\begin{aligned} t_{n,1}^{-1}\cdot \widetilde{t_{b+1,a+1}}\cdot t_{n,1} &= t_{n,1}^{-1}\cdot\left(\prod_{j=a+2}^{b+1} X_{a+1\,j}\right)t_{b+1,a+1}\cdot t_{n,1} \\ &= \left(\prod_{j=a+1}^{b} X_{aj}\right)t_{n,1}^{-1}\cdot t_{b+1,a+1}\cdot t_{n,1} = \left(\prod_{j=a+1}^{b} X_{aj}\right)t_{b,a} = \widetilde{t_{b,a}} \end{aligned}$$

čime je dokazana jednakost (62). Pritom se koristilo svojstvo (61).

(ii) Jednakost (63) proizlazi direktno primjenom definicije cikličke permutacije $t_{n,1}^{-1}$, tj. imamo $\quad t_{n,1}^{-1}\cdot X_{\{a+1,a+2\}}\cdot t_{n,1} = X_{\{a,a+1\}}\cdot \underbrace{t_{n,1}^{-1}\,t_{n,1}}_{=\mathrm{id}} = X_{\{a,\,a+1\}}\,\mathrm{id}$,

što dokazuje jednakost (63). Pritom je $1\le a\le n-1$, $X_{\{a,a+1\}} = X_{a\,a+1}\cdot X_{a+1\,a}$.

Primijetimo da se jednakost (63) može pisati u obliku $\widetilde{t_a}^{\,2} = t_{n,1}^{-1}\cdot \widetilde{t_{a+1}}^{\,2}\cdot t_{n,1}$, gdje je $\widetilde{t_a}^{\,2} = X_{\{a,\,a+1\}}\,\mathrm{id}$ (lema 3.10).

Primjedba 3.26

Uzimajući u obzir lemu 3.10 imamo da se element $\widetilde{\delta_{n-k+1}} \in \mathcal{A}_n$, $1\le k\le n-1$ definiran u 3.21 može pisati u obliku

$$\begin{aligned} \widetilde{\delta_{n-k+1}} &= \left(\mathrm{id} - X_{\{k,k+1\}}\,\widetilde{t_{n,k+1}}\right)\cdot\left(\mathrm{id} - X_{\{k,k+1\}}\,\widetilde{t_{n-1,k+1}}\right)\cdots\left(\mathrm{id} - X_{\{k,k+1\}}\,\widetilde{t_{k+2,k+1}}\right)\cdot\left(\mathrm{id} - X_{\{k,k+1\}}\,\widetilde{t_{k+1,k+1}}\right) \\ &= \prod_{k+1\le s\le n}^{\leftarrow}\left(\mathrm{id} - X_{\{k,k+1\}}\,\widetilde{t_{s,k+1}}\right) \end{aligned} \tag{64}$$

gdje je $\widetilde{t_{k+1,k+1}} = \mathrm{id}$.

Neka je $g \in S_n$ bilo koja permutacija u S_n. Uvodimo oznaku $\mathrm{Des}(g)$ za skup padova permutacije g na sljedeći način

$$\mathrm{Des}(g) := \{1 \le i \le n-1 \mid g(i) > g(i+1)\}. \tag{65}$$

Pritom $\mathrm{des}(g) = \mathrm{Card}(\mathrm{Des}(g))$ označava broj padova.

Propozicija 3.27

Neka je n fiksiran, $1 \le k \le n-1$ i neka je

$$\Delta_{n-k+1}^{n} := \left(id - X_{\{k,k+1\}}\right)\cdot\left(id - X_{\{k,k+1,k+2\}}\right)\cdots\left(id - X_{\{k,k+1,\dots,n\}}\right), \tag{66}$$

$$\widetilde{\varepsilon_{n-k+1}^{n}} := \sum_{g \in S_1^k \times S_{n-k}} \omega_{n-k+1}^{n}(g)\cdot \tilde{g}, \tag{67}$$

gdje je za $g \in S_1^k \times S_{n-k}$:

$$\omega_{n-k+1}^{n}(g) := \prod_{i \in \mathrm{Des}(g^{-1})} X_{\{k,k+1,\dots,i\}}, \tag{68}$$

$$\mathrm{Des}(g^{-1}) = \{k+1 \le i \le n-1 \mid g^{-1}(i) > g^{-1}(i+1)\}. \tag{69}$$

Uvedimo precizniju oznaku

$$\widetilde{\delta_{n-k+1}^{n}} := \widetilde{\delta_{n-k+1}}. \tag{70}$$

Tada se inverz elementa $\widetilde{\delta_{n-k+1}^{n}} \in \mathcal{A}_n$, $1 \le k \le n-1$ označava sa $\widetilde{\delta_{n-k+1}^{n}}^{-1}$ i računa po formuli

$$\boxed{\widetilde{\delta_{n-k+1}^{n}}^{-1} = \left(\Delta_{n-k+1}^{n}\right)^{-1}\cdot \widetilde{\varepsilon_{n-k+1}^{n}}}. \tag{71}$$

Napomena:

Neka je $A = \{k, k+1, \dots, i\}$, $i \in \mathrm{Des}(g^{-1})$.

Tada primjenom definicije 3.2 imamo da je faktor X_A u izrazu (68) dan eksplicitno sa $X_A = \prod_{(a,b) \in A \times A,\ a<b} X_{\{a,b\}}$, gdje je $X_{\{a,b\}} = X_{ab} \cdot X_{ba}$.

Dokaz:

Koristeći oznaku (70) imamo da je izraz (64) dan sa

$$\widetilde{\delta^{n}_{n-k+1}} = \overleftarrow{\prod_{k+1\le s\le n}} \left(id - X_{\{k,k+1\}}\,\widetilde{t_{s,k+1}}\right)$$

$$= \left(id - X_{\{k,k+1\}}\,\widetilde{t_{n,k+1}}\right)\cdot \overleftarrow{\prod_{k+1\le s\le n-1}} \left(id - X_{\{k,k+1\}}\,\widetilde{t_{s,k+1}}\right)$$

ili

$$\widetilde{\delta^{n}_{n-k+1}} = \left(id - X_{\{k,k+1\}}\,\widetilde{t_{n,k+1}}\right)\cdot \widetilde{\delta^{n-1}_{n-k+1}}\,, \tag{72}$$

gdje je

$$\widetilde{\delta^{n-1}_{n-k+1}} = \overleftarrow{\prod_{k+1\le s\le n-1}} \left(id - X_{\{k,k+1\}}\,\widetilde{t_{s,k+1}}\right). \tag{73}$$

Primjenom leme 3.25, prema kojoj imamo da je

$$\widetilde{t_{s,k+1}} = t_{n,1}^{-1}\cdot \widetilde{t_{s+1,k+2}}\cdot t_{n,1}\,,$$

$$X_{\{k,k+1\}}\,id = t_{n,1}^{-1}\cdot X_{\{k+1,k+2\}}\cdot t_{n,1}\,,$$

dobivamo da se izraz (73) može pisati u obliku

$$\widetilde{\delta^{n-1}_{n-k+1}} = \overleftarrow{\prod_{k+1\le s\le n-1}} \left(id - X_{\{k,k+1\}}\,\widetilde{t_{s,k+1}}\right)$$

$$= \overleftarrow{\prod_{k+1\le s\le n-1}} \left(id - t_{n,1}^{-1}\cdot X_{\{k+1,k+2\}}\cdot \underbrace{t_{n,1}\cdot t_{n,1}^{-1}}_{=id}\cdot \widetilde{t_{s+1,k+2}}\cdot t_{n,1}\right)$$

$$= \overleftarrow{\prod_{k+1\le s\le n-1}} t_{n,1}^{-1}\cdot\left(id - X_{\{k+1,k+2\}}\widetilde{t_{s+1,k+2}}\right)\cdot t_{n,1}$$

$$= t_{n,1}^{-1}\cdot\left(\overleftarrow{\prod_{k+1\le s\le n-1}} \left(id - X_{\{k+1,k+2\}}\widetilde{t_{s+1,k+2}}\right)\right)\cdot t_{n,1}$$

$$= t_{n,1}^{-1}\cdot\left(\overleftarrow{\prod_{k+2\le s\le n}} \left(id - X_{\{k+1,k+2\}}\widetilde{t_{s,k+2}}\right)\right)\cdot t_{n,1}\,,$$

tj.

$$\widetilde{\delta^{n-1}_{n-k+1}} = t_{n,1}^{-1}\cdot\left(\overleftarrow{\prod_{k+2\le s\le n}} \left(id - X_{\{k+1,k+2\}}\widetilde{t_{s,k+2}}\right)\right)\cdot t_{n,1}\,. \tag{74}$$

S druge strane, primjenom oznake (70) i izraza (64), možemo pisati

$$\widetilde{\delta^{n}_{n-k}} = \overleftarrow{\prod_{k+2\le s\le n}} \left(id - X_{\{k+1,k+2\}}\widetilde{t_{s,k+2}}\right),$$

što povlači da se izraz (74) može kraće zapisati u obliku

$$\widetilde{\delta^{n-1}_{n-k+1}} = t^{-1}_{n,1} \cdot \widetilde{\delta^{n}_{n-k}} \cdot t_{n,1} \,. \tag{75}$$

Uvrstimo li sada (75) u izraz (72) dobivamo

$$\widetilde{\delta^{n}_{n-k+1}} = \left(\mathrm{id} - X_{\{k,k+1\}}\ \widetilde{t_{n,k+1}}\right) \cdot t^{-1}_{n,1} \cdot \widetilde{\delta^{n}_{n-k}} \cdot t_{n,1}$$

odnosno

$$\widetilde{\delta^{n}_{n-k+1}}^{-1} \cdot \left(\mathrm{id} - X_{\{k,k+1\}}\ \widetilde{t_{n,k+1}}\right) = t^{-1}_{n,1} \cdot \widetilde{\delta^{n}_{n-k}}^{-1} \cdot t_{n,1} \,. \tag{76}$$

Primijetimo da će formula (71) vrijediti, ako vrijedi

$$\left(\Delta^{n}_{n-k+1}\right)^{-1} \cdot \widetilde{\varepsilon^{n}_{n-k+1}} \cdot \left(\mathrm{id} - X_{\{k,k+1\}}\ \widetilde{t_{n,k+1}}\right) = t^{-1}_{n,1} \cdot \left(\Delta^{n}_{n-k}\right)^{-1} \cdot \widetilde{\varepsilon^{n}_{n-k}} \cdot t_{n,1} \,,$$

(što se dobiva primjenom formule (71) na identitet (76)),

odnosno ako vrijedi identitet

$$\widetilde{\varepsilon^{n}_{n-k+1}} \cdot \left(\mathrm{id} - X_{\{k,k+1\}}\ \widetilde{t_{n,k+1}}\right) = \left(\mathrm{id} - X_{\{k,k+1,\ldots,n\}}\right) \cdot \widetilde{\varepsilon^{n-1}_{n-k+1}} \,, \tag{77}$$

gdje je

$$\widetilde{\varepsilon^{n-1}_{n-k+1}} = t^{-1}_{n,1} \cdot \widetilde{\varepsilon^{n}_{n-k}} \cdot t_{n,1} \,,$$

$$\mathrm{id} - X_{\{k,k+1,\ldots,n\}} = \Delta^{n}_{n-k+1} \cdot t^{-1}_{n,1} \cdot \left(\Delta^{n}_{n-k}\right)^{-1} \cdot t_{n,1} \,.$$

Napomena:

Primijetimo da je

$$\begin{aligned} t^{-1}_{n,1} \cdot \left(\Delta^{n}_{n-k}\right)^{-1} \cdot t_{n,1} &= t^{-1}_{n,1} \cdot \left[\left(\mathrm{id} - X_{\{k+1,k+2\}}\right) \cdot \left(\mathrm{id} - X_{\{k+1,k+2,k+3\}}\right) \cdots \left(\mathrm{id} - X_{\{k+1,k+2,\ldots,n\}}\right)\right]^{-1} \cdot t_{n,1} \\ &= \left[\left(\mathrm{id} - X_{\{k,k+1\}}\right) \cdot \left(\mathrm{id} - X_{\{k,k+1,k+2\}}\right) \cdots \left(\mathrm{id} - X_{\{k,k+1,\ldots,n-1\}}\right)\right]^{-1}, \end{aligned}$$

stoga je

$$\begin{aligned} &\Delta^{n}_{n-k+1} \cdot t^{-1}_{n,1} \cdot \left(\Delta^{n}_{n-k}\right)^{-1} \cdot t_{n,1} \\ &\quad = \frac{\left(\mathrm{id} - X_{\{k,k+1\}}\right) \cdot \left(\mathrm{id} - X_{\{k,k+1,k+2\}}\right) \cdots \left(\mathrm{id} - X_{\{k,k+1,\ldots,n-1\}}\right) \cdot \left(\mathrm{id} - X_{\{k,k+1,\ldots,n-1,n\}}\right)}{\left(\mathrm{id} - X_{\{k,k+1\}}\right) \cdot \left(\mathrm{id} - X_{\{k,k+1,k+2\}}\right) \cdots \left(\mathrm{id} - X_{\{k,k+1,\ldots,n-1\}}\right)} \\ &\quad = \mathrm{id} - X_{\{k,k+1,\ldots,n\}} \,. \end{aligned}$$

Formula (71) dokazuje se matematičkom indukcijom. Pritom je dovoljno dokazati da vrijedi identitet (77), koji je ekvivalentan izrazu

$$\widetilde{\varepsilon^{n}_{n-k+1}} = \left(\mathrm{id} - X_{\{k,k+1,\dots,n\}}\right)\cdot \widetilde{\varepsilon^{n-1}_{n-k+1}} \cdot \left(\mathrm{id} - X_{\{k,k+1\}}\,\widetilde{t_{n,k+1}}\right)^{-1}.$$

Slijedi dokaz identiteta (77).

Izračunajmo najprije vrijednost izraza $\widetilde{\varepsilon^{n}_{n-k+1}} \cdot X_{\{k,k+1\}}\,\widetilde{t_{n,k+1}}$. Primjenom (66)–(69) iz propozicije 3.27 dobivamo sljedeće

$$\begin{aligned}
\widetilde{\varepsilon^{n}_{n-k+1}} \cdot X_{\{k,k+1\}}\,\widetilde{t_{n,k+1}} &= \sum_{\sigma\in S_1^k\times S_{n-k}} \omega^{n}_{n-k+1}(\sigma)\cdot\widetilde{\sigma}\cdot X_{\{k,k+1\}}\,\widetilde{t_{n,k+1}} \\
&= \sum_{\sigma\in S_1^k\times S_{n-k}} \omega^{n}_{n-k+1}(\sigma)\cdot X_{\{\sigma(k),\sigma(k+1)\}}\,\widetilde{\sigma}\cdot\widetilde{t_{n,k+1}} \\
&= \sum_{\sigma\in S_1^k\times S_{n-k}} \omega^{n}_{n-k+1}(\sigma)\cdot X_{\{k,\sigma(k+1)\}}\,\widetilde{\sigma}\cdot\widetilde{t_{n,k+1}} \\
&\overset{\text{lema 3.13}}{=} \sum_{\sigma\in S_1^k\times S_{n-k}} \omega^{n}_{n-k+1}(\sigma)\cdot X_{\{k,\sigma(k+1)\}}\left(\prod_{\substack{k+1<i\le n,\\ \sigma(k+1)>\sigma(i)}} X_{\{\sigma(i),\sigma(k+1)\}}\right)\widetilde{\sigma\,t_{n,k+1}} \\
&= \sum_{\sigma\in S_1^k\times S_{n-k}} \omega^{n}_{n-k+1}(\sigma)\cdot X_{\{k,\sigma(k+1)\}}\cdot\prod_{k+1\le j<\sigma(k+1)} X_{\{j,\sigma(k+1)\}}\,\widetilde{\sigma\,t_{n,k+1}} \\
&= \sum_{\sigma\in S_1^k\times S_{n-k}} \omega^{n}_{n-k+1}(\sigma)\cdot\prod_{k\le j<\sigma(k+1)} X_{\{j,\sigma(k+1)\}}\,\widetilde{\sigma\,t_{n,k+1}}
\end{aligned}$$

odnosno

$$\widetilde{\varepsilon^{n}_{n-k+1}} \cdot X_{\{k,k+1\}}\,\widetilde{t_{n,k+1}} = \sum_{\sigma\in S_1^k\times S_{n-k}} \omega^{n}_{n-k+1}(\sigma)\cdot\prod_{k\le j<\sigma(k+1)} X_{\{j,\sigma(k+1)\}}\,\widetilde{\sigma\,t_{n,k+1}}. \tag{78}$$

Iz $\sigma, t_{n,k+1}\in S_1^k\times S_{n-k}\subset S_n$ proizlazi $g=\sigma\,t_{n,k+1}\in S_1^k\times S_{n-k}\subset S_n$.

Pritom iz $\quad g=\sigma\,t_{n,k+1}$,

proizlazi $\quad \sigma = g\,t_{n,k+1}^{-1}$,

gdje je

$$\sigma(k+1) = g\,t_{n,k+1}^{-1}(k+1) = g(n),$$

što se uvrštavanjem u (78) dobiva

$$\widetilde{\varepsilon^{n}_{n-k+1}} \cdot X_{\{k,k+1\}}\,\widetilde{t_{n,k+1}} = \sum_{g\in S_1^k\times S_{n-k}} \omega^{n}_{n-k+1}\left(g\,t_{n,k+1}^{-1}\right)\cdot\prod_{k\le j<g(n)} X_{\{j,g(n)\}}\,\widetilde{g}. \tag{79}$$

Proučimo sada skupove $\mathrm{Des}\left(\left(g\,t_{n,k+1}^{-1}\right)^{-1}\right)=\mathrm{Des}\left(t_{n,k+1}\,g^{-1}\right)$ i $\mathrm{Des}\left(g^{-1}\right)$.

Imamo

$$\mathrm{Des}\left(g^{-1}\right)=\left\{k+1\le i\le n-1 \mid g^{-1}(i)>g^{-1}(i+1)\right\},$$

$$\mathrm{Des}\left(t_{n,k+1}\,g^{-1}\right)=\left\{k+1\le i\le n-1 \mid t_{n,k+1}\,g^{-1}(i)>t_{n,k+1}\,g^{-1}(i+1)\right\}.$$

Primijetimo sljedeće

- Ako je $g^{-1}(i)=n$, onda $i=g(n)\notin \mathrm{Des}\left(t_{n,k+1}\,g^{-1}\right)$ nije točka pada permutacije $g\,t_{n,k+1}^{-1}$, jer je $t_{n,k+1}\,g^{-1}(i)=t_{n,k+1}(n)=k+1$, tj. $\underbrace{t_{n,k+1}\,g^{-1}(i)}_{=k+1}\not> t_{n,k+1}\,g^{-1}(i+1)$.

  S druge strane imamo da je $i=g(n)\in \mathrm{Des}\left(g^{-1}\right)$ točka pada permutacije g, jer je

  $$\underbrace{g^{-1}(i)}_{=n}>g^{-1}(i+1) \quad \text{za svaki} \quad k+1\le i\le n-1;$$

- Ako je $g^{-1}(i+1)=n$, onda je $i=\left(g(n)-1\right)\in \mathrm{Des}\left(t_{n,k+1}\,g^{-1}\right)$ točka pada permutacije $g\,t_{n,k+1}^{-1}$, jer je $t_{n,k+1}\,g^{-1}(i+1)=t_{n,k+1}(n)=k+1$, tj.

  $$t_{n,k+1}\,g^{-1}(i)>\underbrace{t_{n,k+1}\,g^{-1}(i+1)}_{=k+1} \quad \text{za svaki} \quad k+2\le i\le n-1.$$

  S druge strane imamo da $i=\left(g(n)-1\right)\notin \mathrm{Des}\left(g^{-1}\right)$ nije točka pada permutacije g, jer $g^{-1}(i)\not>\underbrace{g^{-1}(i+1)}_{=n}$.

Na osnovu rečenog zaključujemo da je

$$\mathrm{Des}\left(t_{n,k+1}\,g^{-1}\right)=\begin{cases}\left(\mathrm{Des}\left(g^{-1}\right)\setminus\left\{g(n)\right\}\right)\cup\left\{g(n)-1\right\} & \text{ako je} \quad g(n)<n\\ \mathrm{Des}\left(g^{-1}\right)\cup\left\{g(n)-1\right\} & \text{ako je} \quad g(n)=n\end{cases}$$

ili

$$\mathrm{Des}\left(t_{n,k+1}\,g^{-1}\right)=\left(\mathrm{Des}\left(g^{-1}\right)\setminus\left\{g(n)\right\}\right)\cup\left\{g(n)-1\right\} \quad \text{ako je} \quad g(n)\le n. \tag{80}$$

Napomena:

Specijalno za $g(n)=n$ imamo da $g(n)\notin \mathrm{Des}\left(g^{-1}\right)$, što povlači

$$\mathrm{Des}\left(g^{-1}\right)\setminus\left\{g(n)\right\}=\mathrm{Des}\left(g^{-1}\right),$$

jer je $\mathrm{Des}\left(g^{-1}\right)\cap\left\{g(n)\right\}=\varnothing$.

Uzimajući u obzir identitet (80) imamo da je

$$\prod_{i\in \mathrm{Des}\left(t_{n,k+1}\, g^{-1}\right)} X_{\{k,k+1,\dots,i\}} = \prod_{\substack{i\in \mathrm{Des}\left(t_{n,k+1}\, g^{-1}\right)\\ i\neq g(n)-1}} X_{\{k,k+1,\dots,i\}} \cdot \underbrace{X_{\{k,k+1,\dots,g(n)-1\}}}_{\text{za } i=g(n)-1},$$

stoga dobivamo

$$\begin{aligned} \omega^{n}_{n-k+1}\left(g\, t^{-1}_{n,k+1}\right)\cdot \prod_{k\leq j<g(n)} X_{\{j,g(n)\}} &= \prod_{i\in \mathrm{Des}\left(t_{n,k+1}\, g^{-1}\right)} X_{\{k,k+1,\dots,i\}} \cdot \prod_{k\leq j<g(n)} X_{\{j,g(n)\}} \\ &= \prod_{\substack{i\in \mathrm{Des}\left(t_{n,k+1}\, g^{-1}\right)\\ i\neq g(n)-1}} X_{\{k,k+1,\dots,i\}} \cdot X_{\{k,k+1,\dots,g(n)-1\}} \cdot \prod_{k\leq j<g(n)} X_{\{j,g(n)\}} \\ &= \prod_{\substack{i\in \mathrm{Des}\left(t_{n,k+1}\, g^{-1}\right)\\ i\neq g(n)-1}} X_{\{k,k+1,\dots,i\}} \cdot X_{\{k,k+1,\dots,g(n)\}}. \end{aligned} \tag{81}$$

Promotrimo sada desnu stranu identiteta (81).

Ako je $g(n)<n$, onda imamo

$$\prod_{\substack{i\in \mathrm{Des}\left(t_{n,k+1}\, g^{-1}\right)\\ i\neq g(n)-1}} X_{\{k,k+1,\dots,i\}} \cdot X_{\{k,k+1,\dots,g(n)\}} = \prod_{i\in \mathrm{Des}\left(g^{-1}\right)} X_{\{k,k+1,\dots,i\}} = \omega^{n}_{n-k+1}(g).$$

Ako je $g(n)=n$, onda imamo $g(n)\notin \mathrm{Des}\left(g^{-1}\right)$, $g(n)\notin \mathrm{Des}\left(t_{n,k+1}\, g^{-1}\right)$ pa je

$$\begin{aligned} \prod_{\substack{i\in \mathrm{Des}\left(t_{n,k+1}\, g^{-1}\right)\\ i\neq g(n)-1}} X_{\{k,k+1,\dots,i\}} \cdot X_{\{k,k+1,\dots,g(n)\}} &= X_{\{k,k+1,\dots,n\}} \cdot \prod_{i\in \mathrm{Des}\left(g^{-1}\right)} X_{\{k,k+1,\dots,i\}} \\ &= X_{\{k,k+1,\dots,n\}} \cdot \omega^{n}_{n-k+1}(g). \end{aligned}$$

Time dobivamo

$$\omega^{n}_{n-k+1}\left(g\, t^{-1}_{n,k+1}\right)\cdot \prod_{k\leq j<g(n)} X_{\{j,g(n)\}} = \begin{cases} \omega^{n}_{n-k+1}(g) & \text{ako je } g(n)<n \\ X_{\{k,k+1,\dots,n\}} \cdot \omega^{n}_{n-k+1}(g) & \text{ako je } g(n)=n \end{cases}$$

što uvrštavanjem u identitet (79)

$$\begin{aligned} &\widetilde{\varepsilon^{n}_{n-k+1}} \cdot X_{\{k,k+1\}}\, \widetilde{t_{n,k+1}} = \\ &= \begin{cases} \displaystyle\sum_{g\in S^{k}_{1}\times S_{n-k}} \omega^{n}_{n-k+1}(g)\cdot \tilde{g} = \widetilde{\varepsilon^{n}_{n-k+1}} & \text{ako je } g(n)<n \\ \displaystyle\sum_{g\in S^{k}_{1}\times S_{n-k}} X_{\{k,k+1,\dots,n\}} \cdot \omega^{n}_{n-k+1}(g)\cdot \tilde{g} = X_{\{k,k+1,\dots,n\}} \cdot \widetilde{\varepsilon^{n}_{n-k+1}} & \text{ako je } g(n)=n \end{cases} \end{aligned}$$

imamo da je lijeva strana identiteta (77) dana sa

$$\widetilde{\varepsilon^{n}_{n-k+1}} \cdot \left(\mathrm{id} - X_{\{k,k+1\}}\, \widetilde{t_{n,k+1}}\right)$$

$$= \widetilde{\varepsilon^{n}_{n-k+1}} - \widetilde{\varepsilon^{n}_{n-k+1}} \cdot X_{\{k,k+1\}}\, \widetilde{t_{n,k+1}}$$

$$= \left(\sum_{\substack{g \in S_1^k \times S_{n-k}, \\ g(n)<n}} \omega^{n}_{n-k+1}(g) \cdot \tilde{g} + \sum_{\substack{g \in S_1^k \times S_{n-k}, \\ g(n)=n}} \omega^{n}_{n-k+1}(g) \cdot \tilde{g} \right)$$

$$- \left(\sum_{\substack{g \in S_1^k \times S_{n-k}, \\ g(n)<n}} \omega^{n}_{n-k+1}(g) \cdot \tilde{g} + \sum_{\substack{g \in S_1^k \times S_{n-k}, \\ g(n)=n}} X_{\{k,k+1,\dots,n\}} \cdot \omega^{n}_{n-k+1}(g) \cdot \tilde{g} \right)$$

$$= \sum_{\substack{g \in S_1^k \times S_{n-k}, \\ g(n)<n}} \left(\omega^{n}_{n-k+1}(g) \cdot \tilde{g} - \omega^{n}_{n-k+1}(g) \cdot \tilde{g} \right) + \sum_{\substack{g \in S_1^k \times S_{n-k}, \\ g(n)=n}} \left(\mathrm{id} - X_{\{k,k+1,\dots,n\}}\right) \omega^{n}_{n-k+1}(g) \cdot \tilde{g}$$

$$= \left(\mathrm{id} - X_{\{k,k+1,\dots,n\}}\right) \cdot \sum_{\substack{g \in S_1^k \times S_{n-k}, \\ g(n)=n}} \omega^{n}_{n-k+1}(g) \cdot \tilde{g}$$

$$= \left(\mathrm{id} - X_{\{k,k+1,\dots,n\}}\right) \cdot \widetilde{\varepsilon^{n-1}_{n-k+1}}\,.$$

Napomena:

Uočimo da je

$$\sum_{\substack{g' \in S_1^k \times S_{n-k}, \\ g'(n)=n}} \omega^{n}_{n-k+1}(g') \cdot \widetilde{g'} = \sum_{\substack{g' \in S_1^k \times S_{n-k}, \\ g'(n)=n}} \underbrace{t_{n,1}^{-1} \cdot t_{n,1}}_{=\mathrm{id}} \cdot \omega^{n}_{n-k+1}(g') \cdot \widetilde{g'} \cdot \underbrace{t_{n,1}^{-1} \cdot t_{n,1}}_{=\mathrm{id}}$$

$$= \sum_{\substack{g' \in S_1^k \times S_{n-k}, \\ g'(n)=n}} t_{n,1}^{-1} \cdot \underbrace{\left(t_{n,1} \cdot \omega^{n}_{n-k+1}(g') \cdot \widetilde{g'} \cdot t_{n,1}^{-1} \right)}_{=\omega^{n}_{n-k}(g) \cdot \tilde{g},\ g \in S_1^{k+1} \times S_{n-k-1}} \cdot t_{n,1}$$

$$= \sum_{g \in S_1^{k+1} \times S_{n-k-1}} t_{n,1}^{-1} \cdot \omega^{n}_{n-k}(g) \cdot \tilde{g} \cdot t_{n,1}$$

$$= t_{n,1}^{-1} \cdot \sum_{g \in S_1^{k+1} \times S_{n-k-1}} \omega^{n}_{n-k}(g) \cdot \tilde{g} \cdot t_{n,1}$$

$$= t_{n,1}^{-1} \cdot \widetilde{\varepsilon^{n}_{n-k}} \cdot t_{n,1}$$

$$= \widetilde{\varepsilon^{n-1}_{n-k+1}}\,.$$

Time je dokazan identitet (77), a samim time i propozicija 3.27.

Primjedba 3.28

Izračunavanje inverza $\widetilde{\delta_{n-k+1}}^{-1} \in \mathcal{A}_n$, $1 \le k \le n-1$ elementa $\widetilde{\delta_{n-k+1}} \in \mathcal{A}_n$ se provodi kako bi se mogli izračunati inverzi $\widetilde{\beta_{n-k+1}}^{-1} = \widetilde{\gamma_{n-k+1}} \cdot \widetilde{\delta_{n-k+1}}^{-1}$, $1 \le k \le n-1$ (propozicija 3.22), a samim time i analogon Zagier-ove jednoparametarske formule za izračunavanje inverza elementa $\widetilde{\alpha_n} = \sum_{g \in S_n} \widetilde{g}$, koji se faktorizira formulom

$$\widetilde{\alpha_n} = \widetilde{\beta_2} \cdots \widetilde{\beta_{n-1}} \cdot \widetilde{\beta_n}$$

danom u propoziciji 3.19.

Pritom je

$$\widetilde{\alpha_n}^{-1} = \widetilde{\gamma_n} \cdot \widetilde{\delta_n}^{-1} \cdot \widetilde{\gamma_{n-1}} \cdot \widetilde{\delta_{n-1}}^{-1} \cdots \widetilde{\gamma_2} \cdot \widetilde{\delta_2}^{-1}.$$

Više o tome pogledati u [MS1], gdje nije korišten formalizam zakrenutih grupovnih algebri, već faktorizacije na matričnom nivou.

4. Reprezentacija zakrenute grupovne algebre $\mathcal{A}_n$ na težinskim potprostorima $\mathcal{B}_{l_1 l_2 \dots l_n}$ algebre $\mathcal{B}$

U ovom poglavlju uglavnom će se proučavati reprezentacija zakrenute grupovne algebre $\mathcal{A}_n$ na bilo kojem generičkom težinskom potprostoru $\mathcal{B}_Q = \mathcal{B}_{l_1 l_2 \dots l_n}$ (algebre $\mathcal{B}$) pridruženom skupu $Q = \{l_1, l_2, \dots, l_n\} \subseteq \mathcal{N} = \{i_1, i_2, \dots, i_N\}$.

Podsjetimo se, generički težinski potprostor $\mathcal{B}_Q = \mathcal{B}_{l_1 l_2 \dots l_n}$ algebre $\mathcal{B}$ uveden je u drugom poglavlju ove disertacije (vidi str. 18)

Uzimajući u obzir da se svaki multiskup $Q = \{l_1 \le l_2 \le \dots \le l_n\}$ može dobiti iz skupa $Q = \{l_1, l_2, \dots, l_n\}$ poistovjećivanjem nekih elemenata u skupu Q, što ima za posljedicu da se svaki degenerirani potprostor algebre $\mathcal{B}$ može dobiti iz generičkog potprostora, u nastavku ćemo se koncentrirati na proučavanje generičkih potprostora i (zakrenute regularne) reprezentacije algebre $\mathcal{A}_n$ na njima.

Definicija 4.1

Kažemo da je preslikavanje $\mathcal{R} : \mathcal{A}_n \to \mathrm{End}(\mathcal{B}_Q)$ reprezentacija zakrenute grupovne algebre $\mathcal{A}_n$ na težinskom potprostoru $\mathcal{B}_Q$ algebre $\mathcal{B}$ ako je $\mathcal{R}$ homomorfizam.

Specijalno, ako je $\mathcal{B}_Q$ generički težinski potprostor, onda će važan primjer za nas biti zakrenuta regularna reprezentacija $\mathcal{R} : \mathcal{A}_n \to \mathrm{End}(\mathcal{B}_Q)$.

Podsjetimo se, u trećem poglavlju ove disertacije, definirali smo općenitiju grupovnu algebru simetrične grupe S_n s koeficijentima u prstenu polinoma R_n, koju smo označili sa $\mathcal{A}_n = R_n \rtimes \mathbb{C}[S_n]$ i nazvali je zakrenutom grupovnom algebrom.

Pritom smo s $R_n = \mathbb{C}[X_{km} \mid 1 \le k, m \le n]$ označili prsten polinoma u n^2 komutirajućih varijabli X_{km}. $\mathbb{C}[S_n]$ je grupovna algebra simetrične grupe S_n.

U zakrenutoj grupovnoj algebri $\mathcal{A}_n$ je množenje inducirano pravilom:

$$(X_{km} g_1) \cdot (X_{k'm'} g_2) = (X_{km} \cdot g_1 . X_{k'm'})\, g_1 g_2 = \left(X_{km} \cdot X_{g_1(k')\, g_1(m')}\right) g_1 g_2 .$$

U nastavku ćemo dokazati da je preslikavanje $\mathcal{R}: \mathcal{A}_n \to \mathrm{End}(\mathcal{B}_Q)$ homorfizam, tj. da je $\mathcal{R}$ reprezentacija zakrenute grupovne algebre $\mathcal{A}_n$ na generičkom težinskom potprostoru $\mathcal{B}_Q$ algebre $\mathcal{B}$.

Najprije promatrajmo dvije reprezentacije: reprezentaciju ρ algebre polinoma R_n na generičkom težinskom potprostoru $\mathcal{B}_Q$ algebre $\mathcal{B}$ i reprezentaciju R_n grupovne algebre $\mathbb{C}[S_n]$ na istom potprostoru.

Neka je $j_1 j_2 \dots j_n \in \widehat{Q}$, gdje je $\widehat{Q} = S_n Q = \{\sigma(l_1 l_2 \dots l_n) \mid \sigma \in S_n\}$ skup svih permutacija skupa $Q = \{l_1, l_2, \dots, l_n\}$.

Označimo sa Q_{km} dijagonalne operatore na generičkom težinskom potprostoru $\mathcal{B}_Q$ definirane sa:

$$Q_{km}\, e_{j_1 j_2 \cdots j_n} = (Q_{km})_{j_1 j_2 \cdots j_n, j_1 j_2 \cdots j_n}\, e_{j_1 j_2 \cdots j_n} = q_{j_k j_m}\, e_{j_1 j_2 \cdots j_n}. \tag{1}$$

Tada je

$$(Q_{km} \cdot Q_{k'm'})\, e_{j_1 j_2 \cdots j_n} = q_{j_k j_m} q_{j_{k'} j_{m'}}\, e_{j_1 j_2 \cdots j_n} \tag{2}$$

i specijalno je

$$Q_{km}^r\, e_{j_1 j_2 \cdots j_n} = q_{j_k j_m}^r\, e_{j_1 j_2 \cdots j_n}. \tag{3}$$

Definirajmo homomorfizam $\rho: R_n \to \mathrm{End}(\mathcal{B}_Q)$, $R_n = \mathbb{C}[X_{km} \mid 1 \le k, m \le n]$ na generatorima sa

tj.

$$\left.\begin{aligned} \rho(X_{km}) &:= Q_{km}, \qquad (1 \le k, m \le n) \\ \rho(X_{km}) e_{j_1 j_2 \cdots j_n} &= (Q_{km})_{j_1 j_2 \cdots j_n, j_1 j_2 \cdots j_n}\, e_{j_1 j_2 \cdots j_n} = q_{j_k j_m}\, e_{j_1 j_2 \cdots j_n}, \end{aligned}\right\} \tag{4}$$

gdje je dijagonalni operator Q_{km} na $\mathcal{B}_Q$ definiran s (1).

Pritom je

$$\rho(p \cdot p') = \rho(p) \cdot \rho(p'),$$

gdje su $p = p(\dots, X_{km}, \dots)$ i $p' = p'(\dots, X_{km}, \dots)$ polinomi u varijablama X_{km} iz algebre polinoma $R_n = \mathbb{C}[X_{km} \mid 1 \le k, m \le n]$.

Sada ćemo za $g \in S_n$ definirati linearni operator $\mathcal{R}_n : \mathbb{C}[S_n] \to \mathrm{End}(\mathcal{B}_Q)$ sa

$$\mathcal{R}_n(g)\, e_{j_1 j_2 \dots j_n} := e_{j_{g^{-1}(1)} j_{g^{-1}(2)} \dots j_{g^{-1}(n)}} . \tag{5}$$

Specijalno, ako je $g = \mathrm{id}$, onda iz (5) dobivamo

$$\mathcal{R}_n(\mathrm{id})\, e_{j_1 j_2 \dots j_n} = e_{j_1 j_2 \dots j_n} . \tag{6}$$

Propozicija 4.2

Preslikavanje $\mathcal{R}_n : \mathbb{C}[S_n] \to \mathrm{End}(\mathcal{B}_Q)$ je homomorfizam grupovne algebre simetrične grupe S_n u algebru endomorfizama od $\mathcal{B}_Q$.

Dokaz:

Treba dokazati da je preslikavanje $\mathcal{R}_n : \mathbb{C}[S_n] \to \mathrm{End}(\mathcal{B}_Q)$ homomorfizam, tj. da vrijedi

$$\mathcal{R}_n(g_1 g_2)\, e_{j_1 j_2 \dots j_n} = \mathcal{R}_n(g_1) \cdot \mathcal{R}_n(g_2)\, e_{j_1 j_2 \dots j_n} , \tag{7}$$

gdje je $g_1 g_2$ kompozicija permutacija g_1, g_2, tj. $g_1 g_2 = g_1 \circ g_2$.

Neka je $g_1, g_2 \in S_n$. Tada primjenom relacije (5) imamo da je element $\mathcal{R}_n(g_1 g_2) \in \mathrm{End}(\mathcal{B}_Q)$ dan sa

$$\mathcal{R}_n(g_1 g_2)\, e_{j_1 j_2 \dots j_n} = e_{j_{g_2^{-1} g_1^{-1}(1)} j_{g_2^{-1} g_1^{-1}(2)} \dots j_{g_2^{-1} g_1^{-1}(n)}} , \tag{8}$$

ako stavimo da je $g = g_1 g_2$. Pritom je $g^{-1} = (g_1 g_2)^{-1} = g_2^{-1} g_1^{-1}$.

Dokažimo sada jednakost (7). Imamo

$$\mathcal{R}_n(g_1) \cdot \mathcal{R}_n(g_2)\, e_{j_1 j_2 \dots j_n} = \mathcal{R}_n(g_1) \cdot \left(\mathcal{R}_n(g_2)\, e_{j_1 j_2 \dots j_n} \right) \overset{(5)}{=} \mathcal{R}_n(g_1)\, e_{j_{g_2^{-1}(1)} j_{g_2^{-1}(2)} \dots j_{g_2^{-1}(n)}} .$$

Uvedemo li oznaku

$$k_m := j_{g_2^{-1}(m)} \tag{9}$$

za svaki $1 \le m \le n$, tada možemo pisati

$$\mathcal{R}_n(g_1)\, e_{j_{g_2^{-1}(1)} j_{g_2^{-1}(2)} \dots j_{g_2^{-1}(n)}} = \mathcal{R}_n(g_1)\, e_{k_1 k_2 \dots k_n} \overset{(5)}{=} e_{k_{g_1^{-1}(1)} k_{g_1^{-1}(2)} \dots k_{g_1^{-1}(n)}} .$$

Koristeći oznaku (9) zaključujemo da vrijedi

$$k_{g_1^{-1}(r)} = j_{g_2^{-1}\left(g_1^{-1}(r)\right)}, \quad 1 \le r \le n ,$$

čime je

$$\mathrm{e}_{\mathrm{k}_{g_1^{-1}(1)}\mathrm{k}_{g_1^{-1}(2)}\cdots\mathrm{k}_{g_1^{-1}(n)}} = \mathrm{e}_{\mathrm{j}_{g_2^{-1}\left(g_1^{-1}(1)\right)}\mathrm{j}_{g_2^{-1}\left(g_1^{-1}(2)\right)}\cdots\mathrm{j}_{g_2^{-1}\left(g_1^{-1}(n)\right)}} = \mathrm{e}_{\mathrm{j}_{g_2^{-1}g_1^{-1}(1)}\mathrm{j}_{g_2^{-1}g_1^{-1}(2)}\cdots\mathrm{j}_{g_2^{-1}g_1^{-1}(n)}},$$

stoga dobivamo da je

$$R_n(g_1)\cdot R_n(g_2)\,\mathrm{e}_{j_1 j_2\cdots j_n} = \mathrm{e}_{\mathrm{j}_{g_2^{-1}g_1^{-1}(1)}\mathrm{j}_{g_2^{-1}g_1^{-1}(2)}\cdots\mathrm{j}_{g_2^{-1}g_1^{-1}(n)}}. \tag{10}$$

Uspoređivanjem (10) s (8) direktno se dobiva jednakost (7), čime je propozicija 4.2 dokazana.

Sada možemo definirati preslikavanje $\mathcal{R}:\boldsymbol{\mathcal{A}}_n \to \mathrm{End}\left(\boldsymbol{\mathcal{B}}_Q\right)$ stavljajući

$$\mathcal{R}(p\,g) := \rho(p)\cdot R_n(g). \tag{11}$$

Imamo

$$\begin{aligned}
\mathcal{R}\left(p(\ldots,X_{km},\ldots)\,g\right)\mathrm{e}_{j_1 j_2\cdots j_n} &= \rho\left(p(\ldots,X_{km},\ldots)\right)\cdot R_n(g)\,\mathrm{e}_{j_1 j_2\cdots j_n}\\
&\overset{(5)}{=} \rho\left(p(\ldots,X_{km},\ldots)\right)\mathrm{e}_{\mathrm{j}_{g^{-1}(1)}\mathrm{j}_{g^{-1}(2)}\cdots\mathrm{j}_{g^{-1}(n)}}\\
&= p\left(\rho(\ldots,X_{km},\ldots)\right)\mathrm{e}_{\mathrm{j}_{g^{-1}(1)}\mathrm{j}_{g^{-1}(2)}\cdots\mathrm{j}_{g^{-1}(n)}}\\
&= p(\ldots,Q_{km},\ldots)\,\mathrm{e}_{\mathrm{j}_{g^{-1}(1)}\mathrm{j}_{g^{-1}(2)}\cdots\mathrm{j}_{g^{-1}(n)}}\\
&= p\left(\ldots,q_{\mathrm{j}_{g^{-1}(k)}\mathrm{j}_{g^{-1}(m)}},\ldots\right)\mathrm{e}_{\mathrm{j}_{g^{-1}(1)}\mathrm{j}_{g^{-1}(2)}\cdots\mathrm{j}_{g^{-1}(n)}},
\end{aligned}$$

pa je

$$\mathcal{R}\left(p(\ldots,X_{km},\ldots)\,g\right)\mathrm{e}_{j_1 j_2\cdots j_n} = p\left(\ldots,q_{\mathrm{j}_{g^{-1}(k)}\mathrm{j}_{g^{-1}(m)}},\ldots\right)\mathrm{e}_{\mathrm{j}_{g^{-1}(1)}\mathrm{j}_{g^{-1}(2)}\cdots\mathrm{j}_{g^{-1}(n)}}. \tag{12}$$

Specijalno za $g = \mathrm{id}$ imamo:

$$\begin{aligned}
\mathcal{R}\left(p(\ldots,X_{km},\ldots)\,\mathrm{id}\right)\mathrm{e}_{j_1 j_2\cdots j_n} &= \rho\left(p(\ldots,X_{km},\ldots)\right)\cdot R_n(\mathrm{id})\,\mathrm{e}_{j_1 j_2\cdots j_n}\\
&\overset{(6)}{=} \rho\left(p(\ldots,X_{km},\ldots)\right)\mathrm{e}_{j_1 j_2\cdots j_n}\\
&= p\left(\ldots,q_{j_k j_m},\ldots\right)\mathrm{e}_{j_1 j_2\cdots j_n},
\end{aligned}$$

odnosno

$$\mathcal{R}\left(p(\ldots,X_{km},\ldots)\,\mathrm{id}\right)\mathrm{e}_{j_1 j_2\cdots j_n} = p\left(\ldots,q_{j_k j_m},\ldots\right)\mathrm{e}_{j_1 j_2\cdots j_n}. \tag{13}$$

Neka je $p(\ldots,X_{km},\ldots) = p_e = 1 \in R_n$. Tada je

$$\mathcal{R}(p_e\,g)\,\mathrm{e}_{j_1 j_2\cdots j_n} = \rho(p_e)\cdot R_n(g)\,\mathrm{e}_{j_1 j_2\cdots j_n} = \rho(p_e)\,\mathrm{e}_{\mathrm{j}_{g^{-1}(1)}\mathrm{j}_{g^{-1}(2)}\cdots\mathrm{j}_{g^{-1}(n)}}. \tag{14}$$

S druge strane imamo

$$\mathcal{R}\left(p_{e}\,g\right)e_{j_1 j_2\cdots j_n}=\mathcal{R}\left(1\cdot g\right)e_{j_1 j_2\cdots j_n}=\mathcal{R}\left(g\right)e_{j_1 j_2\cdots j_n}$$

$$=R_{n}\left(g\right)e_{j_1 j_2\cdots j_n}=e_{j_{g^{-1}(1)}j_{g^{-1}(2)}\cdots j_{g^{-1}(n)}}\,. \tag{15}$$

Uspoređivanjem (14) s (15) dobivamo

$$\rho\left(p_{e}\right)e_{j_{g^{-1}(1)}j_{g^{-1}(2)}\cdots j_{g^{-1}(n)}}=e_{j_{g^{-1}(1)}j_{g^{-1}(2)}\cdots j_{g^{-1}(n)}}$$

što povlači da je

$$\rho\left(p_{e}\right)=\mathrm{id}\,.$$

Komentar 4.3

Uzimajući u obzir relaciju (2) iz trećeg poglavlja, kojom je dano djelovanje simetrične grupe na prstenu polinoma, tj.

$$g.p\left(\ldots,X_{km},\ldots\right)=p\left(\ldots,X_{g(k)\,g(m)},\ldots\right)\in R_{n}=\mathbb{C}\left[X_{km}\mid 1\le k,m\le n\right]$$

imamo da je

$$\rho\left(g.p\left(\ldots,X_{km},\ldots\right)\right)\in \mathrm{End}\left(\mathcal{B}_{Q}\right),$$

dano sa

$$\rho\left(g.p\left(\ldots,X_{km},\ldots\right)\right)e_{j_1 j_2\cdots j_n}=\rho\left(p\left(\ldots,X_{g(k)\,g(m)},\ldots\right)\right)e_{j_1 j_2\cdots j_n}$$

$$=p\left(\ldots,Q_{g(k)\,g(m)},\ldots\right)e_{j_1 j_2\cdots j_n}=p\left(\ldots,q_{j_{g(k)}j_{g(m)}},\ldots\right)e_{j_1 j_2\cdots j_n}\,.$$

Neka je preslikavanje $\mathcal{R}:\mathcal{A}_{n}\to \mathrm{End}\left(\mathcal{B}_{Q}\right)$ definirano sa (11), gdje je zakrenuta grupovna algebra $\mathcal{A}_{n}=\left\{\sum_{g_i\in S_n} p_i\,g_i \mid p_i\in R_n\right\}$ općenitija grupovna algebra simetrične grupe S_n s koeficijentima u prstenu polinoma R_n.

Tada je element $\mathcal{R}\left(\sum_{g_i\in S_n} p_i\,g_i\right)\in \mathrm{End}\left(\mathcal{B}_{Q}\right)$ dan sa

$$\mathcal{R}\left(\sum_{g_i\in S_n} p_i\,g_i\right)e_{j_1 j_2\cdots j_n}=\sum_{g_i\in S_n}\mathcal{R}\left(p_i\,g_i\right)e_{j_1 j_2\cdots j_n}=\sum_{g_i\in S_n}\rho\left(p_i\left(\ldots,X_{km},\ldots\right)\right)\cdot\left(R_n\left(g_i\right)e_{j_1 j_2\cdots j_n}\right)$$

$$=\sum_{g_i\in S_n}p_i\left(\ldots,Q_{km},\ldots\right)e_{j_{g_i^{-1}(1)}j_{g_i^{-1}(2)}\cdots j_{g_i^{-1}(n)}}$$

$$=\sum_{g_i\in S_n}p_i\left(\ldots,q_{j_{g_i^{-1}(k)}j_{g_i^{-1}(m)}},\ldots\right)e_{j_{g_i^{-1}(1)}j_{g_i^{-1}(2)}\cdots j_{g_i^{-1}(n)}}\,. \tag{16}$$

Uočimo da je relacija (12) pojednostavljeni zapis od (16).

Propozicija 4.4

Preslikavanje $\mathcal{R}:\boldsymbol{\mathcal{A}}_{\mathrm{n}} \to \mathrm{End}\left(\boldsymbol{\mathcal{B}}_{\mathrm{Q}}\right)$ je zakrenuta regularna reprezentacija zakrenute grupovne algebre $\boldsymbol{\mathcal{A}}_{\mathrm{n}}$ na generičkom težinskom potprostoru $\boldsymbol{\mathcal{B}}_{\mathrm{Q}}$ algebre $\boldsymbol{\mathcal{B}}$.

Dokaz:

Treba dokazati da je preslikavanje $\mathcal{R}:\boldsymbol{\mathcal{A}}_{\mathrm{n}} \to \mathrm{End}\left(\boldsymbol{\mathcal{B}}_{\mathrm{Q}}\right)$ homomorfizam, tj. da za svaki $p_1, p_2 \in \mathrm{R}_{\mathrm{n}}$, $\mathrm{g}_1, \mathrm{g}_2 \in \mathrm{S}_{\mathrm{n}}$ vrijedi

$$\mathcal{R}\left(\left(p_1\,\mathrm{g}_1\right)\cdot\left(p_2\,\mathrm{g}_2\right)\right)\mathrm{e}_{\mathrm{j}_1\mathrm{j}_2\cdots\mathrm{j}_{\mathrm{n}}} = \mathcal{R}\left(p_1\,\mathrm{g}_1\right)\cdot\mathcal{R}\left(p_2\,\mathrm{g}_2\right)\mathrm{e}_{\mathrm{j}_1\mathrm{j}_2\cdots\mathrm{j}_{\mathrm{n}}}\,. \tag{17}$$

Dokaz ćemo ilustrirati na generatorima, stoga ćemo dokazati da vrijedi

$$\mathcal{R}\left(\left(\mathrm{X}_{\mathrm{k\,m}}\,\mathrm{g}_1\right)\cdot\left(\mathrm{X}_{\mathrm{k'm'}}\,\mathrm{g}_2\right)\right)\mathrm{e}_{\mathrm{j}_1\mathrm{j}_2\cdots\mathrm{j}_{\mathrm{n}}} = \mathcal{R}\left(\mathrm{X}_{\mathrm{k\,m}}\,\mathrm{g}_1\right)\cdot\mathcal{R}\left(\mathrm{X}_{\mathrm{k'm'}}\,\mathrm{g}_2\right)\mathrm{e}_{\mathrm{j}_1\mathrm{j}_2\cdots\mathrm{j}_{\mathrm{n}}}\,. \tag{18}$$

Pritom ćemo koristiti činjenicu da je u algebri $\boldsymbol{\mathcal{A}}_{\mathrm{n}}$ množenje inducirano pravilom

$$\left(\mathrm{X}_{\mathrm{k\,m}}\,\mathrm{g}_1\right)\cdot\left(\mathrm{X}_{\mathrm{k'm'}}\,\mathrm{g}_2\right) = \left(\mathrm{X}_{\mathrm{k\,m}}\cdot\mathrm{g}_1 \,.\, \mathrm{X}_{\mathrm{k'm'}}\right)\mathrm{g}_1\mathrm{g}_2 = \left(\mathrm{X}_{\mathrm{k\,m}}\cdot\mathrm{X}_{\mathrm{g}_1(\mathrm{k'})\,\mathrm{g}_1(\mathrm{m'})}\right)\mathrm{g}_1\mathrm{g}_2\,. \tag{19}$$

Uzimajući u obzir (19), imamo da je lijeva strana identiteta (18) dana sa

$$\begin{aligned}
\mathcal{R}\left(\left(\mathrm{X}_{\mathrm{k\,m}}\,\mathrm{g}_1\right)\cdot\left(\mathrm{X}_{\mathrm{k'm'}}\,\mathrm{g}_2\right)\right)\mathrm{e}_{\mathrm{j}_1\mathrm{j}_2\cdots\mathrm{j}_{\mathrm{n}}} &= \mathcal{R}\left(\left(\mathrm{X}_{\mathrm{k\,m}}\cdot\mathrm{X}_{\mathrm{g}_1(\mathrm{k'})\,\mathrm{g}_1(\mathrm{m'})}\right)\mathrm{g}_1\mathrm{g}_2\right)\mathrm{e}_{\mathrm{j}_1\mathrm{j}_2\cdots\mathrm{j}_{\mathrm{n}}}\\
&\overset{(11)}{=} \rho\left(\mathrm{X}_{\mathrm{k\,m}}\cdot\mathrm{X}_{\mathrm{g}_1(\mathrm{k'})\,\mathrm{g}_1(\mathrm{m'})}\right)\cdot R_{\mathrm{n}}\left(\mathrm{g}_1\mathrm{g}_2\right)\mathrm{e}_{\mathrm{j}_1\mathrm{j}_2\cdots\mathrm{j}_{\mathrm{n}}}\\
&\overset{(8)}{=} \rho\left(\mathrm{X}_{\mathrm{k\,m}}\cdot\mathrm{X}_{\mathrm{g}_1(\mathrm{k'})\,\mathrm{g}_1(\mathrm{m'})}\right)\mathrm{e}_{\mathrm{j}_{\mathrm{g}_2^{-1}\mathrm{g}_1^{-1}(1)}\mathrm{j}_{\mathrm{g}_2^{-1}\mathrm{g}_1^{-1}(2)}\cdots\,\mathrm{j}_{\mathrm{g}_2^{-1}\mathrm{g}_1^{-1}(\mathrm{n})}}\\
&= \rho\left(\mathrm{X}_{\mathrm{k\,m}}\right)\cdot\rho\left(\mathrm{X}_{\mathrm{g}_1(\mathrm{k'})\,\mathrm{g}_1(\mathrm{m'})}\right)\mathrm{e}_{\mathrm{j}_{\mathrm{g}_2^{-1}\mathrm{g}_1^{-1}(1)}\mathrm{j}_{\mathrm{g}_2^{-1}\mathrm{g}_1^{-1}(2)}\cdots\,\mathrm{j}_{\mathrm{g}_2^{-1}\mathrm{g}_1^{-1}(\mathrm{n})}}\\
&\overset{(4)}{=} \mathrm{q}_{\mathrm{j}_{\mathrm{g}_2^{-1}\mathrm{g}_1^{-1}(\mathrm{k})}\mathrm{j}_{\mathrm{g}_2^{-1}\mathrm{g}_1^{-1}(\mathrm{m})}}\cdot\mathrm{q}_{\mathrm{j}_{\mathrm{g}_2^{-1}\mathrm{g}_1^{-1}\mathrm{g}_1(\mathrm{k'})}\mathrm{j}_{\mathrm{g}_2^{-1}\mathrm{g}_1^{-1}\mathrm{g}_1(\mathrm{m'})}}\,\mathrm{e}_{\mathrm{j}_{\mathrm{g}_2^{-1}\mathrm{g}_1^{-1}(1)}\mathrm{j}_{\mathrm{g}_2^{-1}\mathrm{g}_1^{-1}(2)}\cdots\,\mathrm{j}_{\mathrm{g}_2^{-1}\mathrm{g}_1^{-1}(\mathrm{n})}}\\
&= \mathrm{q}_{\mathrm{j}_{\mathrm{g}_2^{-1}\mathrm{g}_1^{-1}(\mathrm{k})}\mathrm{j}_{\mathrm{g}_2^{-1}\mathrm{g}_1^{-1}(\mathrm{m})}}\cdot\mathrm{q}_{\mathrm{j}_{\mathrm{g}_2^{-1}(\mathrm{k'})}\mathrm{j}_{\mathrm{g}_2^{-1}(\mathrm{m'})}}\,\mathrm{e}_{\mathrm{j}_{\mathrm{g}_2^{-1}\mathrm{g}_1^{-1}(1)}\mathrm{j}_{\mathrm{g}_2^{-1}\mathrm{g}_1^{-1}(2)}\cdots\,\mathrm{j}_{\mathrm{g}_2^{-1}\mathrm{g}_1^{-1}(\mathrm{n})}}
\end{aligned} \tag{20}$$

Raspišimo sada desnu stranu identiteta (18). Uzimajući u obzir da je

$$\mathcal{R}\left(\mathrm{X}_{\mathrm{k\,m}}\,\mathrm{g}_1\right)\cdot\mathcal{R}\left(\mathrm{X}_{\mathrm{k'm'}}\,\mathrm{g}_2\right)\mathrm{e}_{\mathrm{j}_1\mathrm{j}_2\cdots\mathrm{j}_{\mathrm{n}}} = \mathcal{R}\left(\mathrm{X}_{\mathrm{k\,m}}\,\mathrm{g}_1\right)\cdot\left(\mathcal{R}\left(\mathrm{X}_{\mathrm{k'm'}}\,\mathrm{g}_2\right)\mathrm{e}_{\mathrm{j}_1\mathrm{j}_2\cdots\mathrm{j}_{\mathrm{n}}}\right) \tag{21}$$

gdje je

$$\begin{aligned}
\mathcal{R}\left(\mathrm{X}_{\mathrm{k'm'}}\,\mathrm{g}_2\right)\mathrm{e}_{\mathrm{j}_1\mathrm{j}_2\cdots\mathrm{j}_{\mathrm{n}}} &\overset{(11)}{=} \rho\left(\mathrm{X}_{\mathrm{k'm'}}\right)\cdot R_{\mathrm{n}}\left(\mathrm{g}_2\right)\mathrm{e}_{\mathrm{j}_1\mathrm{j}_2\cdots\mathrm{j}_{\mathrm{n}}}\\
&= \mathrm{q}_{\mathrm{j}_{\mathrm{g}_2^{-1}(\mathrm{k'})}\mathrm{j}_{\mathrm{g}_2^{-1}(\mathrm{m'})}}\,\mathrm{e}_{\mathrm{j}_{\mathrm{g}_2^{-1}(1)}\mathrm{j}_{\mathrm{g}_2^{-1}(2)}\cdots\,\mathrm{j}_{\mathrm{g}_2^{-1}(\mathrm{n})}}
\end{aligned} \tag{22}$$

dobivamo

$$
\begin{aligned}
\mathcal{R}\left(X_{km}\,g_1\right)\cdot\mathcal{R}\left(X_{k'm'}\,g_2\right)e_{j_1 j_2 \dots j_n} &= \mathcal{R}\left(X_{km}\,g_1\right)\cdot\left(q_{j_{g_2^{-1}(k')}j_{g_2^{-1}(m')}}\,e_{j_{g_2^{-1}(1)}j_{g_2^{-1}(2)}\dots j_{g_2^{-1}(n)}}\right) \\
&= \mathcal{R}\left(X_{km}\,g_1\right)e_{j_{g_2^{-1}(1)}j_{g_2^{-1}(2)}\dots j_{g_2^{-1}(n)}}\cdot q_{j_{g_2^{-1}(k')}j_{g_2^{-1}(m')}} \\
&\overset{(11)}{=} \rho\left(X_{km}\right)\cdot R_n\left(g_1\right)e_{j_{g_2^{-1}(1)}j_{g_2^{-1}(2)}\dots j_{g_2^{-1}(n)}}\cdot q_{j_{g_2^{-1}(k')}j_{g_2^{-1}(m')}} \\
&= \left(Q_{km}\right)\cdot e_{j_{g_2^{-1}g_1^{-1}(1)}j_{g_2^{-1}g_1^{-1}(2)}\dots j_{g_2^{-1}g_1^{-1}(n)}}\cdot q_{j_{g_2^{-1}(k')}j_{g_2^{-1}(m')}} \\
&= q_{j_{g_2^{-1}g_1^{-1}(k)}j_{g_2^{-1}g_1^{-1}(m)}}\,e_{j_{g_2^{-1}g_1^{-1}(1)}j_{g_2^{-1}g_1^{-1}(2)}\dots j_{g_2^{-1}g_1^{-1}(n)}}\cdot q_{j_{g_2^{-1}(k')}j_{g_2^{-1}(m')}} \\
&= q_{j_{g_2^{-1}g_1^{-1}(k)}j_{g_2^{-1}g_1^{-1}(m)}}\cdot q_{j_{g_2^{-1}(k')}j_{g_2^{-1}(m')}}\,e_{j_{g_2^{-1}g_1^{-1}(1)}j_{g_2^{-1}g_1^{-1}(2)}\dots j_{g_2^{-1}g_1^{-1}(n)}}\,.
\end{aligned}
\tag{23}
$$

U izvodu izraza (23) koristi se identitet

$$\mathcal{R}\left(g_1\right)e_{j_{g_2^{-1}(1)}j_{g_2^{-1}(2)}\dots j_{g_2^{-1}(n)}} = e_{j_{g_2^{-1}g_1^{-1}(1)}j_{g_2^{-1}g_1^{-1}(2)}\dots j_{g_2^{-1}g_1^{-1}(n)}}\,,$$

koji je izveden u dokazu propozicije 4.2.

Uspoređivanjem (20) sa (23) proizlazi identitet (18), na osnovu čega zaključujemo da je preslikavanje $\mathcal{R}$ homomorfizam zakrenute grupovne algebre $\mathcal{A}_n$ u algebru endomorfizama od $\mathcal{B}_Q$. Time je propozicija 4.4 dokazana.

U nastavku će se razmatrati reprezentacija elemenata iz zakrenute grupovne algebre $\mathcal{A}_n$ definiranih u trećem poglavlju.

Lema 4.5

Neka je $\mathcal{R}:\mathcal{A}_n \to \mathrm{End}\left(\mathcal{B}_Q\right)$ zakrenuta regularna reprezentacija zakrenute grupovne algebre $\mathcal{A}_n$ na generičkom težinskom potprostoru $\mathcal{B}_Q$ algebre $\mathcal{B}$.

Tada je element $\mathcal{R}\left(\tilde{g}\right)\in \mathrm{End}\left(\mathcal{B}_Q\right)$ dan sa

$$\mathcal{R}\left(\tilde{g}\right)e_{j_1 j_2 \dots j_n} = \prod_{(a,b)\in I(g)} q_{j_b j_a}\, e_{j_{g^{-1}(1)}j_{g^{-1}(2)}\dots j_{g^{-1}(n)}}\,. \tag{24}$$

Dokaz:

Koristeći definicije 3.3. i 3.1 proizlazi da je $\tilde{g} = \prod_{(a',b')\in I\left(g^{-1}\right)} X_{a'b'}\,g$, stoga je

$$\mathcal{R}\left(\tilde{g}\right)e_{j_1 j_2 \cdots j_n} = \prod_{(a',b')\in I\left(g^{-1}\right)} \rho\left(X_{a'b'}\right)\cdot R_n\left(g\right)e_{j_1 j_2 \cdots j_n}$$

$$= \prod_{(a',b')\in I\left(g^{-1}\right)} Q_{a'b'}\, e_{j_{g^{-1}(1)} j_{g^{-1}(2)} \cdots j_{g^{-1}(n)}}$$

$$= \prod_{(a',b')\in I\left(g^{-1}\right)} q_{j_{g^{-1}(a')} j_{g^{-1}(b')}}\, e_{j_{g^{-1}(1)} j_{g^{-1}(2)} \cdots j_{g^{-1}(n)}} . \tag{25}$$

Ako je $(a',b')\in I\left(g^{-1}\right)$, onda je

$$a' < b', \quad g^{-1}(a') > g^{-1}(b') . \tag{26}$$

Neka je $a = g^{-1}(a'),\ b = g^{-1}(b'),\ \left(\text{tj. } a' = g(a),\ b' = g(b)\right)$.

Tada iz (26) proizlazi

$$g(a) < g(b), \quad a > b . \tag{27}$$

Primijetimo da je (27) ekvivalentno zapisu $b < a$, $g(b) > g(a)$, što povlači da je $(b,a)\in I(g)$. Time se (25) može pisati u obliku

$$\mathcal{R}\left(\tilde{g}\right)e_{j_1 j_2 \cdots j_n} = \prod_{(b,a)\in I(g)} q_{j_a j_b}\, e_{j_{g^{-1}(1)} j_{g^{-1}(2)} \cdots j_{g^{-1}(n)}} = \prod_{(a,b)\in I(g)} q_{j_b j_a}\, e_{j_{g^{-1}(1)} j_{g^{-1}(2)} \cdots j_{g^{-1}(n)}} ,$$

što dokazuje lemu 4.5

✎ Primjenom leme 4.5. lako se može pokazati da je element $\mathcal{R}\left(\widetilde{t_{b,a}}\right)\in \mathrm{End}\left(\mathcal{B}_Q\right)$, $1\le a < b \le n$ dan sa

$$\mathcal{R}\left(\widetilde{t_{b,a}}\right)e_{j_1 j_2 \cdots j_a j_{a+1} j_{a+2} \cdots j_{b-1} j_b \cdots j_n} = \prod_{i=a}^{b-1} q_{j_b j_i}\, e_{j_1 j_2 \cdots j_b j_a j_{a+1} \cdots j_{b-2} j_{b-1} \cdots j_n} . \tag{28}$$

Pritom je $t_{b,a} = t_{a,b}^{-1}$ (vidi notaciju 3.9).

Specijalno, ako je $b = a+1$, onda iz (28) proizlazi da je $\mathcal{R}\left(\tilde{t}_a\right) = \mathcal{R}\left(\widetilde{t_{a+1,a}}\right)\in \mathrm{End}\left(\mathcal{B}_Q\right)$, $1\le a \le n-1$ oblika

$$\mathcal{R}\left(\tilde{t}_a\right)e_{j_1 j_2 \cdots j_a j_{a+1} \cdots j_n} = q_{j_{a+1} j_a}\, e_{j_1 j_2 \cdots j_{a+1} j_a \cdots j_n} , \tag{29}$$

gdje je $\tilde{t}_a = \widetilde{t_{a+1,a}}$ (vidi (17) i (19), str. 80).

Obrazložimo ukratko identitet (28).

Neka je $g = t_{b,a}$. Tada primjenom leme 4.5 dobivamo

$$\mathcal{R}\left(\widetilde{t_{b,a}}\right)e_{j_1 j_2 \cdots j_n} = \prod_{(r,s)\in I\left(t_{b,a}\right)} q_{j_s j_r}\, e_{j_{t_{b,a}^{-1}(1)} j_{t_{b,a}^{-1}(2)} \cdots j_{t_{b,a}^{-1}(n)}} = \prod_{(r,s)\in I\left(t_{b,a}\right)} q_{j_s j_r}\, e_{j_{t_{a,b}(1)} j_{t_{a,b}(2)} \cdots j_{t_{a,b}(n)}} . \tag{30}$$

Koristeći definiciju 3.7 dobivamo

$$t_{a,b}(a)=b,\quad t_{a,b}(i)=i-1,\ a+1\le i\le b,$$

odnosno

$$t_{a,b}(p)=p \quad \text{za svaki } \ 1\le p\le a-1 \ \text{ ili } \ b+1\le p\le n$$

iz čega proizlazi da je $\quad e_{j_{t_{a,b}(1)}j_{t_{a,b}(2)}\cdots j_{t_{a,b}(n)}}=e_{j_{t_{a,b}(1)}j_{t_{a,b}(2)}\cdots j_{t_{a,b}(a)}j_{t_{a,b}(a+1)}j_{t_{a,b}(a+2)}\cdots j_{t_{a,b}(b-1)}j_{t_{a,b}(b)}\cdots j_{t_{a,b}(n)}}$,

odnosno

$$e_{j_{t_{a,b}(1)}j_{t_{a,b}(2)}\cdots j_{t_{a,b}(n)}}=e_{j_1j_2\cdots j_bj_aj_{a+1}\cdots j_{b-2}j_{b-1}\cdots j_n}\,. \tag{31}$$

S druge strane, budući da je skup inverzija cikličke permutacije $t_{b,a}$ dan sa

$$I\left(t_{b,a}\right)=\left\{(a,b),(a+1,b),\ldots,(b-2,b),(b-1,b)\right\},$$

možemo pisati

$$\prod_{(r,s)\in I(t_{b,a})} q_{j_sj_r}=\prod_{i=a}^{b-1}q_{j_bj_i}\,. \tag{32}$$

Uvrštavanjem (31) i (32) u (30) dobiva se jednakost (28).

✎ Izračunajmo sada $\mathcal{R}\left(\widetilde{t}_a^{\,2}\right)$, gdje je $\widetilde{t}_a^{\,2}=X_{\{a,a+1\}}\,\mathrm{id}$ (lema 3.10).

Koristeći propoziciju 4.4, za svaki $1\le a\le n-1$, dobivamo

$$\begin{aligned}
\mathcal{R}\left(\widetilde{t}_a^{\,2}\right)e_{j_1j_2\cdots j_aj_{a+1}\cdots j_n} &= \mathcal{R}\left(\widetilde{t}_a\right)\cdot\mathcal{R}\left(\widetilde{t}_a\right)e_{j_1j_2\cdots j_aj_{a+1}\cdots j_n}\\
&= \mathcal{R}\left(X_{a\,a+1}\,t_a\right)\cdot\mathcal{R}\left(X_{a\,a+1}\,t_a\right)e_{j_1j_2\cdots j_aj_{a+1}\cdots j_n}\\
&= q_{j_{t_a^{-1}t_a^{-1}(a)}j_{t_a^{-1}t_a^{-1}(a+1)}}\,q_{j_{t_a^{-1}(a)}j_{t_a^{-1}(a+1)}}\,e_{j_{t_a^{-1}t_a^{-1}(1)}j_{t_a^{-1}t_a^{-1}(2)}\cdots j_{t_a^{-1}t_a^{-1}(a)}j_{t_a^{-1}t_a^{-1}(a+1)}\cdots j_{t_a^{-1}t_a^{-1}(n)}}\\
&= q_{j_aj_{a+1}}q_{j_{a+1}j_a}\,e_{j_1j_2\cdots j_aj_{a+1}\cdots j_n}=\sigma_{j_aj_{a+1}}\,e_{j_1j_2\cdots j_n}
\end{aligned} \tag{33}$$

gdje je $\sigma_{j_aj_{a+1}}=q_{j_aj_{a+1}}q_{j_{a+1}j_a}$. S druge strane je:

$$\begin{aligned}
\mathcal{R}\left(\widetilde{t}_a^{\,2}\right)e_{j_1j_2\cdots j_n} &= \mathcal{R}\left(X_{\{a,a+1\}}\,\mathrm{id}\right)e_{j_1j_2\cdots j_n}=\rho\left(X_{\{a,a+1\}}\right)\cdot R_n(\mathrm{id})e_{j_1j_2\cdots j_n}\\
&= Q_{\{a,a+1\}}\,e_{j_1j_2\cdots j_n}\,.
\end{aligned}$$

Time je:

$$Q_{\{a,a+1\}}\,e_{j_1j_2\cdots j_n}=\sigma_{j_aj_{a+1}}\,e_{j_1j_2\cdots j_n} \tag{33}$$

Primjedba 4.6

Primjenom propozicije 4.4. i leme 4.5 proizlazi

(1) element $\mathcal{R}\left(\widetilde{t_{b,a}}\right)\in \mathrm{End}\left(\mathcal{B}_Q\right)$ se faktorizira po formuli

$$\mathcal{R}\left(\widetilde{t_{b,a}}\right)e_{j_1j_2\cdots j_n}=\mathcal{R}\left(\prod_{i=a}^{b-1}\widetilde{t}_i\right)e_{j_1j_2\cdots j_n}=\left(\prod_{i=a}^{b-1}\mathcal{R}\left(\widetilde{t}_i\right)\right)e_{j_1j_2\cdots j_n} \tag{34}$$

za svaki $1\le a<b\le n$;

(2) element $\mathcal{R}\left(\widetilde{\alpha_n}\right) \in \mathrm{End}\left(\mathcal{B}_Q\right)$ je dan sa

$$\mathcal{R}\left(\widetilde{\alpha_n}\right) e_{j_1 j_2 \cdots j_n} = \mathcal{R}\left(\sum_{g \in S_n} \tilde{g}\right) e_{j_1 j_2 \cdots j_n} = \sum_{g \in S_n} \mathcal{R}\left(\tilde{g}\right) e_{j_1 j_2 \cdots j_n}$$

$$= \sum_{g \in S_n} \left(\prod_{(a,b) \in I(g)} q_{j_b j_a} \; e_{j_{g^{-1}(1)} j_{g^{-1}(2)} \cdots j_{g^{-1}(n)}} \right). \tag{35}$$

Neka je $x_1 x_2 \ldots x_n$ bilo koji niz, koji se sastoji od n članova $x_1, x_2, \ldots, x_n$.
Uvodimo oznaku

$$\underline{x} := x_1 x_2 \ldots x_n \,. \tag{36}$$

Definicija 4.7

Neka je $g \in S_n$ bilo koja permutacija iz skupa S_n i neka je $\underline{j} = j_1 j_2 \ldots j_n \in \widehat{Q}$.
Definiramo

$$g.\, \underline{j} := j_{g^{-1}(1)} j_{g^{-1}(2)} \cdots j_{g^{-1}(n)} \tag{37}$$

te uvodimo oznaku

$$\underline{k} = g.\, \underline{j}. \tag{38}$$

Pritom je svaki element k_p niza $\underline{k} = k_1 k_2 \ldots k_n$ određen relacijom $k_p = j_{g^{-1}(p)}$, $1 \le p \le n$, stoga $\underline{j} \in \widehat{Q}$ povlači $\underline{k} \in \widehat{Q}$.

Napomena:

Regularna reprezentacija bilo kojeg elementa iz algebre $\mathcal{A}_n$ na generičkom težinskom potporostoru $\mathcal{B}_Q$ je kvadratna matrica reda $n!$, kojoj su retci i stupci indeksirani s elementima baze generičkog težinskog potprostora $\mathcal{B}_Q$.
Ako je potprostor $\mathcal{B}_Q$ degenerirani, onda reprezentacija bilo kojeg elementa iz algebre $\mathcal{A}_n$ na $\mathcal{B}_Q$ je kvadrata matrica reda $\mathrm{Card}\,\widehat{Q} = \#$ permutacija multiskupa Q, čiji retci i stupci su indeksirani s elementima baze $\mathcal{B}_Q$ ($\mathrm{Card}\,\widehat{Q}$ je djeljitelj od $n!$).

Primjedba 4.8

U suglasnosti s definicijom 4.7, neka je $\underline{j} = j_1 j_2 \ldots j_n \in \widehat{Q} = S_n Q$, $g \in S_n$ takva da je $\underline{k} = g.\, \underline{j}$ (tj. $k_p = j_{g^{-1}(p)}$, $1 \le p \le n$). Tada je $\left(\underline{k}, \underline{j}\right)$-ti element matrice $\mathcal{R}\left(\tilde{g}\right)$ jednak

$\prod_{(a,b)\in I(g)} q_{j_b j_a}$. U protivnom, ako je $\underline{k} \neq g\centerdot \underline{j}$, onda je $\left(\underline{k},\underline{j}\right)$-ti element matrice $\mathcal{R}\left(\tilde{g}\right)$ jednak nuli i pišemo

$$\left(\mathcal{R}\left(\tilde{g}\right)\right)_{\underline{k},\underline{j}} = \begin{cases} \prod_{(a,b)\in I(g)} q_{j_b j_a} & \text{ako je } \underline{k} = g\centerdot \underline{j} \\ 0 & \text{inače} \end{cases} \tag{39}$$

$\underline{j},\underline{k} \in \hat{Q}$, $g \in S_n$.

Pod $\left(\underline{k},\underline{j}\right)$-tim elementom matrice $\mathcal{R}\left(\tilde{g}\right)$ misli se na $\left(e_{\underline{k}}, e_{\underline{j}}\right)$-ti element te matrice, tj. promatra se onaj element matrice $\mathcal{R}\left(\tilde{g}\right)$ koji se nalazi na poziciji $e_{\underline{k}}$-tog retka i $e_{\underline{j}}$-tog stupca.

Specijalno ako je $I(g) = \emptyset$, onda je $g = \mathrm{id}$ pa je $\tilde{g} = 1\cdot \mathrm{id} = \mathrm{id}$, stoga iz (39) proizlazi

$$\left(\mathcal{R}\left(\mathrm{id}\right)\right)_{\underline{k},\underline{j}} = \begin{cases} 1 & \text{ako je } \underline{k} = \underline{j} \\ 0 & \text{inače} \end{cases} \tag{40}$$

pa je $\mathcal{R}\left(\mathrm{id}\right) = I$ jedinična matrica.

Uzimajući u obzir identitet (33) imamo da je $\left(\mathcal{R}\left(\tilde{t}_a\right)\right)^2 \in \mathrm{End}\left(\mathcal{B}_Q\right)$, $1 \le a \le n-1$ dijagonalna matrica kojoj je $\underline{j}$-ti dijagonalni element jednak $\sigma_{j_a j_{a+1}}$ i pišemo

$$\left(\left(\mathcal{R}\left(\tilde{t}_a\right)\right)^2\right)_{\underline{j},\underline{j}} = \sigma_{j_a j_{a+1}}. \tag{41}$$

Pritom je $\sigma_{j_a j_{a+1}} = q_{j_a j_{a+1}} q_{j_{a+1} j_a}$.

Propozicija 4.9

Neka je $\mathcal{R} : \mathcal{A}_n \to \mathrm{End}\left(\mathcal{B}_Q\right)$ regularna reprezentacija zakrenute grupovne algebre $\mathcal{A}_n$ na generičkom težinskom potprostoru $\mathcal{B}_Q$ algebre $\mathcal{B}$, $\mathrm{Card}\, Q = n$, $n \ge 2$. Označimo sa

$$\mathcal{A}_Q := \mathcal{R}\left(\widetilde{\alpha_n}\right). \tag{42}$$

Tada je $\left(\underline{k},\underline{j}\right)$-ti element matrice $\mathcal{A}_Q$ oblika

$$\left(\mathcal{A}_Q\right)_{\underline{k},\underline{j}} = \prod_{(a,b)\in I(g)} q_{j_b j_a}, \tag{43}$$

gdje je $\underline{k} = g\centerdot \underline{j}$.

Dokaz:

Uzimajući u obzir da je element $\mathcal{R}\left(\widetilde{\alpha_n}\right) \in \mathrm{End}\left(\mathcal{B}_Q\right)$ dan sa

$$\mathcal{R}\left(\widetilde{\alpha_n}\right) e_{j_1 j_2 \ldots j_n} = \sum_{g \in S_n} \left(\prod_{(a,b) \in I(g)} q_{j_b j_a} \, e_{j_{g^{-1}(1)} j_{g^{-1}(2)} \ldots j_{g^{-1}(n)}} \right)$$

(primjedba 4.6) te da je prema definiciji 4.7 svaki element k_p niza $\underline{k} = k_1 k_2 \ldots k_n$ određen relacijom $k_p = j_{g^{-1}(p)}$, $1 \le p \le n$, gdje je $\underline{j} = j_1 j_2 \ldots j_n \in \widehat{Q}$, $g \in S_n$, zaključujemo da se matrica $\mathcal{A}_Q = \mathcal{R}\left(\widetilde{\alpha_n}\right)$ može zapisati u obliku $\mathcal{A}_Q \, e_{\underline{j}} = \sum_{g \in S_n} \left(\prod_{(a,b) \in I(g)} q_{j_b j_a} \, e_{\underline{k}} \right)$ pa je $\left(\underline{k}, \underline{j}\right)$-ti element matrice $\mathcal{A}_Q$ jednak izrazu $\prod_{(a,b) \in I(g)} q_{j_b j_a}$ i pišemo $\left(\mathcal{A}_Q\right)_{\underline{k},\underline{j}} = \prod_{(a,b) \in I(g)} q_{j_b j_a}$.

Drugim riječima, neka je $\underline{j} \in \widehat{Q}$ fiksiran. Tada svaka permutacija $\underline{k} \in \widehat{Q}$, definirana sa $\underline{k} = g . \underline{j}$, je jednoznačno određena nekom permutacijom $g \in S_n$.

Napomena:

Specijalno, ako je $\mathrm{Card}\, Q = 1$, onda je $\mathcal{A}_Q$ kvadratna matrica 1. reda, kojoj je jedini element jednak jedan (trivijalan slučaj), stoga će se u nastavku podrazumijevati da je $\mathrm{Card}\, Q = n \ge 2$.

Komentar 4.10

Ako je $\mathcal{R} : \mathcal{A}_n \to \mathrm{End}\left(\mathcal{B}_Q\right)$ reprezentacija zakrenute grupovne algebre $\mathcal{A}_n$ na degeneriranom težinskom potprostoru $\mathcal{B}_Q$ algebre $\mathcal{B}$, onda je $\left(\underline{k}, \underline{j}\right)$-ti element matrice $\mathcal{A}_Q$ oblika

$$\left(\mathcal{A}_Q\right)_{\underline{k},\underline{j}} = \sum_{g \in g(\underline{k},\underline{j})} \left(\prod_{(a,b) \in I(g)} q_{j_b j_a} \right), \tag{44}$$

gdje je $g\left(\underline{k}, \underline{j}\right) := \left\{ g \in S_n \mid k_p = j_{g^{-1}(p)} \text{ za svaki } 1 \le p \le n \right\}$.

Primjer 4.11

Promatrajmo $S_3 = \{123, 132, 312, 321, 231, 213\}$ skup svih permutacija tročlanog skupa $\{1, 2, 3\}$.

Neka je tada je

$$g_1 = 123 = id, \qquad g_1^{-1} = 123 = id,$$

$$g_2 = 132, \qquad g_2^{-1} = 132,$$

$$g_3 = 312, \qquad g_3^{-1} = 231,$$

$$g_4 = 321, \qquad g_4^{-1} = 321,$$

$$g_5 = 231, \qquad g_5^{-1} = 312,$$

$$g_6 = 213, \qquad g_6^{-1} = 213.$$

Pripadni skupovi inverzija su:

$$I(g_1) = \varnothing, \qquad I\left(g_1^{-1}\right) = \varnothing,$$

$$I(g_2) = \{(2,3)\}, \qquad I\left(g_2^{-1}\right) = I(g_2) = \{(2,3)\},$$

$$I(g_3) = \{(1,2),(1,3)\}, \qquad I\left(g_3^{-1}\right) = I(g_5) = \{(1,3),(2,3)\},$$

$$I(g_4) = \{(1,2),(1,3),(2,3)\}, \qquad I\left(g_4^{-1}\right) = I(g_4) = \{(1,2),(1,3),(2,3)\},$$

$$I(g_5) = \{(1,3),(2,3)\}, \qquad I\left(g_5^{-1}\right) = I(g_3) = \{(1,2),(1,3)\},$$

$$I(g_6) = \{(1,2)\}, \qquad I\left(g_6^{-1}\right) = I(g_6) = \{(1,2)\}.$$

Svakoj permutaciji $g_i \in S_3$, $1 \le i \le 6$ pridružen je odgovarajući element $\widetilde{g_i} \in \mathcal{A}_3$ definiran sa $\widetilde{g_i} = \prod_{(a',b') \in I\left(g^{-1}\right)} X_{a'b'}\, g_i$, stoga imamo

$$\widetilde{g_1} = 1 \cdot g_1 = id, \qquad \widetilde{g_2} = X_{23}\, g_2,$$

$$\widetilde{g_3} = X_{13}\, X_{23}\, g_3, \qquad \widetilde{g_4} = X_{12}\, X_{13}\, X_{23}\, g_4,$$

$$\widetilde{g_5} = X_{12}\, X_{13}\, g_5, \qquad \widetilde{g_6} = X_{12}\, g_6.$$

Neka je $\mathcal{R} : \mathcal{A}_3 \to \mathrm{End}\left(\mathcal{B}_Q\right)$ regularna reprezentacija algebre $\mathcal{A}_3$ na generičkom težinskom potprostoru $\mathcal{B}_Q = span_{\mathbb{C}}\left\{e_{j_1 j_2 j_n} \mid j_1 j_2 j_3 \in \widehat{Q}\right\}$ pridruženom skupu $Q = \{l_1, l_2, l_3\} \subseteq \mathcal{N}$, gdje je $\widehat{Q} = S_3 Q = \left\{\sigma\left(l_1 l_2 l_3\right) \mid \sigma \in S_3\right\}$.

Tada primjenom leme 4.5 imamo da su elementi $\mathcal{R}(g_i) \in \mathrm{End}\left(\mathcal{B}_Q\right)$, $1 \le i \le 6$ dani sa

$$\mathcal{R}\left(\widetilde{g_i}\right) e_{j_1 j_2 j_3} = \prod_{(a,b) \in I(g_i)} q_{j_b j_a}\, e_{j_{g_i^{-1}(1)} j_{g_i^{-1}(2)} j_{g_i^{-1}(3)}},$$

odnosno

$$\left.\begin{aligned}
\mathcal{R}\left(\widetilde{g_1}\right) e_{j_1j_2j_3} &= 1\, e_{j_1j_2j_3} ,\\
\mathcal{R}\left(\widetilde{g_2}\right) e_{j_1j_2j_3} &= q_{j_3j_2}\, e_{j_1j_3j_2} ,\\
\mathcal{R}\left(\widetilde{g_3}\right) e_{j_1j_2j_3} &= q_{j_3j_1} q_{j_2j_1}\, e_{j_2j_3j_1} ,\\
\mathcal{R}\left(\widetilde{g_4}\right) e_{j_1j_2j_3} &= q_{j_2j_1} q_{j_3j_1} q_{j_3j_2}\, e_{j_3j_2j_1} ,\\
\mathcal{R}\left(\widetilde{g_5}\right) e_{j_1j_2j_3} &= q_{j_3j_1} q_{j_3j_2}\, e_{j_3j_1j_2} ,\\
\mathcal{R}\left(\widetilde{g_6}\right) e_{j_1j_2j_3} &= q_{j_2j_1}\, e_{j_2j_1j_3} ,
\end{aligned}\right\} \tag{45}$$

stoga razlikujemo sljedeće kvadratne matrice reda $3! = 6$

$$\mathcal{R}\left(\widetilde{g_1}\right) = \begin{array}{c} \\ e_{j_1j_2j_3} \\ e_{j_1j_3j_2} \\ e_{j_3j_1j_2} \\ e_{j_3j_2j_1} \\ e_{j_2j_3j_1} \\ e_{j_2j_1j_3} \end{array} \begin{array}{c} \begin{array}{cccccc} e_{j_1j_2j_3} & e_{j_1j_3j_2} & e_{j_3j_1j_2} & e_{j_3j_2j_1} & e_{j_2j_3j_1} & e_{j_2j_1j_3} \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} = I$$

$$\mathcal{R}\left(\widetilde{g_2}\right) = \begin{array}{c} \\ e_{j_1j_2j_3} \\ e_{j_1j_3j_2} \\ e_{j_3j_1j_2} \\ e_{j_3j_2j_1} \\ e_{j_2j_3j_1} \\ e_{j_2j_1j_3} \end{array} \begin{array}{c} \begin{array}{cccccc} e_{j_1j_2j_3} & e_{j_1j_3j_2} & e_{j_3j_1j_2} & e_{j_3j_2j_1} & e_{j_2j_3j_1} & e_{j_2j_1j_3} \end{array} \\ \begin{bmatrix} 0 & q_{j_2j_3} & 0 & 0 & 0 & 0 \\ q_{j_3j_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_{j_1j_2} & 0 & 0 \\ 0 & 0 & q_{j_2j_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q_{j_3j_1} \\ 0 & 0 & 0 & 0 & q_{j_1j_3} & 0 \end{bmatrix} \end{array}$$

$$\mathcal{R}\left(\widetilde{g_3}\right) = \begin{array}{c} \\ e_{j_1j_2j_3} \\ e_{j_1j_3j_2} \\ e_{j_3j_1j_2} \\ e_{j_3j_2j_1} \\ e_{j_2j_3j_1} \\ e_{j_2j_1j_3} \end{array} \begin{array}{c} \begin{array}{cccccc} e_{j_1j_2j_3} & e_{j_1j_3j_2} & e_{j_3j_1j_2} & e_{j_3j_2j_1} & e_{j_2j_3j_1} & e_{j_2j_1j_3} \end{array} \\ \begin{bmatrix} 0 & 0 & q_{j_2j_3}q_{j_1j_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q_{j_3j_2}q_{j_1j_2} \\ 0 & 0 & 0 & 0 & q_{j_1j_2}q_{j_3j_2} & 0 \\ 0 & q_{j_2j_1}q_{j_3j_1} & 0 & 0 & 0 & 0 \\ q_{j_3j_1}q_{j_2j_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_{j_1j_3}q_{j_2j_3} & 0 & 0 \end{bmatrix} \end{array}$$

$$
\mathcal{R}\left(\widetilde{g_4}\right)=
\begin{array}{cc}
 & \begin{array}{cccccc} e_{j_1j_2j_3} & e_{j_1j_3j_2} & e_{j_3j_1j_2} & e_{j_3j_2j_1} & e_{j_2j_3j_1} & e_{j_2j_1j_3} \end{array} \\
\begin{array}{c} e_{j_1j_2j_3} \\ e_{j_1j_3j_2} \\ e_{j_3j_1j_2} \\ e_{j_3j_2j_1} \\ e_{j_2j_3j_1} \\ e_{j_2j_1j_3} \end{array} &
\begin{bmatrix}
0 & 0 & 0 & q_{j_2j_3}q_{j_1j_3}q_{j_1j_2} & 0 & 0 \\
0 & 0 & 0 & 0 & q_{j_3j_2}q_{j_1j_2}q_{j_1j_3} & 0 \\
0 & 0 & 0 & 0 & 0 & q_{j_1j_2}q_{j_3j_2}q_{j_3j_1} \\
q_{j_2j_1}q_{j_3j_1}q_{j_3j_2} & 0 & 0 & 0 & 0 & 0 \\
0 & q_{j_3j_1}q_{j_2j_1}q_{j_2j_3} & 0 & 0 & 0 & 0 \\
0 & 0 & q_{j_1j_3}q_{j_2j_3}q_{j_2j_1} & 0 & 0 & 0
\end{bmatrix}
\end{array}
$$

$$
\mathcal{R}\left(\widetilde{g_5}\right)=
\begin{array}{cc}
 & \begin{array}{cccccc} e_{j_1j_2j_3} & e_{j_1j_3j_2} & e_{j_3j_1j_2} & e_{j_3j_2j_1} & e_{j_2j_3j_1} & e_{j_2j_1j_3} \end{array} \\
\begin{array}{c} e_{j_1j_2j_3} \\ e_{j_1j_3j_2} \\ e_{j_3j_1j_2} \\ e_{j_3j_2j_1} \\ e_{j_2j_3j_1} \\ e_{j_2j_1j_3} \end{array} &
\begin{bmatrix}
0 & 0 & 0 & 0 & q_{j_1j_2}q_{j_1j_3} & 0 \\
0 & 0 & 0 & q_{j_1j_3}q_{j_1j_2} & 0 & 0 \\
q_{j_3j_1}q_{j_3j_2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & q_{j_3j_2}q_{j_3j_1} \\
0 & 0 & q_{j_2j_3}q_{j_2j_1} & 0 & 0 & 0 \\
0 & q_{j_2j_1}q_{j_2j_3} & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}
$$

$$
\mathcal{R}\left(\widetilde{g_6}\right)=
\begin{array}{cc}
 & \begin{array}{cccccc} e_{j_1j_2j_3} & e_{j_1j_3j_2} & e_{j_3j_1j_2} & e_{j_3j_2j_1} & e_{j_2j_3j_1} & e_{j_2j_1j_3} \end{array} \\
\begin{array}{c} e_{j_1j_2j_3} \\ e_{j_1j_3j_2} \\ e_{j_3j_1j_2} \\ e_{j_3j_2j_1} \\ e_{j_2j_3j_1} \\ e_{j_2j_1j_3} \end{array} &
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & q_{j_1j_2} \\
0 & 0 & q_{j_1j_3} & 0 & 0 & 0 \\
0 & q_{j_3j_1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & q_{j_3j_2} & 0 \\
0 & 0 & 0 & q_{j_2j_3} & 0 & 0 \\
q_{j_2j_1} & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}
$$

U suglasnosti sa izrazom (35) imamo

$$
\begin{aligned}
\mathcal{R}\left(\widetilde{\alpha_3}\right)e_{j_1j_2j_3} &= \sum_{g\in S_3}\mathcal{R}\left(\tilde{g}\right)e_{j_1j_2j_3} = \sum_{i=1}^{6}\mathcal{R}\left(\widetilde{g_i}\right)e_{j_1j_2j_3} \\
&= 1\,e_{j_1j_2j_3} + q_{j_3j_2}\,e_{j_1j_3j_2} + q_{j_3j_1}q_{j_3j_2}\,e_{j_3j_1j_2} + q_{j_2j_1}q_{j_3j_1}q_{j_3j_2}\,e_{j_3j_2j_1} + q_{j_3j_1}q_{j_2j_1}\,e_{j_2j_3j_1} + q_{j_2j_1}\,e_{j_2j_1j_3}
\end{aligned}
\tag{46}
$$

pa je matrica $\mathcal{A}_Q = \mathcal{R}\left(\widetilde{\alpha_3}\right)$ dana sa

$$
\mathcal{A}_Q=
\begin{array}{cc}
 & \begin{array}{cccccc} e_{j_1j_2j_3} & e_{j_1j_3j_2} & e_{j_3j_1j_2} & e_{j_3j_2j_1} & e_{j_2j_3j_1} & e_{j_2j_1j_3} \end{array} \\
\begin{array}{c} e_{j_1j_2j_3} \\ e_{j_1j_3j_2} \\ e_{j_3j_1j_2} \\ e_{j_3j_2j_1} \\ e_{j_2j_3j_1} \\ e_{j_2j_1j_3} \end{array} &
\begin{bmatrix}
1 & q_{j_2j_3} & q_{j_2j_3}q_{j_1j_3} & q_{j_2j_3}q_{j_1j_3}q_{j_1j_2} & q_{j_1j_2}q_{j_1j_3} & q_{j_1j_2} \\
q_{j_3j_2} & 1 & q_{j_1j_3} & q_{j_1j_3}q_{j_1j_2} & q_{j_3j_2}q_{j_1j_2}q_{j_1j_3} & q_{j_3j_2}q_{j_1j_2} \\
q_{j_3j_1}q_{j_3j_2} & q_{j_3j_1} & 1 & q_{j_1j_2} & q_{j_1j_2}q_{j_3j_2} & q_{j_1j_2}q_{j_3j_2}q_{j_3j_1} \\
q_{j_2j_1}q_{j_3j_1}q_{j_3j_2} & q_{j_2j_1}q_{j_3j_1} & q_{j_2j_1} & 1 & q_{j_3j_2} & q_{j_3j_2}q_{j_3j_1} \\
q_{j_3j_1}q_{j_2j_1} & q_{j_3j_1}q_{j_2j_1}q_{j_2j_3} & q_{j_2j_3}q_{j_2j_1} & q_{j_2j_3} & 1 & q_{j_3j_1} \\
q_{j_2j_1} & q_{j_2j_1}q_{j_2j_3} & q_{j_1j_3}q_{j_2j_3}q_{j_2j_1} & q_{j_1j_3}q_{j_2j_3} & q_{j_1j_3} & 1
\end{bmatrix}
\end{array}
$$

(usporediti s propozicijom 4.9).

Specijalno, ako je $l_1 = 1$, $l_2 = 2$, $l_3 = 3$, onda je skupu $\mathrm{Q} = \{1,2,3\}$ pridružen generički težinski potprostor $\mathcal{B}_{123} = \mathcal{B}_{\mathrm{Q}} = span_{\mathbb{C}}\{\mathrm{e}_{123}, \mathrm{e}_{132}, \mathrm{e}_{312}, \mathrm{e}_{321}, \mathrm{e}_{231}, \mathrm{e}_{213}\}$. U ovom slučaju se izraz (46) zapisuje u obliku

$$\mathcal{R}\left(\widetilde{\alpha_3}\right)\mathrm{e}_{123} = 1\,\mathrm{e}_{123} + \mathrm{q}_{32}\,\mathrm{e}_{132} + \mathrm{q}_{31}\mathrm{q}_{32}\,\mathrm{e}_{312} + \mathrm{q}_{21}\mathrm{q}_{31}\mathrm{q}_{32}\,\mathrm{e}_{321} + \mathrm{q}_{31}\mathrm{q}_{21}\,\mathrm{e}_{231} + \mathrm{q}_{21}\,\mathrm{e}_{213}, \quad (47)$$

stoga je matrica $\mathcal{A}_{123} := \mathcal{A}_{\mathrm{Q}} = \mathcal{R}\left(\widetilde{\alpha_3}\right)$ dana sa

$$\mathcal{A}_{123} = \begin{array}{c} \\ \mathrm{e}_{123} \\ \mathrm{e}_{132} \\ \mathrm{e}_{312} \\ \mathrm{e}_{321} \\ \mathrm{e}_{231} \\ \mathrm{e}_{213} \end{array}\!\!\begin{array}{c} \begin{array}{cccccc} \mathrm{e}_{123} & \mathrm{e}_{132} & \mathrm{e}_{312} & \mathrm{e}_{321} & \mathrm{e}_{231} & \mathrm{e}_{213} \end{array} \\ \begin{bmatrix} 1 & \mathrm{q}_{23} & \mathrm{q}_{23}\mathrm{q}_{13} & \mathrm{q}_{23}\mathrm{q}_{13}\mathrm{q}_{12} & \mathrm{q}_{12}\mathrm{q}_{13} & \mathrm{q}_{12} \\ \mathrm{q}_{32} & 1 & \mathrm{q}_{13} & \mathrm{q}_{13}\mathrm{q}_{12} & \mathrm{q}_{32}\mathrm{q}_{12}\mathrm{q}_{13} & \mathrm{q}_{32}\mathrm{q}_{12} \\ \mathrm{q}_{31}\mathrm{q}_{32} & \mathrm{q}_{31} & 1 & \mathrm{q}_{12} & \mathrm{q}_{12}\mathrm{q}_{32} & \mathrm{q}_{12}\mathrm{q}_{32}\mathrm{q}_{31} \\ \mathrm{q}_{21}\mathrm{q}_{31}\mathrm{q}_{32} & \mathrm{q}_{21}\mathrm{q}_{31} & \mathrm{q}_{21} & 1 & \mathrm{q}_{32} & \mathrm{q}_{32}\mathrm{q}_{31} \\ \mathrm{q}_{31}\mathrm{q}_{21} & \mathrm{q}_{31}\mathrm{q}_{21}\mathrm{q}_{23} & \mathrm{q}_{23}\mathrm{q}_{21} & \mathrm{q}_{23} & 1 & \mathrm{q}_{31} \\ \mathrm{q}_{21} & \mathrm{q}_{21}\mathrm{q}_{23} & \mathrm{q}_{13}\mathrm{q}_{23}\mathrm{q}_{21} & \mathrm{q}_{13}\mathrm{q}_{23} & \mathrm{q}_{13} & 1 \end{bmatrix} \end{array}.$$

Uočimo da je matrica $\mathcal{A}_{123}$ jednaka Varchenkovoj matrici $\mathcal{B}$ diskriminantnog orijentiranog aranžmana C_2 (primjer 1.12).

Komentar 4.12

Prokomentirajmo ukratko propoziciju 4.9 s obzirom na matricu $\mathcal{A}_{\mathrm{Q}}$ iz primjera 4.11. Neka je $\underline{\mathrm{j}} = \mathrm{j}_1\mathrm{j}_2\mathrm{j}_3 \in \widehat{\mathrm{Q}} = \mathrm{S}_3\mathrm{Q} = \{\sigma(l_1l_2l_3) \mid \sigma \in \mathrm{S}_3\}$ fiksna permutacija u skupu $\widehat{\mathrm{Q}}$, gdje je $\mathrm{Q} = \{l_1, l_2, l_3\} \subseteq \mathcal{N}$.

Tada je relacijom $\underline{\mathrm{k}} = \mathrm{g}.\,\underline{\mathrm{j}} = \mathrm{j}_{\mathrm{g}^{-1}(1)}\mathrm{j}_{\mathrm{g}^{-1}(2)}\mathrm{j}_{\mathrm{g}^{-1}(3)}$ jednoznačno određena svaka permutacija $\underline{\mathrm{k}} \in \widehat{\mathrm{Q}}$ s obzirom na neki $\mathrm{g} \in \mathrm{S}_3$, stoga je $\left(\mathrm{e}_{\underline{\mathrm{k}}}, \mathrm{e}_{\underline{\mathrm{j}}}\right)$-ti element matrice $\mathcal{A}_{\mathrm{Q}}$ jednak $\prod_{(a,b)\in I(g)} \mathrm{q}_{\mathrm{j}_b\mathrm{j}_a}$. Konkretno neka je $\underline{\mathrm{j}} = \mathrm{j}_1\mathrm{j}_2\mathrm{j}_3$, tada dobivamo sljedeće

ako je	onda je	pa je $\left(\mathrm{e}_{\underline{\mathrm{k}}}, \mathrm{e}_{\underline{\mathrm{j}}}\right)$-ti element dan sa
$\mathrm{g} = 123$ ($\mathrm{g}^{-1} = 123$),	$\underline{\mathrm{k}} = \mathrm{j}_1\mathrm{j}_2\mathrm{j}_3$	1,
$\mathrm{g} = 132$ ($\mathrm{g}^{-1} = 132$),	$\underline{\mathrm{k}} = \mathrm{j}_1\mathrm{j}_3\mathrm{j}_2$	$\mathrm{q}_{\mathrm{j}_3\mathrm{j}_2}$,
$\mathrm{g} = 312$ ($\mathrm{g}^{-1} = 231$),	$\underline{\mathrm{k}} = \mathrm{j}_2\mathrm{j}_3\mathrm{j}_1$	$\mathrm{q}_{\mathrm{j}_3\mathrm{j}_1}\mathrm{q}_{\mathrm{j}_2\mathrm{j}_1}$,
$\mathrm{g} = 321$ ($\mathrm{g}^{-1} = 321$),	$\underline{\mathrm{k}} = \mathrm{j}_3\mathrm{j}_2\mathrm{j}_1$	$\mathrm{q}_{\mathrm{j}_2\mathrm{j}_1}\mathrm{q}_{\mathrm{j}_3\mathrm{j}_1}\mathrm{q}_{\mathrm{j}_3\mathrm{j}_2}$,

$g=231 \ \ (g^{-1}=312)$,	$\underline{k}=j_3j_1j_2$	$q_{j_3j_1}q_{j_3j_2}$,
$g=213 \ \ (g^{-1}=213)$,	$\underline{k}=j_2j_1j_3$	$q_{j_2j_1}$.

Navedeni elementi su ujedno elementi prvog stupca matrice $\mathcal{A}_Q$, jer je $\underline{j}=j_1j_2j_3$.

Ispišimo sada elemente $e_{\underline{j}}$-tog stupca matrice $\mathcal{A}_Q$ za $\underline{j}=j_3j_1j_2\in\widehat{Q}$.

Analogno gore navedenom dobivamo

ako je	onda je	pa je $\left(e_{\underline{k}},e_{\underline{j}}\right)$-ti element jednak
$g=123 \ \ (g^{-1}=123)$,	$\underline{k}=j_3j_1j_2$	1,
$g=132 \ \ (g^{-1}=132)$,	$\underline{k}=j_3j_2j_1$	$q_{j_2j_1}$,
$g=312 \ \ (g^{-1}=231)$,	$\underline{k}=j_1j_2j_3$	$q_{j_1j_3}q_{j_2j_3}$,
$g=321 \ \ (g^{-1}=321)$,	$\underline{k}=j_2j_1j_3$	$q_{j_1j_3}q_{j_2j_3}q_{j_2j_1}$,
$g=231 \ \ (g^{-1}=312)$,	$\underline{k}=j_2j_3j_1$	$q_{j_2j_3}q_{j_2j_1}$,
$g=213 \ \ (g^{-1}=213)$,	$\underline{k}=j_1j_3j_2$	$q_{j_1j_3}$.

Na opisani način dobivaju se elementi bilo kojeg stupca matrice $\mathcal{A}_Q$, a samim time i bilo koji element matrice $\mathcal{A}_Q$.

Napomena:

Pretpostavimo da je $l_2=l_1$.

Tada je multiskupu $Q=\{l_1,l_1,l_3\}$ pridruženi težinski potprostor $\mathcal{B}_Q$ degeneriran.

U ovom slučaju imamo da je matrica $\mathcal{A}_Q=\mathcal{R}\left(\widetilde{\alpha_3}\right)\in\operatorname{End}\left(\mathcal{B}_Q\right)$, $Q=l_1l_1l_3$ dana sa

$$
\mathcal{A}_Q=\begin{array}{c c}
 & \begin{array}{ccc} \qquad e_{j_1j_1j_3} \qquad & \qquad e_{j_1j_3j_1} \qquad & \qquad e_{j_3j_1j_1} \qquad \end{array} \\
\begin{array}{c} e_{j_1j_1j_3} \\ e_{j_1j_3j_1} \\ e_{j_3j_1j_1} \end{array} & \begin{bmatrix} 1+q_{j_1j_1} & q_{j_1j_3}\left(1+q_{j_1j_1}\right) & q_{j_1j_3}^2\left(1+q_{j_1j_1}\right) \\ q_{j_3j_1}\left(1+q_{j_1j_1}\right) & 1+q_{j_1j_1}\sigma_{j_1j_3} & q_{j_1j_3}\left(1+q_{j_1j_1}\right) \\ q_{j_3j_1}^2\left(1+q_{j_1j_1}\right) & q_{j_3j_1}\left(1+q_{j_1j_1}\right) & 1+q_{j_1j_1} \end{bmatrix}
\end{array}. \tag{48}
$$

Primijetimo da se specijalizacijom indeksa $j_2=j_1$ u izrazu (46) dobiva

$$
\begin{aligned}
\mathcal{R}\left(\widetilde{\alpha_3}\right)e_{j_1j_1j_3} &= \left(1+q_{j_1j_1}\right)e_{j_1j_1j_3}+\left(q_{j_3j_1}+q_{j_3j_1}q_{j_1j_1}\right)e_{j_1j_3j_1}+\left(q_{j_3j_1}^2+q_{j_1j_1}q_{j_3j_1}^2\right)e_{j_3j_1j_1} \\
&= \left(1+q_{j_1j_1}\right)e_{j_1j_1j_3}+q_{j_3j_1}\left(1+q_{j_1j_1}\right)e_{j_1j_3j_1}+q_{j_3j_1}^2\left(1+q_{j_1j_1}\right)e_{j_3j_1j_1} \qquad (49)
\end{aligned}
$$

te da se analogno dobiva

$$\mathcal{A}_Q\, e_{j_1 j_3 j_1} = q_{j_1 j_3}\left(1+q_{j_1 j_1}\right) e_{j_1 j_1 j_3} + \left(1+q_{j_1 j_1}\sigma_{j_1 j_3}\right) e_{j_1 j_3 j_1} + q_{j_3 j_1}\left(1+q_{j_1 j_1}\right) e_{j_3 j_1 j_1},$$

$$\mathcal{A}_Q\, e_{j_3 j_1 j_1} = q_{j_1 j_3}^2\left(1+q_{j_1 j_1}\right) e_{j_1 j_1 j_3} + q_{j_1 j_3}\left(1+q_{j_1 j_1}\right) e_{j_1 j_3 j_1} + \left(1+q_{j_1 j_1}\right) e_{j_3 j_1 j_1}.$$

Pritom je $\sigma_{j_1 j_3} = q_{j_1 j_3} q_{j_3 j_1}$.

Specijalno, ako je $l_1 = 1$, $l_3 = 3$, tj. $j_1 = 1$ $j_3 = 3$, onda je matrica $\mathcal{A}_{113} := \mathcal{A}_Q$, ($Q = 113$) dana sa (48) oblika

$$\mathcal{A}_{113} = \begin{array}{c} \\ e_{113} \\ e_{131} \\ e_{311} \end{array} \begin{array}{c} \begin{array}{ccc} e_{113} & e_{131} & e_{311} \end{array} \\ \begin{bmatrix} 1+q_{11} & q_{13}\left(1+q_{11}\right) & q_{13}^2\left(1+q_{11}\right) \\ q_{31}\left(1+q_{11}\right) & 1+q_{11}\sigma_{13} & q_{13}\left(1+q_{11}\right) \\ q_{31}^2\left(1+q_{11}\right) & q_{31}\left(1+q_{11}\right) & 1+q_{11} \end{bmatrix} \end{array}.$$

Kažemo da je matrica $\mathcal{A}_{113}$ tzv. degenerirana Varchenkova matrica, jer je reducirana matrica Varchenkove matrice $\mathcal{A}_{123}$.

Prirodno se nameće sljedeći problem:

> kako glasi formula za izračunavanje determinante matrice $\mathcal{A}_Q \in \mathrm{End}\left(\mathcal{B}_Q\right)$ neovisno o tome da li je potprostor $\mathcal{B}_Q$ generički ili degenerirani.

S tom motivacijom će se najprije izvesti formula po kojoj će se faktorizirati matrica $\mathcal{A}_Q \in \mathrm{End}\left(\mathcal{B}_Q\right)$, gdje je $\mathcal{B}_Q$ generički težinski potprostor algebre $\mathcal{B}$.

4. 1. Faktorizacija matrice $\mathcal{A}_Q$ i njene determinante

U nastavku će se izvesti formula za faktorizaciju matrice $\mathcal{A}_Q$ po uzoru na razmatranja provedena u trećem poglavlju. Pritom će se pretpostavljati da je $\mathcal{B}_Q$ generički težinski potprostor algebre $\mathcal{B}$ i da je $\mathcal{A}_Q = \mathcal{R}\left(\widetilde{\alpha_n}\right) \in \mathrm{End}\left(\mathcal{B}_Q\right)$.

Uvodimo oznake

$$T_{b,a} := \mathcal{R}\left(\widetilde{t_{b,a}}\right) \quad \text{za svaki } 1 \le a < b \le n, \tag{50}$$

$$T_a := \mathcal{R}\left(\widetilde{t_a}\right) \quad \text{za svaki } 1 \le a \le n-1. \tag{51}$$

U suglasnosti s gore navedenim imamo sljedeće

(1) $\left(\underline{k},\underline{j}\right)$-ti element cikličke matrice $T_{b,a}$, $1 \le a < b \le n$ je dan sa

$$\left(T_{b,a}\right)_{\underline{k},\underline{j}} = \begin{cases} \prod_{i=a}^{b-1} q_{j_b j_i} & \text{ako je } \underline{k} = t_{b,a} \bullet \underline{j} \\ 0 & \text{inače} \end{cases} \tag{52}$$

Pritom je $t_{b,a} \bullet \underline{j} = t_{b,a} \bullet j_1 j_2 \dots j_a j_{a+1} j_{a+2} \dots j_{b-1} j_b \dots j_n = j_1 j_2 \dots j_b j_a j_{a+1} \dots j_{b-2} j_{b-1} \dots j_n$.

(2) $\left(\underline{k},\underline{j}\right)$-ti element elementarne cikličke matrice T_a, $1 \le a \le n-1$ je dan sa

$$\left(T_a\right)_{\underline{k},\underline{j}} = \begin{cases} q_{j_{a+1} j_a} & \text{ako je } \underline{k} = t_a \bullet \underline{j} \\ 0 & \text{inače} \end{cases} \tag{53}$$

gdje je $t_a \bullet \underline{j} = t_a \bullet j_1 j_2 \dots j_a j_{a+1} \dots j_n = j_1 j_2 \dots j_{a+1} j_a \dots j_n$.

Koristeći gore uvedene oznake imamo da se izraz (34), primjedba 4.6, može pisati u obliku

$$T_{b,a} = \prod_{i=a}^{b-1} T_i , \tag{54}$$

što povlači da je

$$\det T_{b,a} = \prod_{i=a}^{b-1} \det T_i \tag{55}$$

za svaki $1 \le a < b \le n$.

Analogno se primjenom oznake (51) na izraz (33) dobiva

$$T_a^2\, e_{j_1 j_2 \cdots j_n} = \sigma_{j_a j_{a+1}}\, e_{j_1 j_2 \cdots j_n} , \tag{56}$$

ili kraće $$T_a^2\, e_{\underline{j}} = \sigma_{j_a j_{a+1}}\, e_{\underline{j}} , \quad 1 \le a \le n-1$$

Pritom je $T_a^2 = \left(T_a\right)^2$ dijagonalna matrica kojoj je $\underline{j}$-ti dijagonalni element jednak $\sigma_{j_a j_{a+1}}$. Uzimajući u obzir identitet (33) imamo da je $T_a^2 = Q_{\{a,\, a+1\}}$.

Pogledajmo sada reprezentaciju elemenata $\widetilde{\beta_{n-k+1}}$ (definicija 3.18).

Imamo da je element $\mathcal{R}\left(\widetilde{\beta_{n-k+1}}\right) \in \operatorname{End}\left(\mathcal{B}_Q\right)$ dan sa

$$\begin{aligned} \mathcal{R}\left(\widetilde{\beta_{n-k+1}}\right) e_{j_1 j_2 \cdots j_n} &= \mathcal{R}\left(\overleftarrow{\sum_{k+1 \le s \le n}} \widetilde{t_{s,k}} + \mathrm{id} \right) e_{j_1 j_2 \cdots j_n} \\ &= \overleftarrow{\sum_{k+1 \le s \le n}} \mathcal{R}\left(\widetilde{t_{s,k}}\right) + \mathcal{R}\left(\mathrm{id}\right). \end{aligned} \tag{57}$$

Neka je $$B_{Q,\, n-k+1} := \mathcal{R}\left(\widetilde{\beta_{n-k+1}}\right). \tag{58}$$

Tada s obzirom na gore uvedene oznake možemo pisati

$$B_{Q,\,n-k+1} = \overset{\leftarrow}{\sum_{k+1\le s\le n}} T_{s,k} + I \tag{59}$$

ili u raspisanom obliku

$$B_{Q,\,n-k+1} = T_{n,k} + T_{n-1,k} + \cdots + T_{k+2,k} + T_{k+1,k} + I\,. \tag{60}$$

Ponekad ćemo izraz (60) pisati u obliku

$$B_{Q,\,n-k+1} = T_k T_{k+1} \cdots T_{n-2} T_{n-1} + T_k T_{k+1} \cdots T_{n-2} + \cdots + T_k T_{k+1} + T_k + I\,. \tag{61}$$

pri čemu se primijenjuje identitet (54).

Specijalno, iz $\widetilde{t_{k,k}} = \mathrm{id}$ proizlazi da je ciklička matrica $T_{k,k} = \mathcal{R}\left(\widetilde{t_{k,k}}\right) = I$, $1 \le k \le n$ jednaka jediničnoj matrici *I*, stoga se izraz (59) može pisati u obliku

$$B_{Q,\,n-k+1} = \overset{\leftarrow}{\sum_{k\le s\le n}} T_{s,k}\,. \tag{62}$$

Pritom je

$$\begin{aligned} B_{Q,\,n-k+1}\, e_{j_1 j_2 \cdots j_n} &= \overset{\leftarrow}{\sum_{k\le s\le n}} \left(T_{s,k}\, e_{j_1 j_2 \cdots j_n} \right) \\ &= \overset{\leftarrow}{\sum_{k\le s\le n}} \left(\prod_{i=k}^{s-1} q_{j_s j_i}\, e_{j_1 j_2 \cdots j_s j_k j_{k+1} \cdots j_{s-2} j_{s-1} \cdots j_n} \right) \end{aligned} \tag{63}$$

za svaki $1 \le k \le n$.

Napomena:

U nastavku će se uzimati $1 \le k \le n-1$, jer za $k = n$ imamo da je

$$B_{Q,\,1} = \mathcal{R}\left(\widetilde{\beta_1}\right) = \mathcal{R}(\mathrm{id}) = I \quad \text{jedinična matrica.}$$

Ako je $\underline{k} = t_{s,k} \cdot \underline{j} = j_1 j_2 \ldots j_s j_k j_{k+1} \ldots j_{s-2} j_{s-1} \ldots j_n$, onda je $\left(\underline{k}, \underline{j}\right)$-ti element matrice $B_{Q,\,n-k+1}$ jednak izrazu $\prod_{i=k}^{s-1} q_{j_s j_i}$, a u protivnom je jednak nuli, tj.

$$\left(B_{Q,\,n-k+1}\right)_{\underline{k},\underline{j}} = \begin{cases} \prod_{i=k}^{s-1} q_{j_s j_i} & \text{ako je } \underline{k} = t_{s,k} \cdot \underline{j}\,,\ k \le s \le n \\ 0 & \text{inače} \end{cases} \tag{64}$$

$1 \le k \le n-1$.

Specijalno, ako je $s = k$, onda je

$$\underline{k} = t_{k,k} \centerdot \underline{j} = \mathrm{id} \centerdot \underline{j} = \underline{j}, \qquad \prod_{i=k}^{s-1} q_{j_s j_i} = \prod_{i=k}^{k-1} q_{j_s j_i} := 1,$$

što povlači da je $\left(B_{Q,\, n-k+1}\right)_{\underline{j},\underline{j}} = 1$.

Neka je $k = 1$. Tada imamo da je izraz (62) dan sa

$$B_{Q,\, n} = \overset{\leftarrow}{\sum_{1 \le s \le n}} T_{s,1}$$

ili u raspisanom obliku

$$B_{Q,\, n} = T_{n,1} + T_{n-1,1} + \cdots + T_{3,1} + T_{2,1} + T_{1,1}, \tag{65}$$

gdje je $T_{1,1} = I$.

Primjenom identiteta (54) proizlazi da se (65) može pisati u obliku

$$B_{Q,\, n} = T_1 T_2 \cdots T_{n-2} T_{n-1} + T_1 T_2 \cdots T_{n-2} + \cdots + T_1 T_2 + T_1 + I. \tag{66}$$

Dakle, u ovom slučaju (za $k = 1$) imamo da je

$$\begin{aligned} B_{Q,\, n}\, e_{j_1 j_2 \cdots j_n} &= \overset{\leftarrow}{\sum_{1 \le s \le n}} \left(T_{s,1}\, e_{j_1 j_2 \cdots j_n} \right) \\ &= \overset{\leftarrow}{\sum_{1 \le s \le n}} \left(\prod_{i=1}^{s-1} q_{j_s j_i}\, e_{j_s j_1 j_2 \cdots j_{s-2} j_{s-1} \cdots j_n} \right) \end{aligned} \tag{67}$$

ili ekvivalentno

$$B_{Q,\, n}\, e_{\underline{j}} = \overset{\leftarrow}{\sum_{1 \le p \le n}} \left(\prod_{r=1}^{p-1} q_{j_p j_r}\, e_{\underline{k}} \right). \tag{68}$$

Pritom je $\underline{k} = t_{p,1} \centerdot \underline{j} = t_{p,1} . j_1 j_2 ... j_{p-1} j_p j_{p+1} ... j_n = j_p j_1 j_2 ... j_{p-1} j_{p+1} ... j_n$, $1 \le p \le n$.

Jasno, izraz (68) u raspisanom obliku je dan sa

$$\begin{aligned} B_{Q,\, n}\, e_{\underline{j}} &= \prod_{r=1}^{n-1} q_{j_n j_r}\, e_{t_{n,1} . \underline{j}} + \prod_{r=1}^{n-2} q_{j_{n-1} j_r}\, e_{t_{n-1,1} . \underline{j}} + \cdots + q_{j_2 j_1}\, e_{t_{2,1} . \underline{j}} + 1 e_{t_{1,1} . \underline{j}} \\ &= q_{j_n j_1} q_{j_n j_2} \cdots q_{j_n j_{n-2}} q_{j_n j_{n-1}}\, e_{j_n j_1 j_2 \cdots j_{n-2} j_{n-1}} + q_{j_{n-1} j_1} q_{j_{n-1} j_2} \cdots q_{j_{n-1} j_{n-3}} q_{j_{n-1} j_{n-2}}\, e_{j_{n-1} j_1 j_2 \cdots j_{n-3} j_{n-2} j_n} + \\ &\quad \cdots + q_{j_2 j_1}\, e_{j_2 j_1 \cdots j_{n-1} j_n} + 1 e_{j_1 j_2 \cdots j_{n-1} j_n}, \end{aligned} \tag{69}$$

gdje je $\underline{j} = j_1 j_2 ... j_{n-1} j_n$.

Uspoređivanjem matrice $B_{Q,\, n}$ sa matricom B_Q (pridružene operatoru $\partial|_{\mathcal{B}_Q} : \mathcal{B}_Q \to \mathcal{B}_Q$ u monomijalnoj bazi težinskog potprostora $\mathcal{B}_Q$) vidimo da vrijedi sljedeća jednakost

$$\mathrm{B}_{\mathrm{Q,\,n}} = \mathrm{B}_{\mathrm{Q}}, \tag{70}$$

što povlači da je

$$\det \mathrm{B}_{\mathrm{Q,\,n}} = \det \mathrm{B}_{\mathrm{Q}}. \tag{71}$$

Napomena:

Iz jednakosti (71) proizlazi da je za izračunavanje determinante matrice B_{Q} (koja je u direktnoj vezi s izračunavanjem netrivijalnih konstanti u potprostoru $\mathcal{B}_{\mathrm{Q}}$ algebre $\mathcal{B}$) potrebno naći formulu po kojoj će se faktorizirati matrica $\mathrm{B}_{\mathrm{Q,\,n}}$, a samim time i njena determinanta.
S tom motivacijom će se najprije izvesti faktorizacija matrice $\mathrm{B}_{\mathrm{Q,\,n-k+1}}$, $1 \le \mathrm{k} \le \mathrm{n}-1$.
Uzimajući u obzir propoziciju 3.22 imamo da se element $\widetilde{\beta_{\mathrm{n-k+1}}} \in \mathcal{A}_{\mathrm{n}}$, $1 \le \mathrm{k} \le \mathrm{n}-1$ faktorizira po formuli $\widetilde{\beta_{\mathrm{n-k+1}}} = \widetilde{\delta_{\mathrm{n-k+1}}} \cdot \widetilde{\gamma_{\mathrm{n-k+1}}}^{-1}$, pri čemu je

$$\widetilde{\gamma_{\mathrm{n-k+1}}} = \overleftarrow{\prod_{\mathrm{k+1 \le s \le n}}} \left(\mathrm{id} - \widetilde{\mathrm{t}_{\mathrm{s,k}}}\right),$$

$$\widetilde{\delta_{\mathrm{n-k+1}}} = \overleftarrow{\prod_{\mathrm{k+1 \le s \le n}}} \left(\mathrm{id} - \widetilde{\mathrm{t}_{\mathrm{k}}}^{2}\, \widetilde{\mathrm{t}_{\mathrm{s,k+1}}}\right).$$

Time će se matrica $\mathrm{B}_{\mathrm{Q,\,n-k+1}}$ faktorizirati pomoću matrica

$$\mathrm{C}_{\mathrm{Q,\,n-k+1}} := \mathcal{R}\left(\widetilde{\gamma_{\mathrm{n-k+1}}}\right) \quad \mathrm{i} \quad \mathrm{D}_{\mathrm{Q,\,n-k+1}} := \mathcal{R}\left(\widetilde{\delta_{\mathrm{n-k+1}}}\right)$$

po formuli

$$\mathrm{B}_{\mathrm{Q,\,n-k+1}} = \mathrm{D}_{\mathrm{Q,\,n-k+1}} \cdot \mathrm{C}_{\mathrm{Q,\,n-k+1}}^{-1}, \tag{72}$$

gdje je

$$\mathrm{C}_{\mathrm{Q,\,n-k+1}} = \left(I - \mathrm{T}_{\mathrm{n,k}}\right) \cdot \left(I - \mathrm{T}_{\mathrm{n-1,k}}\right) \cdots \left(I - \mathrm{T}_{\mathrm{k+2,k}}\right) \cdot \left(I - \mathrm{T}_{\mathrm{k+1,k}}\right),$$

$$\mathrm{D}_{\mathrm{Q,\,n-k+1}} = \left(I - \mathrm{T}_{\mathrm{k}}^{2} \cdot \mathrm{T}_{\mathrm{n,k+1}}\right) \cdot \left(I - \mathrm{T}_{\mathrm{k}}^{2} \cdot \mathrm{T}_{\mathrm{n-1,k+1}}\right) \cdots \left(I - \mathrm{T}_{\mathrm{k}}^{2} \cdot \mathrm{T}_{\mathrm{k+2,k+1}}\right) \cdot \left(I - \mathrm{T}_{\mathrm{k}}^{2} \cdot \mathrm{T}_{\mathrm{k+1,k+1}}\right),$$

ili kraće

$$\mathrm{C}_{\mathrm{Q,\,n-k+1}} = \overleftarrow{\prod_{\mathrm{k+1 \le s \le n}}} \left(I - \mathrm{T}_{\mathrm{s,k}}\right), \tag{73}$$

$$\mathrm{D}_{\mathrm{Q,\,n-k+1}} = \overleftarrow{\prod_{\mathrm{k+1 \le s \le n}}} \left(I - \mathrm{T}_{\mathrm{k}}^{2} \cdot \mathrm{T}_{\mathrm{s,k+1}}\right). \tag{74}$$

$\mathrm{T}_{\mathrm{k}}^{2} = \mathrm{T}_{\mathrm{k+1,k}}^{2}$ je dijagonalna matrica definirana sa (56).

Uvrštavanjem (73) i (74) u izraz (72) dobivamo

$$\mathrm{B}_{\mathrm{Q},\,n-k+1} = \overleftarrow{\prod_{k+1\le s\le n}} \left(I - \mathrm{T}_k^2 \cdot \mathrm{T}_{s,k+1}\right) \cdot \overrightarrow{\prod_{k+1\le s\le n}} \left(I - \mathrm{T}_{s,k}\right)^{-1} \tag{75}$$

ili u raspisanom obliku

$$\begin{aligned}\mathrm{B}_{\mathrm{Q},\,n-k+1} = {} & \left(I - \mathrm{T}_k^2 \cdot \mathrm{T}_{n,k+1}\right) \cdot \left(I - \mathrm{T}_k^2 \cdot \mathrm{T}_{n-1,k+1}\right) \cdots \left(I - \mathrm{T}_k^2 \cdot \mathrm{T}_{k+2,k+1}\right) \cdot \left(I - \mathrm{T}_k^2 \cdot \mathrm{T}_{k+1,k+1}\right) \\ & \cdot \left(I - \mathrm{T}_{k+1,k}\right)^{-1} \cdot \left(I - \mathrm{T}_{k+2,k}\right)^{-1} \cdots \left(I - \mathrm{T}_{n-1,k}\right)^{-1} \cdot \left(I - \mathrm{T}_{n,k}\right)^{-1}\end{aligned} \tag{76}$$

$1 \le k \le n-1$. Pritom je $\mathrm{T}_{k+1,k+1} = \mathrm{id}$, tj. $I - \mathrm{T}_k^2 \cdot \mathrm{T}_{k+1,k+1} = I - \mathrm{T}_k^2$.

Iz rečenog proizlazi da za izračunavanje determinante matrice $\mathrm{B}_{\mathrm{Q},\,n-k+1}$, $1 \le k \le n-1$ je dovoljno izračunati determinante matrica $\mathrm{D}_{\mathrm{Q},\,n-k+1}$ i $\mathrm{C}_{\mathrm{Q},\,n-k+1}$, tj. determinante matrica $I - \mathrm{T}_{s,k}$ i $I - \mathrm{T}_k^2 \cdot \mathrm{T}_{s,k+1}$, $k+1 \le s \le n$ (vidi lemu 4.1.1).

Imamo

$$\det \mathrm{B}_{\mathrm{Q},\,n-k+1} = \frac{\det \mathrm{D}_{\mathrm{Q},\,n-k+1}}{\det \mathrm{C}_{\mathrm{Q},\,n-k+1}} = \frac{\prod\limits_{k+1\le s\le n} \det\left(I - \mathrm{T}_k^2 \cdot \mathrm{T}_{s,k+1}\right)}{\prod\limits_{k+1\le s\le n} \det\left(I - \mathrm{T}_{s,k}\right)}. \tag{77}$$

Specijalno, ako je $k = 1$, onda je

$$\begin{aligned}\mathrm{B}_{\mathrm{Q},\,n} = {} & \left(I - \mathrm{T}_1^2 \cdot \mathrm{T}_{n,2}\right) \cdot \left(I - \mathrm{T}_1^2 \cdot \mathrm{T}_{n-1,2}\right) \cdots \left(I - \mathrm{T}_1^2 \cdot \mathrm{T}_{3,2}\right) \cdot \left(I - \mathrm{T}_1^2 \cdot \mathrm{T}_{2,2}\right) \\ & \cdot \left(I - \mathrm{T}_{2,1}\right)^{-1} \cdot \left(I - \mathrm{T}_{3,1}\right)^{-1} \cdots \left(I - \mathrm{T}_{n-1,1}\right)^{-1} \cdot \left(I - \mathrm{T}_{n,1}\right)^{-1},\end{aligned} \tag{78}$$

odnosno

$$\det \mathrm{B}_{\mathrm{Q},\,n} = \frac{\prod\limits_{2\le s\le n} \det\left(I - \mathrm{T}_1^2 \cdot \mathrm{T}_{s,2}\right)}{\prod\limits_{2\le s\le n} \det\left(I - \mathrm{T}_{s,1}\right)},$$

gdje je $\mathrm{T}_1^2 = \mathrm{T}_{2,1}^2$, $\mathrm{T}_{2,2} = I$.

Primjenom propozicije 3.19 i svojstva da je $\mathcal{R}$ homorfizam dobivamo da se matrica $\mathcal{A}_\mathrm{Q}$ (propozicija 4.9) faktorizira matricama $\mathrm{B}_{\mathrm{Q},\,n-k+1}$, $1 \le k \le n-1$ po formuli

$$\mathcal{A}_\mathrm{Q} = \overleftarrow{\prod_{1\le k\le n-1}} \mathrm{B}_{\mathrm{Q},\,n-k+1} = \mathrm{B}_{\mathrm{Q},\,2} \cdot \mathrm{B}_{\mathrm{Q},\,3} \cdots \mathrm{B}_{\mathrm{Q},\,n-1} \cdot \mathrm{B}_{\mathrm{Q},\,n}, \tag{79}$$

odnosno

$$\mathcal{A}_Q = \prod_{1\le k\le n-1}^{\leftarrow} \left(D_{Q,\,n-k+1} \cdot C_{Q,\,n-k+1}^{-1} \right),$$

ili

$$\mathcal{A}_Q = \prod_{1\le k\le n-1}^{\leftarrow} \left(\prod_{k+1\le s\le n}^{\leftarrow} \left(I - T_k^2 \cdot T_{s,k+1} \right) \cdot \prod_{k+1\le s\le n}^{\rightarrow} \left(I - T_{s,k} \right)^{-1} \right) \tag{80}$$

pa je

$$\begin{aligned} \det \mathcal{A}_Q &= \prod_{k=1}^{n-1} \det B_{Q,\,n-k+1} \\ &= \prod_{k=1}^{n-1} \frac{\det D_{Q,\,n-k+1}}{\det C_{Q,\,n-k+1}} \\ &= \prod_{k=1}^{n-1} \frac{\prod\limits_{k+1\le s\le n} \det\left(I - T_k^2 \cdot T_{s,k+1} \right)}{\prod\limits_{k+1\le s\le n} \det\left(I - T_{s,k} \right)}. \end{aligned}$$

Lema 4.1.1

Neka je $\mathcal{R} : \mathcal{A}_n \to \operatorname{End}\left(\mathcal{B}_Q\right)$ regularna reprezentacija zakrenute grupovne algebre $\mathcal{A}_n$ na potprostor $\mathcal{B}_Q$ algebre $\mathcal{B}$, gdje je $Q = \{l_1, l_2, ..., l_n\} \subseteq \mathcal{N} = \{i_1, i_2, ..., i_N\}$.
Tada vrijedi

(a) $$\det\left(I - T_{b,a} \right) = \prod_{T \in \binom{Q}{b-a+1}} \left(1 - \sigma_T \right)^{(b-a)!\,(n-b+a-1)!} \tag{81}$$

za svaki $1 \le a < b \le n$,

(b) $$\det\left(I - T_{a-1}^2 \cdot T_{b,a} \right) = \prod_{T \in \binom{Q}{b-a+2}} \left(1 - \sigma_T \right)^{(b-a)!\,(b-a+2)\,(n-b+a-2)!} \tag{82}$$

za svaki $1 < a \le b \le n$, gdje je $T_{a-1}^2 = T_{a,a-1}^2$.

Pritom je $\binom{Q}{m} = \left\{ T \subseteq Q \mid \operatorname{Card}(T) = m \right\}$, $\sigma_T = \prod_{\{i\ne j\}\subset T} \sigma_{ij} = \prod_{i\ne j\in T} q_{ij}$.

Dokaz:

Lema 4.1.1 analogna je lemi 1.9.1 [MS1, Lemma 1.9.1]. Napomenimo da se ovdje umjesto cikličke permutacije $t_{a,b} \in S_n$ promatra njen inverz $t_{b,a} = t_{a,b}^{-1} \in S_n$.

(a) Neka je $H := \langle \mathrm{t}_{\mathrm{b,a}} \rangle \subset \mathrm{S}_\mathrm{n}$ ciklička podgrupa od S_n generirana ciklusom $\mathrm{t}_{\mathrm{b,a}}$ (duljine $\mathrm{b}-\mathrm{a}+1$). Tada je svaka H-orbita na generičkom potprostoru $\mathcal{B}_{\mathrm{Q}}$ dana sa

$$\mathcal{B}_{\mathrm{Q}}^{[\underline{\mathrm{j}}]_{\mathrm{a}}^{\mathrm{b}}} = span_{\mathbb{C}} \left\{ \mathrm{e}_{\mathrm{t}_{\mathrm{b,a}}^{k} \cdot \underline{\mathrm{j}}} \mid 0 \le k \le \mathrm{b}-\mathrm{a} \right\},$$

koja korespondira cikličkoj $\mathrm{t}_{\mathrm{b,a}}$-klasi ekvivalencije $\left[\underline{\mathrm{j}}\right]_{\mathrm{a}}^{\mathrm{b}} = \mathrm{j}_1\mathrm{j}_2 \ldots (\mathrm{j}_\mathrm{a}\mathrm{j}_{\mathrm{a}+1} \ldots \mathrm{j}_{\mathrm{b}-1}\mathrm{j}_\mathrm{b}) \ldots \mathrm{j}_\mathrm{n}$ niza $\underline{\mathrm{j}} = \mathrm{j}_1\mathrm{j}_2 \ldots \mathrm{j}_\mathrm{n} \in \widehat{\mathrm{Q}}$. Imamo da je

$$\mathrm{T}_{\mathrm{b,a}} \left(\mathrm{e}_{\mathrm{t}_{\mathrm{b,a}}^{k} \cdot \underline{\mathrm{j}}} \right) = c_k \, \mathrm{e}_{\mathrm{t}_{\mathrm{b,a}}^{k+1} \cdot \underline{\mathrm{j}}}, \qquad 0 \le k \le \mathrm{b}-\mathrm{a},$$

gdje je

$$\begin{aligned}
c_0 &= \mathrm{q}_{\mathrm{j}_\mathrm{b}\mathrm{j}_\mathrm{a}} \mathrm{q}_{\mathrm{j}_\mathrm{b}\mathrm{j}_{\mathrm{a}+1}} \mathrm{q}_{\mathrm{j}_\mathrm{b}\mathrm{j}_{\mathrm{a}+2}} \cdots \mathrm{q}_{\mathrm{j}_\mathrm{b}\mathrm{j}_{\mathrm{b}-1}}, \\
c_1 &= \mathrm{q}_{\mathrm{j}_{\mathrm{b}-1}\mathrm{j}_\mathrm{b}} \mathrm{q}_{\mathrm{j}_{\mathrm{b}-1}\mathrm{j}_\mathrm{a}} \mathrm{q}_{\mathrm{j}_{\mathrm{b}-1}\mathrm{j}_{\mathrm{a}+1}} \cdots \mathrm{q}_{\mathrm{j}_{\mathrm{b}-1}\mathrm{j}_{\mathrm{b}-2}}, \\
c_2 &= \mathrm{q}_{\mathrm{j}_{\mathrm{b}-2}\mathrm{j}_{\mathrm{b}-1}} \mathrm{q}_{\mathrm{j}_{\mathrm{b}-2}\mathrm{j}_\mathrm{b}} \mathrm{q}_{\mathrm{j}_{\mathrm{b}-2}\mathrm{j}_\mathrm{a}} \cdots \mathrm{q}_{\mathrm{j}_{\mathrm{b}-2}\mathrm{j}_{\mathrm{b}-3}}, \\
&\vdots \\
c_{\mathrm{b}-\mathrm{a}-1} &= \mathrm{q}_{\mathrm{j}_{\mathrm{a}+1}\mathrm{j}_{\mathrm{a}+2}} \mathrm{q}_{\mathrm{j}_{\mathrm{a}+1}\mathrm{j}_{\mathrm{a}+3}} \mathrm{q}_{\mathrm{j}_{\mathrm{a}+1}\mathrm{j}_{\mathrm{a}+4}} \cdots \mathrm{q}_{\mathrm{j}_{\mathrm{a}+1}\mathrm{j}_\mathrm{a}}. \\
c_{\mathrm{b}-\mathrm{a}} &= \mathrm{q}_{\mathrm{j}_\mathrm{a}\mathrm{j}_{\mathrm{a}+1}} \mathrm{q}_{\mathrm{j}_\mathrm{a}\mathrm{j}_{\mathrm{a}+2}} \mathrm{q}_{\mathrm{j}_\mathrm{a}\mathrm{j}_{\mathrm{a}+3}} \cdots \mathrm{q}_{\mathrm{j}_\mathrm{a}\mathrm{j}_\mathrm{b}}.
\end{aligned}$$

Napomena:

Uzimajući u obzir da je $(\underline{\mathrm{k}}, \underline{\mathrm{j}})$-ti (tj. $(\mathrm{e}_{\underline{\mathrm{k}}}, \mathrm{e}_{\underline{\mathrm{j}}})$-ti) element cikličke matrice $\mathrm{T}_{\mathrm{b,a}}$, $1 \le \mathrm{a} < \mathrm{b} \le \mathrm{n}$ dan izrazom (52), koji je ekvivalentan izrazu

$$\mathrm{T}_{\mathrm{b,a}} \, \mathrm{e}_{\underline{\mathrm{j}}} = \mathrm{q}_{\mathrm{j}_\mathrm{b}\mathrm{j}_\mathrm{a}} \mathrm{q}_{\mathrm{j}_\mathrm{b}\mathrm{j}_{\mathrm{a}+1}} \mathrm{q}_{\mathrm{j}_\mathrm{b}\mathrm{j}_{\mathrm{a}+2}} \cdots \mathrm{q}_{\mathrm{j}_\mathrm{b}\mathrm{j}_{\mathrm{b}-1}} \, \mathrm{e}_{\underline{\mathrm{k}}} = c_0 \, \mathrm{e}_{\underline{\mathrm{k}}}$$

gdje je $\underline{\mathrm{j}} = \mathrm{t}_{\mathrm{b,a}}^{0} \cdot \underline{\mathrm{j}} = \mathrm{j}_1\mathrm{j}_2 \ldots \mathrm{j}_\mathrm{a}\mathrm{j}_{\mathrm{a}+1}\mathrm{j}_{\mathrm{a}+2} \ldots \mathrm{j}_{\mathrm{b}-1}\mathrm{j}_\mathrm{b} \ldots \mathrm{j}_\mathrm{n}$, $\underline{\mathrm{k}} = \mathrm{t}_{\mathrm{b,a}}^{1} \cdot \underline{\mathrm{j}} = \mathrm{j}_1\mathrm{j}_2 \ldots \mathrm{j}_\mathrm{b}\mathrm{j}_\mathrm{a}\mathrm{j}_{\mathrm{a}+1} \ldots \mathrm{j}_{\mathrm{b}-2}\mathrm{j}_{\mathrm{b}-1} \ldots \mathrm{j}_\mathrm{n}$.

Nadalje, imamo da je

$$\begin{aligned}
\left(\mathrm{T}_{\mathrm{b,a}}\right)^2 \mathrm{e}_{\underline{\mathrm{j}}} &= \mathrm{T}_{\mathrm{b,a}} \left(c_0 \, \mathrm{e}_{\underline{\mathrm{k}}} \right) = c_0 \, \mathrm{T}_{\mathrm{b,a}} \left(\mathrm{e}_{\mathrm{t}_{\mathrm{b,a}}^{1} \cdot \underline{\mathrm{j}}} \right) = c_0 \, \mathrm{T}_{\mathrm{b,a}} \, \mathrm{e}_{\mathrm{j}_1\mathrm{j}_2 \cdots \mathrm{j}_\mathrm{b}\mathrm{j}_\mathrm{a}\mathrm{j}_{\mathrm{a}+1} \cdots \mathrm{j}_{\mathrm{b}-2}\mathrm{j}_{\mathrm{b}-1} \cdots \mathrm{j}_\mathrm{n}} \\
&= c_0 \, \mathrm{q}_{\mathrm{j}_{\mathrm{b}-1}\mathrm{j}_\mathrm{b}} \mathrm{q}_{\mathrm{j}_{\mathrm{b}-1}\mathrm{j}_\mathrm{a}} \mathrm{q}_{\mathrm{j}_{\mathrm{b}-1}\mathrm{j}_{\mathrm{a}+1}} \cdots \mathrm{q}_{\mathrm{j}_{\mathrm{b}-1}\mathrm{j}_{\mathrm{b}-2}} \, \mathrm{e}_{\mathrm{j}_1\mathrm{j}_2 \cdots \mathrm{j}_{\mathrm{b}-1}\mathrm{j}_\mathrm{b}\mathrm{j}_\mathrm{a} \cdots \mathrm{j}_{\mathrm{b}-3}\mathrm{j}_{\mathrm{b}-2} \cdots \mathrm{j}_\mathrm{n}} \\
&= c_0 \, c_1 \, \mathrm{e}_{\mathrm{t}_{\mathrm{b,a}}^{2} \cdot \underline{\mathrm{j}}},
\end{aligned}$$

$$\begin{aligned}
\left(\mathrm{T}_{\mathrm{b,a}}\right)^3 \mathrm{e}_{\underline{\mathrm{j}}} &= \mathrm{T}_{\mathrm{b,a}} \left(c_0 \, c_1 \, \mathrm{e}_{\mathrm{t}_{\mathrm{b,a}}^{2} \cdot \underline{\mathrm{j}}} \right) = c_0 \, c_1 \, \mathrm{T}_{\mathrm{b,a}} \left(\mathrm{e}_{\mathrm{t}_{\mathrm{b,a}}^{2} \cdot \underline{\mathrm{j}}} \right) = c_0 \, c_1 \, \mathrm{T}_{\mathrm{b,a}} \, \mathrm{e}_{\mathrm{j}_1\mathrm{j}_2 \cdots \mathrm{j}_{\mathrm{b}-1}\mathrm{j}_\mathrm{b}\mathrm{j}_\mathrm{a} \cdots \mathrm{j}_{\mathrm{b}-3}\mathrm{j}_{\mathrm{b}-2} \cdots \mathrm{j}_\mathrm{n}} \\
&= c_0 \, c_1 \, \mathrm{q}_{\mathrm{j}_{\mathrm{b}-2}\mathrm{j}_{\mathrm{b}-1}} \mathrm{q}_{\mathrm{j}_{\mathrm{b}-2}\mathrm{j}_\mathrm{b}} \mathrm{q}_{\mathrm{j}_{\mathrm{b}-2}\mathrm{j}_\mathrm{a}} \cdots \mathrm{q}_{\mathrm{j}_{\mathrm{b}-2}\mathrm{j}_{\mathrm{b}-3}} \, \mathrm{e}_{\mathrm{j}_1\mathrm{j}_2 \cdots \mathrm{j}_{\mathrm{b}-2}\mathrm{j}_{\mathrm{b}-1}\mathrm{j}_\mathrm{b} \cdots \mathrm{j}_{\mathrm{b}-4}\mathrm{j}_{\mathrm{b}-3} \cdots \mathrm{j}_\mathrm{n}} \\
&= c_0 \, c_1 \, c_2 \, \mathrm{e}_{\mathrm{t}_{\mathrm{b,a}}^{3} \cdot \underline{\mathrm{j}}},
\end{aligned}$$

$$\vdots$$

$$\left(T_{b,a}\right)^{b-a} e_{\underline{j}} = c_0\, c_1\, c_2 \cdots c_{b-a-2}\, q_{j_{a+1}j_{a+2}} q_{j_{a+1}j_{a+3}} q_{j_{a+1}j_{a+4}} \cdots q_{j_{a+1}j_a}\, e_{j_1 j_2 \cdots j_{a+1} j_{a+2} j_{a+3} \cdots j_b j_a \cdots j_n}$$

$$= c_0\, c_1\, c_2 \cdots c_{b-a-2}\, c_{b-a-1}\, e_{t_{b,a}^{b-a} \cdot \underline{j}} \ ,$$

$$\left(T_{b,a}\right)^{b-a+1} e_{\underline{j}} = c_0\, c_1\, c_2 \cdots c_{b-a-2}\, c_{b-a-1}\, q_{j_a j_{a+1}} q_{j_a j_{a+2}} q_{j_a j_{a+3}} \cdots q_{j_a j_b}\, e_{j_1 j_2 \cdots j_a j_{a+1} j_{a+2} \cdots j_{b-1} j_b \cdots j_n}$$

$$= c_0\, c_1\, c_2 \cdots c_{b-a-2}\, c_{b-a-1}\, c_{b-a}\, e_{t_{b,a}^{b-a+1} \cdot \underline{j}}$$

$$= c_0\, c_1\, c_2 \cdots c_{b-a-2}\, c_{b-a-1}\, c_{b-a}\, e_{\underline{j}} \ ,$$

Iz rečenog proizlazi da se k-ta, $1 \le k \le b-a+1$ potencija cikličke matrice $T_{b,a}$ može pisati u obliku

$$\left(T_{b,a}\right)^{k} e_{\underline{j}} = c_0\, c_1\, c_2 \cdots c_{k-1}\, e_{t_{b,a}^{k} \cdot \underline{j}} \tag{83}$$

gdje je $\underline{j} \in \widehat{Q}$.

Budući da je $T_{b,a} | \mathcal{B}_Q^{[j]_a^b}$ ciklički operator (koji je poistovjećen s cikličkom matricom), dobivamo

$$\det\left(\left(I - T_{b,a}\right) | \mathcal{B}_Q^{[j]_a^b}\right) = 1 - c_0\, c_1\, c_2 \cdots c_{b-a} = 1 - \prod_{i \ne j \in T = \{j_a, j_{a+1}, \ldots, j_b\}} q_{ij}$$

$$= 1 - \prod_{\{i,j\} \subset T = \{j_a, j_{a+1}, \ldots, j_b\}} \sigma_{ij}$$

$$= 1 - \sigma_T \ . \tag{84}$$

Pritom je $\sigma_{ij} = q_{ij} q_{ji}$, $1 \le i < j \le n$.

S obzirom na skup $T = \{j_a, j_{a+1}, \ldots, j_b\}$, Card $T = b-a+1$ imamo

$$(b-a)!\ \left(n-(b-a+1)\right)! = (b-a)!\ (n-b+a-1)!$$

H-orbita za koje determinanta (84) prima vrijednost $1-\sigma_T$, stoga se dobiva da je

$\det\left(I - T_{b,a}\right) = \prod_{T \in \binom{Q}{b-a+1}} \left(1 - \sigma_T\right)^{(b-a)!\,(n-b+a-1)!}$, čime je dokazana formula (81).

(b) analogno se dokazuje kao i (a) s time da u ovom slučaju imamo

$$T_{a-1}^2 \cdot T_{b,a}\left(e_{t_{b,a}^{k} \cdot \underline{j}}\right) = d_k\, e_{t_{b,a}^{k+1} \cdot \underline{j}}, \qquad 0 \le k \le b-a \,,$$

gdje je

$$d_0 = \sigma_{j_{a-1} j_b}\, c_0 = \sigma_{j_{a-1} j_b}\, q_{j_b j_a} q_{j_b j_{a+1}} q_{j_b j_{a+2}} \cdots q_{j_b j_{b-1}} \,,$$

$$d_1 = \sigma_{j_{a-1} j_{b-1}}\, c_1 = \sigma_{j_{a-1} j_{b-1}}\, q_{j_{b-1} j_b} q_{j_{b-1} j_a} q_{j_{b-1} j_{a+1}} \cdots q_{j_{b-1} j_{b-2}} \,,$$

$$d_2 = \sigma_{j_{a-1}j_{b-2}}\, c_2 = \sigma_{j_{a-1}j_{b-2}}\, q_{j_{b-2}j_{b-1}} q_{j_{b-2}j_b} q_{j_{b-2}j_a} \cdots q_{j_{b-2}j_{b-3}}\,,$$
$$\vdots$$
$$d_{b-a-1} = \sigma_{j_{a-1}j_{a+1}}\, c_{b-a-1} = \sigma_{j_{a-1}j_{a+1}}\, q_{j_{a+1}j_{a+2}} q_{j_{a+1}j_{a+3}} q_{j_{a+1}j_{a+4}} \cdots q_{j_{a+1}j_a}\,,$$
$$d_{b-a} = \sigma_{j_{a-1}j_a}\, c_{b-a} = \sigma_{j_{a-1}j_a}\, q_{j_a j_{a+1}} q_{j_a j_{a+2}} q_{j_a j_{a+3}} \cdots q_{j_a j_b}\,.$$

Napomena:

Analogno gore navedenom lako se pokazuje da se k-ta potencija cikličke matrice $T_{a-1}^2 \cdot T_{b,a}$, $1 \le k \le b-a+1$ može pisati u obliku

$$\left(T_{a-1}^2 \cdot T_{b,a}\right)^k e_{\underline{j}} = d_0\, d_1\, d_2 \cdots d_{k-1}\, e_{t_{b,a}^k \cdot \underline{j}} \tag{85}$$

gdje je $\underline{j} \in \widehat{Q}$.

Dobivamo

$$\begin{aligned}\det\left(\left(I - T_{a-1}^2 \cdot T_{b,a}\right)|\mathcal{B}_Q^{[j]_a^b}\right) &= 1 - d_0\, d_1 \cdots d_{b-a} = 1 - \prod_{i \ne j \in T = \{j_{a-1}, j_a, j_{a+1}, \dots, j_b\}} q_{ij} \\ &= 1 - \prod_{\{i,j\} \subset T = \{j_{a-1}, j_a, j_{a+1}, \dots, j_b\}} \sigma_{ij} \\ &= 1 - \sigma_T\,.\end{aligned}$$

S obzirom na skup $T = \{j_{a-1}, j_a, j_{a+1}, \dots, j_b\}$, $\operatorname{Card} T = b-a+2$ imamo

$$(b-a)!\,(b-a+2)(n-(b-a+2))! = (b-a)!\,(b-a+2)(n-b+a-2)!$$

H-orbita za koje $\det\left(\left(I - T_{a-1}^2 \cdot T_{b,a}\right)|\mathcal{B}_Q^{[j]_a^b}\right)$ prima vrijednost $1-\sigma_T$.

Time je dokazana formula (82).

Teorem 4.1.2

Neka je $\mathcal{R} : \mathcal{A}_n \to \operatorname{End}(\mathcal{B}_Q)$ regularna reprezentacija zakrenute grupovne algebre $\mathcal{A}_n$ na generičkom težinskom potprostoru $\mathcal{B}_Q$ algebre $\mathcal{B}$. Tada se

(*i*) determinanta matrice $\mathcal{A}_Q$ računa po formuli

$$\det \mathcal{A}_Q = \prod_{m=2}^{n} \prod_{T \in \binom{Q}{m}} (1-\sigma_T)^{(m-2)!\,(n-m+1)!}\,, \tag{86}$$

(*ii*) determinanta matrice $B_{Q,\,n-k+1}$, $1 \le k \le n-1$ računa po formuli

$$\det B_{Q,\,n-k+1} = \prod_{m=2}^{n-k+1} \prod_{T \in \binom{Q}{m}} (1-\sigma_T)^{(m-2)!\,(n-m)!}\,. \tag{87}$$

Pritom je $\sigma_T = \prod_{\{i\neq j\}\subset T} \sigma_{ij} = \prod_{i\neq j\in T} q_{ij}$, $\binom{Q}{m} = \left\{ T \subseteq Q \mid \mathrm{Card}(T) = m \right\}$.

Teorem 4.1.2 je analogan teoremu 1.9.2 u [MS1].

Dokaz:

Teorem se dokazuje primjenom leme 4.1.1. Imamo sljedeće

$$\det C_{Q,\,n-k+1} = \prod_{s=k+1}^{n} \det\left(I - T_{s,k}\right) = \prod_{s=k+1}^{n} \prod_{T\in\binom{Q}{s-k+1}} \left(1-\sigma_T\right)^{(s-k)!\,(n-s+k-1)!}$$

$$= \prod_{m=2}^{n-k+1} \prod_{T\in\binom{Q}{m}} \left(1-\sigma_T\right)^{(m-1)!\,(n-m)!}, \tag{88}$$

$$\det D_{Q,\,n-k+1} = \prod_{s=k+1}^{n} \det\left(I - T_k^2 \cdot T_{s,k+1}\right) = \prod_{s=k+1}^{n} \prod_{T\in\binom{Q}{s-k+1}} \left(1-\sigma_T\right)^{(s-k-1)!\,(s-k+1)\,(n-s+k-1)!}$$

$$= \prod_{m=2}^{n-k+1} \prod_{T\in\binom{Q}{m}} \left(1-\sigma_T\right)^{(m-2)!\,m\,(n-m)!}, \tag{89}$$

pa je

$$\det B_{Q,\,n-k+1} = \prod_{m=2}^{n-k+1} \prod_{T\in\binom{Q}{m}} \left(1-\sigma_T\right)^{(m-2)!\,(n-m)!}. \tag{90}$$

Ako je $k=1$, onda iz formule (90) proizlazi

$$\det B_{Q,\,n} = \prod_{m=2}^{n} \prod_{T\in\binom{Q}{m}} \left(1-\sigma_T\right)^{(m-2)!\,(n-m)!}. \tag{91}$$

Uvrštavanjem (90) u (79) dobivamo

$$\det \mathcal{A}_Q = \prod_{k=1}^{n-1} \det B_{Q,\,n-k+1} = \prod_{k=1}^{n-1} \prod_{m=2}^{n-k+1} \prod_{T\in\binom{Q}{m}} \left(1-\sigma_T\right)^{(m-2)!\,(n-m)!}$$

$$= \prod_{m=2}^{n} \prod_{T\in\binom{Q}{m}} \left(1-\sigma_T\right)^{(m-2)!\,(n-m)!\,(n-m+1)} = \prod_{m=2}^{n} \prod_{T\in\binom{Q}{m}} \left(1-\sigma_T\right)^{(m-2)!\,(n-m+1)!}.$$

Time je dokaz teorema 4.1.2 završen.

U sljedećem primjeru obrazložiti će se navedene formule te će se izračunati determinanta matrice $\mathcal{A}_{\mathrm{Q}}$ iz primjera 4.11.

Primjer 4.1.3

U primjeru 4.11 pokazali smo da je matrica $\mathcal{A}_{\mathrm{Q}} = \mathcal{R}\left(\widetilde{\alpha_3}\right) = \sum_{\mathrm{g} \in \mathrm{S}_3} \mathcal{R}\left(\widetilde{\mathrm{g}}\right)$ dana sa

$$
\mathcal{A}_{\mathrm{Q}} = \begin{array}{c} \\ e_{j_1j_2j_3} \\ e_{j_1j_3j_2} \\ e_{j_3j_1j_2} \\ e_{j_3j_2j_1} \\ e_{j_2j_3j_1} \\ e_{j_2j_1j_3} \end{array}
\begin{array}{c}
\begin{array}{cccccc} e_{j_1j_2j_3} & e_{j_1j_3j_2} & e_{j_3j_1j_2} & e_{j_3j_2j_1} & e_{j_2j_3j_1} & e_{j_2j_1j_3} \end{array} \\
\begin{bmatrix}
1 & q_{j_2j_3} & q_{j_2j_3} q_{j_1j_3} & q_{j_2j_3} q_{j_1j_3} q_{j_1j_2} & q_{j_1j_2} q_{j_1j_3} & q_{j_1j_2} \\
q_{j_3j_2} & 1 & q_{j_1j_3} & q_{j_1j_3} q_{j_1j_2} & q_{j_3j_2} q_{j_1j_2} q_{j_1j_3} & q_{j_3j_2} q_{j_1j_2} \\
q_{j_3j_1} q_{j_3j_2} & q_{j_3j_1} & 1 & q_{j_1j_2} & q_{j_1j_2} q_{j_3j_2} & q_{j_1j_2} q_{j_3j_2} q_{j_3j_1} \\
q_{j_2j_1} q_{j_3j_1} q_{j_3j_2} & q_{j_2j_1} q_{j_3j_1} & q_{j_2j_1} & 1 & q_{j_3j_2} & q_{j_3j_2} q_{j_3j_1} \\
q_{j_3j_1} q_{j_2j_1} & q_{j_3j_1} q_{j_2j_1} q_{j_2j_3} & q_{j_2j_3} q_{j_2j_1} & q_{j_2j_3} & 1 & q_{j_3j_1} \\
q_{j_2j_1} & q_{j_2j_1} q_{j_2j_3} & q_{j_1j_3} q_{j_2j_3} q_{j_2j_1} & q_{j_1j_3} q_{j_2j_3} & q_{j_1j_3} & 1
\end{bmatrix}
\end{array}
$$

Pritom je $\mathcal{R} : \boldsymbol{\mathcal{A}}_3 \to \mathrm{End}\left(\boldsymbol{\mathcal{B}}_{\mathrm{Q}}\right)$ regularna reprezentacija zakrenute grupovne algebre $\boldsymbol{\mathcal{A}}_3$ na generičkom težinskom potprostoru $\boldsymbol{\mathcal{B}}_{\mathrm{Q}}$ pridruženog skupu $\mathrm{Q} = \{l_1, l_2, l_3\} \subseteq \mathcal{N}$ te je $\mathrm{n} = \mathrm{Card}\ \mathrm{Q} = 3$.

Matricu $\mathcal{A}_{\mathrm{Q}}$ možemo faktorizirati matricama $\mathrm{B}_{\mathrm{Q},2} = \mathcal{R}\left(\widetilde{\beta_2}\right)$ i $\mathrm{B}_{\mathrm{Q},3} = \mathcal{R}\left(\widetilde{\beta_3}\right)$ primjenom formule $\mathcal{A}_{\mathrm{Q}} = \mathrm{B}_{\mathrm{Q},2} \cdot \mathrm{B}_{\mathrm{Q},3}$, pri čemu iz identiteta (59) proizlazi da je

$$
\begin{aligned}
\mathrm{B}_{\mathrm{Q},3} &= \mathrm{T}_{3,1} + \mathrm{T}_{2,1} + I \qquad && \text{ako je } \mathrm{k} = 1, \\
\mathrm{B}_{\mathrm{Q},2} &= \mathrm{T}_{3,2} + I && \text{ako je } \mathrm{k} = 2.
\end{aligned}
$$

Koristeći identitet (63) dobivamo

$$
\begin{aligned}
\mathrm{B}_{\mathrm{Q},3}\, e_{j_1j_2j_3} &= q_{j_3j_1} q_{j_3j_2}\, e_{j_3j_1j_2} + q_{j_2j_1}\, e_{j_2j_1j_3} + 1\, e_{j_1j_2j_3}, \\
\mathrm{B}_{\mathrm{Q},2}\, e_{j_1j_2j_3} &= q_{j_3j_2}\, e_{j_1j_3j_2} + 1\, e_{j_1j_2j_3}
\end{aligned}
$$

ili

$$
\mathrm{B}_{\mathrm{Q},2} = \begin{array}{c} \\ e_{j_1j_2j_3} \\ e_{j_1j_3j_2} \\ e_{j_3j_1j_2} \\ e_{j_3j_2j_1} \\ e_{j_2j_3j_1} \\ e_{j_2j_1j_3} \end{array}
\begin{array}{c}
\begin{array}{cccccc} e_{j_1j_2j_3} & e_{j_1j_3j_2} & e_{j_3j_1j_2} & e_{j_3j_2j_1} & e_{j_2j_3j_1} & e_{j_2j_1j_3} \end{array} \\
\begin{bmatrix}
1 & q_{j_2j_3} & 0 & 0 & 0 & 0 \\
q_{j_3j_2} & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & q_{j_1j_2} & 0 & 0 \\
0 & 0 & q_{j_2j_1} & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & q_{j_3j_1} \\
0 & 0 & 0 & 0 & q_{j_1j_3} & 1
\end{bmatrix}
\end{array}
\tag{92}
$$

$$
B_{Q,3}=\begin{array}{cc}
 & \begin{array}{cccccc} e_{j_1j_2j_3} & e_{j_1j_3j_2} & e_{j_3j_1j_2} & e_{j_3j_2j_1} & e_{j_2j_3j_1} & e_{j_2j_1j_3} \end{array} \\
\begin{array}{c} e_{j_1j_2j_3} \\ e_{j_1j_3j_2} \\ e_{j_3j_1j_2} \\ e_{j_3j_2j_1} \\ e_{j_2j_3j_1} \\ e_{j_2j_1j_3} \end{array} &
\begin{bmatrix}
1 & 0 & 0 & 0 & q_{j_1j_2}q_{j_1j_3} & q_{j_1j_2} \\
0 & 1 & q_{j_1j_3} & q_{j_1j_2}q_{j_1j_3} & 0 & 0 \\
q_{j_3j_1}q_{j_3j_2} & q_{j_3j_1} & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & q_{j_3j_2} & q_{j_3j_1}q_{j_3j_2} \\
0 & 0 & q_{j_2j_1}q_{j_2j_3} & q_{j_2j_3} & 1 & 0 \\
q_{j_2j_1} & q_{j_2j_1}q_{j_2j_3} & 0 & 0 & 0 & 1
\end{bmatrix}
\end{array} \tag{93}
$$

Uzimajući u obzir prethodno provedena razmatranja imamo da je

$$B_{Q,2}=D_{Q,2}\cdot C_{Q,2}^{-1},$$

$$B_{Q,3}=D_{Q,3}\cdot C_{Q,3}^{-1},$$

tj.

$$D_{Q,2}=I-T_{3,2}^{2},$$

$$C_{Q,2}=I-T_{3,2},$$

$$D_{Q,3}=\left(I-T_{2,1}^{2}T_{3,2}\right)\left(I-T_{2,1}^{2}\right),$$

$$C_{Q,3}=\left(I-T_{3,1}\right)\left(I-T_{2,1}\right),$$

pa je

$$\left.\begin{aligned}
B_{Q,2}&=\left(I-T_{3,2}^{2}\right)\left(I-T_{3,2}\right)^{-1},\\
B_{Q,3}&=\left(I-T_{2,1}^{2}T_{3,2}\right)\left(I-T_{2,1}^{2}\right)\left(I-T_{2,1}\right)^{-1}\left(I-T_{3,1}\right)^{-1}.
\end{aligned}\right\} \tag{94}$$

Budući da je

$$T_{3,2}^{2}\,e_{j_1j_2j_3}=q_{j_3j_2}\,q_{j_2j_3}\,e_{j_1j_2j_3}=\sigma_{j_2j_3}\,e_{j_1j_2j_3},$$

$$T_{3,2}\,e_{j_1j_2j_3}=q_{j_3j_2}\,e_{j_1j_3j_2},$$

odnosno

$$T_{2,1}^{2}T_{3,2}\,e_{j_1j_2j_3}=q_{j_3j_1}q_{j_1j_3}q_{j_3j_2}\,e_{j_1j_3j_2}=\sigma_{j_1j_3}\,q_{j_3j_2}\,e_{j_1j_3j_2},$$

$$T_{2,1}^{2}\,e_{j_1j_2j_3}=q_{j_2j_1}\,q_{j_1j_2}\,e_{j_1j_2j_3}=\sigma_{j_1j_2}\,e_{j_1j_2j_3},$$

$$T_{2,1}\,e_{j_1j_2j_3}=q_{j_2j_1}\,e_{j_2j_1j_3},$$

$$T_{3,1}\,e_{j_1j_2j_3}=q_{j_3j_1}q_{j_3j_2}\,e_{j_3j_1j_2},$$

dobivamo sljedeće matrice (kojima ćemo determinante izračunavati primjenom leme 4.1.1)

$$
I - \mathrm{T}_{3,2}^{2} = \begin{array}{cc}
 & \begin{array}{cccccc} e_{j_1j_2j_3} & e_{j_1j_3j_2} & e_{j_3j_1j_2} & e_{j_3j_2j_1} & e_{j_2j_3j_1} & e_{j_2j_1j_3} \end{array} \\
\begin{array}{c} e_{j_1j_2j_3} \\ e_{j_1j_3j_2} \\ e_{j_3j_1j_2} \\ e_{j_3j_2j_1} \\ e_{j_2j_3j_1} \\ e_{j_2j_1j_3} \end{array} &
\begin{bmatrix}
1-\sigma_{j_2j_3} & 0 & 0 & 0 & 0 & 0 \\
0 & 1-\sigma_{j_2j_3} & 0 & 0 & 0 & 0 \\
0 & 0 & 1-\sigma_{j_1j_2} & 0 & 0 & 0 \\
0 & 0 & 0 & 1-\sigma_{j_1j_2} & 0 & 0 \\
0 & 0 & 0 & 0 & 1-\sigma_{j_1j_3} & 0 \\
0 & 0 & 0 & 0 & 0 & 1-\sigma_{j_1j_3}
\end{bmatrix}
\end{array}
$$

$$
\det\left(I - \mathrm{T}_{3,2}^{2}\right) = \left(1-\sigma_{j_1j_2}\right)^2 \left(1-\sigma_{j_1j_3}\right)^2 \left(1-\sigma_{j_2j_3}\right)^2
$$

$$
I - \mathrm{T}_{3,2} = \begin{array}{cc}
 & \begin{array}{cccccc} e_{j_1j_2j_3} & e_{j_1j_3j_2} & e_{j_3j_1j_2} & e_{j_3j_2j_1} & e_{j_2j_3j_1} & e_{j_2j_1j_3} \end{array} \\
\begin{array}{c} e_{j_1j_2j_3} \\ e_{j_1j_3j_2} \\ e_{j_3j_1j_2} \\ e_{j_3j_2j_1} \\ e_{j_2j_3j_1} \\ e_{j_2j_1j_3} \end{array} &
\begin{bmatrix}
1 & -q_{j_2j_3} & 0 & 0 & 0 & 0 \\
-q_{j_3j_2} & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -q_{j_1j_2} & 0 & 0 \\
0 & 0 & -q_{j_2j_1} & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -q_{j_3j_1} \\
0 & 0 & 0 & 0 & -q_{j_1j_3} & 1
\end{bmatrix}
\end{array}
$$

$$
\det\left(I - \mathrm{T}_{3,2}\right) = \left(1-\sigma_{j_1j_2}\right)\left(1-\sigma_{j_1j_3}\right)\left(1-\sigma_{j_2j_3}\right)
$$

$$
I - \mathrm{T}_{2,1}^{2}\mathrm{T}_{3,2} = \begin{array}{cc}
 & \begin{array}{cccccc} e_{j_1j_2j_3} & e_{j_1j_3j_2} & e_{j_3j_1j_2} & e_{j_3j_2j_1} & e_{j_2j_3j_1} & e_{j_2j_1j_3} \end{array} \\
\begin{array}{c} e_{j_1j_2j_3} \\ e_{j_1j_3j_2} \\ e_{j_3j_1j_2} \\ e_{j_3j_2j_1} \\ e_{j_2j_3j_1} \\ e_{j_2j_1j_3} \end{array} &
\begin{bmatrix}
1 & -\sigma_{j_1j_2}q_{j_2j_3} & 0 & 0 & 0 & 0 \\
-\sigma_{j_1j_3}q_{j_3j_2} & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -\sigma_{j_1j_3}q_{j_1j_2} & 0 & 0 \\
0 & 0 & -\sigma_{j_2j_3}q_{j_2j_1} & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -\sigma_{j_2j_3}q_{j_3j_1} \\
0 & 0 & 0 & 0 & -\sigma_{j_1j_2}q_{j_1j_3} & 1
\end{bmatrix}
\end{array}
$$

$$
\det\left(I - \mathrm{T}_{2,1}^{2}\mathrm{T}_{3,2}\right) = \left(1-\sigma_{j_1j_2j_3}\right)^3
$$

$$
I-\mathrm{T}_{2,1}^{2}=\begin{array}{rl}
 & \begin{array}{cccccc} \mathrm{e}_{j_1j_2j_3} & \mathrm{e}_{j_1j_3j_2} & \mathrm{e}_{j_3j_1j_2} & \mathrm{e}_{j_3j_2j_1} & \mathrm{e}_{j_2j_3j_1} & \mathrm{e}_{j_2j_1j_3} \end{array} \\
\begin{array}{c} \mathrm{e}_{j_1j_2j_3} \\ \mathrm{e}_{j_1j_3j_2} \\ \mathrm{e}_{j_3j_1j_2} \\ \mathrm{e}_{j_3j_2j_1} \\ \mathrm{e}_{j_2j_3j_1} \\ \mathrm{e}_{j_2j_1j_3} \end{array} & \left[\begin{array}{cccccc}
1-\sigma_{j_1j_2} & 0 & 0 & 0 & 0 & 0 \\
0 & 1-\sigma_{j_1j_3} & 0 & 0 & 0 & 0 \\
0 & 0 & 1-\sigma_{j_1j_3} & 0 & 0 & 0 \\
0 & 0 & 0 & 1-\sigma_{j_2j_3} & 0 & 0 \\
0 & 0 & 0 & 0 & 1-\sigma_{j_2j_3} & 0 \\
0 & 0 & 0 & 0 & 0 & 1-\sigma_{j_1j_2}
\end{array}\right]
\end{array}
$$

$$
\det\left(I-\mathrm{T}_{2,1}^{2}\right)=\left(1-\sigma_{j_1j_2}\right)^{2}\left(1-\sigma_{j_1j_3}\right)^{2}\left(1-\sigma_{j_2j_3}\right)^{2}
$$

$$
I-\mathrm{T}_{2,1}=\begin{array}{rl}
 & \begin{array}{cccccc} \mathrm{e}_{j_1j_2j_3} & \mathrm{e}_{j_1j_3j_2} & \mathrm{e}_{j_3j_1j_2} & \mathrm{e}_{j_3j_2j_1} & \mathrm{e}_{j_2j_3j_1} & \mathrm{e}_{j_2j_1j_3} \end{array} \\
\begin{array}{c} \mathrm{e}_{j_1j_2j_3} \\ \mathrm{e}_{j_1j_3j_2} \\ \mathrm{e}_{j_3j_1j_2} \\ \mathrm{e}_{j_3j_2j_1} \\ \mathrm{e}_{j_2j_3j_1} \\ \mathrm{e}_{j_2j_1j_3} \end{array} & \left[\begin{array}{cccccc}
1 & 0 & 0 & 0 & 0 & -\mathrm{q}_{j_1j_2} \\
0 & 1 & -\mathrm{q}_{j_1j_3} & 0 & 0 & 0 \\
0 & -\mathrm{q}_{j_3j_1} & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -\mathrm{q}_{j_3j_2} & 0 \\
0 & 0 & 0 & -\mathrm{q}_{j_2j_3} & 1 & 0 \\
-\mathrm{q}_{j_2j_1} & 0 & 0 & 0 & 0 & 1
\end{array}\right]
\end{array}
$$

$$
\det\left(I-\mathrm{T}_{2,1}\right)=\left(1-\sigma_{j_1j_2}\right)\left(1-\sigma_{j_1j_3}\right)\left(1-\sigma_{j_2j_3}\right)
$$

$$
I-\mathrm{T}_{3,1}=\begin{array}{rl}
 & \begin{array}{cccccc} \mathrm{e}_{j_1j_2j_3} & \mathrm{e}_{j_1j_3j_2} & \mathrm{e}_{j_3j_1j_2} & \mathrm{e}_{j_3j_2j_1} & \mathrm{e}_{j_2j_3j_1} & \mathrm{e}_{j_2j_1j_3} \end{array} \\
\begin{array}{c} \mathrm{e}_{j_1j_2j_3} \\ \mathrm{e}_{j_1j_3j_2} \\ \mathrm{e}_{j_3j_1j_2} \\ \mathrm{e}_{j_3j_2j_1} \\ \mathrm{e}_{j_2j_3j_1} \\ \mathrm{e}_{j_2j_1j_3} \end{array} & \left[\begin{array}{cccccc}
1 & 0 & 0 & 0 & -\mathrm{q}_{j_1j_2}\mathrm{q}_{j_1j_3} & 0 \\
0 & 1 & 0 & -\mathrm{q}_{j_1j_2}\mathrm{q}_{j_1j_3} & 0 & 0 \\
-\mathrm{q}_{j_3j_1}\mathrm{q}_{j_3j_2} & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -\mathrm{q}_{j_3j_1}\mathrm{q}_{j_3j_2} \\
0 & 0 & -\mathrm{q}_{j_2j_1}\mathrm{q}_{j_2j_3} & 0 & 1 & 0 \\
0 & -\mathrm{q}_{j_2j_1}\mathrm{q}_{j_2j_3} & 0 & 0 & 0 & 1
\end{array}\right]
\end{array}
$$

$$
\det\left(I-\mathrm{T}_{3,1}\right)=\left(1-\sigma_{j_1j_2j_3}\right)^{2}
$$

Jednostavnim izračunavanjem, lako se može pokazati da vrijedi identitet (94). Izračunajmo sada determinante matrica $\mathrm{B}_{\mathrm{Q},2}$, $\mathrm{B}_{\mathrm{Q},3}$ i $\mathcal{A}_{\mathrm{Q}}$.

Iz gore navedenog proizlazi da je

$$\begin{aligned}
\det \mathrm{B}_{\mathrm{Q},\,2} &= \frac{\det \mathrm{D}_{\mathrm{Q},\,2}}{\det \mathrm{C}_{\mathrm{Q},\,2}} \\
&= \frac{\det\left(I - \mathrm{T}_2^2\right)}{\det\left(I - \mathrm{T}_2\right)} \\
&= \left(1-\sigma_{\mathrm{j}_1\mathrm{j}_2}\right)\left(1-\sigma_{\mathrm{j}_1\mathrm{j}_3}\right)\left(1-\sigma_{\mathrm{j}_2\mathrm{j}_3}\right),
\end{aligned}$$

$$\begin{aligned}
\det \mathrm{B}_{\mathrm{Q},\,3} &= \frac{\det \mathrm{D}_{\mathrm{Q},\,3}}{\det \mathrm{C}_{\mathrm{Q},\,3}} \\
&= \frac{\det\left(I - \mathrm{T}_1^2\,\mathrm{T}_2\right)\cdot \det\left(I - \mathrm{T}_1^2\right)}{\det\left(I - \mathrm{T}_1\right)\cdot \det\left(I - \mathrm{T}_1\mathrm{T}_2\right)} \\
&= \left(1-\sigma_{\mathrm{j}_1\mathrm{j}_2}\right)\left(1-\sigma_{\mathrm{j}_1\mathrm{j}_3}\right)\left(1-\sigma_{\mathrm{j}_2\mathrm{j}_3}\right)\left(1-\sigma_{\mathrm{j}_1\mathrm{j}_2\mathrm{j}_3}\right),
\end{aligned}$$

tj.

$$\det \mathcal{A}_{\mathrm{Q}} = \left(1-\sigma_{\mathrm{j}_1\mathrm{j}_2}\right)^2\left(1-\sigma_{\mathrm{j}_1\mathrm{j}_3}\right)^2\left(1-\sigma_{\mathrm{j}_2\mathrm{j}_3}\right)^2\left(1-\sigma_{\mathrm{j}_1\mathrm{j}_2\mathrm{j}_3}\right).$$

Pritom je $\sigma_{\mathrm{ijk}} = \sigma_{\mathrm{ij}}\sigma_{\mathrm{ik}}\sigma_{\mathrm{jk}}$, $\sigma_{\mathrm{ij}} = \mathrm{q}_{\mathrm{ij}}\mathrm{q}_{\mathrm{ji}}$.

Napomena:

Vrijednost determinante matrice $\mathcal{A}_{\mathrm{Q}}$ mogli smo direktno izračunati primjenom formule (86) iz teorema 4.1.2.

Pretpostavimo da je $l_2 = l_1$. Tada je multiskupu $\mathrm{Q} = \{l_1, l_1, l_3\}$ pridružen degenerirani težinski potprostor $\mathcal{B}_{\mathrm{Q}}$.

U ovom slučaju imamo da je matrica $\mathcal{A}_{\mathrm{Q}} = \mathcal{R}\left(\widetilde{\alpha_3}\right) \in \mathrm{End}\left(\mathcal{B}_{\mathrm{Q}}\right)$, $\mathrm{Q} = l_1 l_1 l_3$ dana sa

$$\mathcal{A}_{l_1l_1l_3} = \mathcal{A}_{\mathrm{Q}} = \begin{array}{c} \\ \mathrm{e}_{\mathrm{j}_1\mathrm{j}_1\mathrm{j}_3} \\ \mathrm{e}_{\mathrm{j}_1\mathrm{j}_3\mathrm{j}_1} \\ \mathrm{e}_{\mathrm{j}_3\mathrm{j}_1\mathrm{j}_1} \end{array} \begin{array}{c} \begin{array}{ccc} \quad\mathrm{e}_{\mathrm{j}_1\mathrm{j}_1\mathrm{j}_3}\quad & \quad\mathrm{e}_{\mathrm{j}_1\mathrm{j}_3\mathrm{j}_1}\quad & \quad\mathrm{e}_{\mathrm{j}_3\mathrm{j}_1\mathrm{j}_1}\quad \end{array} \\ \begin{bmatrix} 1+\mathrm{q}_{\mathrm{j}_1\mathrm{j}_1} & \mathrm{q}_{\mathrm{j}_1\mathrm{j}_3}\left(1+\mathrm{q}_{\mathrm{j}_1\mathrm{j}_1}\right) & \mathrm{q}^2_{\mathrm{j}_1\mathrm{j}_3}\left(1+\mathrm{q}_{\mathrm{j}_1\mathrm{j}_1}\right) \\ \mathrm{q}_{\mathrm{j}_3\mathrm{j}_1}\left(1+\mathrm{q}_{\mathrm{j}_1\mathrm{j}_1}\right) & 1+\mathrm{q}_{\mathrm{j}_1\mathrm{j}_1}\sigma_{\mathrm{j}_1\mathrm{j}_3} & \mathrm{q}_{\mathrm{j}_1\mathrm{j}_3}\left(1+\mathrm{q}_{\mathrm{j}_1\mathrm{j}_1}\right) \\ \mathrm{q}^2_{\mathrm{j}_3\mathrm{j}_1}\left(1+\mathrm{q}_{\mathrm{j}_1\mathrm{j}_1}\right) & \mathrm{q}_{\mathrm{j}_3\mathrm{j}_1}\left(1+\mathrm{q}_{\mathrm{j}_1\mathrm{j}_1}\right) & 1+\mathrm{q}_{\mathrm{j}_1\mathrm{j}_1} \end{bmatrix} \end{array}.$$

Analogno kao i u generičkom slučaju imamo da se matrica $\mathcal{A}_{l_1l_1l_3}$ može faktorizirati odgovarajućim matricama $\mathrm{B}_{l_1l_1l_3,\,2} = \mathcal{R}\left(\widetilde{\beta_2}\right)$ i $\mathrm{B}_{l_1l_1l_3,\,3} = \mathcal{R}\left(\widetilde{\beta_3}\right)$

primjenom formule $\mathcal{A}_{l_1l_1l_3} = \mathrm{B}_{l_1l_1l_3,\,2} \cdot \mathrm{B}_{l_1l_1l_3,\,3}$, gdje je $$B_{l_1 l_1 l_3,\,3} = T_{3,1} + T_{2,1} + I\,, \qquad B_{l_1 l_1 l_3,\,2} = T_{3,2} + I\,. \qquad (95)$$

Uzimajući u obzir da je $\underline{j} = j_1 j_2 j_3 \in \widehat{Q} = S_3 Q$, možemo pisati

$$\left.\begin{aligned} T_{3,1}\, e_{j_1j_2j_3} &= q_{j_3j_1} q_{j_3j_2}\, e_{j_3j_1j_2}\,, \\ T_{2,1}\, e_{j_1j_2j_3} &= q_{j_2j_1}\, e_{j_2j_1j_3}\,, \\ T_{3,2}\, e_{j_1j_2j_3} &= q_{j_3j_2}\, e_{j_1j_3j_2}\,. \end{aligned}\right\} \qquad (96)$$

Identifikacijom indeksa j_2 s indeksom j_1 u identitetima (96) te uvrštavanjem u izraze (95), pri čemu je $I = 1\, e_{j_1j_1j_3}$, dobivamo da je

$$B_{l_1l_1l_3,\,3}\, e_{j_1j_1j_3} = q^2_{j_3j_1}\, e_{j_3j_1j_1} + q_{j_1j_1}\, e_{j_1j_1j_3} + 1\, e_{j_1j_1j_3} = \left(1 + q_{j_1j_1}\right) e_{j_1j_1j_3} + q^2_{j_3j_1}\, e_{j_3j_1j_1}\,,$$

$$B_{l_1l_1l_3,\,2}\, e_{j_1j_1j_3} = q_{j_3j_1}\, e_{j_1j_3j_1} + 1\, e_{j_1j_1j_3} = 1\, e_{j_1j_1j_3} + q_{j_3j_1}\, e_{j_1j_3j_1}$$

tj.

$$B_{l_1l_1l_3,\,2} = \begin{array}{cc} & \begin{array}{ccc} e_{j_1j_1j_3} & e_{j_1j_3j_1} & e_{j_3j_1j_1} \end{array} \\ \begin{array}{c} e_{j_1j_1j_3} \\ e_{j_1j_3j_1} \\ e_{j_3j_1j_1} \end{array} & \begin{bmatrix} 1 & q_{j_1j_3} & 0 \\ q_{j_3j_1} & 1 & 0 \\ 0 & 0 & 1 + q_{j_1j_1} \end{bmatrix} \end{array}$$

$$B_{l_1l_1l_3,\,3} = \begin{array}{cc} & \begin{array}{ccc} e_{j_1j_1j_3} & e_{j_1j_3j_1} & e_{j_3j_1j_1} \end{array} \\ \begin{array}{c} e_{j_1j_1j_3} \\ e_{j_1j_3j_1} \\ e_{j_3j_1j_1} \end{array} & \begin{bmatrix} 1 + q_{j_1j_1} & q_{j_1j_1} q_{j_1j_3} & 0 \\ 0 & 1 & q_{j_1j_3}\left(1 + q_{j_1j_1}\right) \\ q^2_{j_3j_1} & q_{j_3j_1} & 1 \end{bmatrix} \end{array}$$

Nadalje, imamo da je

$$\left.\begin{aligned} B_{l_1l_1l_3,\,2} &= \left(I - T^2_{3,2}\right)\left(I - T_{3,2}\right)^{-1}, \\ B_{l_1l_1l_3,\,3} &= \left(I - T^2_{2,1}T_{3,2}\right)\left(I - T^2_{2,1}\right)\left(I - T_{2,1}\right)^{-1}\left(I - T_{3,1}\right)^{-1}, \end{aligned}\right\} \qquad (97)$$

gdje je

$$\left.\begin{aligned} T^2_{3,2}\, e_{j_1j_2j_3} &= q_{j_3j_2}\, q_{j_2j_3}\, e_{j_1j_2j_3} = \sigma_{j_2j_3}\, e_{j_1j_2j_3}\,, \\ T_{3,2}\, e_{j_1j_2j_3} &= q_{j_3j_2}\, e_{j_1j_3j_2}\,, \\ T^2_{2,1}T_{3,2}\, e_{j_1j_2j_3} &= q_{j_3j_1} q_{j_1j_3} q_{j_3j_2}\, e_{j_1j_3j_2} = \sigma_{j_1j_3}\, q_{j_3j_2}\, e_{j_1j_3j_2}\,, \\ T^2_{2,1}\, e_{j_1j_2j_3} &= q_{j_2j_1}\, q_{j_1j_2}\, e_{j_1j_2j_3} = \sigma_{j_1j_2}\, e_{j_1j_2j_3}\,, \\ T_{2,1}\, e_{j_1j_2j_3} &= q_{j_2j_1}\, e_{j_2j_1j_3}\,, \\ T_{3,1}\, e_{j_1j_2j_3} &= q_{j_3j_1} q_{j_3j_2}\, e_{j_3j_1j_2}\,. \end{aligned}\right\} \qquad (98)$$

Identifikacijom indeksa j_2 s indeksom j_1 u identitetima (98) dobivamo sljedeće matrice

$$I-T_{3,2}^2 = \begin{array}{cc} & \begin{array}{ccc} e_{j_1j_1j_3} & e_{j_1j_3j_1} & e_{j_3j_1j_1} \end{array} \\ \begin{array}{c} e_{j_1j_1j_3} \\ e_{j_1j_3j_1} \\ e_{j_3j_1j_1} \end{array} & \begin{bmatrix} 1-\sigma_{j_1j_3} & 0 & 0 \\ 0 & 1-\sigma_{j_1j_3} & 0 \\ 0 & 0 & 1-q_{j_1j_1}^2 \end{bmatrix} \end{array} \qquad \det\left(I-T_{3,2}^2\right)=\left(1-q_{j_1j_1}^2\right)\left(1-\sigma_{j_1j_3}\right)^2$$

$$I-T_{3,2} = \begin{array}{cc} & \begin{array}{ccc} e_{j_1j_1j_3} & e_{j_1j_3j_1} & e_{j_3j_1j_1} \end{array} \\ \begin{array}{c} e_{j_1j_1j_3} \\ e_{j_1j_3j_1} \\ e_{j_3j_1j_1} \end{array} & \begin{bmatrix} 1 & -q_{j_1j_3} & 0 \\ -q_{j_3j_1} & 1 & 0 \\ 0 & 0 & 1-q_{j_1j_1} \end{bmatrix} \end{array} \qquad \det\left(I-T_{3,2}\right)=\left(1-q_{j_1j_1}\right)\left(1-\sigma_{j_1j_3}\right)$$

$$I-T_{2,1}^2T_{3,2} = \begin{array}{cc} & \begin{array}{ccc} e_{j_1j_1j_3} & e_{j_1j_3j_1} & e_{j_3j_1j_1} \end{array} \\ \begin{array}{c} e_{j_1j_1j_3} \\ e_{j_1j_3j_1} \\ e_{j_3j_1j_1} \end{array} & \begin{bmatrix} 1 & -q_{j_1j_3}q_{j_1j_1}^2 & 0 \\ -\sigma_{j_1j_3}q_{j_3j_1} & 1 & 0 \\ 0 & 0 & 1-q_{j_1j_1}\sigma_{j_1j_3} \end{bmatrix} \end{array}$$

$$\det\left(I-T_{2,1}^2T_{3,2}\right)=\left(1-q_{j_1j_1}\sigma_{j_1j_3}\right)\left(1-q_{j_1j_1}^2\sigma_{j_1j_3}^2\right)$$

$$I-T_{2,1}^2 = \begin{array}{cc} & \begin{array}{ccc} e_{j_1j_1j_3} & e_{j_1j_3j_1} & e_{j_3j_1j_1} \end{array} \\ \begin{array}{c} e_{j_1j_1j_3} \\ e_{j_1j_3j_1} \\ e_{j_3j_1j_1} \end{array} & \begin{bmatrix} 1-q_{j_1j_1}^2 & 0 & 0 \\ 0 & 1-\sigma_{j_1j_3} & 0 \\ 0 & 0 & 1-\sigma_{j_1j_3} \end{bmatrix} \end{array} \qquad \det\left(I-T_{2,1}^2\right)=\left(1-q_{j_1j_1}^2\right)\left(1-\sigma_{j_1j_3}\right)^2$$

$$I-T_{2,1} = \begin{array}{cc} & \begin{array}{ccc} e_{j_1j_1j_3} & e_{j_1j_3j_1} & e_{j_3j_1j_1} \end{array} \\ \begin{array}{c} e_{j_1j_1j_3} \\ e_{j_1j_3j_1} \\ e_{j_3j_1j_1} \end{array} & \begin{bmatrix} 1-q_{j_1j_1} & 0 & 0 \\ 0 & 1 & -q_{j_1j_3} \\ 0 & -q_{j_3j_1} & 1 \end{bmatrix} \end{array} \qquad \det\left(I-T_{2,1}\right)=\left(1-q_{j_1j_1}\right)\left(1-\sigma_{j_1j_3}\right)$$

$$I-T_{3,1} = \begin{array}{cc} & \begin{array}{ccc} e_{j_1j_1j_3} & e_{j_1j_3j_1} & e_{j_3j_1j_1} \end{array} \\ \begin{array}{c} e_{j_1j_1j_3} \\ e_{j_1j_3j_1} \\ e_{j_3j_1j_1} \end{array} & \begin{bmatrix} 1 & -q_{j_1j_1}q_{j_1j_3} & 0 \\ 0 & 1 & -q_{j_1j_1}q_{j_1j_3} \\ -q_{j_3j_1}^2 & 0 & 1 \end{bmatrix} \end{array} \qquad \det\left(I-T_{3,1}\right)=\left(1-q_{j_1j_1}^2\sigma_{j_1j_3}^2\right)$$

pa je

$$\det B_{l_1 l_1 l_3,\,2}=\frac{\det\left(I-T_2^2\right)}{\det\left(I-T_2\right)}=\left(1+q_{j_1 j_1}\right)\left(1-\sigma_{j_1 j_3}\right),$$

$$\det B_{l_1 l_1 l_3,\,3}=\frac{\det\left(I-T_1^2\,T_2\right)\cdot\det\left(I-T_1^2\right)}{\det\left(I-T_1\right)\cdot\det\left(I-T_1 T_2\right)}=\left(1+q_{j_1 j_1}\right)\left(1-\sigma_{j_1 j_3}\right)\left(1-q_{j_1 j_1}\sigma_{j_1 j_3}\right),$$

odnosno

$$\det \mathcal{A}_{l_1 l_1 l_3}=\left(1+q_{j_1 j_1}\right)^2\left(1-\sigma_{j_1 j_3}\right)^2\left(1-q_{j_1 j_1}\sigma_{j_1 j_3}\right).$$

Primjedba 4.1.4

Ako je $Q=\{l_1,l_2,...,l_n\}\subseteq\mathcal{N}$ podskup od $\mathcal{N}=\{i_1,i_2,...,i_N\}$, onda se determinante matrica $\mathcal{A}_Q$ i $B_{Q,\,n-k+1}$, $1\le k\le n-1$ faktoriziraju formulama (86) i (87) iz teorema 4.1.2. U protivnom, tj. ako je $Q=\{l_1\le l_2\le...\le l_n\}$, $l_j\in\mathcal{N}$ multiskup, onda nema jedinstvene formule (ovisi o izboru multiskupa Q) za faktorizaciju $\det\mathcal{A}_Q$ i $\det B_{Q,\,n-k+1}$, $1\le k\le n-1$.

Napomenimo sljedeće. Uzimajući u obzir da je

$$\det\mathcal{A}_Q=\overleftarrow{\prod_{1\le k\le n-1}}\det B_{Q,\,n-k+1}=\det B_{Q,\,2}\cdot\det B_{Q,\,3}\cdots\det B_{Q,\,n-1}\cdot\det B_{Q,\,n}=\prod_{k=2}^{n}\det B_{Q,\,k},$$

gdje je
$$\det B_{Q,\,k}=\prod_{m=2}^{k}\prod_{T\in\binom{Q}{m}}\left(1-\sigma_T\right)^{(m-2)!\,(n-m)!},\quad 2\le k\le n$$

u nastavku će se radi jednostavnosti zapisa umjesto $\det B_{Q,\,n-k+1}$, $1\le k\le n-1$ pisati (tj. izračunavati) $\det B_{Q,\,k}$, $2\le k\le n$.

Specijalno, ako je $l=l_1=l_2=...=l_n$, onda imamo multiskup $Q=\{l^n\}$, $l\in\mathcal{N}$. U ovom slučaju je $\mathcal{A}_Q\left(=B_{Q,\,k},\ 2\le k\le n\right)$ 1×1 matrica, kojoj je jedini element jednak $[n]_{q_{ll}}$, gdje je $[n]_q=\sum_{r=0}^{n-1}q^r=1+q+q^2+...+q^{n-1}$, $n\ge1$, stoga je

$$\det\mathcal{A}_Q=[n]_{q_{ll}}.\tag{99}$$

Koristeći svojstvo da je $1-q^n = (1-q)\cdot\left(1+q+q^2+...+q^{n-1}\right)$

$$= (1-q)\cdot[n]_q ,$$

imamo da se formula (99) možemo pisati u obliku $\det \mathcal{A}_Q = \dfrac{1-q_{ll}^n}{1-q_{ll}}$, $q_{ll}\neq 1$, $l\in\mathcal{N}$.

Neka je $l_3 = l_4 = ... = l_n = l_1$, $l_1 \neq l_2$.

Tada imamo multiskup $Q=\left\{l_1^{n-1}, l_2\right\}$ pa je $\mathcal{A}_Q$ $n\times n$ matrica, gdje je $n = \mathrm{Card}\,Q$. Dobivamo

$$\det B_{Q,\,n} = \det B_{l_1^{n-1}l_2,\,n} = [n-1]_{q_{l_1l_1}}!\ \prod_{j=0}^{n-2}\left(1-q_{l_1l_1}^{j}\sigma_{l_1l_2}\right) \tag{100}$$

odnosno

$$\det B_{Q,\,k} = \det B_{l_1^{n-1}l_2,\,k} = [n-1]_{q_{l_1l_1}}!\ \prod_{j=0}^{k-2}\left(1-q_{l_1l_1}^{j}\sigma_{l_1l_2}\right) \tag{101}$$

$2\le k\le n$ pa je

$$\det \mathcal{A}_Q = \det \mathcal{A}_{l_1^{n-1}l_2} = \prod_{k=2}^{n}\det B_{l_1^{n-1}l_2,\,k}$$

$$= \prod_{k=2}^{n}\left([n-1]_{q_{l_1l_1}}!\ \prod_{j=0}^{k-2}\left(1-q_{l_1l_1}^{j}\sigma_{l_1l_2}\right)\right)$$

$$= \left([n-1]_{q_{l_1l_1}}!\right)^{n-1}\cdot\prod_{j=0}^{n-2}\left(1-q_{l_1l_1}^{j}\sigma_{l_1l_2}\right)^{n-j-1}. \tag{102}$$

Pritom je $[k]_q = \sum_{r=0}^{k-1} q^r = 1+q+q^2+...+q^{k-1}$.

Slijedi prikaz formula za izračunavanje determinante matrice $B_{Q,\,n}\left(=B_Q\right)$ s obzirom na $Q=\left\{l_1^{n-2}, l_2^2\right\}$ i $Q=\left\{l_1^{n-2}, l_2, l_3\right\}$.

(*i*) Neka je $l_4 = l_5 = ... = l_n = l_1$, $l_3 = l_2$, $l_1\neq l_2$.

Tada imamo multiskup $Q=\left\{l_1^{n-2}, l_2^2\right\}$, pri čemu je

$$\det B_{Q,\,n} = \det B_{l_1^{n-2}l_2^2,\,n}$$

$$= \prod_{j=1}^{n-2}[j]_{q_{l_1l_1}}!\cdot\left(1-q_{l_1l_1}^{n-2-j}\sigma_{l_1l_2}\right)^{j}\ \prod_{m=0}^{n-2}\left(1+(-1)^m q_{l_1l_1}^{\frac{m\cdot(m-1)}{2}} q_{l_2l_2}\sigma_{l_1l_2}^{m}\right). \tag{103}$$

(*ii*) Neka je $l_4 = l_5 = \ldots = l_n = l_1$, $l_3 \neq l_2$, $l_1 \neq l_2$.

Tada imamo multiskup $Q = \{ l_1^{n-2}, l_2, l_3 \}$, pri čemu je

$$\det B_{Q,n} = \det B_{l_1^{n-2} l_2 l_3, n}$$

$$= \prod_{i=1}^{n-2} \left([i]_{q_{l_1 l_1}} ! \right)^2 \prod_{j=2}^{3} \left(1 - q_{l_1 l_1}^{n-2-i} \sigma_{l_1 l_j} \right)^i \prod_{m=0}^{n-2} \left(1 - q_{l_1 l_1}^{m\cdot(m-1)} \sigma_{l_1 l_2}^m \sigma_{l_1 l_3}^m \sigma_{l_2 l_3} \right). \quad (104)$$

Primjedba 4.1.5

Neka je $Q = \{ l_1, l_2, \ldots, l_n \} \subseteq \mathcal{N}$ podskup od $\mathcal{N} = \{ i_1, i_2, \ldots, i_N \}$.

Tada primjenom identiteta (72) specijalno za $k = 1$ proizlazi

$$B_{Q,n} = D_{Q,n} \cdot C_{Q,n}^{-1},$$

što ćemo pisati u obliku

$$B_{Q,n} \cdot C_{Q,n} = D_{Q,n}. \quad (105)$$

Pritom je

$$C_{Q,n} = \left(I - T_{n,1} \right) \cdot \left(I - T_{n-1,1} \right) \cdots \left(I - T_{3,1} \right) \cdot \left(I - T_{2,1} \right),$$

$$D_{Q,n} = \left(I - T_{2,1}^2 T_{n,2} \right) \cdot \left(I - T_{2,1}^2 T_{n-1,2} \right) \cdots \left(I - T_{2,1}^2 T_{3,2} \right) \cdot \left(I - T_{2,1}^2 T_{2,2} \right),$$

stoga se (105) može pisati u obliku

$$B_{Q,n} \cdot \left(I - T_{n,1} \right) \cdot \left(I - T_{n-1,1} \right) \cdots \left(I - T_{3,1} \right) \cdot \left(I - T_{2,1} \right) =$$

$$= \left(I - T_{2,1}^2 T_{n,2} \right) \cdot \left(I - T_{2,1}^2 T_{n-1,2} \right) \cdots \left(I - T_{2,1}^2 T_{3,2} \right) \cdot \left(I - T_{2,1}^2 T_{2,2} \right)$$

ili

$$B_{Q,n} \cdot C_{Q,n} = \left(I - T_{2,1}^2 T_{n,2} \right) \cdot \left(I - T_{2,1}^2 T_{n-1,2} \right) \cdots \left(I - T_{2,1}^2 T_{3,2} \right) \cdot \left(I - T_{2,1}^2 T_{2,2} \right) \quad (106)$$

Koristeći lemu 4.1.1 dobivamo

$$\det \left(I - T_{2,1}^2 T_{n,2} \right) = \prod_{T \in \binom{Q}{n}} \left(1 - \sigma_T \right)^{n\cdot(n-2)!} = \left(1 - \sigma_{l_1 l_2 \ldots l_n} \right)^{n\cdot(n-2)!},$$

$$\det \left(I - T_{2,1}^2 T_{n-1,2} \right) = \prod_{T \in \binom{Q}{n-1}} \left(1 - \sigma_T \right)^{(n-1)\cdot(n-3)!} = \prod_{1 \le j_1 < j_2 < \cdots < j_{n-1} \le n} \left(1 - \sigma_{l_{j_1} l_{j_2} \ldots l_{j_{n-1}}} \right)^{(n-1)\cdot(n-3)!},$$

$$\det \left(I - T_{2,1}^2 T_{n-2,2} \right) = \prod_{T \in \binom{Q}{n-2}} \left(1 - \sigma_T \right)^{(n-2)\cdot(n-4)!\cdot 2!} = \prod_{1 \le j_1 < j_2 < \cdots < j_{n-2} \le n} \left(1 - \sigma_{l_{j_1} l_{j_2} \ldots l_{j_{n-2}}} \right)^{(n-2)\cdot(n-4)!\cdot 2!}$$

$$\vdots$$

$$\det\left(I - T_{2,1}^2 T_{3,2}\right) = \prod_{T \in \binom{Q}{3}} \left(1-\sigma_T\right)^{3\cdot(n-3)!} = \prod_{1\le j_1<j_2<j_3\le n} \left(1-\sigma_{l_{j_1} l_{j_2} l_{j_3}}\right)^{3\cdot(n-3)!},$$

$$\det\left(I - T_{2,1}^2 T_{2,2}\right) = \det\left(I - T_{2,1}^2\right) = \prod_{T \in \binom{Q}{2}} \left(1-\sigma_T\right)^{2\cdot(n-2)!} = \prod_{1\le j_1<j_2\le n} \left(1-\sigma_{l_{j_1} l_{j_2}}\right)^{2\cdot(n-2)!},$$

gdje je $\binom{Q}{m} = \left\{T \subseteq Q \mid \mathrm{Card}\left(T\right) = m\right\}$, $\sigma_T = \prod_{\{i,j\}\subset T} \sigma_{ij}$.

4. 2. Prikaz konstante pomoću iteriranih komutatora

Koristeći činjenice

- rastav algebre $\mathcal{B} = \mathbb{C}\langle e_{i_1}, e_{i_2}, ..., e_{i_N}\rangle$ po multihomogenim komponentama glasi

$$\mathcal{B} = \bigoplus_{\substack{n\ge 0 \\ l_1\le l_2\le ...\le l_n,\, l_j\in\mathcal{N}=\{i_1,i_2,...,i_N\}}} \mathcal{B}_{l_1 l_2 \dots l_n},$$

  gdje se svaki težinski potprostor $\mathcal{B}_Q = \mathcal{B}_{l_1 l_2 \dots l_n}$ algebre $\mathcal{B}$ naziva generičkim ako je $Q = \left\{l_1, l_2, ..., l_n\right\}$ skup, odnosno degeneriranim ako je Q multiskup koji nije skup,
- konstante u bilo kojem degeneriranom težinskom potprostoru mogu se konstruirati iz konstanti nekog generičkog težinskog potprostora,

dovodi nas do zaključka da se problem nalaženja prostora konstanti u algebri $\mathcal{B}$ svodi na traženje konstanti u bilo kojem generičkom težinskom potprostoru $\mathcal{B}_Q = \mathcal{B}_{l_1 l_2 \dots l_n}$, što je ekvivalentno nalaženju jezgre operatora $\mathbf{B}_Q = \partial\,|_{\mathcal{B}_Q} : \mathcal{B}_Q \to \mathcal{B}_Q$ (vidi 2.2.).

U nastavku će se detaljnije proučavati jezgra operatora $\mathbf{B}_Q$, kojemu je u monomijalnoj bazi težinskog potprostora $\mathcal{B}_Q$ pridružena matrica B_Q takva da je $B_Q = B_{Q,n}$, gdje je $B_{Q,n}$ n-ti faktor matrice $\mathcal{A}_Q$ (vidi odjeljak 4.1, identiteti (70) i (79)), stoga ćemo operator $\mathbf{B}_Q$ poistovjetiti s matricom $B_{Q,n}$ (analogno ćemo matrice $C_{Q,n}$, $D_{Q,n}$, $T_{b,a}$, $1 \le a < b \le n$, $T_{a,a-1}^2 T_{b,a}$, $1 < a \le b \le n$ razmatrati kao operatore).

Dakle, predmet sljedećeg proučavanja biti će određivanje jezgre operatora $B_{Q,n}$, tj. $Ker\left(B_{Q,n}\right)$, $n \ge 2$.

Pretpostavimo da je $X \in Ker\left(\mathrm{B}_{\mathrm{Q,\,n}}\right)$. Tada je $\mathrm{B}_{\mathrm{Q,\,n}} \cdot X = 0$.

Želimo odrediti vektor (stupac) X. Uzimajući u obzir identitet (106) imamo da je

$$\mathrm{B}_{\mathrm{Q,\,n}} \cdot \mathrm{C}_{\mathrm{Q,\,n}} = \left(I - \mathrm{T}_{2,1}^{2}\,\mathrm{T}_{\mathrm{n},2}\right)\cdot\left(I - \mathrm{T}_{2,1}^{2}\,\mathrm{T}_{\mathrm{n}-1,2}\right)\cdots\left(I - \mathrm{T}_{2,1}^{2}\,\mathrm{T}_{3,2}\right)\cdot\left(I - \mathrm{T}_{2,1}^{2}\,\mathrm{T}_{2,2}\right),$$

$$\mathrm{B}_{\mathrm{Q,\,n}} \cdot \mathrm{C}_{\mathrm{Q,\,n}} \cdot \left(I - \mathrm{T}_{2,1}^{2}\,\mathrm{T}_{2,2}\right)^{-1} = \left(I - \mathrm{T}_{2,1}^{2}\,\mathrm{T}_{\mathrm{n},2}\right)\cdot\left(I - \mathrm{T}_{2,1}^{2}\,\mathrm{T}_{\mathrm{n}-1,2}\right)\cdots\left(I - \mathrm{T}_{2,1}^{2}\,\mathrm{T}_{3,2}\right),$$

$$\mathrm{B}_{\mathrm{Q,\,n}} \cdot \mathrm{C}_{\mathrm{Q,\,n}} \cdot \left(I - \mathrm{T}_{2,1}^{2}\,\mathrm{T}_{2,2}\right)^{-1} \cdot \left(I - \mathrm{T}_{2,1}^{2}\,\mathrm{T}_{3,2}\right)^{-1} = \left(I - \mathrm{T}_{2,1}^{2}\,\mathrm{T}_{\mathrm{n},2}\right)\cdot\left(I - \mathrm{T}_{2,1}^{2}\,\mathrm{T}_{\mathrm{n}-1,2}\right)\cdots\left(I - \mathrm{T}_{2,1}^{2}\,\mathrm{T}_{4,2}\right),$$

$$\vdots$$

$$\mathrm{B}_{\mathrm{Q,\,n}} \cdot \mathrm{C}_{\mathrm{Q,\,n}} \cdot \left(I - \mathrm{T}_{2,1}^{2}\,\mathrm{T}_{2,2}\right)^{-1} \cdot \left(I - \mathrm{T}_{2,1}^{2}\,\mathrm{T}_{3,2}\right)^{-1} \cdots \left(I - \mathrm{T}_{2,1}^{2}\,\mathrm{T}_{\mathrm{n}-1,2}\right)^{-1} = \left(I - \mathrm{T}_{2,1}^{2}\,\mathrm{T}_{\mathrm{n},2}\right),$$

odnosno

$$\left.\begin{aligned}
&\mathrm{B}_{\mathrm{Q,\,n}} \cdot \mathrm{C}_{\mathrm{Q,\,n}} \cdot U_1 \\
&\qquad = \left(I - \mathrm{T}_{2,1}^{2}\,\mathrm{T}_{\mathrm{n},2}\right)\cdot\left(I - \mathrm{T}_{2,1}^{2}\,\mathrm{T}_{\mathrm{n}-1,2}\right)\cdots\left(I - \mathrm{T}_{2,1}^{2}\,\mathrm{T}_{3,2}\right)\cdot\left(I - \mathrm{T}_{2,1}^{2}\,\mathrm{T}_{2,2}\right)\cdot U_1, \\
&\mathrm{B}_{\mathrm{Q,\,n}} \cdot \mathrm{C}_{\mathrm{Q,\,n}} \cdot \left(I - \mathrm{T}_{2,1}^{2}\,\mathrm{T}_{2,2}\right)^{-1} \cdot U_2 \\
&\qquad = \left(I - \mathrm{T}_{2,1}^{2}\,\mathrm{T}_{\mathrm{n},2}\right)\cdot\left(I - \mathrm{T}_{2,1}^{2}\,\mathrm{T}_{\mathrm{n}-1,2}\right)\cdots\left(I - \mathrm{T}_{2,1}^{2}\,\mathrm{T}_{3,2}\right)\cdot U_2, \\
&\mathrm{B}_{\mathrm{Q,\,n}} \cdot \mathrm{C}_{\mathrm{Q,\,n}} \cdot \left(I - \mathrm{T}_{2,1}^{2}\,\mathrm{T}_{2,2}\right)^{-1} \cdot \left(I - \mathrm{T}_{2,1}^{2}\,\mathrm{T}_{3,2}\right)^{-1} \cdot U_3 \\
&\qquad = \left(I - \mathrm{T}_{2,1}^{2}\,\mathrm{T}_{\mathrm{n},2}\right)\cdot\left(I - \mathrm{T}_{2,1}^{2}\,\mathrm{T}_{\mathrm{n}-1,2}\right)\cdots\left(I - \mathrm{T}_{2,1}^{2}\,\mathrm{T}_{4,2}\right)\cdot U_3, \\
&\quad\vdots \\
&\mathrm{B}_{\mathrm{Q,\,n}} \cdot \mathrm{C}_{\mathrm{Q,\,n}} \cdot \left(I - \mathrm{T}_{2,1}^{2}\,\mathrm{T}_{2,2}\right)^{-1} \cdot \left(I - \mathrm{T}_{2,1}^{2}\,\mathrm{T}_{3,2}\right)^{-1} \cdots \left(I - \mathrm{T}_{2,1}^{2}\,\mathrm{T}_{\mathrm{n}-1,2}\right)^{-1} \cdot U_{\mathrm{n}-1} \\
&\qquad = \left(I - \mathrm{T}_{2,1}^{2}\,\mathrm{T}_{\mathrm{n},2}\right)\cdot U_{\mathrm{n}-1},
\end{aligned}\right\} \tag{107}$$

gdje je $U_{\mathrm{j}} \neq 0$, $1 \leq \mathrm{j} \leq \mathrm{n} - 1$.

Pretpostavimo da je $\quad U_{\mathrm{j}} \in Ker\left(I - \mathrm{T}_{2,1}^{2}\,\mathrm{T}_{\mathrm{j}+1,2}\right), \quad 1 \leq \mathrm{j} \leq \mathrm{n} - 1$.

Tada je $\quad \left(I - \mathrm{T}_{2,1}^{2}\,\mathrm{T}_{\mathrm{j}+1,2}\right)\cdot U_{\mathrm{j}} = 0$.

Naime, iz ... proizlazi

$$\begin{aligned}
U_1 &\in Ker\left(I - \mathrm{T}_{2,1}^{2}\,\mathrm{T}_{2,2}\right) & \left(I - \mathrm{T}_{2,1}^{2}\,\mathrm{T}_{2,2}\right)\cdot U_1 &= 0, \\
U_2 &\in Ker\left(I - \mathrm{T}_{2,1}^{2}\,\mathrm{T}_{3,2}\right) & \left(I - \mathrm{T}_{2,1}^{2}\,\mathrm{T}_{3,2}\right)\cdot U_2 &= 0, \\
U_3 &\in Ker\left(I - \mathrm{T}_{2,1}^{2}\,\mathrm{T}_{4,2}\right) & \left(I - \mathrm{T}_{2,1}^{2}\,\mathrm{T}_{4,2}\right)\cdot U_3 &= 0, \\
&\vdots \\
U_{\mathrm{n}-1} &\in Ker\left(I - \mathrm{T}_{2,1}^{2}\,\mathrm{T}_{\mathrm{n},2}\right) & \left(I - \mathrm{T}_{2,1}^{2}\,\mathrm{T}_{\mathrm{n},2}\right)\cdot U_{\mathrm{n}-1} &= 0,
\end{aligned}$$

što uvrštavanjem u identitete (107) povlači da je

$$\left.\begin{aligned}
&\mathrm{B}_{\mathrm{Q,\,n}}\cdot\mathrm{C}_{\mathrm{Q,\,n}}\cdot U_1=0\,,\\
&\mathrm{B}_{\mathrm{Q,\,n}}\cdot\mathrm{C}_{\mathrm{Q,\,n}}\cdot\left(I-\mathrm{T}_{2,1}^2\,\mathrm{T}_{2,2}\right)^{-1}\cdot U_2=0\,,\\
&\mathrm{B}_{\mathrm{Q,\,n}}\cdot\mathrm{C}_{\mathrm{Q,\,n}}\cdot\left(I-\mathrm{T}_{2,1}^2\,\mathrm{T}_{2,2}\right)^{-1}\cdot\left(I-\mathrm{T}_{2,1}^2\,\mathrm{T}_{3,2}\right)^{-1}\cdot U_3=0\,,\\
&\quad\vdots\\
&\mathrm{B}_{\mathrm{Q,\,n}}\cdot\mathrm{C}_{\mathrm{Q,\,n}}\cdot\left(I-\mathrm{T}_{2,1}^2\,\mathrm{T}_{2,2}\right)^{-1}\cdot\left(I-\mathrm{T}_{2,1}^2\,\mathrm{T}_{3,2}\right)^{-1}\cdots\left(I-\mathrm{T}_{2,1}^2\,\mathrm{T}_{\mathrm{n-1},2}\right)^{-1}\cdot U_{\mathrm{n-1}}=0\,.
\end{aligned}\right\}\qquad(108)$$

Ako je $U_{\mathrm{j}}\in Ker\left(I-\mathrm{T}_{2,1}^2\,\mathrm{T}_{\mathrm{j+1},2}\right)$, onda je $X=W_{\mathrm{j}}\cdot U_{\mathrm{j}}\in Ker\left(\mathrm{B}_{\mathrm{Q,\,n}}\right)$, $1\le\mathrm{j}\le\mathrm{n}-1$.

Pritom je $W_1=\mathrm{C}_{\mathrm{Q,\,n}}$, $W_{\mathrm{j}}=\mathrm{C}_{\mathrm{Q,\,n}}\cdot\prod_{2\le\mathrm{i}\le\mathrm{j}}\left(I-\mathrm{T}_{2,1}^2\,\mathrm{T}_{\mathrm{i},2}\right)^{-1}$, $2\le\mathrm{j}\le\mathrm{n}-1$.

Drugim riječima, imamo

ako je onda je

$$\begin{aligned}
&U_1\in Ker\left(I-\mathrm{T}_{2,1}^2\,\mathrm{T}_{2,2}\right), && X=\mathrm{C}_{\mathrm{Q,\,n}}\cdot U_1\\
&U_2\in Ker\left(I-\mathrm{T}_{2,1}^2\,\mathrm{T}_{3,2}\right), && X=\mathrm{C}_{\mathrm{Q,\,n}}\cdot\left(I-\mathrm{T}_{2,1}^2\,\mathrm{T}_{2,2}\right)^{-1}\cdot U_2\\
&U_3\in Ker\left(I-\mathrm{T}_{2,1}^2\,\mathrm{T}_{4,2}\right), && X=\mathrm{C}_{\mathrm{Q,\,n}}\cdot\left(I-\mathrm{T}_{2,1}^2\,\mathrm{T}_{2,2}\right)^{-1}\cdot\left(I-\mathrm{T}_{2,1}^2\,\mathrm{T}_{3,2}\right)^{-1}\cdot U_3\\
&\quad\vdots\\
&U_{\mathrm{n-1}}\in Ker\left(I-\mathrm{T}_{2,1}^2\,\mathrm{T}_{\mathrm{n},2}\right), && X=\mathrm{C}_{\mathrm{Q,\,n}}\cdot\left(I-\mathrm{T}_{2,1}^2\,\mathrm{T}_{2,2}\right)^{-1}\cdots\left(I-\mathrm{T}_{2,1}^2\,\mathrm{T}_{\mathrm{n-1},2}\right)^{-1}\cdot U_{\mathrm{n-1}}\,.
\end{aligned}$$

Na osnovu rečenog proizlazi sljedeća propozicija.

Propozicija 4.2.1

Ako je $U_1\in Ker\left(I-\mathrm{T}_{2,1}^2\,\mathrm{T}_{2,2}\right)$, onda se $X\in Ker\left(\mathrm{B}_{\mathrm{Q,\,n}}\right)$ računa po formuli

$$X=\mathrm{C}_{\mathrm{Q,\,n}}\cdot U_1\,.\qquad(109)$$

Ako je $U_{\mathrm{j}}\in Ker\left(I-\mathrm{T}_{2,1}^2\,\mathrm{T}_{\mathrm{j+1},2}\right)$, $2\le\mathrm{j}\le\mathrm{n}-1$, onda se $X\in Ker\left(\mathrm{B}_{\mathrm{Q,\,n}}\right)$ računa po formuli

$$X=\mathrm{C}_{\mathrm{Q,\,n}}\cdot\prod_{2\le\mathrm{i}\le\mathrm{j}}\left(I-\mathrm{T}_{2,1}^2\,\mathrm{T}_{\mathrm{i},2}\right)^{-1}\cdot U_{\mathrm{j}}\,.\qquad(110)$$

Pritom je $\mathrm{C}_{\mathrm{Q,\,n}}=\left(I-\mathrm{T}_{\mathrm{n},1}\right)\cdot\left(I-\mathrm{T}_{\mathrm{n-1},1}\right)\cdots\left(I-\mathrm{T}_{3,1}\right)\cdot\left(I-\mathrm{T}_{2,1}\right)$,

$\mathrm{Q}=\{l_1,l_2,\ldots,l_{\mathrm{n}}\}\subseteq\mathcal{N}=\{\mathrm{i}_1,\mathrm{i}_2,\ldots,\mathrm{i}_{\mathrm{N}}\}$.

Od posebnog je interesa vektor $U_{n-1} \in Ker\left(I - T_{2,1}^2 T_{n,2}\right)$, stoga će se u nastavku provoditi razmatranja za određivanje vektora $X \in Ker\left(B_{Q,n}\right)$ uz pretpostavku da je $U_{n-1} \in Ker\left(I - T_{2,1}^2 T_{n,2}\right)$. Primjenom propozicije 4.2.1, ako je $U_{n-1} \in Ker\left(I - T_{2,1}^2 T_{n,2}\right)$,

onda je $$X = C_{Q,n} \cdot \left(I - T_{2,1}^2 T_{2,2}\right)^{-1} \cdots \left(I - T_{2,1}^2 T_{n-1,2}\right)^{-1} \cdot U_{n-1} .$$

Pritom je $$\left(I - T_{2,1}^2 T_{n,2}\right) \cdot U_{n-1} = 0 .$$

Prirodno se nameće sljedeće pitanje:

kako glase vektori koji razapinju $Ker\left(I - T_{2,1}^2 T_{n,2}\right)$, odnosno čemu je jednaka baza jezgre operatora $I - T_{2,1}^2 T_{n,2}$?

Prije nego li se pozabavimo odgovorom na postavljeno pitanje, podsjetimo se prethodno navedenog.

Skupu $Q = \{l_1, l_2, ..., l_n\} \subseteq \mathcal{N} = \{i_1, i_2, ..., i_N\}$ pridružuje se generički težinski potprostor $\mathcal{B}_Q = \mathcal{B}_{l_1 l_2 \cdots l_n}$. Pritom je

$$\mathcal{B}_Q = \left\{ e_{\underline{j}} = e_{j_1 j_2 \cdots j_n} = e_{j_1} e_{j_2} \cdots e_{j_n} \mid \underline{j} = j_1 j_2 ... j_n \in \widehat{Q} \right\}$$

baza od $\mathcal{B}_Q$, gdje je $\widehat{Q} = S_n Q = \left\{ \sigma\left(l_1 l_2 ... l_n\right) \mid \sigma \in S_n \right\}$ skup svih permutacija skupa Q. $\dim \mathcal{B}_Q = \text{Card } \widehat{Q} = n!$

Koristeći izraz (85) imamo da je

$$\left(T_{2,1}^2 T_{i,2}\right)^k e_{\underline{j}} = d_0\, d_1\, d_2 \cdots d_{k-1}\, e_{t_{i,2}^k \cdot \underline{j}} , \tag{111}$$

$2 \le i \le n$, $1 \le k \le i-1$, $\underline{j} \in \widehat{Q}$, $e_{\underline{j}} \in \mathcal{B}_Q$, gdje je

$$d_0 = \sigma_{j_1 j_i}\, c_0 = \sigma_{j_1 j_i}\, q_{j_i j_2} q_{j_i j_3} q_{j_i j_4} \cdots q_{j_i j_{i-2}} q_{j_i j_{i-1}} ,$$

$$d_1 = \sigma_{j_1 j_{i-1}}\, c_1 = \sigma_{j_1 j_{i-1}}\, q_{j_{i-1} j_i} q_{j_{i-1} j_2} q_{j_{i-1} j_3} \cdots q_{j_{i-1} j_{i-3}} q_{j_{i-1} j_{i-2}} ,$$

$$d_2 = \sigma_{j_1 j_{i-2}}\, c_2 = \sigma_{j_1 j_{i-2}}\, q_{j_{i-2} j_{i-1}} q_{j_{i-2} j_i} q_{j_{i-2} j_2} \cdots q_{j_{i-2} j_{i-4}} q_{j_{i-2} j_{i-3}} ,$$

$$\vdots$$

$$d_{i-3} = \sigma_{j_1 j_3}\, c_{i-3} = \sigma_{j_1 j_3}\, q_{j_3 j_4} q_{j_3 j_5} q_{j_3 j_6} \cdots q_{j_3 j_i} q_{j_3 j_2} ,$$

$$d_{i-2} = \sigma_{j_1 j_2}\, c_{i-2} = \sigma_{j_1 j_2}\, q_{j_2 j_3} q_{j_2 j_4} q_{j_2 j_5} \cdots q_{j_2 j_{i-1}} q_{j_2 j_i} .$$

Pritom za	imamo da je desna strana identiteta (111) dana sa
$k=1$	$d_0 = \sigma_{j_1 j_i}\, q_{j_i j_2} q_{j_i j_3} q_{j_i j_4} \cdots q_{j_i j_{i-2}} q_{j_i j_{i-1}}$
$k=2$	$d_0 \cdot d_1 = \sigma_{j_1 j_{i-1} j_i}\, q_{j_i j_2} q_{j_i j_3} \cdots q_{j_i j_{i-2}}\, q_{j_{i-1} j_2} q_{j_{i-1} j_3} \cdots q_{j_{i-1} j_{i-2}}$
$k=3$	$d_0 \cdot d_1 \cdot d_2 = \sigma_{j_1 j_{i-2} j_{i-1} j_i}\, q_{j_i j_2} q_{j_i j_3} \cdots q_{j_i j_{i-3}}\, q_{j_{i-1} j_2} q_{j_{i-1} j_3} \cdots q_{j_{i-1} j_{i-3}}\, q_{j_{i-2} j_2} q_{j_{i-2} j_3} \cdots q_{j_{i-2} j_{i-3}}$
$\vdots$	
$k=i-2$	$d_0 \cdot d_1 \cdots d_{i-3} = \sigma_{j_1 j_3 \cdots j_{i-2} j_{i-1} j_i}\, q_{j_i j_2} q_{j_{i-1} j_2} \cdots q_{j_5 j_2} q_{j_4 j_2} q_{j_3 j_2}$
$k=i-1$	$d_0 \cdot d_1 \cdots d_{i-2} = \sigma_{j_1 j_3 \cdots j_{i-2} j_{i-1} j_i}\, \sigma_{j_i j_2} \sigma_{j_{i-1} j_2} \cdots \sigma_{j_5 j_2} \sigma_{j_4 j_2} \sigma_{j_3 j_2} = \sigma_{j_1 j_2 j_3 \cdots j_{i-2} j_{i-1} j_i}$

$2 \le i \le n$.

Dakle, iz (111) proizlazi

$$\left(T_{2,1}^2\, T_{i,2}\right)^{i-1} e_{\underline{j}} = d_0\, d_1\, d_2 \cdots d_{i-2}\, e_{t_{i,2}^{i-1} \cdot \underline{j}} = \sigma_{j_1 j_2 \dots j_i}\, e_{\underline{j}}, \tag{112}$$

gdje je $e_{t_{i,2}^{i-1} \cdot \underline{j}} = e_{\underline{j}}$. Koristeći identitet (14) iz odjeljka 2.1, imamo da je

$$\sigma_{j_1 j_2 \dots j_i} = \prod_{1 \le p < r \le i} \sigma_{j_p j_r}\ . \tag{113}$$

Specijalno, ako je $i = n$, onda iz identiteta (112) proizlazi da za svaki bazični monom $e_{\underline{j}} \in \mathcal{B}_Q$, $Q = \{l_1, l_2, \dots, l_n\} \subseteq \mathcal{N} = \{i_1, i_2, \dots, i_N\}$ vrijedi da je

$$\left(T_{2,1}^2\, T_{n,2}\right)^{n-1} e_{\underline{j}} = \sigma_{j_1 j_2 \dots j_n}\, e_{\underline{j}} = \sigma_{l_1 l_2 \dots l_n}\, e_{\underline{j}}, \tag{114}$$

stoga je

$$\left(I - \left(T_{2,1}^2\, T_{n,2}\right)^{n-1}\right) e_{\underline{j}} = \left(1 - \sigma_{l_1 l_2 \dots l_n}\right) e_{\underline{j}}, \tag{115}$$

gdje je $Q_{\{1,2,\dots,n\}}\, e_{\underline{j}} = \sigma_{j_1 j_2 \dots j_n}\, e_{\underline{j}} = \sigma_{l_1 l_2 \dots l_n}\, e_{\underline{j}}$. S druge strane imamo

$$\left(I - T_{2,1}^2\, T_{n,2}\right) \cdot \left(I + T_{2,1}^2\, T_{n,2} + \left(T_{2,1}^2\, T_{n,2}\right)^2 + \cdots + \left(T_{2,1}^2\, T_{n,2}\right)^{n-2}\right) e_{\underline{j}} = \left(I - \left(T_{2,1}^2\, T_{n,2}\right)^{n-1}\right) e_{\underline{j}},$$

odnosno primjenom identiteta (115) dobivamo

$$\left(I - T_{2,1}^2\, T_{n,2}\right) \cdot \left(I + T_{2,1}^2\, T_{n,2} + \left(T_{2,1}^2\, T_{n,2}\right)^2 + \cdots + \left(T_{2,1}^2\, T_{n,2}\right)^{n-2}\right) e_{\underline{j}} = \left(1 - \sigma_{l_1 l_2 \dots l_n}\right) e_{\underline{j}}. \tag{116}$$

Pretpostavimo da je $\sigma_{l_1 l_2 \dots l_n} = 1$. Tada iz (116) proizlazi

$$\left(I - T_{2,1}^2\, T_{n,2}\right) \cdot \left(I + T_{2,1}^2\, T_{n,2} + \left(T_{2,1}^2\, T_{n,2}\right)^2 + \cdots + \left(T_{2,1}^2\, T_{n,2}\right)^{n-2}\right) e_{\underline{j}} = 0, \tag{117}$$

gdje je

$$U_{\underline{j}} = \left(I + T_{2,1}^2 T_{n,2} + \left(T_{2,1}^2 T_{n,2}\right)^2 + \cdots + \left(T_{2,1}^2 T_{n,2}\right)^{n-2}\right) e_{\underline{j}} \ , \tag{118}$$

$U_{\underline{j}} \in Ker\left(I - T_{2,1}^2 T_{n,2}\right)$ za svaki $\underline{j} = j_1 j_2 ... j_n \in \widehat{Q} = S_n Q$. Pritom je $Q = \{l_1, l_2, ..., l_n\}$.

Budući da je Card $\widehat{Q} = n!$, zaključujemo:

ako je $\sigma_{l_1 l_2 \dots l_n} = 1$, onda je izrazom (118) dano $n!$ vektora $U_{\underline{j}}$, $\underline{j} \in \widehat{Q} = S_n Q$, koji razapinju jezgru operatora $I - T_{2,1}^2 T_{n,2}$.

U protivnom, za $\sigma_{l_1 l_2 \dots l_n} \neq 1$ imamo $U_{\underline{j}} \notin Ker\left(I - T_{2,1}^2 T_{n,2}\right)$, $\underline{j} \in \widehat{Q} = S_n Q$.

Zanima nas koliko ima linearno nezavisnih vektora, koji će činiti bazu jezgre operatora $I - T_{2,1}^2 T_{n,2}$ uz uvjet da je $\sigma_{l_1 l_2 \dots l_n} = 1$?

Primjedba 4.2.2

Detaljnim izračunavanjem vektora $U_{\underline{j}} \in Ker\left(I - T_{2,1}^2 T_{n,2}\right)$ danih izrazom (118) za svaki $\underline{j} \in S_n Q$, gdje je $Q = \{l_1, l_2, ..., l_n\} \subseteq \mathcal{N} = \{i_1, i_2, ..., i_N\}$, dolazi se do sljedećeg rezultata. Ako je $\sigma_{l_1 l_2 \dots l_n} = 1$, onda

(1) dimenzija jezgre operatora $I - T_{2,1}^2 T_{n,2}$ jednaka je $n \cdot \dfrac{(n-1)!}{n-1} = n \cdot (n-2)!$

(2) linearno nezavisni vektori (koji čine bazu jezgre operatora $I - T_{2,1}^2 T_{n,2}$) su

$$\left.\begin{aligned} U_{l_1 l_2 j_3 j_4 \dots j_n} &= \left(I + T_{2,1}^2 T_{n,2} + \left(T_{2,1}^2 T_{n,2}\right)^2 + \cdots + \left(T_{2,1}^2 T_{n,2}\right)^{n-2}\right) e_{l_1 l_2 j_3 j_4 \dots j_n} \ , \\ U_{l_k l_1 i_3 i_4 \dots i_n} &= \left(I + T_{2,1}^2 T_{n,2} + \left(T_{2,1}^2 T_{n,2}\right)^2 + \cdots + \left(T_{2,1}^2 T_{n,2}\right)^{n-2}\right) e_{l_k l_1 i_3 i_4 \dots i_n} \ , \end{aligned}\right\} \tag{119}$$

za svaki $2 \le k \le n$.

Pritom je

$j_3 j_4 ... j_n \in S_{n-2} Q'$, $\quad Q' = Q \setminus \{l_1, l_2\} = \{l_3, l_4, ..., l_n\}$,

$i_3 i_4 ... i_n \in S_{n-2} Q''$, $\quad Q'' = Q \setminus \{l_1, l_k\} = \left\{l_2, l_3, ..., l_{k-1}, \widehat{l_k}, l_{k+1}, ..., l_n\right\}$, $\ 2 \le k \le n$

gdje je $\quad S_{n-2} Q' = \{\sigma(l_3 l_4 ... l_n) \mid \sigma \in S_{n-2}\}$

$S_{n-2} Q'' = \left\{\sigma\left(l_2 l_3 ... l_{k-1} \widehat{l_k} l_{k+1} ... l_n\right) \mid \sigma \in S_{n-2}\right\}$, $\ 2 \le k \le n$.

Primjeri 4.2.3

Obrazložimo navedene tvrdnje iz primjedbe 4.2.2.

(1) Neka je skupu $\mathrm{Q}=\{1,2,3\}$ pridružen generički težinski potprostor $\mathcal{B}_{\mathrm{Q}}=\mathcal{B}_{123}$.

Pritom je $\mathcal{B}_{\mathrm{Q}}=\{e_{123},e_{132},e_{312},e_{321},e_{231},e_{213}\}$ baza od $\mathcal{B}_{\mathrm{Q}}$.

Pretpostavimo da je $\sigma_{123}=1$.

Tada primjenom identiteta (118) imamo da vektori

$$U_{123}=\left(I+\mathrm{T}_{2,1}^{2}\,\mathrm{T}_{3,2}\right)e_{123}=e_{123}+q_{32}\sigma_{13}\,e_{132},$$

$$U_{132}=\left(I+\mathrm{T}_{2,1}^{2}\,\mathrm{T}_{3,2}\right)e_{132}=e_{132}+q_{23}\sigma_{12}\,e_{123}=q_{23}\sigma_{12}\,e_{123}+e_{132},$$

$$U_{312}=\left(I+\mathrm{T}_{2,1}^{2}\,\mathrm{T}_{3,2}\right)e_{312}=e_{312}+q_{21}\sigma_{23}\,e_{321},$$

$$U_{321}=\left(I+\mathrm{T}_{2,1}^{2}\,\mathrm{T}_{3,2}\right)e_{321}=e_{321}+q_{12}\sigma_{13}\,e_{312}=q_{12}\sigma_{13}\,e_{312}+e_{321},$$

$$U_{231}=\left(I+\mathrm{T}_{2,1}^{2}\,\mathrm{T}_{3,2}\right)e_{231}=e_{231}+q_{13}\sigma_{12}\,e_{213},$$

$$U_{213}=\left(I+\mathrm{T}_{2,1}^{2}\,\mathrm{T}_{3,2}\right)e_{213}=e_{213}+q_{31}\sigma_{23}\,e_{231}=q_{31}\sigma_{23}\,e_{231}+e_{213}$$

razapinju jezgru operatora $I-\mathrm{T}_{2,1}^{2}\,\mathrm{T}_{3,2}$.

Jasno, dane vektore možemo pisati i ovako:

$$U_{123}=[1,q_{32}\sigma_{13},0,0,0,0],\qquad U_{132}=[q_{23}\sigma_{12},1,0,0,0,0],$$

$$U_{312}=[0,0,1,q_{21}\sigma_{23},0,0],\qquad U_{321}=[0,0,q_{12}\sigma_{13},1,0,0],$$

$$U_{231}=[0,0,0,0,1,q_{13}\sigma_{12}],\qquad U_{213}=[0,0,0,0,q_{31}\sigma_{23},1].$$

Lako se vidi da su vektori U_{123} i U_{132}; U_{312} i U_{321}; U_{231} i U_{213} linearno zavisni, tj. da vrijedi

$$U_{132}=q_{23}\sigma_{12}\,U_{123},\quad \text{odnosno}\quad U_{123}=q_{32}\sigma_{13}\,U_{132},$$

$$U_{321}=q_{12}\sigma_{13}\,U_{312},\quad \text{odnosno}\quad U_{312}=q_{21}\sigma_{23}\,U_{321},$$

$$U_{213}=q_{31}\sigma_{23}\,U_{231},\quad \text{odnosno}\quad U_{231}=q_{13}\sigma_{12}\,U_{213},$$

ako je $\sigma_{123}=1$.

Dobili smo sljedeće:

Ako je $\sigma_{123}=1$, onda su vektori U_{123}, U_{312} i U_{231} linearno nezavisni, ali isto tako i vektori U_{123}, U_{213} i U_{312} su linearno nezavisni. Time možemo uzeti da je $\mathscr{B}=\{U_{123},U_{213},U_{312}\}$ baza jezgre operatora $I-\mathrm{T}_{2,1}^{2}\,\mathrm{T}_{3,2}$, ako je $\sigma_{123}=1$.

(2) Neka je skupu $Q=\{1,2,3,4\}$ pridružen generički težinski potprostor $\mathcal{B}_{1234}$.

Pritom je $\mathcal{B}_{1234}=\{e_{1234}, e_{1243}, e_{1324}, e_{1342}, e_{1423}, e_{1432}, e_{2134}, e_{2143}, e_{2314}, e_{2341}, e_{2413}, e_{2431},$

$e_{3124}, e_{3142}, e_{3214}, e_{3241}, e_{3412}, e_{3421}, e_{4123}, e_{4132}, e_{4213}, e_{4231}, e_{4312}, e_{4321}\}$ baza od $\mathcal{B}_{1234}$.

Pretpostavimo da je $\sigma_{1234}=1$.

Tada primjenom identiteta (118) razlikujemo $4!=24$ vektora oblika

$$U_{\underline{j}}=\left(I+T_{2,1}^2\,T_{4,2}+\left(T_{2,1}^2\,T_{4,2}\right)^2\right)e_{\underline{j}} \qquad \text{za svaki } \underline{j}=j_1j_2j_3j_4\in S_4,$$

koji razapinju jezgru operatora $I-T_{2,1}^2\,T_{4,2}$.

Pritom je

$$U_{1234}=\left(I+T_{2,1}^2\,T_{4,2}+\left(T_{2,1}^2\,T_{4,2}\right)^2\right)e_{1234}=e_{1234}+q_{42}q_{43}\sigma_{14}\,e_{1423}+q_{32}q_{42}\sigma_{134}\,e_{1342},$$

$$U_{1243}=\left(I+T_{2,1}^2\,T_{4,2}+\left(T_{2,1}^2\,T_{4,2}\right)^2\right)e_{1243}=e_{1243}+q_{32}q_{34}\sigma_{13}\,e_{1324}+q_{42}q_{32}\sigma_{134}\,e_{1432},$$

$$U_{1324}=\left(I+T_{2,1}^2\,T_{4,2}+\left(T_{2,1}^2\,T_{4,2}\right)^2\right)e_{1324}=e_{1324}+q_{43}q_{42}\sigma_{14}\,e_{1432}+q_{23}q_{43}\sigma_{124}\,e_{1243},$$

$$U_{1342}=\left(I+T_{2,1}^2\,T_{4,2}+\left(T_{2,1}^2\,T_{4,2}\right)^2\right)e_{1342}=e_{1342}+q_{23}q_{24}\sigma_{12}\,e_{1234}+q_{43}q_{23}\sigma_{124}\,e_{1423},$$

$$U_{1423}=\left(I+T_{2,1}^2\,T_{4,2}+\left(T_{2,1}^2\,T_{4,2}\right)^2\right)e_{1423}=e_{1423}+q_{34}q_{32}\sigma_{13}\,e_{1342}+q_{24}q_{34}\sigma_{123}\,e_{1234},$$

$$U_{1432}=\left(I+T_{2,1}^2\,T_{4,2}+\left(T_{2,1}^2\,T_{4,2}\right)^2\right)e_{1432}=e_{1432}+q_{24}q_{23}\sigma_{12}\,e_{1243}+q_{34}q_{24}\sigma_{123}\,e_{1324},$$

$$U_{2134}=\left(I+T_{2,1}^2\,T_{4,2}+\left(T_{2,1}^2\,T_{4,2}\right)^2\right)e_{2134}=e_{2134}+q_{41}q_{43}\sigma_{24}\,e_{2413}+q_{31}q_{41}\sigma_{234}\,e_{2341},$$

$$U_{2143}=\left(I+T_{2,1}^2\,T_{4,2}+\left(T_{2,1}^2\,T_{4,2}\right)^2\right)e_{2143}=e_{2143}+q_{31}q_{34}\sigma_{23}\,e_{2314}+q_{41}q_{31}\sigma_{234}\,e_{2431},$$

$$U_{2314}=\left(I+T_{2,1}^2\,T_{4,2}+\left(T_{2,1}^2\,T_{4,2}\right)^2\right)e_{2314}=e_{2314}+q_{43}q_{41}\sigma_{24}\,e_{2431}+q_{13}q_{43}\sigma_{124}\,e_{2143},$$

$$U_{2341}=\left(I+T_{2,1}^2\,T_{4,2}+\left(T_{2,1}^2\,T_{4,2}\right)^2\right)e_{2341}=e_{2341}+q_{13}q_{14}\sigma_{12}\,e_{2134}+q_{43}q_{13}\sigma_{124}\,e_{2413},$$

$$U_{2413}=\left(I+T_{2,1}^2\,T_{4,2}+\left(T_{2,1}^2\,T_{4,2}\right)^2\right)e_{2413}=e_{2413}+q_{34}q_{31}\sigma_{23}\,e_{2341}+q_{14}q_{34}\sigma_{123}\,e_{2134},$$

$$U_{2431}=\left(I+T_{2,1}^2\,T_{4,2}+\left(T_{2,1}^2\,T_{4,2}\right)^2\right)e_{2431}=e_{2431}+q_{14}q_{13}\sigma_{12}\,e_{2143}+q_{34}q_{14}\sigma_{123}\,e_{2314},$$

$$U_{3124} = \left(I + \mathrm{T}_{2,1}^2\,\mathrm{T}_{4,2} + \left(\mathrm{T}_{2,1}^2\,\mathrm{T}_{4,2}\right)^2\right)\mathrm{e}_{3124} = \mathrm{e}_{3124} + \mathrm{q}_{41}\mathrm{q}_{42}\sigma_{34}\,\mathrm{e}_{3412} + \mathrm{q}_{21}\mathrm{q}_{41}\sigma_{234}\,\mathrm{e}_{3241},$$

$$U_{3142} = \left(I + \mathrm{T}_{2,1}^2\,\mathrm{T}_{4,2} + \left(\mathrm{T}_{2,1}^2\,\mathrm{T}_{4,2}\right)^2\right)\mathrm{e}_{3142} = \mathrm{e}_{3142} + \mathrm{q}_{21}\mathrm{q}_{24}\sigma_{23}\,\mathrm{e}_{3214} + \mathrm{q}_{41}\mathrm{q}_{21}\sigma_{234}\,\mathrm{e}_{3421},$$

$$U_{3214} = \left(I + \mathrm{T}_{2,1}^2\,\mathrm{T}_{4,2} + \left(\mathrm{T}_{2,1}^2\,\mathrm{T}_{4,2}\right)^2\right)\mathrm{e}_{3214} = \mathrm{e}_{3214} + \mathrm{q}_{42}\mathrm{q}_{41}\sigma_{34}\,\mathrm{e}_{3421} + \mathrm{q}_{12}\mathrm{q}_{42}\sigma_{134}\,\mathrm{e}_{3142},$$

$$U_{3241} = \left(I + \mathrm{T}_{2,1}^2\,\mathrm{T}_{4,2} + \left(\mathrm{T}_{2,1}^2\,\mathrm{T}_{4,2}\right)^2\right)\mathrm{e}_{3241} = \mathrm{e}_{3241} + \mathrm{q}_{12}\mathrm{q}_{14}\sigma_{13}\,\mathrm{e}_{3124} + \mathrm{q}_{42}\mathrm{q}_{12}\sigma_{134}\,\mathrm{e}_{3412},$$

$$U_{3412} = \left(I + \mathrm{T}_{2,1}^2\,\mathrm{T}_{4,2} + \left(\mathrm{T}_{2,1}^2\,\mathrm{T}_{4,2}\right)^2\right)\mathrm{e}_{3412} = \mathrm{e}_{3412} + \mathrm{q}_{24}\mathrm{q}_{21}\sigma_{23}\,\mathrm{e}_{3241} + \mathrm{q}_{14}\mathrm{q}_{24}\sigma_{123}\,\mathrm{e}_{3124},$$

$$U_{3421} = \left(I + \mathrm{T}_{2,1}^2\,\mathrm{T}_{4,2} + \left(\mathrm{T}_{2,1}^2\,\mathrm{T}_{4,2}\right)^2\right)\mathrm{e}_{3421} = \mathrm{e}_{3421} + \mathrm{q}_{14}\mathrm{q}_{12}\sigma_{13}\,\mathrm{e}_{3142} + \mathrm{q}_{24}\mathrm{q}_{14}\sigma_{123}\,\mathrm{e}_{3214},$$

$$U_{4123} = \left(I + \mathrm{T}_{2,1}^2\,\mathrm{T}_{4,2} + \left(\mathrm{T}_{2,1}^2\,\mathrm{T}_{4,2}\right)^2\right)\mathrm{e}_{4123} = \mathrm{e}_{4123} + \mathrm{q}_{31}\mathrm{q}_{32}\sigma_{34}\,\mathrm{e}_{4312} + \mathrm{q}_{21}\mathrm{q}_{31}\sigma_{234}\,\mathrm{e}_{4231},$$

$$U_{4132} = \left(I + \mathrm{T}_{2,1}^2\,\mathrm{T}_{4,2} + \left(\mathrm{T}_{2,1}^2\,\mathrm{T}_{4,2}\right)^2\right)\mathrm{e}_{4132} = \mathrm{e}_{4132} + \mathrm{q}_{21}\mathrm{q}_{23}\sigma_{24}\,\mathrm{e}_{4213} + \mathrm{q}_{31}\mathrm{q}_{21}\sigma_{234}\,\mathrm{e}_{4321},$$

$$U_{4213} = \left(I + \mathrm{T}_{2,1}^2\,\mathrm{T}_{4,2} + \left(\mathrm{T}_{2,1}^2\,\mathrm{T}_{4,2}\right)^2\right)\mathrm{e}_{4213} = \mathrm{e}_{4213} + \mathrm{q}_{32}\mathrm{q}_{31}\sigma_{34}\,\mathrm{e}_{4321} + \mathrm{q}_{12}\mathrm{q}_{32}\sigma_{134}\,\mathrm{e}_{4132},$$

$$U_{4231} = \left(I + \mathrm{T}_{2,1}^2\,\mathrm{T}_{4,2} + \left(\mathrm{T}_{2,1}^2\,\mathrm{T}_{4,2}\right)^2\right)\mathrm{e}_{4231} = \mathrm{e}_{4231} + \mathrm{q}_{12}\mathrm{q}_{13}\sigma_{14}\,\mathrm{e}_{4123} + \mathrm{q}_{32}\mathrm{q}_{12}\sigma_{134}\,\mathrm{e}_{4312},$$

$$U_{4312} = \left(I + \mathrm{T}_{2,1}^2\,\mathrm{T}_{4,2} + \left(\mathrm{T}_{2,1}^2\,\mathrm{T}_{4,2}\right)^2\right)\mathrm{e}_{4312} = \mathrm{e}_{4312} + \mathrm{q}_{23}\mathrm{q}_{21}\sigma_{24}\,\mathrm{e}_{4231} + \mathrm{q}_{13}\mathrm{q}_{23}\sigma_{124}\,\mathrm{e}_{4123},$$

$$U_{4321} = \left(I + \mathrm{T}_{2,1}^2\,\mathrm{T}_{4,2} + \left(\mathrm{T}_{2,1}^2\,\mathrm{T}_{4,2}\right)^2\right)\mathrm{e}_{4321} = \mathrm{e}_{4321} + \mathrm{q}_{13}\mathrm{q}_{12}\sigma_{14}\,\mathrm{e}_{4132} + \mathrm{q}_{23}\mathrm{q}_{13}\sigma_{124}\,\mathrm{e}_{4213},$$

Ako je $\sigma_{1234} = 1$, onda imamo samo osam $(= 4 \cdot 2!)$ linearno nezavisnih vektora.

Lako se pokaže da uz uvjet $\sigma_{1234} = 1$ vrijedi

$$U_{1324} = \mathrm{q}_{23}\mathrm{q}_{43}\sigma_{124}\,U_{1243}, \qquad U_{2314} = \mathrm{q}_{13}\mathrm{q}_{43}\sigma_{124}\,U_{2143},$$

$$U_{1342} = \mathrm{q}_{23}\mathrm{q}_{24}\sigma_{12}\,U_{1234}, \qquad U_{2341} = \mathrm{q}_{13}\mathrm{q}_{14}\sigma_{12}\,U_{2134},$$

$$U_{1423} = \mathrm{q}_{24}\mathrm{q}_{34}\sigma_{123}\,U_{1234}, \qquad U_{2413} = \mathrm{q}_{14}\mathrm{q}_{34}\sigma_{123}\,U_{2134},$$

$$U_{1432} = \mathrm{q}_{24}\mathrm{q}_{23}\sigma_{12}\,U_{1243}, \qquad U_{2431} = \mathrm{q}_{14}\mathrm{q}_{13}\sigma_{12}\,U_{2143},$$

$$U_{3214} = \mathrm{q}_{12}\mathrm{q}_{42}\sigma_{134}\,U_{3142}, \qquad U_{4213} = \mathrm{q}_{12}\mathrm{q}_{32}\sigma_{134}\,U_{4132},$$

$$U_{3241} = \mathrm{q}_{12}\mathrm{q}_{14}\sigma_{13}\,U_{3124}, \qquad U_{4231} = \mathrm{q}_{12}\mathrm{q}_{13}\sigma_{14}\,U_{4123},$$

$$U_{3412} = \mathrm{q}_{14}\mathrm{q}_{24}\sigma_{123}\,U_{3124}, \qquad U_{4312} = \mathrm{q}_{13}\mathrm{q}_{23}\sigma_{124}\,U_{4123},$$

$$U_{3421} = \mathrm{q}_{14}\mathrm{q}_{12}\sigma_{13}\,U_{3142}, \qquad U_{4321} = \mathrm{q}_{13}\mathrm{q}_{12}\sigma_{14}\,U_{4132},$$

stoga možemo uzeti da je

$$\mathscr{B} = \{U_{1234}, U_{1243}, U_{2134}, U_{2143}, U_{3124}, U_{3142}, U_{4123}, U_{4132}\}$$

baza jezgre operatora $I - \mathrm{T}_{2,1}^2 \mathrm{T}_{4,2}$, ako je $\sigma_{1234} = 1$.

Odredimo sada vektore $X_{\underline{\mathrm{j}}} \in Ker\left(\mathrm{B}_{\mathrm{Q,\,n}}\right)$ uz uvjet $\sigma_{l_1 l_2 \dots l_{\mathrm{n}}} = 1$.

Neka je $\sigma_{l_1 l_2 \dots l_{\mathrm{n}}} = 1$ i neka su $U_{\underline{\mathrm{j}}} \in Ker\left(I - \mathrm{T}_{2,1}^2 \mathrm{T}_{\mathrm{n},2}\right)$ za svaki $\underline{\mathrm{j}} = \mathrm{j}_1 \mathrm{j}_2 \dots \mathrm{j}_{\mathrm{n}} \in \widehat{\mathrm{Q}} = \mathrm{S}_{\mathrm{n}} \mathrm{Q}$. Tada primjenom propozicije 4.2.1 imamo da se vektori $X_{\underline{\mathrm{j}}} \in Ker\left(\mathrm{B}_{\mathrm{Q,\,n}}\right)$ računaju po formuli (danom izrazom (110) za $\mathrm{j} = \mathrm{n} - 1$)

$$\begin{aligned} X_{\underline{\mathrm{j}}} &= \mathrm{C}_{\mathrm{Q,\,n}} \cdot \prod_{2 \le \mathrm{i} \le \mathrm{n}-1} \left(I - \mathrm{T}_{2,1}^2 \mathrm{T}_{\mathrm{i},2}\right)^{-1} \cdot U_{\underline{\mathrm{j}}} \\ &= \mathrm{C}_{\mathrm{Q,\,n}} \cdot \left(I - \mathrm{T}_{2,1}^2 \mathrm{T}_{2,2}\right)^{-1} \cdot \left(I - \mathrm{T}_{2,1}^2 \mathrm{T}_{3,2}\right)^{-1} \cdots \left(I - \mathrm{T}_{2,1}^2 \mathrm{T}_{\mathrm{n}-1,2}\right)^{-1} \cdot U_{\underline{\mathrm{j}}} \end{aligned} \tag{120}$$

tj.

$$\begin{aligned} X_{\underline{\mathrm{j}}} = \mathrm{C}_{\mathrm{Q,\,n}} &\cdot \left(I - \mathrm{T}_{2,1}^2 \mathrm{T}_{2,2}\right)^{-1} \cdot \left(I - \mathrm{T}_{2,1}^2 \mathrm{T}_{3,2}\right)^{-1} \cdots \\ &\left(I - \mathrm{T}_{2,1}^2 \mathrm{T}_{\mathrm{n}-1,2}\right)^{-1} \cdot \left(I + \mathrm{T}_{2,1}^2 \mathrm{T}_{\mathrm{n},2} + \left(\mathrm{T}_{2,1}^2 \mathrm{T}_{\mathrm{n},2}\right)^2 + \cdots + \left(\mathrm{T}_{2,1}^2 \mathrm{T}_{\mathrm{n},2}\right)^{\mathrm{n}-2}\right) \mathrm{e}_{\underline{\mathrm{j}}} \end{aligned} \tag{121}$$

za svaki $\underline{\mathrm{j}} \in \widehat{\mathrm{Q}} = \mathrm{S}_{\mathrm{n}} \mathrm{Q}$.

Identitet (121) dobiva se uvrštavanjem vrijednosti vektora $U_{\underline{\mathrm{j}}}$, $\underline{\mathrm{j}} \in \widehat{\mathrm{Q}} = \mathrm{S}_{\mathrm{n}} \mathrm{Q}$, danih izrazom (118) u identitet (120).

Uzimajući u obzir identitet (116), kojeg možemo pisati u obliku

$$\left(I + \mathrm{T}_{2,1}^2 \mathrm{T}_{\mathrm{n},2} + \left(\mathrm{T}_{2,1}^2 \mathrm{T}_{\mathrm{n},2}\right)^2 + \cdots + \left(\mathrm{T}_{2,1}^2 \mathrm{T}_{\mathrm{n},2}\right)^{\mathrm{n}-2}\right) \mathrm{e}_{\underline{\mathrm{j}}} = \left(I - \mathrm{T}_{2,1}^2 \mathrm{T}_{\mathrm{n},2}\right)^{-1} \cdot \left(1 - \sigma_{l_1 l_2 \dots l_{\mathrm{n}}}\right) \mathrm{e}_{\underline{\mathrm{j}}},$$

odnosno

$$\left(I + \mathrm{T}_{2,1}^2 \mathrm{T}_{\mathrm{n},2} + \left(\mathrm{T}_{2,1}^2 \mathrm{T}_{\mathrm{n},2}\right)^2 + \cdots + \left(\mathrm{T}_{2,1}^2 \mathrm{T}_{\mathrm{n},2}\right)^{\mathrm{n}-2}\right) \mathrm{e}_{\underline{\mathrm{j}}} = \left(I - \mathrm{T}_{2,1}^2 \mathrm{T}_{\mathrm{n},2}\right)^{-1} \cdot \left(I - \mathrm{Q}_{\{1,2,\dots,\mathrm{n}\}}\right) \mathrm{e}_{\underline{\mathrm{j}}} \tag{122}$$

te uvrštavanjem (122) u (121) dobivamo

$$\begin{aligned} X_{\underline{\mathrm{j}}} &= \mathrm{C}_{\mathrm{Q,\,n}} \cdot \left(I - \mathrm{T}_{2,1}^2 \mathrm{T}_{2,2}\right)^{-1} \cdot \left(I - \mathrm{T}_{2,1}^2 \mathrm{T}_{3,2}\right)^{-1} \cdots \left(I - \mathrm{T}_{2,1}^2 \mathrm{T}_{\mathrm{n}-1,2}\right)^{-1} \cdot \left(I - \mathrm{T}_{2,1}^2 \mathrm{T}_{\mathrm{n},2}\right)^{-1} \cdot \left(I - \mathrm{Q}_{\{1,2,\dots,\mathrm{n}\}}\right) \mathrm{e}_{\underline{\mathrm{j}}} \\ &= \mathrm{C}_{\mathrm{Q,\,n}} \cdot \left(\left(I - \mathrm{T}_{2,1}^2 \mathrm{T}_{\mathrm{n},2}\right) \cdot \left(I - \mathrm{T}_{2,1}^2 \mathrm{T}_{\mathrm{n}-1,2}\right) \cdots \left(I - \mathrm{T}_{2,1}^2 \mathrm{T}_{3,2}\right) \cdot \left(I - \mathrm{T}_{2,1}^2 \mathrm{T}_{2,2}\right)\right)^{-1} \cdot \left(I - \mathrm{Q}_{\{1,2,\dots,\mathrm{n}\}}\right) \mathrm{e}_{\underline{\mathrm{j}}} \end{aligned}$$

tj.

$$X_{\underline{\mathrm{j}}} = \left(\mathrm{C}_{\mathrm{Q,\,n}} \cdot \mathrm{D}_{\mathrm{Q,\,n}}^{-1} \cdot \left(I - \mathrm{Q}_{\{1,2,\dots,\mathrm{n}\}}\right)\right) \mathrm{e}_{\underline{\mathrm{j}}}, \tag{123}$$

gdje je $\mathrm{D}_{\mathrm{Q,\,n}} = \left(I - \mathrm{T}_{2,1}^2 \mathrm{T}_{\mathrm{n},2}\right) \cdot \left(I - \mathrm{T}_{2,1}^2 \mathrm{T}_{\mathrm{n}-1,2}\right) \cdots \left(I - \mathrm{T}_{2,1}^2 \mathrm{T}_{3,2}\right) \cdot \left(I - \mathrm{T}_{2,1}^2 \mathrm{T}_{2,2}\right)$.

Dakle, ako je $\sigma_{l_1 l_2 \cdots l_n} = 1$ i $U_{\underline{j}} \in Ker\left(I - T_{2,1}^2\, T_{n,2}\right)$, $\underline{j} \in \widehat{Q} = S_n Q$, onda za svaki bazični monom $e_{\underline{j}} \in \mathcal{B}_Q$, $Q = \{l_1, l_2, ..., l_n\} \subseteq \mathcal{N}$ (gdje je $\mathcal{B}_Q = \left\{e_{\underline{j}} \mid \underline{j} = j_1 j_2 ... j_n \in \widehat{Q}\right\}$ baza od $\boldsymbol{\mathcal{B}}_Q$) imamo da vektori $X_{\underline{j}}$ (kojih ima $n!$) danih identitetom (123) razapinju jezgru operatora $B_{Q,\,n}$. S druge strane, ako je $\sigma_{l_1 l_2 \cdots l_n} = 1$, onda bi se na prvi pogled iz identiteta (123) moglo zaključiti da su vektori $X_{\underline{j}}$ za svaki $\underline{j} \in \widehat{Q} = S_n Q$ trivijalni elementi jezgre operatora $B_{Q,\,n}$, stoga se na prirodan način nameću pitanja:

- kako glasi formula za izračunavanje operatora $D_{Q,\,n}^{-1}$,
- da li su vektori $X_{\underline{j}}$, $\underline{j} \in \widehat{Q} = S_n Q$ netrivijalni elementi jezgre operatora $B_{Q,\,n}$,
- koliko ima linearno nezavisnih vektora $X_{\underline{j}} \in Ker\left(B_{Q,\,n}\right)$ te koji će vektori činiti bazu jezgre operatora $B_{Q,\,n}$?

Primjedba 4.2.4

Uzimajući u obzir da je

$$D_{Q,\,n} = \left(I - T_{2,1}^2\, T_{n,2}\right) \cdot \left(I - T_{2,1}^2\, T_{n-1,2}\right) \cdots \left(I - T_{2,1}^2\, T_{3,2}\right) \cdot \left(I - T_{2,1}^2\, T_{2,2}\right) = \overleftarrow{\prod_{2 \le i \le n}} \left(I - T_{2,1}^2\, T_{i,2}\right)$$

možemo pisati

$$\begin{aligned} D_{Q,\,n}^{-1} &= \left(I - T_{2,1}^2\, T_{2,2}\right)^{-1} \cdot \left(I - T_{2,1}^2\, T_{3,2}\right)^{-1} \cdots \left(I - T_{2,1}^2\, T_{n-1,2}\right)^{-1} \cdot \left(I - T_{2,1}^2\, T_{n,2}\right)^{-1} \\ &= \prod_{i=2}^{n} \left(I - T_{2,1}^2\, T_{i,2}\right)^{-1} . \end{aligned}$$

Pretpostavimo da je $\sigma_{j_1 j_2 \cdots j_i} \neq 1$, $2 \le i \le n$. Tada se $\left(I - T_{2,1}^2\, T_{i,2}\right)^{-1}$ inverz cikličkog operatora $\left(I - T_{2,1}^2\, T_{i,2}\right)$, $2 \le i \le n$ izračunava po formuli

$$\left(I - T_{2,1}^2\, T_{i,2}\right)^{-1} = \frac{1}{I - Q_{\{1,2,\ldots,i\}}} \cdot \left(I + T_{2,1}^2\, T_{i,2} + \left(T_{2,1}^2\, T_{i,2}\right)^2 + \cdots + \left(T_{2,1}^2\, T_{i,2}\right)^{i-2}\right). \quad (124)$$

Primijetimo da identitet (124) direktno proizlazi iz identiteta

$$\left(I - T_{2,1}^2\, T_{i,2}\right) \cdot \left(I + T_{2,1}^2\, T_{i,2} + \left(T_{2,1}^2\, T_{i,2}\right)^2 + \cdots + \left(T_{2,1}^2\, T_{i,2}\right)^{i-2}\right) e_{\underline{j}} = \left(I - \left(T_{2,1}^2\, T_{i,2}\right)^{i-1}\right) e_{\underline{j}},$$

pri čemu primjenom identiteta (112) imamo da je

$$\left(I - \left(T_{2,1}^2\, T_{i,2}\right)^{i-1}\right) e_{\underline{j}} = \left(I - Q_{\{1,2,\ldots,i\}}\right) e_{\underline{j}} = \left(1 - \sigma_{j_1 j_2 \cdots j_i}\right) e_{\underline{j}},$$

gdje je $\sigma_{j_1 j_2 \cdots j_i}$ dan relacijom (113).

Slijedi izračunavanje $D_{Q,\,n}^{-1}$.

U suglasnosti s gore definiranim $D_{Q,\,n-k+1} = \mathcal{R}\left(\widetilde{\delta_{n-k+1}}\right)$, $1 \le k \le n-1$, gdje su elementi $\widetilde{\delta_{n-k+1}} \in \mathcal{A}_n$ definirani u 3.21 te s uvedenom oznakom (70) u propoziciji 3.27, tj. $\widetilde{\delta_{n-k+1}^n} = \widetilde{\delta_{n-k+1}}$, možemo pisati $D_{Q,\,n-k+1}^n = \mathcal{R}\left(\widetilde{\delta_{n-k+1}^n}\right) = \mathcal{R}\left(\widetilde{\delta_{n-k+1}}\right)$.

Nadalje, u propoziciji 3.27 dokazali smo da se za fiksiran n može izračunati inverz elementa $\widetilde{\delta_{n-k+1}^n} \in \mathcal{A}_n$, $1 \le k \le n-1$ po formuli $\widetilde{\delta_{n-k+1}^n}^{\,-1} = \left(\Delta_{n-k+1}^n\right)^{-1} \cdot \widetilde{\varepsilon_{n-k+1}^n}$. To nas navodi na ideju da će se reprezentacijom dane formule dobiti formula za izračunavanje $\left(D_{Q,\,n-k+1}^n\right)^{-1}$ za bilo koji $1 \le k \le n-1$. Pritom će se koristi svojstvo da je preslikavanje $\mathcal{R} : \mathcal{A}_n \to \mathrm{End}\left(\mathcal{B}_Q\right)$ homomorfizam algebre $\mathcal{A}_n$ u algebru endomorfizama od $\mathcal{B}_Q$ (propozicija 4.4).

Neka je n fiksiran i $1 \le k \le n-1$. Primjenom identiteta (33), str.123 imamo da za bilo koji $k+1 \le i \le n$ vrijedi $Q_{\{k,k+1,\dots,i\}} = \mathcal{R}\left(X_{\{k,k+1,\dots,i\}}\,\mathrm{id}\right)$.

Reprezentacijom elemenata iz zakrenute grupovne algebre, uvedenih u propoziciji 3.27, imamo

$$\begin{aligned}\mathfrak{S}_{n-k+1}^n &= \mathcal{R}\left(\Delta_{n-k+1}^n\right) = \mathcal{R}\left(\left(\mathrm{id} - X_{\{k,k+1\}}\right)\cdot\left(\mathrm{id} - X_{\{k,k+1,k+2\}}\right)\cdots\left(\mathrm{id} - X_{\{k,k+1,\dots,n\}}\right)\right)\\ &= \left(I - Q_{\{k,k+1\}}\right)\cdot\left(I - Q_{\{k,k+1,k+2\}}\right)\cdots\left(I - Q_{\{k,k+1,\dots,n\}}\right),\end{aligned}$$

$$\begin{aligned}W_{n-k+1}^n(g) &= \mathcal{R}\left(\omega_{n-k+1}^n(g)\right) = \prod_{i \in \mathrm{Des}\left(g^{-1}\right)} \mathcal{R}\left(X_{\{k,k+1,\dots,i\}}\right)\\ &= \prod_{i \in \mathrm{Des}\left(g^{-1}\right)} Q_{\{k,k+1,\dots,i\}}\,,\end{aligned}$$

$$\begin{aligned}E_{Q,\,n-k+1}^n &= \mathcal{R}\left(\widetilde{\varepsilon_{n-k+1}^n}\right) = \sum_{g \in S_1^k \times S_{n-k}} \mathcal{R}\left(\omega_{n-k+1}^n(g)\cdot\tilde{g}\right) = \sum_{g \in S_1^k \times S_{n-k}} \mathcal{R}\left(\omega_{n-k+1}^n(g)\right)\cdot\mathcal{R}\left(\tilde{g}\right)\\ &= \sum_{g \in S_1^k \times S_{n-k}} W_{n-k+1}^n(g)\cdot G\,,\end{aligned}$$

gdje za $g \in S_1^k \times S_{n-k}$ je

$$G = \mathcal{R}\left(\tilde{g}\right),$$

$$\mathrm{Des}\left(g^{-1}\right) = \left\{k+1 \le i \le n-1 \mid g^{-1}(i) > g^{-1}(i+1)\right\}.$$

Propozicija 4.2.5

Neka je n fiksiran, $1 \le k \le n-1$ i neka je

$$\mathfrak{S}^n_{n-k+1} = \left(I - Q_{\{k,k+1\}}\right)\cdot\left(I - Q_{\{k,k+1,k+2\}}\right)\cdots\left(I - Q_{\{k,k+1,\dots,n-1\}}\right)\cdot\left(I - Q_{\{k,k+1,\dots,n\}}\right),$$

$$E^n_{Q,\,n-k+1} = \sum_{g\in S^k_1\times S_{n-k}} W^n_{n-k+1}(g)\cdot G,$$

gdje za $g \in S^k_1 \times S_{n-k}$ je

$$W^n_{n-k+1}(g) = \prod_{i\in \mathrm{Des}\left(g^{-1}\right)} Q_{\{k,k+1,\dots,i\}},$$

$$\mathrm{Des}\left(g^{-1}\right) = \left\{k+1 \le i \le n-1 \mid g^{-1}(i) > g^{-1}(i+1)\right\}.$$

Tada se inverz operatora $D^n_{Q,\,n-k+1}$, $1 \le k \le n-1$ računa po formuli

$$\left(D^n_{Q,\,n-k+1}\right)^{-1} = \left(\mathfrak{S}^n_{n-k+1}\right)^{-1}\cdot E^n_{Q,\,n-k+1}. \tag{125}$$

Primjedba 4.2.6

Specijalno, ako je $k=1$ (n fiksiran), onda iz formule (125) proizlazi da se inverz operatora $D_{Q,\,n}$ može izračunati na sljedeći način

$$D^{-1}_{Q,\,n} = \mathfrak{S}^{-1}_n\cdot E_{Q,\,n}. \tag{126}$$

Pritom je

$$D^{-1}_{Q,\,n} = \left(D^n_{Q,\,n}\right)^{-1},\ \ \mathfrak{S}^{-1}_n = \left(\mathfrak{S}^n_n\right)^{-1},\ \ E_{Q,\,n} = E^n_{Q,\,n},\ \ W_n(g) = W^n_n(g).$$

Uz uvedene oznake, u ovom slučaju imamo da je

$$\mathfrak{S}_n = \left(I - Q_{\{1,2\}}\right)\cdot\left(I - Q_{\{1,2,3\}}\right)\cdots\left(I - Q_{\{1,2,\dots,n-1\}}\right)\cdot\left(I - Q_{\{1,2,\dots,n\}}\right),$$

$$E_{Q,\,n} = \sum_{g\in S^1_1\times S_{n-1}} W_n(g)\cdot G,$$

te za $g \in S^1_1\times S_{n-1} \subset S_n$ je

$$W_n(g) = \prod_{i\in \mathrm{Des}\left(g^{-1}\right)} Q_{\{1,2,\dots,i\}},$$

$$\mathrm{Des}\left(g^{-1}\right) = \left\{2 \le i \le n-1 \mid g^{-1}(i) > g^{-1}(i+1)\right\}.$$

Ponovimo, u drugom poglavlju definirali smo iterirane komutatore $Y_{i_1 i_2 \cdots i_p}$, $1 \le p \le N$, $N \in \mathbb{N}_0 = \{0, 1, 2, ...\}$ na sljedeći način

$$\left.\begin{aligned} Y_{i_1} &= e_{i_1}, \\ Y_{i_1 i_2 \cdots i_p} &= \left[Y_{i_1 i_2 \cdots i_{p-1}}, e_{i_p} \right]_{q_{i_p i_1} q_{i_p i_2} \cdots q_{i_p i_{p-1}}} \\ &= Y_{i_1 i_2 \cdots i_{p-1}} e_{i_p} - q_{i_p i_1} q_{i_p i_2} \cdots q_{i_p i_{p-1}} e_{i_p} Y_{i_1 i_2 \cdots i_{p-1}}. \end{aligned}\right\} \tag{127}$$

Lema 4.2.7

Neka je $Q = \{l_1, l_2, ..., l_n\} \subseteq \mathcal{N} = \{i_1, i_2, ..., i_N\}$ i neka je $\underline{j} = j_1 j_2 ... j_n \in \widehat{Q} = S_n Q$, gdje je $\widehat{Q} = S_n Q = \{\sigma(l_1 l_2 ... l_n) \mid \sigma \in S_n\}$ skup svih permutacija skupa Q.

Tada za iterirane komutatore $Y_{i_1 i_2 \cdots i_n}$, $n \ge 2$ vrijedi

$$Y_{j_1 j_2 \cdots j_n} = C_{Q,n}\, e_{j_1 j_2 \cdots j_n}, \tag{128}$$

gdje je

$$C_{Q,n} = (I - T_{n,1}) \cdot (I - T_{n-1,1}) \cdots (I - T_{3,1}) \cdot (I - T_{2,1}) \qquad \text{(primjedba 4.1.5).}$$

Lema 4.2.6 je analogna korolaru 1 [MPS, Corollary 1, *i)*].

Dokaz:

Uzimajući u obzir prethodno provedena razmatranja imamo da je

$$\begin{aligned} I\, e_{j_1 j_2 j_3 \cdots j_{n-1} j_n} &= 1\, e_{j_1 j_2 j_3 \cdots j_{n-1} j_n} \\ T_{2,1}\, e_{j_1 j_2 j_3 \cdots j_{n-1} j_n} &= q_{j_2 j_1}\, e_{j_2 j_1 j_3 \cdots j_{n-1} j_n}, \\ T_{3,1}\, e_{j_1 j_2 j_3 \cdots j_{n-1} j_n} &= q_{j_3 j_1} q_{j_3 j_2}\, e_{j_3 j_1 j_2 \cdots j_{n-1} j_n}, \\ &\vdots \\ T_{n-1,1}\, e_{j_1 j_2 j_3 \cdots j_{n-1} j_n} &= q_{j_{n-1} j_1} q_{j_{n-1} j_2} \cdots q_{j_{n-1} j_{n-2}}\, e_{j_{n-1} j_1 j_2 \cdots j_{n-2} j_n}, \\ T_{n,1}\, e_{j_1 j_2 j_3 \cdots j_{n-1} j_n} &= q_{j_n j_1} q_{j_n j_2} \cdots q_{j_n j_{n-2}} q_{j_n j_{n-1}}\, e_{j_n j_1 j_2 \cdots j_{n-2} j_{n-1}}. \end{aligned}$$

Time dobivamo

$$\begin{aligned} (I - T_{2,1})\, e_{j_1 j_2 j_3 \cdots j_{n-1} j_n} &= 1\, e_{j_1 j_2 j_3 \cdots j_{n-1} j_n} - q_{j_2 j_1}\, e_{j_2 j_1 j_3 \cdots j_{n-1} j_n} = \\ &= \left(e_{j_1 j_2} - q_{j_2 j_1}\, e_{j_2 j_1} \right) e_{j_3 j_4 \cdots j_{n-1} j_n} = \left[e_{j_1}, e_{j_2} \right]_{q_{j_2 j_1}} e_{j_3 j_4 \cdots j_{n-1} j_n} \\ &= Y_{j_1 j_2}\, e_{j_3 j_4 \cdots j_{n-1} j_n}, \end{aligned}$$

$$\left(I-\mathrm{T}_{3,1}\right)\cdot\left(I-\mathrm{T}_{2,1}\right)\mathrm{e}_{j_1j_2j_3\cdots j_{n-1}j_n}=\left(I-\mathrm{T}_{3,1}\right)\cdot\mathrm{Y}_{j_1j_2}\,\mathrm{e}_{j_3j_4\cdots j_{n-1}j_n}$$

$$=\left(I-\mathrm{T}_{3,1}\right)\cdot\left(\mathrm{e}_{j_1j_2}-\mathrm{q}_{j_2j_1}\,\mathrm{e}_{j_2j_1}\right)\mathrm{e}_{j_3j_4\cdots j_{n-1}j_n}$$

$$=\left(I-\mathrm{T}_{3,1}\right)\cdot\left(\mathrm{e}_{j_1j_2j_3}-\mathrm{q}_{j_2j_1}\,\mathrm{e}_{j_2j_1j_3}\right)\mathrm{e}_{j_4j_5\cdots j_{n-1}j_n}$$

$$=\left(\mathrm{e}_{j_1j_2j_3}-\mathrm{q}_{j_2j_1}\,\mathrm{e}_{j_2j_1j_3}-\mathrm{q}_{j_3j_1}\mathrm{q}_{j_3j_2}\,\mathrm{e}_{j_3j_1j_2}+\mathrm{q}_{j_3j_2}\mathrm{q}_{j_3j_1}\mathrm{q}_{j_2j_1}\,\mathrm{e}_{j_3j_2j_1}\right)\mathrm{e}_{j_4j_5\cdots j_{n-1}j_n}$$

$$=\left[\mathrm{Y}_{j_1j_2},\mathrm{e}_{j_3}\right]_{\mathrm{q}_{j_3j_1}\mathrm{q}_{j_3j_2}}\mathrm{e}_{j_4j_5\cdots j_{n-1}j_n}=\mathrm{Y}_{j_1j_2j_3}\,\mathrm{e}_{j_4j_5\cdots j_{n-1}j_n}\,.$$

Analogno se jednostavnim raspisivanjem može pokazati da je

$$\left(I-\mathrm{T}_{n-1,1}\right)\cdots\left(I-\mathrm{T}_{3,1}\right)\cdot\left(I-\mathrm{T}_{2,1}\right)\mathrm{e}_{j_1j_2j_3\cdots j_{n-1}j_n}=\left[\mathrm{Y}_{j_1j_2\cdots j_{n-2}},\mathrm{e}_{j_{n-1}}\right]_{\mathrm{q}_{j_{n-1}j_1}\mathrm{q}_{j_{n-1}j_2}\cdots\mathrm{q}_{j_{n-1}j_{n-2}}}\mathrm{e}_{j_n}$$

$$=\mathrm{Y}_{j_1j_2\cdots j_{n-2}j_{n-1}}\,\mathrm{e}_{j_n}$$

odnosno

$$\left(I-\mathrm{T}_{n,1}\right)\cdot\left(I-\mathrm{T}_{n-1,1}\right)\cdots\left(I-\mathrm{T}_{3,1}\right)\cdot\left(I-\mathrm{T}_{2,1}\right)\mathrm{e}_{j_1j_2j_3\cdots j_{n-1}j_n}=\mathrm{Y}_{j_1j_2\cdots j_{n-1}j_n}\,,$$

što povlači da je

$$\mathrm{C}_{\mathrm{Q},\,n}\,\mathrm{e}_{j_1j_2\cdots j_n}=\mathrm{Y}_{j_1j_2\cdots j_n}\,,$$

čime je dokazana lema 4.2.7.

Propozicija 4.2.8

Neka je $\mathcal{B}_{\mathrm{Q}}=\left\{\mathrm{e}_{\underline{j}}=\mathrm{e}_{j_1j_2\cdots j_n}=\mathrm{e}_{j_1}\mathrm{e}_{j_2}\cdots\mathrm{e}_{j_n}\mid \underline{j}=j_1j_2...j_n\in\widehat{\mathrm{Q}}\right\}$ baza generičkog težinskog potprostora $\boldsymbol{\mathcal{B}}_{\mathrm{Q}}=\boldsymbol{\mathcal{B}}_{l_1l_2\cdots l_n}$ i neka je zadovoljen uvjet kocikličnosti:

$$\sigma_{l_1l_2\cdots l_n}=1,\qquad \sigma_{j_1j_2\cdots j_i}\neq 1\quad \text{za svaki }\ 2\le i<n\,,\tag{129}$$

gdje je $\sigma_{j_1j_2\cdots j_i}=\prod\limits_{1\le p<r\le i}\sigma_{j_pj_r}$.

Ako je

$$X_{\underline{j}}=\left(\mathrm{C}_{\mathrm{Q},\,n}\cdot\mathfrak{S}_{n-1}^{-1}\cdot\mathrm{E}_{\mathrm{Q},\,n}\right)\mathrm{e}_{\underline{j}},\qquad \mathrm{e}_{\underline{j}}\in\mathcal{B}_{\mathrm{Q}}\,,\tag{130}$$

onda je $X_{\underline{j}}\in Ker\left(\mathrm{B}_{\mathrm{Q},\,n}\right)$.

Pritom je $\mathfrak{S}_{n-1}=\left(I-\mathrm{Q}_{\{1,2\}}\right)\cdot\left(I-\mathrm{Q}_{\{1,2,3\}}\right)\cdots\left(I-\mathrm{Q}_{\{1,2,\ldots,n-2\}}\right)\cdot\left(I-\mathrm{Q}_{\{1,2,\ldots,n-1\}}\right)$,

$$\mathrm{E}_{\mathrm{Q},\,n}=\sum_{g\in S_1^1\times S_{n-1}}\mathrm{W}_n\left(g\right)\cdot\mathrm{G}\,,\quad \mathrm{W}_n\left(g\right)=\prod_{i\in\mathrm{Des}\left(g^{-1}\right)}\mathrm{Q}_{\{1,2,\ldots,i\}}\,,$$

$$\mathrm{Des}\left(g^{-1}\right)=\left\{2\le i\le n-1\mid\ g^{-1}\left(i\right)>g^{-1}\left(i+1\right)\right\}.$$

Dokaz:

Pretpostavimo da je zadovoljen uvjet kocikličnosti (129) te da je

$$X_{\underline{j}} = \left(C_{Q,n} \cdot \mathfrak{S}_{n-1}^{-1} \cdot E_{Q,n}\right) e_{\underline{j}}, \qquad e_{\underline{j}} \in \mathcal{B}_Q .$$

Uzimajući u obzir da je

$$\mathfrak{S}_n = \left(1 - Q_{\{1,2\}}\right) \cdot \left(I - Q_{\{1,2,3\}}\right) \cdots \left(I - Q_{\{1,2,\ldots,n-1\}}\right) \cdot \left(I - Q_{\{1,2,\ldots,n\}}\right),$$

$$\mathfrak{S}_{n-1} = \left(1 - Q_{\{1,2\}}\right) \cdot \left(I - Q_{\{1,2,3\}}\right) \cdots \left(I - Q_{\{1,2,\ldots,n-2\}}\right) \cdot \left(I - Q_{\{1,2,\ldots,n-1\}}\right),$$

lako se vidi da vrijedi $\left(I - Q_{\{1,2,\ldots,n\}}\right) \cdot \mathfrak{S}_n^{-1} = \mathfrak{S}_{n-1}^{-1}$. Time imamo:

$$\begin{aligned} X_{\underline{j}} = \left(C_{Q,n} \cdot \mathfrak{S}_{n-1}^{-1} \cdot E_{Q,n}\right) e_{\underline{j}} &= \left(C_{Q,n} \cdot \left(I - Q_{\{1,2,\ldots,n\}}\right) \cdot \mathfrak{S}_n^{-1} \cdot E_{Q,n}\right) e_{\underline{j}} \\ &\overset{(126)}{=} \left(C_{Q,n} \cdot \left(I - Q_{\{1,2,\ldots,n\}}\right) \cdot D_{Q,n}^{-1}\right) e_{\underline{j}} \\ &= \left(C_{Q,n} \cdot D_{Q,n}^{-1} \cdot \left(I - Q_{\{1,2,\ldots,n\}}\right)\right) e_{\underline{j}} . \end{aligned}$$

S obzirom na prethodno navedeno imamo da iz $X_{\underline{j}} = \left(C_{Q,n} \cdot D_{Q,n}^{-1} \cdot \left(I - Q_{\{1,2,\ldots,n\}}\right)\right) e_{\underline{j}}$ proizlazi $X_{\underline{j}} \in Ker\left(B_{Q,n}\right)$, čime je dokaz propozicije 4.2.8 gotov.

Propozicijom 4.2.8 dokazalo se sljedeće.

Ako vrijedi uvjet kocikličnosti, dan sa (129), onda su vektori $X_{\underline{j}}$ za svaki $\underline{j} \in \widehat{Q} = S_n Q$ (tj. za svaki bazični monom $e_{\underline{j}} \in \mathcal{B}_Q$) netrivijalni elementi jezgre operatora $B_{Q,n}$. Drugim rječima, imamo $n!$ vektora $X_{\underline{j}}$ danih identitetom (130), koji razapinju jezgru operatora $B_{Q,n}$. Međutim oni nisu svi linearno nezavisni.

Uzimajući u obzir primjedbu 4.2.2 zaključujemo da će $n \cdot (n-2)!$ vektora danih sa

$$X_{l_1 l_2 j_3 j_4 \cdots j_n} = \left(C_{Q,n} \cdot \mathfrak{S}_{n-1}^{-1} \cdot E_{Q,n}\right) e_{l_1 l_2 j_3 j_4 \cdots j_n} \tag{131}$$

$$X_{l_k l_1 i_3 i_4 \cdots i_n} = \left(C_{Q,n} \cdot \mathfrak{S}_{n-1}^{-1} \cdot E_{Q,n}\right) e_{l_k l_1 i_3 i_4 \cdots i_n}, \ 2 \le k \le n \tag{132}$$

također razapinjati jezgru operatora $B_{Q,n}$.

Pritom je $\quad j_3 j_4 \ldots j_n \in S_{n-2} Q', \qquad Q' = Q \setminus \{l_1, l_2\} = \{l_3, l_4, \ldots, l_n\},$

$$i_3 i_4 \ldots i_n \in S_{n-2} Q'', \qquad Q'' = Q \setminus \{l_1, l_k\} = \left\{l_2, l_3, \ldots, l_{k-1}, \widehat{l_k}, l_{k+1}, \ldots, l_n\right\},$$

gdje je

$$S_{n-2} Q' = \left\{\sigma\left(l_3 l_4 \ldots l_n\right) \mid \sigma \in S_{n-2}\right\}, \ S_{n-2} Q'' = \left\{\sigma\left(l_2 l_3 \ldots l_{k-1} \widehat{l_k} l_{k+1} \ldots l_n\right) \mid \sigma \in S_{n-2}\right\}, \ 2 \le k \le n .$$

Postavlja se pitanje:

da li su svi vektori dani identitetima (131) i (132) linearno nezavisni vektori, tj. da li oni čine bazu jezgre operatora $B_{Q,n}$?

Detaljnim izračunavanjem pokazalo se da svi vektori nisu linearno nezavisni te da za svaki $2 \leq k \leq n$ vektori oblika (132) su linearno zavisni s vektorima oblika (131).
S druge strane, svi vektori koji su dani identitetom (131) su linearno nezavisni, stoga će se u nastavku uzimati kao elementi baze jezgre operatora $B_{Q,n}$.

Na osnovu izloženog možemo izreći sljedeću propoziciju.

Propozicija 4.2.9

Neka je skupu $Q = \{l_1, l_2, ..., l_n\} \subseteq \mathcal{N} = \{i_1, i_2, ..., i_N\}$ pridružen generički težinski potprostor $\mathcal{B}_Q = \mathcal{B}_{l_1 l_2 \cdots l_n} = span_{\mathbb{C}} \left\{ e_{j_1 j_2 \cdots j_n} \mid j_1 j_2 ... j_n \in \widehat{Q} = S_n Q \right\}$ i neka je zadovoljen uvjet kocikličnosti, tj. $\sigma_{l_1 l_2 \cdots l_n} = 1, \quad \sigma_{j_1 j_2 \cdots j_i} \neq 1$ za svaki $2 \leq i < n$.

Pretpostavimo da je $Q' = Q \setminus \{l_1, l_2\} = \{l_3, l_4, ..., l_n\}$, gdje su l_1 i l_2 fiksni.

Tada je

$$\mathcal{B} = \left\{ X_{l_1 l_2 j_3 j_4 \cdots j_n} \mid X_{l_1 l_2 j_3 j_4 \cdots j_n} = \left(C_{Q,n} \cdot \mathfrak{S}_{n-1}^{-1} \cdot E_{Q,n} \right) e_{l_1 l_2 j_3 j_4 \cdots j_n},\ j_3 j_4 ... j_n \in S_{n-2} Q' \right\} \quad (133)$$

baza jezgre operatora $B_{Q,n}$.

Pritom je $S_{n-2} Q' = \left\{ \sigma\left(l_3 l_4 ... l_n \right) \mid \sigma \in S_{n-2} \right\}$ skup svih permutacija skupa Q' ,

$\mathrm{Card}\, Q' = \dim \mathcal{B} = (n-2)!$.

Napomena:

Propozicija 4.2.9 u suglasnosti je s Fronsdal-ovim radom [F2] u kojemu je dokazano da je dimenzija prostora konstanti u generičkom težinskom potprostoru jednaka $(n-2)!$ ako je zadovoljen uvjet kocikličnosti.

Ako je zadovoljen uvjet kocikličnosti, tj. ako je $\sigma_{l_1 l_2 \cdots l_n} = 1$, $\sigma_{j_1 j_2 \cdots j_i} \neq 1$ za svaki $2 \leq i < n$, onda je izračunavanje konstanti C_Q u generičkom težinskom potprostoru $\mathcal{B}_Q = \mathcal{B}_{l_1 l_2 \cdots l_n}$ ekvivalentno traženju jezgre operatora $B_{Q,n}$. Na osnovu toga imamo da su netrivijalne bazične konstante u generičkom težinskom potprostoru $\mathcal{B}_Q = \mathcal{B}_{l_1 l_2 \cdots l_n}$ ujedno elementi baza jezgre operatora $B_{Q,n}$, dane izrazom (133).

Iz tog će se razloga konstante u potprostoru $\mathcal{B}_{Q} = \mathcal{B}_{l_1 l_2 \cdots l_n}$ označavati sa $C_{l_1 l_2 j_3 j_4 \cdots j_n}$ za svaki $j_3 j_4 ... j_n \in S_{n-2} Q'$, gdje je $S_{n-2} Q' = \{ \sigma (l_3 l_4 ... l_n) \mid \sigma \in S_{n-2} \}$ skup svih permutacija skupa $Q' = Q \setminus \{ l_1, l_2 \} = \{ l_3, l_4, ..., l_n \}$.

Dakle, imamo

$$\begin{aligned} C_{l_1 l_2 j_3 j_4 \cdots j_n} &= \left(C_{Q,n} \cdot \mathfrak{S}_{n-1}^{-1} \cdot E_{Q,n} \right) e_{l_1 l_2 j_3 j_4 \cdots j_n} \\ &= \left(\mathfrak{S}_{n-1}^{-1} \cdot E_{Q,n} \right) Y_{l_1 l_2 j_3 j_4 \cdots j_n} . \end{aligned} \tag{134}$$

Pritom se koristila lema 4.2.7 prema kojoj imamo da je $Y_{l_1 l_2 j_3 j_4 \cdots j_n} = C_{Q,n} \, e_{l_1 l_2 j_3 j_4 \cdots j_n}$.

Na osnovu rečenog imamo sljedeći teorem.

Teorem 4.2.10

Neka je $\mathcal{B}_{Q} = \mathcal{B}_{l_1 l_2 \cdots l_n} = span_{\mathbb{C}} \{ e_{j_1 j_2 \cdots j_n} \mid j_1 j_2 ... j_n \in \widehat{Q} \}$ generički težinski potprostor pridružen skupu $Q = \{ l_1, l_2, ..., l_n \} \subseteq \mathcal{N} = \{ i_1, i_2, ..., i_N \}$, $\text{Card}\, Q = n$, $n \geq 2$ i neka je zadovoljen uvjet kocikličnosti, tj. $\sigma_{l_1 l_2 \cdots l_n} = 1$, $\sigma_{j_1 j_2 \cdots j_i} \neq 1$ za svaki $2 \leq i < n$.

Pretpostavimo da je $Q' = Q \setminus \{ l_1, l_2 \} = \{ l_3, l_4, ..., l_n \}$.

Tada u potprostoru $\mathcal{B}_{Q}$ postoji $(n-2)!$ netrivijalnih baznih konstanti $C_{l_1 l_2 j_3 j_4 \cdots j_n}$, takvih da je

$$C_{l_1 l_2 j_3 j_4 \cdots j_n} = \left(\mathfrak{S}_{n-1}^{-1} \cdot E_{Q,n} \right) Y_{l_1 l_2 j_3 j_4 \cdots j_n} , \tag{135}$$

$j_3 j_4 ... j_n \in S_{n-2} Q'$.

Pritom je $\mathfrak{S}_{n-1} = \left(I - Q_{\{1,2\}} \right) \cdot \left(I - Q_{\{1,2,3\}} \right) \cdots \left(I - Q_{\{1,2,\ldots,n-1\}} \right)$,

$$E_{Q,n} = \sum_{g \in S_1^1 \times S_{n-1}} W_n (g) \cdot G ,$$

$$W_n (g) = \prod_{i \in Des(g^{-1})} Q_{\{1,2,\ldots,i\}} ,$$

$$Des(g^{-1}) = \{ 2 \leq i \leq n-1 \mid g^{-1}(i) > g^{-1}(i+1) \}.$$

Napomena:

Primijetimo da za svaki $j_3 j_4 ... j_n \in S_{n-2} Q'$, gdje je $Q' = Q \setminus \{ l_1, l_2 \} = \{ l_3, l_4, ..., l_n \}$ imamo da je

$$
\begin{aligned}
C_{l_1 l_2 j_3 j_4 \cdots j_n} &= \left(C_{Q,n} \cdot \mathfrak{S}_{n-1}^{-1} \cdot E_{Q,n}\right) e_{l_1 l_2 j_3 j_4 \cdots j_n} \\
&= C_{Q,n} \cdot \left(\mathfrak{S}_{n-1}^{-1} \cdot E_{Q,n}\right) e_{l_1 l_2 j_3 j_4 \cdots j_n} \\
&= C_{Q,n} \cdot \frac{\sum\limits_{g \in S_1^1 \times S_{n-1}} \left(\prod\limits_{i \in \mathrm{Des}\left(g^{-1}\right)} Q_{\{1,2,\ldots,i\}} \right) \cdot \mathcal{R}\left(\tilde{g}\right)}{\left(I - Q_{\{1,2\}}\right) \cdot \left(I - Q_{\{1,2,3\}}\right) \cdots \left(I - Q_{\{1,2,\ldots,n-1\}}\right)} e_{l_1 l_2 j_3 j_4 \cdots j_n} \\
&= \frac{\sum\limits_{g \in S_1^1 \times S_{n-1}} \left(\prod\limits_{i \in \mathrm{Des}\left(g^{-1}\right)} Q_{\{1,2,\ldots,i\}} \right) \cdot \mathcal{R}\left(\tilde{g}\right)}{\left(I - Q_{\{1,2\}}\right) \cdot \left(I - Q_{\{1,2,3\}}\right) \cdots \left(I - Q_{\{1,2,\ldots,n-1\}}\right)} Y_{l_1 l_2 j_3 j_4 \cdots j_n} \\
&= \left(\mathfrak{S}_{n-1}^{-1} \cdot E_{Q,n}\right) Y_{l_1 l_2 j_3 j_4 \cdots j_n} ,
\end{aligned}
$$

stoga je identitet (135) u raspisanom obliku dan sa

$$
C_{l_1 l_2 j_3 j_4 \cdots j_n} = \frac{\sum\limits_{g \in S_1^1 \times S_{n-1}} \left(\prod\limits_{i \in \mathrm{Des}\left(g^{-1}\right)} Q_{\{1,2,\ldots,i\}} \right) \cdot \mathcal{R}\left(\tilde{g}\right)}{\left(I - Q_{\{1,2\}}\right) \cdot \left(I - Q_{\{1,2,3\}}\right) \cdots \left(I - Q_{\{1,2,\ldots,n-1\}}\right)} Y_{l_1 l_2 j_3 j_4 \cdots j_n} , \qquad (136)
$$

gdje $g \in S_1^1 \times S_{n-1} \subset S_n$ (tj. g fiksira prvi indeks).

Desna strana identiteta (136) sastoji se od $(n-1)!$ članova.

Drugim riječima, svaka bazična konstanta $C_{l_1 l_2 j_3 j_4 \cdots j_n}$ može se prikazati kao linearna kombinacija od $(n-1)!$ iteriranih komutatora $Y_{l_1 l_2 j_3 j_4 \cdots j_n}$ takvih da im je prvi indeks l_1 fiksiran, a preostalih $n-1$ indeksa $l_2, j_3, j_4, \ldots, j_n$ permutiraju.

Primjer 4.2.11

Skupu $Q = \{l_1, l_2\} \subseteq \mathcal{N} = \{i_1, i_2, \ldots, i_N\}$ pridružen je generički težinski potprostor $\mathcal{B}_Q = \mathcal{B}_{l_1 l_2} = span_{\mathbb{C}} \left\{ e_{j_1 j_2} \mid j_1 j_2 \in \hat{Q} \right\} = span_{\mathbb{C}} \left\{ e_{l_1 l_2}, e_{l_2 l_1} \right\}$.

Pretpostavimo da je zadovoljen uvjet kocikličnosti, tj. da je $\sigma_{l_1 l_2} = 1$.

Tada primjenom identiteta (135) iz teorema 4.2.10 dobivamo da u potprostoru $\mathcal{B}_{l_1 l_2}$ postoji samo jedna (jer je $0! = 1$) netrivijalna bazična konstanta $C_{l_1 l_2}$ takva da je

$$
C_{l_1 l_2} = Y_{l_1 l_2} . \qquad (137)
$$

Uvodimo standarne oznake

$$\left.\begin{aligned} x^{*} &:= \frac{1}{1-x} \\ x^{+} &:= \frac{x}{1-x} \end{aligned}\right\} \tag{138}$$

Primjer 4.2.12

Neka je $\mathcal{B}_{\mathrm{Q}} = \mathcal{B}_{l_1 l_2 l_3} = span_{\mathbb{C}}\left\{ \mathrm{e}_{\mathrm{j}_1\mathrm{j}_2\mathrm{j}_3} \mid \mathrm{j}_1\mathrm{j}_2\mathrm{j}_3 \in \widehat{\mathrm{Q}} \right\}$ generički težinski potprostor pridružen skupu $\mathrm{Q} = \{l_1, l_2, l_3\} \subseteq \mathcal{N} = \{\mathrm{i}_1, \mathrm{i}_2, ..., \mathrm{i}_\mathrm{N}\}$.

Pretpostavimo da je uvjet kocikličnosti zadovoljen, tj. da je

$$\sigma_{l_1 l_2 l_3} = 1, \quad \sigma_{l_i l_j} \neq 1, \ 1 \leq \mathrm{i} < \mathrm{j} \leq 3. \tag{139}$$

Tada primjenom identiteta (135) dobivamo da u potprostoru $\mathcal{B}_{l_1 l_2 l_3}$ postoji samo jedna (jer je $1! = 1$) netrivijalna bazična konstanta $C_{l_1 l_2 l_3}$ takva da je

$$C_{l_1 l_2 l_3} = \frac{1}{1-\sigma_{l_1 l_2}} \mathrm{Y}_{l_1 l_2 l_3} + \frac{\mathrm{q}_{l_3 l_2} \sigma_{l_1 l_3}}{1-\sigma_{l_1 l_3}} \mathrm{Y}_{l_1 l_3 l_2} \tag{140}$$

ili primjenom standardnih oznaka (138)

$$C_{l_1 l_2 l_3} = \sigma^{*}_{l_1 l_2} \mathrm{Y}_{l_1 l_2 l_3} + \mathrm{q}_{l_3 l_2} \sigma^{+}_{l_1 l_3} \mathrm{Y}_{l_1 l_3 l_2}. \tag{141}$$

Napomena:

Specijalno za $l_1 = 1$, $l_2 = 2$, $l_3 = 3$, onda je uvjet kocikličnosti (139) dan sa

$$\sigma_{123} = 1, \quad \sigma_{12} \neq 1, \quad \sigma_{13} \neq 1, \quad \sigma_{23} \neq 1,$$

pa je netrivijalna bazična konstanta C_{123} u potprostoru $\mathcal{B}_{123}$ oblika

$$C_{123} = \frac{1}{1-\sigma_{12}} \mathrm{Y}_{123} + \frac{\mathrm{q}_{32}\sigma_{13}}{1-\sigma_{13}} \mathrm{Y}_{132}. \tag{142}$$

Podsjetimo se, u odjeljku 2.2.3. pokazali smo da se netrivijalna bazična konstanta C_{123} u potprostoru $\mathcal{B}_{123}$ uz zadovoljen uvjet kocikličnosti može zapisati u obliku

$$C_{123} = \frac{1-\sigma_{13}}{q_{31}}\left(e_{123} + q_{21}q_{31}q_{32}\, e_{321}\right) + \frac{1-\sigma_{12}}{q_{12}}\left(e_{231} + q_{32}q_{12}q_{13}\, e_{132}\right)$$

$$+\frac{1-\sigma_{23}}{q_{23}}\left(e_{312} + q_{13}q_{23}q_{21}\, e_{213}\right)$$

ili kraće

$$C_{123} = \sum_{cyc} \frac{1-\sigma_{12}}{q_{12}}\left(e_{231} + q_{32}q_{12}q_{13}\, e_{132}\right) \tag{143}$$

gdje $\sum\limits_{cyc}$ označuje cikličku sumu. Pritom je $\sigma_{ji} = \sigma_{ij}$.

Jednostavnim izračunavanjem pokazuje se da vrijedi identitet

$$\frac{1}{1-\sigma_{12}}\, Y_{123} + \frac{q_{32}\sigma_{13}}{1-\sigma_{13}}\, Y_{132} = \sum_{cyc} \frac{1-\sigma_{12}}{q_{12}}\left(e_{231} + q_{32}q_{12}q_{13}\, e_{132}\right),$$

što se i podrazumijeva da bi moralo vrijediti budući da je C_{123} netrivijalna bazična konstanta u generičkom težinskom potprostoru $\mathcal{B}_{123}$.

Primjer 4.2.13

Neka je $\mathcal{B}_{Q} = \mathcal{B}_{l_1 l_2 l_3 l_4} = span_{\mathbb{C}}\left\{ e_{j_1 j_2 j_3 j_4} \mid j_1 j_2 j_3 j_4 \in \widehat{Q} \right\}$ generički težinski potprostor pridružen skupu $Q = \{l_1, l_2, l_3, l_4\} \subseteq \mathcal{N} = \{i_1, i_2, ..., i_N\}$.

Pretpostavimo da je uvjet kocikličnosti zadovoljen, tj. da je

$$\sigma_{l_1 l_2 l_3 l_4} = 1, \quad \sigma_{l_i l_j l_k} \neq 1, \quad \sigma_{l_i l_j} \neq 1, \quad 1 \leq i < j < k \leq 4.$$

Tada je dimenzija prostora konstanti u potprostoru $\mathcal{B}_{l_1 l_2 l_3 l_4}$ jednaka $(4-2)! = 2$.

Primjenom identiteta (135) dobivamo sljedeće dvije bazične konstante

$$C_{l_1 l_2 l_3 l_4} = \frac{1}{\left(1-\sigma_{l_1 l_2}\right)\left(1-\sigma_{l_1 l_2 l_3}\right)}\, Y_{l_1 l_2 l_3 l_4} + \frac{q_{l_4 l_2} q_{l_4 l_3} \sigma_{l_1 l_4}}{\left(1-\sigma_{l_1 l_4}\right)\left(1-\sigma_{l_1 l_2 l_4}\right)}\, Y_{l_1 l_4 l_2 l_3} + \frac{q_{l_3 l_2} q_{l_4 l_2} \sigma_{l_1 l_3 l_4}}{\left(1-\sigma_{l_1 l_3}\right)\left(1-\sigma_{l_1 l_3 l_4}\right)}\, Y_{l_1 l_3 l_4 l_2}$$

$$+ \frac{q_{l_3 l_2} \sigma_{l_1 l_3}}{\left(1-\sigma_{l_1 l_3}\right)\left(1-\sigma_{l_1 l_2 l_3}\right)}\, Y_{l_1 l_3 l_2 l_4} + \frac{q_{l_4 l_3} \sigma_{l_1 l_2 l_4}}{\left(1-\sigma_{l_1 l_2}\right)\left(1-\sigma_{l_1 l_2 l_4}\right)}\, Y_{l_1 l_2 l_4 l_3} + \frac{q_{l_3 l_2} q_{l_4 l_2} q_{l_4 l_3} \sigma_{l_1 l_4} \sigma_{l_1 l_3 l_4}}{\left(1-\sigma_{l_1 l_4}\right)\left(1-\sigma_{l_1 l_3 l_4}\right)}\, Y_{l_1 l_4 l_3 l_2},$$

$$C_{l_1 l_2 l_4 l_3} = \frac{1}{\left(1-\sigma_{l_1 l_2}\right)\left(1-\sigma_{l_1 l_2 l_4}\right)}\, Y_{l_1 l_2 l_4 l_3} + \frac{q_{l_3 l_2} q_{l_3 l_4} \sigma_{l_1 l_3}}{\left(1-\sigma_{l_1 l_3}\right)\left(1-\sigma_{l_1 l_2 l_3}\right)}\, Y_{l_1 l_3 l_2 l_4} + \frac{q_{l_3 l_2} q_{l_4 l_2} \sigma_{l_1 l_3 l_4}}{\left(1-\sigma_{l_1 l_4}\right)\left(1-\sigma_{l_1 l_3 l_4}\right)}\, Y_{l_1 l_4 l_3 l_2}$$

$$+ \frac{q_{l_4 l_2} \sigma_{l_1 l_4}}{\left(1-\sigma_{l_1 l_4}\right)\left(1-\sigma_{l_1 l_2 l_4}\right)}\, Y_{l_1 l_4 l_2 l_3} + \frac{q_{l_3 l_4} \sigma_{l_1 l_2 l_3}}{\left(1-\sigma_{l_1 l_2}\right)\left(1-\sigma_{l_1 l_2 l_3}\right)}\, Y_{l_1 l_2 l_3 l_4} + \frac{q_{l_3 l_2} q_{l_3 l_4} q_{l_4 l_2} \sigma_{l_1 l_3} \sigma_{l_1 l_3 l_4}}{\left(1-\sigma_{l_1 l_3}\right)\left(1-\sigma_{l_1 l_3 l_4}\right)}\, Y_{l_1 l_3 l_4 l_2},$$

čije ćemo članove zapisati u leksikografskom poretku

$$
\begin{aligned}
C_{l_1l_2l_3l_4} &= \frac{1}{\left(1-\sigma_{l_1l_2}\right)\left(1-\sigma_{l_1l_2l_3}\right)}\,\mathrm{Y}_{l_1l_2l_3l_4} + \frac{\mathrm{q}_{l_4l_3}\sigma_{l_1l_2l_4}}{\left(1-\sigma_{l_1l_2}\right)\left(1-\sigma_{l_1l_2l_4}\right)}\,\mathrm{Y}_{l_1l_2l_4l_3} \\
&+ \frac{\mathrm{q}_{l_3l_2}\sigma_{l_1l_3}}{\left(1-\sigma_{l_1l_3}\right)\left(1-\sigma_{l_1l_2l_3}\right)}\,\mathrm{Y}_{l_1l_3l_2l_4} + \frac{\mathrm{q}_{l_3l_2}\mathrm{q}_{l_4l_2}\sigma_{l_1l_3l_4}}{\left(1-\sigma_{l_1l_3}\right)\left(1-\sigma_{l_1l_3l_4}\right)}\,\mathrm{Y}_{l_1l_3l_4l_2} \\
&+ \frac{\mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_3}\sigma_{l_1l_4}}{\left(1-\sigma_{l_1l_4}\right)\left(1-\sigma_{l_1l_2l_4}\right)}\,\mathrm{Y}_{l_1l_4l_2l_3} + \frac{\mathrm{q}_{l_3l_2}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_3}\sigma_{l_1l_4}\sigma_{l_1l_3l_4}}{\left(1-\sigma_{l_1l_4}\right)\left(1-\sigma_{l_1l_3l_4}\right)}\,\mathrm{Y}_{l_1l_4l_3l_2}\,, \qquad (144)
\end{aligned}
$$

$$
\begin{aligned}
C_{l_1l_2l_4l_3} &= \frac{\mathrm{q}_{l_3l_4}\sigma_{l_1l_2l_3}}{\left(1-\sigma_{l_1l_2}\right)\left(1-\sigma_{l_1l_2l_3}\right)}\,\mathrm{Y}_{l_1l_2l_3l_4} + \frac{1}{\left(1-\sigma_{l_1l_2}\right)\left(1-\sigma_{l_1l_2l_4}\right)}\,\mathrm{Y}_{l_1l_2l_4l_3} \\
&+ \frac{\mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_4}\sigma_{l_1l_3}}{\left(1-\sigma_{l_1l_3}\right)\left(1-\sigma_{l_1l_2l_3}\right)}\,\mathrm{Y}_{l_1l_3l_2l_4} + \frac{\mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_4}\mathrm{q}_{l_4l_2}\sigma_{l_1l_3}\sigma_{l_1l_3l_4}}{\left(1-\sigma_{l_1l_3}\right)\left(1-\sigma_{l_1l_3l_4}\right)}\,\mathrm{Y}_{l_1l_3l_4l_2} \\
&+ \frac{\mathrm{q}_{l_4l_2}\sigma_{l_1l_4}}{\left(1-\sigma_{l_1l_4}\right)\left(1-\sigma_{l_1l_2l_4}\right)}\,\mathrm{Y}_{l_1l_4l_2l_3} + \frac{\mathrm{q}_{l_3l_2}\mathrm{q}_{l_4l_2}\sigma_{l_1l_3l_4}}{\left(1-\sigma_{l_1l_4}\right)\left(1-\sigma_{l_1l_3l_4}\right)}\,\mathrm{Y}_{l_1l_4l_3l_2}\,. \qquad (145)
\end{aligned}
$$

Primjenom standardnih oznaka (138) imamo da se bazične konstante dane identitetima (144) i (145) mogu zapisati na sljedeći način

$$
\left.
\begin{aligned}
C_{l_1l_2l_3l_4} &= \sigma^{*}_{l_1l_2}\sigma^{*}_{l_1l_2l_3}\,\mathrm{Y}_{l_1l_2l_3l_4} + \mathrm{q}_{l_4l_3}\sigma^{*}_{l_1l_2}\sigma^{+}_{l_1l_2l_4}\,\mathrm{Y}_{l_1l_2l_4l_3} + \mathrm{q}_{l_3l_2}\sigma^{+}_{l_1l_3}\sigma^{*}_{l_1l_2l_3}\,\mathrm{Y}_{l_1l_3l_2l_4} \\
&+ \mathrm{q}_{l_3l_2}\mathrm{q}_{l_4l_2}\sigma^{*}_{l_1l_3}\sigma^{+}_{l_1l_3l_4}\,\mathrm{Y}_{l_1l_3l_4l_2} + \mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_3}\sigma^{+}_{l_1l_4}\sigma^{*}_{l_1l_2l_4}\,\mathrm{Y}_{l_1l_4l_2l_3} + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_3}\sigma^{+}_{l_1l_4}\sigma^{+}_{l_1l_3l_4}\,\mathrm{Y}_{l_1l_4l_3l_2} \\
C_{l_1l_2l_4l_3} &= \mathrm{q}_{l_3l_4}\sigma^{*}_{l_1l_2}\sigma^{+}_{l_1l_2l_3}\,\mathrm{Y}_{l_1l_2l_3l_4} + \sigma^{*}_{l_1l_2}\sigma^{*}_{l_1l_2l_4}\,\mathrm{Y}_{l_1l_2l_4l_3} + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_4}\sigma^{+}_{l_1l_3}\sigma^{*}_{l_1l_2l_3}\,\mathrm{Y}_{l_1l_3l_2l_4} \\
&+ \mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_4}\mathrm{q}_{l_4l_2}\sigma^{+}_{l_1l_3}\sigma^{+}_{l_1l_3l_4}\,\mathrm{Y}_{l_1l_3l_4l_2} + \mathrm{q}_{l_4l_2}\sigma^{+}_{l_1l_4}\sigma^{*}_{l_1l_2l_4}\,\mathrm{Y}_{l_1l_4l_2l_3} + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_4l_2}\sigma^{*}_{l_1l_4}\sigma^{+}_{l_1l_3l_4}\,\mathrm{Y}_{l_1l_4l_3l_2}
\end{aligned}
\right\} (146)
$$

Usporediti dobivene konstante s bazičnim konstantama dobivenih u odjeljku 2.2.4.

Primjer 4.2.14

Skupu $\mathrm{Q} = \{l_1, l_2, l_3, l_4, l_5\} \subseteq \mathcal{N} = \{\mathrm{i}_1, \mathrm{i}_2, \ldots, \mathrm{i}_\mathrm{N}\}$ pridružen je generički težinski potprostor $\boldsymbol{\mathcal{B}}_\mathrm{Q} = \boldsymbol{\mathcal{B}}_{l_1l_2l_3l_4l_5} = span_{\mathbb{C}}\left\{\mathrm{e}_{\mathrm{j}_1\mathrm{j}_2\mathrm{j}_3\mathrm{j}_4\mathrm{j}_5} \mid \mathrm{j}_1\mathrm{j}_2\mathrm{j}_3\mathrm{j}_4\mathrm{j}_5 \in \widehat{\mathrm{Q}}\right\}$.

Pritom je $\widehat{\mathrm{Q}} = \mathrm{S}_5\mathrm{Q} = \left\{\sigma\left(l_1l_2l_3l_4l_5\right) \mid \sigma \in \mathrm{S}_5\right\}$, Card $\widehat{\mathrm{Q}} = 5! = 120$.

Pretpostavimo da je uvjet kocikličnosti zadovoljen, tj. da je

$$\sigma_{l_1l_2l_3l_4l_5}=1, \quad \sigma_{l_il_jl_kl_m}\neq 1, \quad \sigma_{l_il_jl_k}\neq 1, \quad \sigma_{l_il_j}\neq 1, \quad 1\leq \mathrm{i}<\mathrm{j}<\mathrm{k}<\mathrm{m}\leq 5 .$$

Tada je dimenzija prostora konstanti u potprostoru $\mathcal{B}_{l_1l_2l_3l_4l_5}$ jednaka $(5-2)!=6$.

Primjenom identiteta (135) dobivamo šest baznih konstanti. Pritom se svaka od njih sastoji se od $4!=24$ člana.

Konkretno, imamo da je prva bazična konstanta dana sa

$$\begin{aligned}
C_{l_1l_2l_3l_4l_5} = {} & \frac{1}{\left(1-\sigma_{l_1l_2}\right)\left(1-\sigma_{l_1l_2l_3}\right)\left(1-\sigma_{l_1l_2l_3l_4}\right)}\mathrm{Y}_{l_1l_2l_3l_4l_5} + \frac{\mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_3}\mathrm{q}_{l_5l_4}\sigma_{l_1l_5}}{\left(1-\sigma_{l_1l_5}\right)\left(1-\sigma_{l_1l_2l_5}\right)\left(1-\sigma_{l_1l_2l_3l_5}\right)}\mathrm{Y}_{l_1l_5l_2l_3l_4} \\
& + \frac{\mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_3}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_3}\sigma_{l_1l_4l_5}}{\left(1-\sigma_{l_1l_4}\right)\left(1-\sigma_{l_1l_4l_5}\right)\left(1-\sigma_{l_1l_2l_4l_5}\right)}\mathrm{Y}_{l_1l_4l_5l_2l_3} + \frac{\mathrm{q}_{l_3l_2}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_5l_2}\sigma_{l_1l_3l_4l_5}}{\left(1-\sigma_{l_1l_3}\right)\left(1-\sigma_{l_1l_3l_4}\right)\left(1-\sigma_{l_1l_3l_4l_5}\right)}\mathrm{Y}_{l_1l_3l_4l_5l_2} \\
& + \frac{\mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_3}\sigma_{l_1l_4}}{\left(1-\sigma_{l_1l_4}\right)\left(1-\sigma_{l_1l_2l_4}\right)\left(1-\sigma_{l_1l_2l_3l_4}\right)}\mathrm{Y}_{l_1l_4l_2l_3l_5} + \frac{\mathrm{q}_{l_3l_2}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_4}\sigma_{l_1l_3l_5}}{\left(1-\sigma_{l_1l_3}\right)\left(1-\sigma_{l_1l_3l_5}\right)\left(1-\sigma_{l_1l_2l_3l_5}\right)}\mathrm{Y}_{l_1l_3l_5l_2l_4} \\
& + \frac{\mathrm{q}_{l_4l_3}\mathrm{q}_{l_5l_3}\sigma_{l_1l_2l_4l_5}}{\left(1-\sigma_{l_1l_2}\right)\left(1-\sigma_{l_1l_2l_4}\right)\left(1-\sigma_{l_1l_2l_4l_5}\right)}\mathrm{Y}_{l_1l_2l_4l_5l_3} + \frac{\mathrm{q}_{l_3l_2}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_3}\mathrm{q}_{l_5l_4}\sigma_{l_1l_5}\sigma_{l_1l_3l_4l_5}}{\left(1-\sigma_{l_1l_5}\right)\left(1-\sigma_{l_1l_3l_5}\right)\left(1-\sigma_{l_1l_3l_4l_5}\right)}\mathrm{Y}_{l_1l_5l_3l_4l_2} \\
& + \frac{\mathrm{q}_{l_3l_2}\mathrm{q}_{l_4l_2}\sigma_{l_1l_3l_4}}{\left(1-\sigma_{l_1l_3}\right)\left(1-\sigma_{l_1l_3l_4}\right)\left(1-\sigma_{l_1l_2l_3l_4}\right)}\mathrm{Y}_{l_1l_3l_4l_2l_5} + \frac{\mathrm{q}_{l_5l_4}\sigma_{l_1l_2l_3l_5}}{\left(1-\sigma_{l_1l_2}\right)\left(1-\sigma_{l_1l_2l_3}\right)\left(1-\sigma_{l_1l_2l_3l_5}\right)}\mathrm{Y}_{l_1l_2l_3l_5l_4} \\
& + \frac{\mathrm{q}_{l_4l_3}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_3}\mathrm{q}_{l_5l_4}\sigma_{l_1l_5}\sigma_{l_1l_2l_4l_5}}{\left(1-\sigma_{l_1l_5}\right)\left(1-\sigma_{l_1l_2l_5}\right)\left(1-\sigma_{l_1l_2l_4l_5}\right)}\mathrm{Y}_{l_1l_5l_2l_4l_3} + \frac{\mathrm{q}_{l_3l_2}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_3}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_3}\sigma_{l_1l_4l_5}\sigma_{l_1l_3l_4l_5}}{\left(1-\sigma_{l_1l_4}\right)\left(1-\sigma_{l_1l_4l_5}\right)\left(1-\sigma_{l_1l_3l_4l_5}\right)}\mathrm{Y}_{l_1l_4l_5l_3l_2} \\
& + \frac{\mathrm{q}_{l_3l_2}\sigma_{l_1l_3}}{\left(1-\sigma_{l_1l_3}\right)\left(1-\sigma_{l_1l_2l_3}\right)\left(1-\sigma_{l_1l_2l_3l_4}\right)}\mathrm{Y}_{l_1l_3l_2l_4l_5} + \frac{\mathrm{q}_{l_5l_3}\mathrm{q}_{l_5l_4}\sigma_{l_1l_2l_5}}{\left(1-\sigma_{l_1l_2}\right)\left(1-\sigma_{l_1l_2l_5}\right)\left(1-\sigma_{l_1l_2l_3l_5}\right)}\mathrm{Y}_{l_1l_2l_5l_3l_4} \\
& + \frac{\mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_3}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_3}\mathrm{q}_{l_5l_4}\sigma_{l_1l_5}\sigma_{l_1l_4l_5}}{\left(1-\sigma_{l_1l_5}\right)\left(1-\sigma_{l_1l_4l_5}\right)\left(1-\sigma_{l_1l_2l_4l_5}\right)}\mathrm{Y}_{l_1l_5l_4l_2l_3} + \frac{\mathrm{q}_{l_3l_2}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_3}\mathrm{q}_{l_5l_2}\sigma_{l_1l_4}\sigma_{l_1l_3l_4l_5}}{\left(1-\sigma_{l_1l_4}\right)\left(1-\sigma_{l_1l_3l_4}\right)\left(1-\sigma_{l_1l_3l_4l_5}\right)}\mathrm{Y}_{l_1l_4l_3l_5l_2} \\
& + \frac{\mathrm{q}_{l_4l_3}\sigma_{l_1l_2l_4}}{\left(1-\sigma_{l_1l_2}\right)\left(1-\sigma_{l_1l_2l_4}\right)\left(1-\sigma_{l_1l_2l_3l_4}\right)}\mathrm{Y}_{l_1l_2l_4l_3l_5} + \frac{\mathrm{q}_{l_3l_2}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_3}\mathrm{q}_{l_5l_4}\sigma_{l_1l_5}\sigma_{l_1l_3l_5}}{\left(1-\sigma_{l_1l_5}\right)\left(1-\sigma_{l_1l_3l_5}\right)\left(1-\sigma_{l_1l_2l_3l_5}\right)}\mathrm{Y}_{l_1l_5l_3l_2l_4} \\
& + \frac{\mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_3}\mathrm{q}_{l_5l_3}\sigma_{l_1l_4}\sigma_{l_1l_2l_4l_5}}{\left(1-\sigma_{l_1l_4}\right)\left(1-\sigma_{l_1l_2l_4}\right)\left(1-\sigma_{l_1l_2l_4l_5}\right)}\mathrm{Y}_{l_1l_4l_2l_5l_3} + \frac{\mathrm{q}_{l_3l_2}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_4}\sigma_{l_1l_3l_5}\sigma_{l_1l_3l_4l_5}}{\left(1-\sigma_{l_1l_3}\right)\left(1-\sigma_{l_1l_3l_5}\right)\left(1-\sigma_{l_1l_3l_4l_5}\right)}\mathrm{Y}_{l_1l_3l_5l_4l_2} \\
& + \frac{\mathrm{q}_{l_3l_2}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_3}\sigma_{l_1l_4}\sigma_{l_1l_3l_4}}{\left(1-\sigma_{l_1l_4}\right)\left(1-\sigma_{l_1l_3l_4}\right)\left(1-\sigma_{l_1l_2l_3l_4}\right)}\mathrm{Y}_{l_1l_4l_3l_2l_5} + \frac{\mathrm{q}_{l_3l_2}\mathrm{q}_{l_5l_4}\sigma_{l_1l_3}\sigma_{l_1l_2l_3l_5}}{\left(1-\sigma_{l_1l_3}\right)\left(1-\sigma_{l_1l_2l_3}\right)\left(1-\sigma_{l_1l_2l_3l_5}\right)}\mathrm{Y}_{l_1l_3l_2l_5l_4} \\
& + \frac{\mathrm{q}_{l_4l_3}\mathrm{q}_{l_5l_3}\mathrm{q}_{l_5l_4}\sigma_{l_1l_2l_5}\sigma_{l_1l_2l_4l_5}}{\left(1-\sigma_{l_1l_2}\right)\left(1-\sigma_{l_1l_2l_5}\right)\left(1-\sigma_{l_1l_2l_4l_5}\right)}\mathrm{Y}_{l_1l_2l_5l_4l_3} + \frac{\mathrm{q}_{l_3l_2}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_3}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_3}\mathrm{q}_{l_5l_4}\sigma_{l_1l_5}\sigma_{l_1l_4l_5}\sigma_{l_1l_3l_4l_5}}{\left(1-\sigma_{l_1l_5}\right)\left(1-\sigma_{l_1l_4l_5}\right)\left(1-\sigma_{l_1l_3l_4l_5}\right)}\mathrm{Y}_{l_1l_5l_4l_3l_2} ,
\end{aligned}$$

tj. primjenom standardnih oznaka (138) imamo da je

$$\begin{aligned}
C_{l_1l_2l_3l_4l_5} = {} & \sigma^*_{l_1l_2}\sigma^*_{l_1l_2l_3}\sigma^*_{l_1l_2l_3l_4}\,\mathrm{Y}_{l_1l_2l_3l_4l_5} + \mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_3}\mathrm{q}_{l_5l_4}\sigma^+_{l_1l_5}\sigma^*_{l_1l_2l_5}\sigma^*_{l_1l_2l_3l_5}\,\mathrm{Y}_{l_1l_5l_2l_3l_4} \\
& + \mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_3}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_3}\sigma^*_{l_1l_4}\sigma^+_{l_1l_4l_5}\sigma^*_{l_1l_2l_4l_5}\,\mathrm{Y}_{l_1l_4l_5l_2l_3} + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_5l_2}\sigma^*_{l_1l_3}\sigma^*_{l_1l_3l_4}\sigma^+_{l_1l_3l_4l_5}\,\mathrm{Y}_{l_1l_3l_4l_5l_2} \\
& + \mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_3}\sigma^+_{l_1l_4}\sigma^*_{l_1l_2l_4}\sigma^*_{l_1l_2l_3l_4}\,\mathrm{Y}_{l_1l_4l_2l_3l_5} + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_4}\sigma^*_{l_1l_3}\sigma^+_{l_1l_3l_5}\sigma^*_{l_1l_2l_3l_5}\,\mathrm{Y}_{l_1l_3l_5l_2l_4} \\
& + \mathrm{q}_{l_4l_3}\mathrm{q}_{l_5l_3}\sigma^*_{l_1l_2}\sigma^*_{l_1l_2l_4}\sigma^+_{l_1l_2l_4l_5}\,\mathrm{Y}_{l_1l_2l_4l_5l_3} + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_3}\mathrm{q}_{l_5l_4}\sigma^+_{l_1l_5}\sigma^*_{l_1l_3l_5}\sigma^+_{l_1l_3l_4l_5}\,\mathrm{Y}_{l_1l_5l_3l_4l_2} \\
& + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_4l_2}\sigma^*_{l_1l_3}\sigma^+_{l_1l_3l_4}\sigma^*_{l_1l_2l_3l_4}\,\mathrm{Y}_{l_1l_3l_4l_2l_5} + \mathrm{q}_{l_5l_4}\sigma^*_{l_1l_2}\sigma^*_{l_1l_2l_3}\sigma^+_{l_1l_2l_3l_5}\,\mathrm{Y}_{l_1l_2l_3l_5l_4} \\
& + \mathrm{q}_{l_4l_3}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_3}\mathrm{q}_{l_5l_4}\sigma^+_{l_1l_5}\sigma^*_{l_1l_2l_5}\sigma^+_{l_1l_2l_4l_5}\,\mathrm{Y}_{l_1l_5l_2l_4l_3} + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_3}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_3}\sigma^*_{l_1l_4}\sigma^+_{l_1l_4l_5}\sigma^+_{l_1l_3l_4l_5}\,\mathrm{Y}_{l_1l_4l_5l_3l_2} \\
& + \mathrm{q}_{l_3l_2}\sigma^+_{l_1l_3}\sigma^*_{l_1l_2l_3}\sigma^*_{l_1l_2l_3l_4}\,\mathrm{Y}_{l_1l_3l_2l_4l_5} + \mathrm{q}_{l_5l_3}\mathrm{q}_{l_5l_4}\sigma^*_{l_1l_2}\sigma^+_{l_1l_2l_5}\sigma^*_{l_1l_2l_3l_5}\,\mathrm{Y}_{l_1l_2l_5l_3l_4} \\
& + \mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_3}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_3}\mathrm{q}_{l_5l_4}\sigma^+_{l_1l_5}\sigma^+_{l_1l_4l_5}\sigma^*_{l_1l_2l_4l_5}\,\mathrm{Y}_{l_1l_5l_4l_2l_3} + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_3}\mathrm{q}_{l_5l_2}\sigma^+_{l_1l_4}\sigma^*_{l_1l_3l_4}\sigma^+_{l_1l_3l_4l_5}\,\mathrm{Y}_{l_1l_4l_3l_5l_2} \\
& + \mathrm{q}_{l_4l_3}\sigma^*_{l_1l_2}\sigma^+_{l_1l_2l_4}\sigma^*_{l_1l_2l_3l_4}\,\mathrm{Y}_{l_1l_2l_4l_3l_5} + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_3}\mathrm{q}_{l_5l_4}\sigma^+_{l_1l_5}\sigma^+_{l_1l_3l_5}\sigma^*_{l_1l_2l_3l_5}\,\mathrm{Y}_{l_1l_5l_3l_2l_4} \\
& + \mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_3}\mathrm{q}_{l_5l_3}\sigma^+_{l_1l_4}\sigma^*_{l_1l_2l_4}\sigma^+_{l_1l_2l_4l_5}\,\mathrm{Y}_{l_1l_4l_2l_5l_3} + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_4}\sigma^*_{l_1l_3}\sigma^+_{l_1l_3l_5}\sigma^+_{l_1l_3l_4l_5}\,\mathrm{Y}_{l_1l_3l_5l_4l_2} \\
& + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_3}\sigma^+_{l_1l_4}\sigma^+_{l_1l_3l_4}\sigma^*_{l_1l_2l_3l_4}\,\mathrm{Y}_{l_1l_4l_3l_2l_5} + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_5l_4}\sigma^+_{l_1l_3}\sigma^*_{l_1l_2l_3}\sigma^+_{l_1l_2l_3l_5}\,\mathrm{Y}_{l_1l_3l_2l_5l_4} \\
& + \mathrm{q}_{l_4l_3}\mathrm{q}_{l_5l_3}\mathrm{q}_{l_5l_4}\sigma^*_{l_1l_2}\sigma^+_{l_1l_2l_5}\sigma^+_{l_1l_2l_4l_5}\,\mathrm{Y}_{l_1l_2l_5l_4l_3} + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_3}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_3}\mathrm{q}_{l_5l_4}\sigma^+_{l_1l_5}\sigma^+_{l_1l_4l_5}\sigma^+_{l_1l_3l_4l_5}\,\mathrm{Y}_{l_1l_5l_4l_3l_2}\,.
\end{aligned} \tag{147}$$

Analogno dobivamo preostalih pet baznih konstanti tako da se za prvi član u identitetu (147) umjesto člana $\sigma^*_{l_1l_2}\sigma^*_{l_1l_2l_3}\sigma^*_{l_1l_2l_3l_4}\,\mathrm{Y}_{l_1l_2l_3l_4l_5}$ redom uzmu sljedeći članovi:

$$\sigma^*_{l_1l_2}\sigma^*_{l_1l_2l_3}\sigma^*_{l_1l_2l_3l_5}\,\mathrm{Y}_{l_1l_2l_3l_5l_4}\,,\quad \sigma^*_{l_1l_2}\sigma^*_{l_1l_2l_4}\sigma^*_{l_1l_2l_3l_4}\,\mathrm{Y}_{l_1l_2l_4l_3l_5}\,,\quad \sigma^*_{l_1l_2}\sigma^*_{l_1l_2l_4}\sigma^*_{l_1l_2l_4l_5}\,\mathrm{Y}_{l_1l_2l_4l_5l_3}\,,$$
$$\sigma^*_{l_1l_2}\sigma^*_{l_1l_2l_5}\sigma^*_{l_1l_2l_3l_5}\,\mathrm{Y}_{l_1l_2l_5l_3l_4}\,,\quad \sigma^*_{l_1l_2}\sigma^*_{l_1l_2l_5}\sigma^*_{l_1l_2l_4l_5}\,\mathrm{Y}_{l_1l_2l_5l_4l_3}\,,$$

a preostala 23 člana dobivaju se redoslijedom permutiranja indeksa kako je dano u identitetu (147). Drugim riječima, preostale pet bazične konstante dobivaju se iz prve bazične konstante uz zamjenu nekih indeksa u svakom članu identiteta (147).
Konkretno

- drugu bazičnu konstantu dobivamo ako je $l_4 := l_5,\ l_5 := l_4$,
- treću bazičnu konstantu dobivamo ako je $l_3 := l_4,\ l_4 := l_3$,
- četvrtu bazičnu konstantu dobivamo ako je $l_3 := l_4,\ l_4 := l_5,\ l_5 := l_3$,
- petu bazičnu konstantu dobivamo ako je $l_3 := l_5,\ l_4 := l_3,\ l_5 := l_4$,
- šestu bazičnu konstantu dobivamo ako je $l_3 := l_5,\ l_5 := l_3$.

$$
\begin{aligned}
C_{l_1l_2l_3l_5l_4} = {} & \sigma^{*}_{l_1l_2}\sigma^{*}_{l_1l_2l_3}\sigma^{*}_{l_1l_2l_3l_5}\ \mathrm{Y}_{l_1l_2l_3l_5l_4} + \mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_3}\mathrm{q}_{l_4l_5}\sigma^{+}_{l_1l_4}\sigma^{*}_{l_1l_2l_4}\sigma^{*}_{l_1l_2l_3l_4}\ \mathrm{Y}_{l_1l_4l_2l_3l_5} \\
& + \mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_3}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_3}\sigma^{*}_{l_1l_5}\sigma^{+}_{l_1l_4l_5}\sigma^{*}_{l_1l_2l_4l_5}\ \mathrm{Y}_{l_1l_5l_4l_2l_3} + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_4l_2}\sigma^{*}_{l_1l_3}\sigma^{*}_{l_1l_3l_5}\sigma^{+}_{l_1l_3l_4l_5}\ \mathrm{Y}_{l_1l_3l_5l_4l_2} \\
& + \mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_3}\sigma^{+}_{l_1l_5}\sigma^{*}_{l_1l_2l_5}\sigma^{*}_{l_1l_2l_3l_5}\ \mathrm{Y}_{l_1l_5l_2l_3l_4} + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_5}\sigma^{*}_{l_1l_3}\sigma^{+}_{l_1l_3l_4}\sigma^{*}_{l_1l_2l_3l_4}\ \mathrm{Y}_{l_1l_3l_4l_2l_5} \\
& + \mathrm{q}_{l_5l_3}\mathrm{q}_{l_4l_3}\sigma^{*}_{l_1l_2}\sigma^{*}_{l_1l_2l_5}\sigma^{+}_{l_1l_2l_4l_5}\ \mathrm{Y}_{l_1l_2l_5l_4l_3} + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_3}\mathrm{q}_{l_4l_5}\sigma^{+}_{l_1l_4}\sigma^{*}_{l_1l_3l_4}\sigma^{+}_{l_1l_3l_4l_5}\ \mathrm{Y}_{l_1l_4l_3l_5l_2} \\
& + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_5l_2}\sigma^{*}_{l_1l_3}\sigma^{+}_{l_1l_3l_5}\sigma^{*}_{l_1l_2l_3l_5}\ \mathrm{Y}_{l_1l_3l_5l_2l_4} + \mathrm{q}_{l_4l_5}\sigma^{*}_{l_1l_2}\sigma^{*}_{l_1l_2l_3}\sigma^{+}_{l_1l_2l_3l_4}\ \mathrm{Y}_{l_1l_2l_3l_4l_5} \\
& + \mathrm{q}_{l_5l_3}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_3}\mathrm{q}_{l_4l_5}\sigma^{+}_{l_1l_4}\sigma^{*}_{l_1l_2l_4}\sigma^{+}_{l_1l_2l_4l_5}\ \mathrm{Y}_{l_1l_4l_2l_5l_3} + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_3}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_3}\sigma^{*}_{l_1l_5}\sigma^{+}_{l_1l_4l_5}\sigma^{+}_{l_1l_3l_4l_5}\ \mathrm{Y}_{l_1l_5l_4l_3l_2} \\
& + \mathrm{q}_{l_3l_2}\sigma^{+}_{l_1l_3}\sigma^{*}_{l_1l_2l_3}\sigma^{*}_{l_1l_2l_3l_5}\ \mathrm{Y}_{l_1l_3l_2l_5l_4} + \mathrm{q}_{l_4l_3}\mathrm{q}_{l_4l_5}\sigma^{*}_{l_1l_2}\sigma^{+}_{l_1l_2l_4}\sigma^{*}_{l_1l_2l_3l_4}\ \mathrm{Y}_{l_1l_2l_4l_3l_5} \\
& + \mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_3}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_3}\mathrm{q}_{l_4l_5}\sigma^{+}_{l_1l_4}\sigma^{+}_{l_1l_4l_5}\sigma^{*}_{l_1l_2l_4l_5}\ \mathrm{Y}_{l_1l_4l_5l_2l_3} + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_3}\mathrm{q}_{l_4l_2}\sigma^{+}_{l_1l_5}\sigma^{*}_{l_1l_3l_5}\sigma^{+}_{l_1l_3l_4l_5}\ \mathrm{Y}_{l_1l_5l_3l_4l_2} \\
& + \mathrm{q}_{l_5l_3}\sigma^{*}_{l_1l_2}\sigma^{+}_{l_1l_2l_5}\sigma^{*}_{l_1l_2l_3l_5}\ \mathrm{Y}_{l_1l_2l_5l_3l_4} + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_3}\mathrm{q}_{l_4l_5}\sigma^{+}_{l_1l_4}\sigma^{+}_{l_1l_3l_4}\sigma^{*}_{l_1l_2l_3l_4}\ \mathrm{Y}_{l_1l_4l_3l_2l_5} \\
& + \mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_3}\mathrm{q}_{l_4l_3}\sigma^{+}_{l_1l_5}\sigma^{*}_{l_1l_2l_5}\sigma^{+}_{l_1l_2l_4l_5}\ \mathrm{Y}_{l_1l_5l_2l_4l_3} + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_5}\sigma^{*}_{l_1l_3}\sigma^{+}_{l_1l_3l_4}\sigma^{+}_{l_1l_3l_4l_5}\ \mathrm{Y}_{l_1l_3l_4l_5l_2} \\
& + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_3}\sigma^{+}_{l_1l_5}\sigma^{+}_{l_1l_3l_5}\sigma^{*}_{l_1l_2l_3l_5}\ \mathrm{Y}_{l_1l_5l_3l_2l_4} + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_4l_5}\sigma^{+}_{l_1l_3}\sigma^{*}_{l_1l_2l_3}\sigma^{+}_{l_1l_2l_3l_4}\ \mathrm{Y}_{l_1l_3l_2l_4l_5} \\
& + \mathrm{q}_{l_5l_3}\mathrm{q}_{l_4l_3}\mathrm{q}_{l_4l_5}\sigma^{*}_{l_1l_2}\sigma^{+}_{l_1l_2l_4}\sigma^{+}_{l_1l_2l_4l_5}\ \mathrm{Y}_{l_1l_2l_4l_5l_3} + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_3}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_3}\mathrm{q}_{l_4l_5}\sigma^{+}_{l_1l_4}\sigma^{+}_{l_1l_4l_5}\sigma^{+}_{l_1l_3l_4l_5}\ \mathrm{Y}_{l_1l_4l_5l_3l_2}\,.
\end{aligned}
\tag{148}
$$

$$
\begin{aligned}
C_{l_1l_2l_4l_3l_5} = {} & \sigma^{*}_{l_1l_2}\sigma^{*}_{l_1l_2l_4}\sigma^{*}_{l_1l_2l_3l_4}\ \mathrm{Y}_{l_1l_2l_4l_3l_5} + \mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_4}\mathrm{q}_{l_5l_3}\sigma^{+}_{l_1l_5}\sigma^{*}_{l_1l_2l_5}\sigma^{*}_{l_1l_2l_4l_5}\ \mathrm{Y}_{l_1l_5l_2l_4l_3} \\
& + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_4}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_4}\sigma^{*}_{l_1l_3}\sigma^{+}_{l_1l_3l_5}\sigma^{*}_{l_1l_2l_3l_5}\ \mathrm{Y}_{l_1l_3l_5l_2l_4} + \mathrm{q}_{l_4l_2}\mathrm{q}_{l_3l_2}\mathrm{q}_{l_5l_2}\sigma^{*}_{l_1l_4}\sigma^{*}_{l_1l_3l_4}\sigma^{+}_{l_1l_3l_4l_5}\ \mathrm{Y}_{l_1l_4l_3l_5l_2} \\
& + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_4}\sigma^{+}_{l_1l_3}\sigma^{*}_{l_1l_2l_3}\sigma^{*}_{l_1l_2l_3l_4}\ \mathrm{Y}_{l_1l_3l_2l_4l_5} + \mathrm{q}_{l_4l_2}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_3}\sigma^{*}_{l_1l_4}\sigma^{+}_{l_1l_4l_5}\sigma^{*}_{l_1l_2l_4l_5}\ \mathrm{Y}_{l_1l_4l_5l_2l_3} \\
& + \mathrm{q}_{l_3l_4}\mathrm{q}_{l_5l_4}\sigma^{*}_{l_1l_2}\sigma^{*}_{l_1l_2l_3}\sigma^{+}_{l_1l_2l_3l_5}\ \mathrm{Y}_{l_1l_2l_3l_5l_4} + \mathrm{q}_{l_4l_2}\mathrm{q}_{l_3l_2}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_4}\mathrm{q}_{l_5l_3}\sigma^{+}_{l_1l_5}\sigma^{*}_{l_1l_4l_5}\sigma^{+}_{l_1l_3l_4l_5}\ \mathrm{Y}_{l_1l_5l_4l_3l_2} \\
& + \mathrm{q}_{l_4l_2}\mathrm{q}_{l_3l_2}\sigma^{*}_{l_1l_4}\sigma^{+}_{l_1l_3l_4}\sigma^{*}_{l_1l_2l_3l_4}\ \mathrm{Y}_{l_1l_4l_3l_2l_5} + \mathrm{q}_{l_5l_3}\sigma^{*}_{l_1l_2}\sigma^{*}_{l_1l_2l_4}\sigma^{+}_{l_1l_2l_4l_5}\ \mathrm{Y}_{l_1l_2l_4l_5l_3} \\
& + \mathrm{q}_{l_3l_4}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_4}\mathrm{q}_{l_5l_3}\sigma^{+}_{l_1l_5}\sigma^{*}_{l_1l_2l_5}\sigma^{+}_{l_1l_2l_3l_5}\ \mathrm{Y}_{l_1l_5l_2l_3l_4} + \mathrm{q}_{l_4l_2}\mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_4}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_4}\sigma^{*}_{l_1l_3}\sigma^{+}_{l_1l_3l_5}\sigma^{+}_{l_1l_3l_4l_5}\ \mathrm{Y}_{l_1l_3l_5l_4l_2} \\
& + \mathrm{q}_{l_4l_2}\sigma^{+}_{l_1l_4}\sigma^{*}_{l_1l_2l_4}\sigma^{*}_{l_1l_2l_3l_4}\ \mathrm{Y}_{l_1l_4l_2l_3l_5} + \mathrm{q}_{l_5l_4}\mathrm{q}_{l_5l_3}\sigma^{*}_{l_1l_2}\sigma^{+}_{l_1l_2l_5}\sigma^{*}_{l_1l_2l_4l_5}\ \mathrm{Y}_{l_1l_2l_5l_4l_3} \\
& + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_4}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_4}\mathrm{q}_{l_5l_3}\sigma^{+}_{l_1l_5}\sigma^{+}_{l_1l_3l_5}\sigma^{*}_{l_1l_2l_3l_5}\ \mathrm{Y}_{l_1l_5l_3l_2l_4} + \mathrm{q}_{l_4l_2}\mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_4}\mathrm{q}_{l_5l_2}\sigma^{+}_{l_1l_3}\sigma^{*}_{l_1l_3l_4}\sigma^{+}_{l_1l_3l_4l_5}\ \mathrm{Y}_{l_1l_3l_4l_5l_2} \\
& + \mathrm{q}_{l_3l_4}\sigma^{*}_{l_1l_2}\sigma^{+}_{l_1l_2l_3}\sigma^{*}_{l_1l_2l_3l_4}\ \mathrm{Y}_{l_1l_2l_3l_4l_5} + \mathrm{q}_{l_4l_2}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_4}\mathrm{q}_{l_5l_3}\sigma^{+}_{l_1l_5}\sigma^{+}_{l_1l_4l_5}\sigma^{*}_{l_1l_2l_4l_5}\ \mathrm{Y}_{l_1l_5l_4l_2l_3} \\
& + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_4}\mathrm{q}_{l_5l_4}\sigma^{+}_{l_1l_3}\sigma^{*}_{l_1l_2l_3}\sigma^{+}_{l_1l_2l_3l_5}\ \mathrm{Y}_{l_1l_3l_2l_5l_4} + \mathrm{q}_{l_4l_2}\mathrm{q}_{l_3l_2}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_3}\sigma^{*}_{l_1l_4}\sigma^{+}_{l_1l_4l_5}\sigma^{+}_{l_1l_3l_4l_5}\ \mathrm{Y}_{l_1l_4l_5l_3l_2} \\
& + \mathrm{q}_{l_4l_2}\mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_4}\sigma^{+}_{l_1l_3}\sigma^{+}_{l_1l_3l_4}\sigma^{*}_{l_1l_2l_3l_4}\ \mathrm{Y}_{l_1l_3l_4l_2l_5} + \mathrm{q}_{l_4l_2}\mathrm{q}_{l_5l_3}\sigma^{+}_{l_1l_4}\sigma^{*}_{l_1l_2l_4}\sigma^{+}_{l_1l_2l_4l_5}\ \mathrm{Y}_{l_1l_4l_2l_5l_3} \\
& + \mathrm{q}_{l_3l_4}\mathrm{q}_{l_5l_4}\mathrm{q}_{l_5l_3}\sigma^{*}_{l_1l_2}\sigma^{+}_{l_1l_2l_5}\sigma^{+}_{l_1l_2l_3l_5}\ \mathrm{Y}_{l_1l_2l_5l_3l_4} + \mathrm{q}_{l_4l_2}\mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_4}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_4}\mathrm{q}_{l_5l_3}\sigma^{+}_{l_1l_5}\sigma^{+}_{l_1l_3l_5}\sigma^{+}_{l_1l_3l_4l_5}\ \mathrm{Y}_{l_1l_5l_3l_4l_2}\,.
\end{aligned}
\tag{149}
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\begin{aligned}
C_{l_1l_2l_4l_5l_3} = {} & \sigma^{*}_{l_1l_2}\sigma^{*}_{l_1l_2l_4}\sigma^{*}_{l_1l_2l_4l_5}\,\mathrm{Y}_{l_1l_2l_4l_5l_3} + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_4}\mathrm{q}_{l_3l_5}\sigma^{+}_{l_1l_3}\sigma^{*}_{l_1l_2l_3}\sigma^{*}_{l_1l_2l_3l_4}\,\mathrm{Y}_{l_1l_3l_2l_4l_5} \\
& + \mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_4}\mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_4}\sigma^{*}_{l_1l_5}\sigma^{+}_{l_1l_3l_5}\sigma^{*}_{l_1l_2l_3l_5}\,\mathrm{Y}_{l_1l_5l_3l_2l_4} + \mathrm{q}_{l_4l_2}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_3l_2}\sigma^{*}_{l_1l_4}\sigma^{*}_{l_1l_4l_5}\sigma^{+}_{l_1l_3l_4l_5}\,\mathrm{Y}_{l_1l_4l_5l_3l_2} \\
& + \mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_4}\sigma^{+}_{l_1l_5}\sigma^{*}_{l_1l_2l_5}\sigma^{*}_{l_1l_2l_4l_5}\,\mathrm{Y}_{l_1l_5l_2l_4l_3} + \mathrm{q}_{l_4l_2}\mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_5}\sigma^{*}_{l_1l_4}\sigma^{+}_{l_1l_3l_4}\sigma^{*}_{l_1l_2l_3l_4}\,\mathrm{Y}_{l_1l_4l_3l_2l_5} \\
& + \mathrm{q}_{l_5l_4}\mathrm{q}_{l_3l_4}\sigma^{*}_{l_1l_2}\sigma^{*}_{l_1l_2l_5}\sigma^{+}_{l_1l_2l_3l_5}\,\mathrm{Y}_{l_1l_2l_5l_3l_4} + \mathrm{q}_{l_4l_2}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_4}\mathrm{q}_{l_3l_5}\sigma^{+}_{l_1l_3}\sigma^{*}_{l_1l_3l_4}\sigma^{+}_{l_1l_3l_4l_5}\,\mathrm{Y}_{l_1l_3l_4l_5l_2} \\
& + \mathrm{q}_{l_4l_2}\mathrm{q}_{l_5l_2}\sigma^{*}_{l_1l_4}\sigma^{+}_{l_1l_4l_5}\sigma^{*}_{l_1l_2l_4l_5}\,\mathrm{Y}_{l_1l_4l_5l_2l_3} + \mathrm{q}_{l_3l_5}\sigma^{*}_{l_1l_2}\sigma^{*}_{l_1l_2l_4}\sigma^{+}_{l_1l_2l_3l_4}\,\mathrm{Y}_{l_1l_2l_4l_3l_5} \\
& + \mathrm{q}_{l_5l_4}\mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_4}\mathrm{q}_{l_3l_5}\sigma^{+}_{l_1l_3}\sigma^{*}_{l_1l_2l_3}\sigma^{+}_{l_1l_2l_3l_5}\,\mathrm{Y}_{l_1l_3l_2l_5l_4} + \mathrm{q}_{l_4l_2}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_4}\mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_4}\sigma^{*}_{l_1l_5}\sigma^{+}_{l_1l_3l_5}\sigma^{+}_{l_1l_3l_4l_5}\,\mathrm{Y}_{l_1l_5l_3l_4l_2} \\
& + \mathrm{q}_{l_4l_2}\sigma^{+}_{l_1l_4}\sigma^{*}_{l_1l_2l_4}\sigma^{*}_{l_1l_2l_4l_5}\,\mathrm{Y}_{l_1l_4l_2l_5l_3} + \mathrm{q}_{l_3l_4}\mathrm{q}_{l_3l_5}\sigma^{*}_{l_1l_2}\sigma^{+}_{l_1l_2l_3}\sigma^{*}_{l_1l_2l_3l_4}\,\mathrm{Y}_{l_1l_2l_3l_4l_5} \\
& + \mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_4}\mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_4}\mathrm{q}_{l_3l_5}\sigma^{+}_{l_1l_3}\sigma^{+}_{l_1l_3l_5}\sigma^{*}_{l_1l_2l_3l_5}\,\mathrm{Y}_{l_1l_3l_5l_2l_4} + \mathrm{q}_{l_4l_2}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_4}\mathrm{q}_{l_3l_2}\sigma^{+}_{l_1l_5}\sigma^{*}_{l_1l_4l_5}\sigma^{+}_{l_1l_3l_4l_5}\,\mathrm{Y}_{l_1l_5l_4l_3l_2} \\
& + \mathrm{q}_{l_5l_4}\sigma^{*}_{l_1l_2}\sigma^{+}_{l_1l_2l_5}\sigma^{*}_{l_1l_2l_4l_5}\,\mathrm{Y}_{l_1l_2l_5l_4l_3} + \mathrm{q}_{l_4l_2}\mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_4}\mathrm{q}_{l_3l_5}\sigma^{+}_{l_1l_3}\sigma^{+}_{l_1l_3l_4}\sigma^{*}_{l_1l_2l_3l_4}\,\mathrm{Y}_{l_1l_3l_4l_2l_5} \\
& + \mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_4}\mathrm{q}_{l_3l_4}\sigma^{+}_{l_1l_5}\sigma^{*}_{l_1l_2l_5}\sigma^{+}_{l_1l_2l_3l_5}\,\mathrm{Y}_{l_1l_5l_2l_3l_4} + \mathrm{q}_{l_4l_2}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_5}\sigma^{*}_{l_1l_4}\sigma^{+}_{l_1l_3l_4}\sigma^{+}_{l_1l_3l_4l_5}\,\mathrm{Y}_{l_1l_4l_3l_5l_2} \\
& + \mathrm{q}_{l_4l_2}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_4}\sigma^{+}_{l_1l_5}\sigma^{+}_{l_1l_4l_5}\sigma^{*}_{l_1l_2l_4l_5}\,\mathrm{Y}_{l_1l_5l_4l_2l_3} + \mathrm{q}_{l_4l_2}\mathrm{q}_{l_3l_5}\sigma^{+}_{l_1l_4}\sigma^{*}_{l_1l_2l_4}\sigma^{+}_{l_1l_2l_3l_4}\,\mathrm{Y}_{l_1l_4l_2l_3l_5} \\
& + \mathrm{q}_{l_5l_4}\mathrm{q}_{l_3l_4}\mathrm{q}_{l_3l_5}\sigma^{*}_{l_1l_2}\sigma^{+}_{l_1l_2l_3}\sigma^{+}_{l_1l_2l_3l_5}\,\mathrm{Y}_{l_1l_2l_3l_5l_4} + \mathrm{q}_{l_4l_2}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_4}\mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_4}\mathrm{q}_{l_3l_5}\sigma^{+}_{l_1l_3}\sigma^{+}_{l_1l_3l_5}\sigma^{+}_{l_1l_3l_4l_5}\,\mathrm{Y}_{l_1l_3l_5l_4l_2}\,.
\end{aligned}
\tag{150}
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\begin{aligned}
C_{l_1l_2l_5l_3l_4} = {} & \sigma^{*}_{l_1l_2}\sigma^{*}_{l_1l_2l_5}\sigma^{*}_{l_1l_2l_3l_5}\,\mathrm{Y}_{l_1l_2l_5l_3l_4} + \mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_5}\mathrm{q}_{l_4l_3}\sigma^{+}_{l_1l_4}\sigma^{*}_{l_1l_2l_4}\sigma^{*}_{l_1l_2l_4l_5}\,\mathrm{Y}_{l_1l_4l_2l_5l_3} \\
& + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_5}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_5}\sigma^{*}_{l_1l_3}\sigma^{+}_{l_1l_3l_4}\sigma^{*}_{l_1l_2l_3l_4}\,\mathrm{Y}_{l_1l_3l_4l_2l_5} + \mathrm{q}_{l_5l_2}\mathrm{q}_{l_3l_2}\mathrm{q}_{l_4l_2}\sigma^{*}_{l_1l_5}\sigma^{*}_{l_1l_3l_5}\sigma^{+}_{l_1l_3l_4l_5}\,\mathrm{Y}_{l_1l_5l_3l_4l_2} \\
& + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_5}\sigma^{+}_{l_1l_3}\sigma^{*}_{l_1l_2l_3}\sigma^{*}_{l_1l_2l_3l_5}\,\mathrm{Y}_{l_1l_3l_2l_5l_4} + \mathrm{q}_{l_5l_2}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_3}\sigma^{*}_{l_1l_5}\sigma^{+}_{l_1l_4l_5}\sigma^{*}_{l_1l_2l_4l_5}\,\mathrm{Y}_{l_1l_5l_4l_2l_3} \\
& + \mathrm{q}_{l_3l_5}\mathrm{q}_{l_4l_5}\sigma^{*}_{l_1l_2}\sigma^{*}_{l_1l_2l_3}\sigma^{+}_{l_1l_2l_3l_4}\,\mathrm{Y}_{l_1l_2l_3l_4l_5} + \mathrm{q}_{l_5l_2}\mathrm{q}_{l_3l_2}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_5}\mathrm{q}_{l_4l_3}\sigma^{+}_{l_1l_4}\sigma^{*}_{l_1l_4l_5}\sigma^{+}_{l_1l_3l_4l_5}\,\mathrm{Y}_{l_1l_4l_5l_3l_2} \\
& + \mathrm{q}_{l_5l_2}\mathrm{q}_{l_3l_2}\sigma^{*}_{l_1l_5}\sigma^{+}_{l_1l_3l_5}\sigma^{*}_{l_1l_2l_3l_5}\,\mathrm{Y}_{l_1l_5l_3l_2l_4} + \mathrm{q}_{l_4l_3}\sigma^{*}_{l_1l_2}\sigma^{*}_{l_1l_2l_5}\sigma^{+}_{l_1l_2l_4l_5}\,\mathrm{Y}_{l_1l_2l_5l_4l_3} \\
& + \mathrm{q}_{l_3l_5}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_5}\mathrm{q}_{l_4l_3}\sigma^{+}_{l_1l_4}\sigma^{*}_{l_1l_2l_4}\sigma^{+}_{l_1l_2l_3l_4}\,\mathrm{Y}_{l_1l_4l_2l_3l_5} + \mathrm{q}_{l_5l_2}\mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_5}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_5}\sigma^{*}_{l_1l_3}\sigma^{+}_{l_1l_3l_4}\sigma^{+}_{l_1l_3l_4l_5}\,\mathrm{Y}_{l_1l_3l_4l_5l_2} \\
& + \mathrm{q}_{l_5l_2}\sigma^{+}_{l_1l_5}\sigma^{*}_{l_1l_2l_5}\sigma^{*}_{l_1l_2l_3l_5}\,\mathrm{Y}_{l_1l_5l_2l_3l_4} + \mathrm{q}_{l_4l_5}\mathrm{q}_{l_4l_3}\sigma^{*}_{l_1l_2}\sigma^{+}_{l_1l_2l_4}\sigma^{*}_{l_1l_2l_4l_5}\,\mathrm{Y}_{l_1l_2l_4l_5l_3} \\
& + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_5}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_5}\mathrm{q}_{l_4l_3}\sigma^{+}_{l_1l_4}\sigma^{+}_{l_1l_3l_4}\sigma^{*}_{l_1l_2l_3l_4}\,\mathrm{Y}_{l_1l_4l_3l_2l_5} + \mathrm{q}_{l_5l_2}\mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_5}\mathrm{q}_{l_4l_2}\sigma^{+}_{l_1l_3}\sigma^{*}_{l_1l_3l_5}\sigma^{+}_{l_1l_3l_4l_5}\,\mathrm{Y}_{l_1l_3l_5l_4l_2} \\
& + \mathrm{q}_{l_3l_5}\sigma^{*}_{l_1l_2}\sigma^{+}_{l_1l_2l_3}\sigma^{*}_{l_1l_2l_3l_5}\,\mathrm{Y}_{l_1l_2l_3l_5l_4} + \mathrm{q}_{l_5l_2}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_5}\mathrm{q}_{l_4l_3}\sigma^{+}_{l_1l_4}\sigma^{+}_{l_1l_4l_5}\sigma^{*}_{l_1l_2l_4l_5}\,\mathrm{Y}_{l_1l_4l_5l_2l_3} \\
& + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_5}\mathrm{q}_{l_4l_5}\sigma^{+}_{l_1l_3}\sigma^{*}_{l_1l_2l_3}\sigma^{+}_{l_1l_2l_3l_4}\,\mathrm{Y}_{l_1l_3l_2l_4l_5} + \mathrm{q}_{l_5l_2}\mathrm{q}_{l_3l_2}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_3}\sigma^{*}_{l_1l_5}\sigma^{+}_{l_1l_4l_5}\sigma^{+}_{l_1l_3l_4l_5}\,\mathrm{Y}_{l_1l_5l_4l_3l_2} \\
& + \mathrm{q}_{l_5l_2}\mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_5}\sigma^{+}_{l_1l_3}\sigma^{+}_{l_1l_3l_5}\sigma^{*}_{l_1l_2l_3l_5}\,\mathrm{Y}_{l_1l_3l_5l_2l_4} + \mathrm{q}_{l_5l_2}\mathrm{q}_{l_4l_3}\sigma^{+}_{l_1l_5}\sigma^{*}_{l_1l_2l_5}\sigma^{+}_{l_1l_2l_4l_5}\,\mathrm{Y}_{l_1l_5l_2l_4l_3} \\
& + \mathrm{q}_{l_3l_5}\mathrm{q}_{l_4l_5}\mathrm{q}_{l_4l_3}\sigma^{*}_{l_1l_2}\sigma^{+}_{l_1l_2l_4}\sigma^{+}_{l_1l_2l_3l_4}\,\mathrm{Y}_{l_1l_2l_4l_3l_5} + \mathrm{q}_{l_5l_2}\mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_5}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_5}\mathrm{q}_{l_4l_3}\sigma^{+}_{l_1l_4}\sigma^{+}_{l_1l_3l_4}\sigma^{+}_{l_1l_3l_4l_5}\,\mathrm{Y}_{l_1l_4l_3l_5l_2}\,.
\end{aligned}
\tag{151}
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\begin{aligned}
C_{l_1l_2l_5l_4l_3} = {}& \sigma^{*}_{l_1l_2}\sigma^{*}_{l_1l_2l_5}\sigma^{*}_{l_1l_2l_4l_5}\,\mathrm{Y}_{l_1l_2l_5l_4l_3} + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_5}\mathrm{q}_{l_3l_4}\sigma^{+}_{l_1l_3}\sigma^{*}_{l_1l_2l_3}\sigma^{*}_{l_1l_2l_3l_5}\,\mathrm{Y}_{l_1l_3l_2l_5l_4} \\
& + \mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_5}\mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_5}\sigma^{*}_{l_1l_4}\sigma^{+}_{l_1l_3l_4}\sigma^{*}_{l_1l_2l_3l_4}\,\mathrm{Y}_{l_1l_4l_3l_2l_5} + \mathrm{q}_{l_5l_2}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_3l_2}\sigma^{*}_{l_1l_5}\sigma^{*}_{l_1l_4l_5}\sigma^{+}_{l_1l_3l_4l_5}\,\mathrm{Y}_{l_1l_5l_4l_3l_2} \\
& + \mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_5}\sigma^{+}_{l_1l_4}\sigma^{*}_{l_1l_2l_4}\sigma^{*}_{l_1l_2l_4l_5}\,\mathrm{Y}_{l_1l_4l_2l_5l_3} + \mathrm{q}_{l_5l_2}\mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_4}\sigma^{*}_{l_1l_5}\sigma^{+}_{l_1l_3l_5}\sigma^{*}_{l_1l_2l_3l_5}\,\mathrm{Y}_{l_1l_5l_3l_2l_4} \\
& + \mathrm{q}_{l_4l_5}\mathrm{q}_{l_3l_5}\sigma^{*}_{l_1l_2}\sigma^{*}_{l_1l_2l_4}\sigma^{+}_{l_1l_2l_3l_4}\,\mathrm{Y}_{l_1l_2l_4l_3l_5} + \mathrm{q}_{l_5l_2}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_5}\mathrm{q}_{l_3l_4}\sigma^{+}_{l_1l_3}\sigma^{*}_{l_1l_3l_5}\sigma^{+}_{l_1l_3l_4l_5}\,\mathrm{Y}_{l_1l_3l_5l_4l_2} \\
& + \mathrm{q}_{l_5l_2}\mathrm{q}_{l_4l_2}\sigma^{*}_{l_1l_5}\sigma^{+}_{l_1l_4l_5}\sigma^{*}_{l_1l_2l_4l_5}\,\mathrm{Y}_{l_1l_5l_4l_2l_3} + \mathrm{q}_{l_3l_4}\sigma^{*}_{l_1l_2}\sigma^{*}_{l_1l_2l_5}\sigma^{+}_{l_1l_2l_3l_5}\,\mathrm{Y}_{l_1l_2l_5l_3l_4} \\
& + \mathrm{q}_{l_4l_5}\mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_5}\mathrm{q}_{l_3l_4}\sigma^{+}_{l_1l_3}\sigma^{*}_{l_1l_2l_3}\sigma^{+}_{l_1l_2l_3l_4}\,\mathrm{Y}_{l_1l_3l_2l_4l_5} + \mathrm{q}_{l_5l_2}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_5}\mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_5}\sigma^{*}_{l_1l_4}\sigma^{+}_{l_1l_3l_4}\sigma^{+}_{l_1l_3l_4l_5}\,\mathrm{Y}_{l_1l_4l_3l_5l_2} \\
& + \mathrm{q}_{l_5l_2}\sigma^{+}_{l_1l_5}\sigma^{*}_{l_1l_2l_5}\sigma^{*}_{l_1l_2l_4l_5}\,\mathrm{Y}_{l_1l_5l_2l_4l_3} + \mathrm{q}_{l_3l_5}\mathrm{q}_{l_3l_4}\sigma^{*}_{l_1l_2}\sigma^{+}_{l_1l_2l_3}\sigma^{*}_{l_1l_2l_3l_5}\,\mathrm{Y}_{l_1l_2l_3l_5l_4} \\
& + \mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_5}\mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_5}\mathrm{q}_{l_3l_4}\sigma^{+}_{l_1l_3}\sigma^{+}_{l_1l_3l_4}\sigma^{*}_{l_1l_2l_3l_4}\,\mathrm{Y}_{l_1l_3l_4l_2l_5} + \mathrm{q}_{l_5l_2}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_5}\mathrm{q}_{l_3l_2}\sigma^{+}_{l_1l_4}\sigma^{*}_{l_1l_4l_5}\sigma^{+}_{l_1l_3l_4l_5}\,\mathrm{Y}_{l_1l_4l_5l_3l_2} \\
& + \mathrm{q}_{l_4l_5}\sigma^{*}_{l_1l_2}\sigma^{+}_{l_1l_2l_4}\sigma^{*}_{l_1l_2l_4l_5}\,\mathrm{Y}_{l_1l_2l_4l_5l_3} + \mathrm{q}_{l_5l_2}\mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_5}\mathrm{q}_{l_3l_4}\sigma^{+}_{l_1l_3}\sigma^{+}_{l_1l_3l_5}\sigma^{*}_{l_1l_2l_3l_5}\,\mathrm{Y}_{l_1l_3l_5l_2l_4} \\
& + \mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_5}\mathrm{q}_{l_3l_5}\sigma^{+}_{l_1l_4}\sigma^{*}_{l_1l_2l_4}\sigma^{+}_{l_1l_2l_3l_4}\,\mathrm{Y}_{l_1l_4l_2l_3l_5} + \mathrm{q}_{l_5l_2}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_4}\sigma^{*}_{l_1l_5}\sigma^{+}_{l_1l_3l_5}\sigma^{+}_{l_1l_3l_4l_5}\,\mathrm{Y}_{l_1l_5l_3l_4l_2} \\
& + \mathrm{q}_{l_5l_2}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_5}\sigma^{+}_{l_1l_4}\sigma^{+}_{l_1l_4l_5}\sigma^{*}_{l_1l_2l_4l_5}\,\mathrm{Y}_{l_1l_4l_5l_2l_3} + \mathrm{q}_{l_5l_2}\mathrm{q}_{l_3l_4}\sigma^{+}_{l_1l_5}\sigma^{*}_{l_1l_2l_5}\sigma^{+}_{l_1l_2l_3l_5}\,\mathrm{Y}_{l_1l_5l_2l_3l_4} \\
& + \mathrm{q}_{l_4l_5}\mathrm{q}_{l_3l_5}\mathrm{q}_{l_3l_4}\sigma^{*}_{l_1l_2}\sigma^{+}_{l_1l_2l_3}\sigma^{+}_{l_1l_2l_3l_4}\,\mathrm{Y}_{l_1l_2l_3l_4l_5} + \mathrm{q}_{l_5l_2}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_5}\mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_5}\mathrm{q}_{l_3l_4}\sigma^{+}_{l_1l_3}\sigma^{+}_{l_1l_3l_4}\sigma^{+}_{l_1l_3l_4l_5}\,\mathrm{Y}_{l_1l_3l_4l_5l_2} \,.
\end{aligned}
\tag{152}
$$

Zapišimo sada bazične konstante dane identitetima (147) - (152) u leksikografskom poretku njihovih članova:

$$
\begin{aligned}
C_{l_1l_2l_3l_4l_5} = {}& \sigma^{*}_{l_1l_2}\sigma^{*}_{l_1l_2l_3}\sigma^{*}_{l_1l_2l_3l_4}\,\mathrm{Y}_{l_1l_2l_3l_4l_5} + \mathrm{q}_{l_5l_4}\sigma^{*}_{l_1l_2}\sigma^{*}_{l_1l_2l_3}\sigma^{+}_{l_1l_2l_3l_5}\,\mathrm{Y}_{l_1l_2l_3l_5l_4} \\
& + \mathrm{q}_{l_4l_3}\sigma^{*}_{l_1l_2}\sigma^{+}_{l_1l_2l_4}\sigma^{*}_{l_1l_2l_3l_4}\,\mathrm{Y}_{l_1l_2l_4l_3l_5} + \mathrm{q}_{l_4l_3}\mathrm{q}_{l_5l_3}\sigma^{*}_{l_1l_2}\sigma^{*}_{l_1l_2l_4}\sigma^{+}_{l_1l_2l_4l_5}\,\mathrm{Y}_{l_1l_2l_4l_5l_3} \\
& + \mathrm{q}_{l_5l_3}\mathrm{q}_{l_5l_4}\sigma^{*}_{l_1l_2}\sigma^{+}_{l_1l_2l_5}\sigma^{*}_{l_1l_2l_3l_5}\,\mathrm{Y}_{l_1l_2l_5l_3l_4} + \mathrm{q}_{l_4l_3}\mathrm{q}_{l_5l_3}\mathrm{q}_{l_5l_4}\sigma^{*}_{l_1l_2}\sigma^{+}_{l_1l_2l_5}\sigma^{+}_{l_1l_2l_4l_5}\,\mathrm{Y}_{l_1l_2l_5l_4l_3} \\
& + \mathrm{q}_{l_3l_2}\sigma^{+}_{l_1l_3}\sigma^{*}_{l_1l_2l_3}\sigma^{*}_{l_1l_2l_3l_4}\,\mathrm{Y}_{l_1l_3l_2l_4l_5} + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_5l_4}\sigma^{+}_{l_1l_3}\sigma^{*}_{l_1l_2l_3}\sigma^{+}_{l_1l_2l_3l_5}\,\mathrm{Y}_{l_1l_3l_2l_5l_4} \\
& + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_4l_2}\sigma^{*}_{l_1l_3}\sigma^{+}_{l_1l_3l_4}\sigma^{*}_{l_1l_2l_3l_4}\,\mathrm{Y}_{l_1l_3l_4l_2l_5} + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_5l_2}\sigma^{*}_{l_1l_3}\sigma^{*}_{l_1l_3l_4}\sigma^{+}_{l_1l_3l_4l_5}\,\mathrm{Y}_{l_1l_3l_4l_5l_2} \\
& + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_4}\sigma^{*}_{l_1l_3}\sigma^{+}_{l_1l_3l_5}\sigma^{*}_{l_1l_2l_3l_5}\,\mathrm{Y}_{l_1l_3l_5l_2l_4} + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_4}\sigma^{*}_{l_1l_3}\sigma^{+}_{l_1l_3l_5}\sigma^{+}_{l_1l_3l_4l_5}\,\mathrm{Y}_{l_1l_3l_5l_4l_2} \\
& + \mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_3}\sigma^{+}_{l_1l_4}\sigma^{*}_{l_1l_2l_4}\sigma^{*}_{l_1l_2l_3l_4}\,\mathrm{Y}_{l_1l_4l_2l_3l_5} + \mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_3}\mathrm{q}_{l_5l_3}\sigma^{+}_{l_1l_4}\sigma^{*}_{l_1l_2l_4}\sigma^{+}_{l_1l_2l_4l_5}\,\mathrm{Y}_{l_1l_4l_2l_5l_3} \\
& + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_3}\sigma^{+}_{l_1l_4}\sigma^{+}_{l_1l_3l_4}\sigma^{*}_{l_1l_2l_3l_4}\,\mathrm{Y}_{l_1l_4l_3l_2l_5} + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_3}\mathrm{q}_{l_5l_2}\sigma^{+}_{l_1l_4}\sigma^{*}_{l_1l_3l_4}\sigma^{+}_{l_1l_3l_4l_5}\,\mathrm{Y}_{l_1l_4l_3l_5l_2} \\
& + \mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_3}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_3}\sigma^{*}_{l_1l_4}\sigma^{+}_{l_1l_4l_5}\sigma^{*}_{l_1l_2l_4l_5}\,\mathrm{Y}_{l_1l_4l_5l_2l_3} + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_3}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_3}\sigma^{*}_{l_1l_4}\sigma^{+}_{l_1l_4l_5}\sigma^{+}_{l_1l_3l_4l_5}\,\mathrm{Y}_{l_1l_4l_5l_3l_2} \\
& + \mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_3}\mathrm{q}_{l_5l_4}\sigma^{+}_{l_1l_5}\sigma^{*}_{l_1l_2l_5}\sigma^{*}_{l_1l_2l_3l_5}\,\mathrm{Y}_{l_1l_5l_2l_3l_4} + \mathrm{q}_{l_4l_3}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_3}\mathrm{q}_{l_5l_4}\sigma^{+}_{l_1l_5}\sigma^{*}_{l_1l_2l_5}\sigma^{+}_{l_1l_2l_4l_5}\,\mathrm{Y}_{l_1l_5l_2l_4l_3} \\
& + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_3}\mathrm{q}_{l_5l_4}\sigma^{+}_{l_1l_5}\sigma^{+}_{l_1l_3l_5}\sigma^{*}_{l_1l_2l_3l_5}\,\mathrm{Y}_{l_1l_5l_3l_2l_4} + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_3}\mathrm{q}_{l_5l_4}\sigma^{+}_{l_1l_5}\sigma^{*}_{l_1l_3l_5}\sigma^{+}_{l_1l_3l_4l_5}\,\mathrm{Y}_{l_1l_5l_3l_4l_2} \\
& + \mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_3}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_3}\mathrm{q}_{l_5l_4}\sigma^{+}_{l_1l_5}\sigma^{+}_{l_1l_4l_5}\sigma^{*}_{l_1l_2l_4l_5}\,\mathrm{Y}_{l_1l_5l_4l_2l_3} + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_3}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_3}\mathrm{q}_{l_5l_4}\sigma^{+}_{l_1l_5}\sigma^{+}_{l_1l_4l_5}\sigma^{+}_{l_1l_3l_4l_5}\,\mathrm{Y}_{l_1l_5l_4l_3l_2}
\end{aligned}
$$

$$
\begin{aligned}
C_{l_1l_2l_3l_5l_4} = {} & q_{l_4l_5}\sigma^{*}_{l_1l_2}\sigma^{*}_{l_1l_2l_3}\sigma^{+}_{l_1l_2l_3l_4}\ Y_{l_1l_2l_3l_4l_5} + \sigma^{*}_{l_1l_2}\sigma^{*}_{l_1l_2l_3}\sigma^{*}_{l_1l_2l_3l_5}\ Y_{l_1l_2l_3l_5l_4} \\
& + q_{l_4l_3}q_{l_4l_5}\sigma^{*}_{l_1l_2}\sigma^{+}_{l_1l_2l_4}\sigma^{*}_{l_1l_2l_3l_4}\ Y_{l_1l_2l_4l_3l_5} + q_{l_4l_3}q_{l_4l_5}q_{l_5l_3}\sigma^{*}_{l_1l_2}\sigma^{+}_{l_1l_2l_4}\sigma^{+}_{l_1l_2l_4l_5}\ Y_{l_1l_2l_4l_5l_3} \\
& + q_{l_5l_3}\sigma^{*}_{l_1l_2}\sigma^{+}_{l_1l_2l_5}\sigma^{*}_{l_1l_2l_3l_5}\ Y_{l_1l_2l_5l_3l_4} + q_{l_4l_3}q_{l_5l_3}\sigma^{*}_{l_1l_2}\sigma^{*}_{l_1l_2l_5}\sigma^{+}_{l_1l_2l_4l_5}\ Y_{l_1l_2l_5l_4l_3} \\
& + q_{l_3l_2}q_{l_4l_5}\sigma^{+}_{l_1l_3}\sigma^{*}_{l_1l_2l_3}\sigma^{+}_{l_1l_2l_3l_4}\ Y_{l_1l_3l_2l_4l_5} + q_{l_3l_2}\sigma^{+}_{l_1l_3}\sigma^{*}_{l_1l_2l_3}\sigma^{*}_{l_1l_2l_3l_5}\ Y_{l_1l_3l_2l_5l_4} \\
& + q_{l_3l_2}q_{l_4l_2}q_{l_4l_5}\sigma^{*}_{l_1l_3}\sigma^{+}_{l_1l_3l_4}\sigma^{*}_{l_1l_2l_3l_4}\ Y_{l_1l_3l_4l_2l_5} + q_{l_3l_2}q_{l_4l_2}q_{l_4l_5}q_{l_5l_2}\sigma^{*}_{l_1l_3}\sigma^{+}_{l_1l_3l_4}\sigma^{+}_{l_1l_3l_4l_5}\ Y_{l_1l_3l_4l_5l_2} \\
& + q_{l_3l_2}q_{l_5l_2}\sigma^{*}_{l_1l_3}\sigma^{+}_{l_1l_3l_5}\sigma^{*}_{l_1l_2l_3l_5}\ Y_{l_1l_3l_5l_2l_4} + q_{l_3l_2}q_{l_4l_2}q_{l_5l_2}\sigma^{*}_{l_1l_3}\sigma^{*}_{l_1l_3l_5}\sigma^{+}_{l_1l_3l_4l_5}\ Y_{l_1l_3l_5l_4l_2} \\
& + q_{l_4l_2}q_{l_4l_3}q_{l_4l_5}\sigma^{+}_{l_1l_4}\sigma^{*}_{l_1l_2l_4}\sigma^{*}_{l_1l_2l_3l_4}\ Y_{l_1l_4l_2l_3l_5} + q_{l_4l_2}q_{l_4l_3}q_{l_4l_5}q_{l_5l_3}\sigma^{+}_{l_1l_4}\sigma^{*}_{l_1l_2l_4}\sigma^{+}_{l_1l_2l_4l_5}\ Y_{l_1l_4l_2l_5l_3} \\
& + q_{l_3l_2}q_{l_4l_2}q_{l_4l_3}q_{l_4l_5}\sigma^{+}_{l_1l_4}\sigma^{+}_{l_1l_3l_4}\sigma^{*}_{l_1l_2l_3l_4}\ Y_{l_1l_4l_3l_2l_5} + q_{l_3l_2}q_{l_4l_2}q_{l_4l_3}q_{l_4l_5}q_{l_5l_2}\sigma^{+}_{l_1l_4}\sigma^{*}_{l_1l_3l_4}\sigma^{+}_{l_1l_3l_4l_5}\ Y_{l_1l_4l_3l_5l_2} \\
& + q_{l_4l_2}q_{l_4l_3}q_{l_4l_5}q_{l_5l_2}q_{l_5l_3}\sigma^{+}_{l_1l_4}\sigma^{+}_{l_1l_4l_5}\sigma^{*}_{l_1l_2l_4l_5}\ Y_{l_1l_4l_5l_2l_3} + q_{l_3l_2}q_{l_4l_2}q_{l_4l_3}q_{l_4l_5}q_{l_5l_2}q_{l_5l_3}\sigma^{+}_{l_1l_4}\sigma^{+}_{l_1l_4l_5}\sigma^{+}_{l_1l_3l_4l_5}\ Y_{l_1l_4l_5l_3l_2} \\
& + q_{l_5l_2}q_{l_5l_3}\sigma^{+}_{l_1l_5}\sigma^{*}_{l_1l_2l_5}\sigma^{*}_{l_1l_2l_3l_5}\ Y_{l_1l_5l_2l_3l_4} + q_{l_4l_3}q_{l_5l_2}q_{l_5l_3}\sigma^{+}_{l_1l_5}\sigma^{*}_{l_1l_2l_5}\sigma^{+}_{l_1l_2l_4l_5}\ Y_{l_1l_5l_2l_4l_3} \\
& + q_{l_3l_2}q_{l_5l_2}q_{l_5l_3}\sigma^{+}_{l_1l_5}\sigma^{+}_{l_1l_3l_5}\sigma^{*}_{l_1l_2l_3l_5}\ Y_{l_1l_5l_3l_2l_4} + q_{l_3l_2}q_{l_4l_2}q_{l_5l_2}q_{l_5l_3}\sigma^{+}_{l_1l_5}\sigma^{*}_{l_1l_3l_5}\sigma^{+}_{l_1l_3l_4l_5}\ Y_{l_1l_5l_3l_4l_2} \\
& + q_{l_4l_2}q_{l_4l_3}q_{l_5l_2}q_{l_5l_3}\sigma^{*}_{l_1l_5}\sigma^{+}_{l_1l_4l_5}\sigma^{*}_{l_1l_2l_4l_5}\ Y_{l_1l_5l_4l_2l_3} + q_{l_3l_2}q_{l_4l_2}q_{l_4l_3}q_{l_5l_2}q_{l_5l_3}\sigma^{*}_{l_1l_5}\sigma^{+}_{l_1l_4l_5}\sigma^{+}_{l_1l_3l_4l_5}\ Y_{l_1l_5l_4l_3l_2}
\end{aligned}
$$

$$
\begin{aligned}
C_{l_1l_2l_4l_3l_5} = {} & q_{l_3l_4}\sigma^{*}_{l_1l_2}\sigma^{+}_{l_1l_2l_3}\sigma^{*}_{l_1l_2l_3l_4}\ Y_{l_1l_2l_3l_4l_5} + q_{l_3l_4}q_{l_5l_4}\sigma^{*}_{l_1l_2}\sigma^{*}_{l_1l_2l_3}\sigma^{+}_{l_1l_2l_3l_5}\ Y_{l_1l_2l_3l_5l_4} \\
& + \sigma^{*}_{l_1l_2}\sigma^{*}_{l_1l_2l_4}\sigma^{*}_{l_1l_2l_3l_4}\ Y_{l_1l_2l_4l_3l_5} + q_{l_5l_3}\sigma^{*}_{l_1l_2}\sigma^{*}_{l_1l_2l_4}\sigma^{+}_{l_1l_2l_4l_5}\ Y_{l_1l_2l_4l_5l_3} \\
& + q_{l_3l_4}q_{l_5l_3}q_{l_5l_4}\sigma^{*}_{l_1l_2}\sigma^{+}_{l_1l_2l_5}\sigma^{+}_{l_1l_2l_3l_5}\ Y_{l_1l_2l_5l_3l_4} + q_{l_5l_3}q_{l_5l_4}\sigma^{*}_{l_1l_2}\sigma^{+}_{l_1l_2l_5}\sigma^{*}_{l_1l_2l_4l_5}\ Y_{l_1l_2l_5l_4l_3} \\
& + q_{l_3l_2}q_{l_3l_4}\sigma^{+}_{l_1l_3}\sigma^{*}_{l_1l_2l_3}\sigma^{*}_{l_1l_2l_3l_4}\ Y_{l_1l_3l_2l_4l_5} + q_{l_3l_2}q_{l_3l_4}q_{l_5l_4}\sigma^{+}_{l_1l_3}\sigma^{*}_{l_1l_2l_3}\sigma^{+}_{l_1l_2l_3l_5}\ Y_{l_1l_3l_2l_5l_4} \\
& + q_{l_3l_2}q_{l_3l_4}q_{l_4l_2}\sigma^{+}_{l_1l_3}\sigma^{+}_{l_1l_3l_4}\sigma^{*}_{l_1l_2l_3l_4}\ Y_{l_1l_3l_4l_2l_5} + q_{l_3l_2}q_{l_3l_4}q_{l_4l_2}q_{l_5l_2}\sigma^{+}_{l_1l_3}\sigma^{*}_{l_1l_3l_4}\sigma^{+}_{l_1l_3l_4l_5}\ Y_{l_1l_3l_4l_5l_2} \\
& + q_{l_3l_2}q_{l_3l_4}q_{l_5l_2}q_{l_5l_4}\sigma^{*}_{l_1l_3}\sigma^{+}_{l_1l_3l_5}\sigma^{*}_{l_1l_2l_3l_5}\ Y_{l_1l_3l_5l_2l_4} + q_{l_3l_2}q_{l_3l_4}q_{l_4l_2}q_{l_5l_2}q_{l_5l_4}\sigma^{*}_{l_1l_3}\sigma^{+}_{l_1l_3l_5}\sigma^{+}_{l_1l_3l_4l_5}\ Y_{l_1l_3l_5l_4l_2} \\
& + q_{l_4l_2}\sigma^{+}_{l_1l_4}\sigma^{*}_{l_1l_2l_4}\sigma^{*}_{l_1l_2l_3l_4}\ Y_{l_1l_4l_2l_3l_5} + q_{l_4l_2}q_{l_5l_3}\sigma^{+}_{l_1l_4}\sigma^{*}_{l_1l_2l_4}\sigma^{+}_{l_1l_2l_4l_5}\ Y_{l_1l_4l_2l_5l_3} \\
& + q_{l_3l_2}q_{l_4l_2}\sigma^{*}_{l_1l_4}\sigma^{+}_{l_1l_3l_4}\sigma^{*}_{l_1l_2l_3l_4}\ Y_{l_1l_4l_3l_2l_5} + q_{l_3l_2}q_{l_4l_2}q_{l_5l_2}\sigma^{*}_{l_1l_4}\sigma^{*}_{l_1l_3l_4}\sigma^{+}_{l_1l_3l_4l_5}\ Y_{l_1l_4l_3l_5l_2} \\
& + q_{l_4l_2}q_{l_5l_2}q_{l_5l_3}\sigma^{*}_{l_1l_4}\sigma^{+}_{l_1l_4l_5}\sigma^{*}_{l_1l_2l_4l_5}\ Y_{l_1l_4l_5l_2l_3} + q_{l_3l_2}q_{l_4l_2}q_{l_5l_2}q_{l_5l_3}\sigma^{*}_{l_1l_4}\sigma^{+}_{l_1l_4l_5}\sigma^{+}_{l_1l_3l_4l_5}\ Y_{l_1l_4l_5l_3l_2} \\
& + q_{l_3l_4}q_{l_5l_2}q_{l_5l_3}q_{l_5l_4}\sigma^{+}_{l_1l_5}\sigma^{*}_{l_1l_2l_5}\sigma^{+}_{l_1l_2l_3l_5}\ Y_{l_1l_5l_2l_3l_4} + q_{l_5l_2}q_{l_5l_3}q_{l_5l_4}\sigma^{+}_{l_1l_5}\sigma^{*}_{l_1l_2l_5}\sigma^{*}_{l_1l_2l_4l_5}\ Y_{l_1l_5l_2l_4l_3} \\
& + q_{l_3l_2}q_{l_3l_4}q_{l_5l_2}q_{l_5l_3}q_{l_5l_4}\sigma^{+}_{l_1l_5}\sigma^{+}_{l_1l_3l_5}\sigma^{*}_{l_1l_2l_3l_5}\ Y_{l_1l_5l_3l_2l_4} + q_{l_3l_2}q_{l_3l_4}q_{l_4l_2}q_{l_5l_2}q_{l_5l_3}q_{l_5l_4}\sigma^{+}_{l_1l_5}\sigma^{+}_{l_1l_3l_5}\sigma^{+}_{l_1l_3l_4l_5}\ Y_{l_1l_5l_3l_4l_2} \\
& + q_{l_4l_2}q_{l_5l_2}q_{l_5l_3}q_{l_5l_4}\sigma^{+}_{l_1l_5}\sigma^{+}_{l_1l_4l_5}\sigma^{*}_{l_1l_2l_4l_5}\ Y_{l_1l_5l_4l_2l_3} + q_{l_3l_2}q_{l_4l_2}q_{l_5l_2}q_{l_5l_3}q_{l_5l_4}\sigma^{+}_{l_1l_5}\sigma^{*}_{l_1l_4l_5}\sigma^{+}_{l_1l_3l_4l_5}\ Y_{l_1l_5l_4l_3l_2}
\end{aligned}
$$

$$
\begin{aligned}
C_{l_1l_2l_4l_5l_3} = {} & \mathrm{q}_{l_3l_4}\mathrm{q}_{l_3l_5}\sigma^{*}_{l_1l_2}\sigma^{+}_{l_1l_2l_3}\sigma^{*}_{l_1l_2l_3l_4}\ \mathrm{Y}_{l_1l_2l_3l_4l_5} + \mathrm{q}_{l_3l_4}\mathrm{q}_{l_3l_5}\mathrm{q}_{l_5l_4}\sigma^{*}_{l_1l_2}\sigma^{+}_{l_1l_2l_3}\sigma^{+}_{l_1l_2l_3l_5}\ \mathrm{Y}_{l_1l_2l_3l_5l_4} \\
& + \mathrm{q}_{l_3l_5}\sigma^{*}_{l_1l_2}\sigma^{*}_{l_1l_2l_4}\sigma^{+}_{l_1l_2l_3l_4}\ \mathrm{Y}_{l_1l_2l_4l_3l_5} + \sigma^{*}_{l_1l_2}\sigma^{*}_{l_1l_2l_4}\sigma^{*}_{l_1l_2l_4l_5}\ \mathrm{Y}_{l_1l_2l_4l_5l_3} \\
& + \mathrm{q}_{l_3l_4}\mathrm{q}_{l_5l_4}\sigma^{*}_{l_1l_2}\sigma^{*}_{l_1l_2l_5}\sigma^{+}_{l_1l_2l_3l_5}\ \mathrm{Y}_{l_1l_2l_5l_3l_4} + \mathrm{q}_{l_5l_4}\sigma^{*}_{l_1l_2}\sigma^{+}_{l_1l_2l_5}\sigma^{*}_{l_1l_2l_4l_5}\ \mathrm{Y}_{l_1l_2l_5l_4l_3} \\
& + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_4}\mathrm{q}_{l_3l_5}\sigma^{+}_{l_1l_3}\sigma^{*}_{l_1l_2l_3}\sigma^{*}_{l_1l_2l_3l_4}\ \mathrm{Y}_{l_1l_3l_2l_4l_5} + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_4}\mathrm{q}_{l_3l_5}\mathrm{q}_{l_5l_4}\sigma^{+}_{l_1l_3}\sigma^{*}_{l_1l_2l_3}\sigma^{+}_{l_1l_2l_3l_5}\ \mathrm{Y}_{l_1l_3l_2l_5l_4} \\
& + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_4}\mathrm{q}_{l_3l_5}\mathrm{q}_{l_4l_2}\sigma^{+}_{l_1l_3}\sigma^{+}_{l_1l_3l_4}\sigma^{*}_{l_1l_2l_3l_4}\ \mathrm{Y}_{l_1l_3l_4l_2l_5} + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_4}\mathrm{q}_{l_3l_5}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_5l_2}\sigma^{+}_{l_1l_3}\sigma^{*}_{l_1l_3l_4}\sigma^{+}_{l_1l_3l_4l_5}\ \mathrm{Y}_{l_1l_3l_4l_5l_2} \\
& + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_4}\mathrm{q}_{l_3l_5}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_4}\sigma^{+}_{l_1l_3}\sigma^{+}_{l_1l_3l_5}\sigma^{*}_{l_1l_2l_3l_5}\ \mathrm{Y}_{l_1l_3l_5l_2l_4} + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_4}\mathrm{q}_{l_3l_5}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_4}\sigma^{+}_{l_1l_3}\sigma^{+}_{l_1l_3l_5}\sigma^{+}_{l_1l_3l_4l_5}\ \mathrm{Y}_{l_1l_3l_5l_4l_2} \\
& + \mathrm{q}_{l_3l_5}\mathrm{q}_{l_4l_2}\sigma^{+}_{l_1l_4}\sigma^{*}_{l_1l_2l_4}\sigma^{+}_{l_1l_2l_3l_4}\ \mathrm{Y}_{l_1l_4l_2l_3l_5} + \mathrm{q}_{l_4l_2}\sigma^{+}_{l_1l_4}\sigma^{*}_{l_1l_2l_4}\sigma^{*}_{l_1l_2l_4l_5}\ \mathrm{Y}_{l_1l_4l_2l_5l_3} \\
& + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_5}\mathrm{q}_{l_4l_2}\sigma^{*}_{l_1l_4}\sigma^{+}_{l_1l_3l_4}\sigma^{*}_{l_1l_2l_3l_4}\ \mathrm{Y}_{l_1l_4l_3l_2l_5} + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_5}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_5l_2}\sigma^{*}_{l_1l_4}\sigma^{+}_{l_1l_3l_4}\sigma^{+}_{l_1l_3l_4l_5}\ \mathrm{Y}_{l_1l_4l_3l_5l_2} \\
& + \mathrm{q}_{l_4l_2}\mathrm{q}_{l_5l_2}\sigma^{*}_{l_1l_4}\sigma^{+}_{l_1l_4l_5}\sigma^{*}_{l_1l_2l_4l_5}\ \mathrm{Y}_{l_1l_4l_5l_2l_3} + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_5l_2}\sigma^{*}_{l_1l_4}\sigma^{*}_{l_1l_4l_5}\sigma^{+}_{l_1l_3l_4l_5}\ \mathrm{Y}_{l_1l_4l_5l_3l_2} \\
& + \mathrm{q}_{l_3l_4}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_4}\sigma^{+}_{l_1l_5}\sigma^{*}_{l_1l_2l_5}\sigma^{+}_{l_1l_2l_3l_5}\ \mathrm{Y}_{l_1l_5l_2l_3l_4} + \mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_4}\sigma^{+}_{l_1l_5}\sigma^{*}_{l_1l_2l_5}\sigma^{*}_{l_1l_2l_4l_5}\ \mathrm{Y}_{l_1l_5l_2l_4l_3} \\
& + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_4}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_4}\sigma^{*}_{l_1l_5}\sigma^{+}_{l_1l_3l_5}\sigma^{*}_{l_1l_2l_3l_5}\ \mathrm{Y}_{l_1l_5l_3l_2l_4} + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_4}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_4}\sigma^{*}_{l_1l_5}\sigma^{+}_{l_1l_3l_5}\sigma^{+}_{l_1l_3l_4l_5}\ \mathrm{Y}_{l_1l_5l_3l_4l_2} \\
& + \mathrm{q}_{l_4l_2}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_4}\sigma^{+}_{l_1l_5}\sigma^{+}_{l_1l_4l_5}\sigma^{*}_{l_1l_2l_4l_5}\ \mathrm{Y}_{l_1l_5l_4l_2l_3} + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_5l_2}\mathrm{q}_{l_5l_4}\sigma^{+}_{l_1l_5}\sigma^{*}_{l_1l_4l_5}\sigma^{+}_{l_1l_3l_4l_5}\ \mathrm{Y}_{l_1l_5l_4l_3l_2}
\end{aligned}
$$

$$
\begin{aligned}
C_{l_1l_2l_5l_3l_4} = {} & \mathrm{q}_{l_3l_5}\mathrm{q}_{l_4l_5}\sigma^{*}_{l_1l_2}\sigma^{*}_{l_1l_2l_3}\sigma^{+}_{l_1l_2l_3l_4}\ \mathrm{Y}_{l_1l_2l_3l_4l_5} + \mathrm{q}_{l_3l_5}\sigma^{*}_{l_1l_2}\sigma^{+}_{l_1l_2l_3}\sigma^{*}_{l_1l_2l_3l_5}\ \mathrm{Y}_{l_1l_2l_3l_5l_4} \\
& + \mathrm{q}_{l_3l_5}\mathrm{q}_{l_4l_3}\mathrm{q}_{l_4l_5}\sigma^{*}_{l_1l_2}\sigma^{+}_{l_1l_2l_4}\sigma^{+}_{l_1l_2l_3l_4}\ \mathrm{Y}_{l_1l_2l_4l_3l_5} + \mathrm{q}_{l_4l_3}\mathrm{q}_{l_4l_5}\sigma^{*}_{l_1l_2}\sigma^{+}_{l_1l_2l_4}\sigma^{*}_{l_1l_2l_4l_5}\ \mathrm{Y}_{l_1l_2l_4l_5l_3} \\
& + \sigma^{*}_{l_1l_2}\sigma^{*}_{l_1l_2l_5}\sigma^{*}_{l_1l_2l_3l_5}\ \mathrm{Y}_{l_1l_2l_5l_3l_4} + \mathrm{q}_{l_4l_3}\sigma^{*}_{l_1l_2}\sigma^{*}_{l_1l_2l_5}\sigma^{+}_{l_1l_2l_4l_5}\ \mathrm{Y}_{l_1l_2l_5l_4l_3} \\
& + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_5}\mathrm{q}_{l_4l_5}\sigma^{+}_{l_1l_3}\sigma^{*}_{l_1l_2l_3}\sigma^{+}_{l_1l_2l_3l_4}\ \mathrm{Y}_{l_1l_3l_2l_4l_5} + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_5}\sigma^{+}_{l_1l_3}\sigma^{*}_{l_1l_2l_3}\sigma^{*}_{l_1l_2l_3l_5}\ \mathrm{Y}_{l_1l_3l_2l_5l_4} \\
& + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_5}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_5}\sigma^{*}_{l_1l_3}\sigma^{+}_{l_1l_3l_4}\sigma^{*}_{l_1l_2l_3l_4}\ \mathrm{Y}_{l_1l_3l_4l_2l_5} + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_5}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_5}\mathrm{q}_{l_5l_2}\sigma^{*}_{l_1l_3}\sigma^{+}_{l_1l_3l_4}\sigma^{+}_{l_1l_3l_4l_5}\ \mathrm{Y}_{l_1l_3l_4l_5l_2} \\
& + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_5}\mathrm{q}_{l_5l_2}\sigma^{+}_{l_1l_3}\sigma^{+}_{l_1l_3l_5}\sigma^{*}_{l_1l_2l_3l_5}\ \mathrm{Y}_{l_1l_3l_5l_2l_4} + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_5}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_5l_2}\sigma^{+}_{l_1l_3}\sigma^{*}_{l_1l_3l_5}\sigma^{+}_{l_1l_3l_4l_5}\ \mathrm{Y}_{l_1l_3l_5l_4l_2} \\
& + \mathrm{q}_{l_3l_5}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_3}\mathrm{q}_{l_4l_5}\sigma^{+}_{l_1l_4}\sigma^{*}_{l_1l_2l_4}\sigma^{+}_{l_1l_2l_3l_4}\ \mathrm{Y}_{l_1l_4l_2l_3l_5} + \mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_3}\mathrm{q}_{l_4l_5}\sigma^{+}_{l_1l_4}\sigma^{*}_{l_1l_2l_4}\sigma^{*}_{l_1l_2l_4l_5}\ \mathrm{Y}_{l_1l_4l_2l_5l_3} \\
& + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_5}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_3}\mathrm{q}_{l_4l_5}\sigma^{+}_{l_1l_4}\sigma^{+}_{l_1l_3l_4}\sigma^{*}_{l_1l_2l_3l_4}\ \mathrm{Y}_{l_1l_4l_3l_2l_5} + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_3l_5}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_3}\mathrm{q}_{l_4l_5}\mathrm{q}_{l_5l_2}\sigma^{+}_{l_1l_4}\sigma^{+}_{l_1l_3l_4}\sigma^{+}_{l_1l_3l_4l_5}\ \mathrm{Y}_{l_1l_4l_3l_5l_2} \\
& + \mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_3}\mathrm{q}_{l_4l_5}\mathrm{q}_{l_5l_2}\sigma^{+}_{l_1l_4}\sigma^{+}_{l_1l_4l_5}\sigma^{*}_{l_1l_2l_4l_5}\ \mathrm{Y}_{l_1l_4l_5l_2l_3} + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_3}\mathrm{q}_{l_4l_5}\mathrm{q}_{l_5l_2}\sigma^{+}_{l_1l_4}\sigma^{*}_{l_1l_4l_5}\sigma^{+}_{l_1l_3l_4l_5}\ \mathrm{Y}_{l_1l_4l_5l_3l_2} \\
& + \mathrm{q}_{l_5l_2}\sigma^{+}_{l_1l_5}\sigma^{*}_{l_1l_2l_5}\sigma^{*}_{l_1l_2l_3l_5}\ \mathrm{Y}_{l_1l_5l_2l_3l_4} + \mathrm{q}_{l_4l_3}\mathrm{q}_{l_5l_2}\sigma^{+}_{l_1l_5}\sigma^{*}_{l_1l_2l_5}\sigma^{+}_{l_1l_2l_4l_5}\ \mathrm{Y}_{l_1l_5l_2l_4l_3} \\
& + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_5l_2}\sigma^{*}_{l_1l_5}\sigma^{+}_{l_1l_3l_5}\sigma^{*}_{l_1l_2l_3l_5}\ \mathrm{Y}_{l_1l_5l_3l_2l_4} + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_5l_2}\sigma^{*}_{l_1l_5}\sigma^{*}_{l_1l_3l_5}\sigma^{+}_{l_1l_3l_4l_5}\ \mathrm{Y}_{l_1l_5l_3l_4l_2} \\
& + \mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_3}\mathrm{q}_{l_5l_2}\sigma^{*}_{l_1l_5}\sigma^{+}_{l_1l_4l_5}\sigma^{*}_{l_1l_2l_4l_5}\ \mathrm{Y}_{l_1l_5l_4l_2l_3} + \mathrm{q}_{l_3l_2}\mathrm{q}_{l_4l_2}\mathrm{q}_{l_4l_3}\mathrm{q}_{l_5l_2}\sigma^{*}_{l_1l_5}\sigma^{+}_{l_1l_4l_5}\sigma^{+}_{l_1l_3l_4l_5}\ \mathrm{Y}_{l_1l_5l_4l_3l_2}
\end{aligned}
$$

$$
\begin{aligned}
C_{l_1l_2l_5l_4l_3} = {} & q_{l_3l_4} q_{l_3l_5} q_{l_4l_5} \sigma^*_{l_1l_2} \sigma^+_{l_1l_2l_3} \sigma^+_{l_1l_2l_3l_4} \, Y_{l_1l_2l_3l_4l_5} + q_{l_3l_4} q_{l_3l_5} \sigma^*_{l_1l_2} \sigma^+_{l_1l_2l_3} \sigma^*_{l_1l_2l_3l_5} \, Y_{l_1l_2l_3l_5l_4} \\
& + q_{l_3l_5} q_{l_4l_5} \sigma^*_{l_1l_2} \sigma^*_{l_1l_2l_4} \sigma^+_{l_1l_2l_3l_4} \, Y_{l_1l_2l_4l_3l_5} + q_{l_4l_5} \sigma^*_{l_1l_2} \sigma^+_{l_1l_2l_4} \sigma^*_{l_1l_2l_4l_5} \, Y_{l_1l_2l_4l_5l_3} \\
& + q_{l_3l_4} \sigma^*_{l_1l_2} \sigma^*_{l_1l_2l_5} \sigma^+_{l_1l_2l_3l_5} \, Y_{l_1l_2l_5l_3l_4} + \sigma^*_{l_1l_2} \sigma^*_{l_1l_2l_5} \sigma^*_{l_1l_2l_4l_5} \, Y_{l_1l_2l_5l_4l_3} \\
& + q_{l_3l_2} q_{l_3l_4} q_{l_3l_5} q_{l_4l_5} \sigma^+_{l_1l_3} \sigma^*_{l_1l_2l_3} \sigma^+_{l_1l_2l_3l_4} \, Y_{l_1l_3l_2l_4l_5} + q_{l_3l_2} q_{l_3l_4} q_{l_3l_5} \sigma^+_{l_1l_3} \sigma^*_{l_1l_2l_3} \sigma^*_{l_1l_2l_3l_5} \, Y_{l_1l_3l_2l_5l_4} \\
& + q_{l_3l_2} q_{l_3l_4} q_{l_3l_5} q_{l_4l_2} q_{l_4l_5} \sigma^+_{l_1l_3} \sigma^+_{l_1l_3l_4} \sigma^*_{l_1l_2l_3l_4} \, Y_{l_1l_3l_4l_2l_5} + q_{l_3l_2} q_{l_3l_4} q_{l_3l_5} q_{l_4l_2} q_{l_4l_5} q_{l_5l_2} \sigma^+_{l_1l_3} \sigma^+_{l_1l_3l_4} \sigma^+_{l_1l_3l_4l_5} \, Y_{l_1l_3l_4l_5l_2} \\
& + q_{l_3l_2} q_{l_3l_4} q_{l_3l_5} q_{l_5l_2} \sigma^+_{l_1l_3} \sigma^+_{l_1l_3l_5} \sigma^*_{l_1l_2l_3l_5} \, Y_{l_1l_3l_5l_2l_4} + q_{l_3l_2} q_{l_3l_4} q_{l_3l_5} q_{l_4l_2} q_{l_5l_2} \sigma^+_{l_1l_3} \sigma^*_{l_1l_3l_5} \sigma^+_{l_1l_3l_4l_5} \, Y_{l_1l_3l_5l_4l_2} \\
& + q_{l_3l_5} q_{l_4l_2} q_{l_4l_5} \sigma^+_{l_1l_4} \sigma^*_{l_1l_2l_4} \sigma^+_{l_1l_2l_3l_4} \, Y_{l_1l_4l_2l_3l_5} + q_{l_4l_2} q_{l_4l_5} \sigma^+_{l_1l_4} \sigma^*_{l_1l_2l_4} \sigma^*_{l_1l_2l_4l_5} \, Y_{l_1l_4l_2l_5l_3} \\
& + q_{l_3l_2} q_{l_3l_5} q_{l_4l_2} q_{l_4l_5} \sigma^*_{l_1l_4} \sigma^+_{l_1l_3l_4} \sigma^*_{l_1l_2l_3l_4} \, Y_{l_1l_4l_3l_2l_5} + q_{l_3l_2} q_{l_3l_5} q_{l_4l_2} q_{l_4l_5} q_{l_5l_2} \sigma^*_{l_1l_4} \sigma^+_{l_1l_3l_4} \sigma^+_{l_1l_3l_4l_5} \, Y_{l_1l_4l_3l_5l_2} \\
& + q_{l_4l_2} q_{l_4l_5} q_{l_5l_2} \sigma^+_{l_1l_4} \sigma^+_{l_1l_4l_5} \sigma^*_{l_1l_2l_4l_5} \, Y_{l_1l_4l_5l_2l_3} + q_{l_3l_2} q_{l_4l_2} q_{l_4l_5} q_{l_5l_2} \sigma^+_{l_1l_4} \sigma^*_{l_1l_4l_5} \sigma^+_{l_1l_3l_4l_5} \, Y_{l_1l_4l_5l_3l_2} \\
& + q_{l_3l_4} q_{l_5l_2} \sigma^+_{l_1l_5} \sigma^*_{l_1l_2l_5} \sigma^+_{l_1l_2l_3l_5} \, Y_{l_1l_5l_2l_3l_4} + q_{l_5l_2} \sigma^+_{l_1l_5} \sigma^*_{l_1l_2l_5} \sigma^*_{l_1l_2l_4l_5} \, Y_{l_1l_5l_2l_4l_3} \\
& + q_{l_3l_2} q_{l_3l_4} q_{l_5l_2} \sigma^*_{l_1l_5} \sigma^+_{l_1l_3l_5} \sigma^*_{l_1l_2l_3l_5} \, Y_{l_1l_5l_3l_2l_4} + q_{l_3l_2} q_{l_3l_4} q_{l_4l_2} q_{l_5l_2} \sigma^*_{l_1l_5} \sigma^+_{l_1l_3l_5} \sigma^+_{l_1l_3l_4l_5} \, Y_{l_1l_5l_3l_4l_2} \\
& + q_{l_4l_2} q_{l_5l_2} \sigma^*_{l_1l_5} \sigma^+_{l_1l_4l_5} \sigma^*_{l_1l_2l_4l_5} \, Y_{l_1l_5l_4l_2l_3} + q_{l_3l_2} q_{l_4l_2} q_{l_5l_2} \sigma^*_{l_1l_5} \sigma^*_{l_1l_4l_5} \sigma^+_{l_1l_3l_4l_5} \, Y_{l_1l_5l_4l_3l_2}
\end{aligned}
$$

Primjedba 4.2.15

Na opisani način, primjenom teorema 4.2.10, mogu se opisati bazične konstante u bilo kojem generičkom težinskom potprostoru $\mathcal{B}_{\mathrm{Q}} = \mathcal{B}_{l_1 l_2 \cdots l_n}$ (algebre $\mathcal{B}$) pridruženom skupu $\mathrm{Q} = \{l_1, l_2, ..., l_n\}$, $\mathrm{Q} \subseteq \mathcal{N} = \{i_1, i_2, ..., i_N\}$ ako je zadovoljen uvjet kocikličnosti, tj. ako vrijedi

$$\sigma_{l_1 l_2 \cdots l_n} = 1, \qquad \sigma_{j_1 j_2 \cdots j_i} \neq 1 \quad \text{za svaki} \quad 2 \leq i < n .$$

Ako je $\sigma_{l_1 l_2 \cdots l_n} \neq 1$, onda u težinskom potprostoru nema konstanti.

Zaključak

U ovoj disertaciji se detaljno izračunavaju konstante u težinskim potprostorima $\mathcal{B}_{\mathrm{Q}}$, $1 \le \mathrm{Card\,Q} \le 4$ algebre $\mathcal{B}$, pri čemu se pokazuje da u potprostorima $\mathcal{B}_{\mathrm{Q}}$, $\mathrm{Card\,Q} = 1$ nema netrivijalnih konstanti.

$\mathcal{B} = \mathbb{C}\langle \mathrm{e}_{\mathrm{i}_1}, \mathrm{e}_{\mathrm{i}_2}, \ldots, \mathrm{e}_{\mathrm{i}_N} \rangle$ je slobodna $\mathbb{C}$-algebra s jedinicom (gdje je stupanj svakog generatora $\mathrm{e}_{\mathrm{i}_s}$ jednak jedan) u kojoj je uvedena **q**-diferencijalna struktura nizom linearnih (diferencijalnih) operatora $\partial_{\mathrm{i}_1}, \partial_{\mathrm{i}_2}, \ldots, \partial_{\mathrm{i}_N} : \mathcal{B} \to \mathcal{B}$ zadanih sa

$$\partial_{\mathrm{i}}(1) = 0\,, \qquad \partial_{\mathrm{i}}\left(\mathrm{e}_{\mathrm{j}}\right) = \delta_{\mathrm{ij}}\,,$$

$$\partial_{\mathrm{i}}\left(\mathrm{e}_{\mathrm{j}}\,\mathrm{x}\right) = \delta_{\mathrm{ij}}\,\mathrm{x} + \mathrm{q}_{\mathrm{ij}}\,\mathrm{e}_{\mathrm{j}}\,\partial_{\mathrm{i}}\,\mathrm{x} \quad \text{za svaki } \mathrm{x} \in \mathcal{B},\ \mathrm{i},\mathrm{j} \in \mathcal{N}.$$

Pritom je $\mathcal{N} = \{\mathrm{i}_1, \mathrm{i}_2, \ldots, \mathrm{i}_N\}$ fiksirani podskup skupa nenegativnih cijelih brojeva.

Općenito je rastav algebre $\mathcal{B}$ po multihomogenim komponentama dan sa

$$\mathcal{B} = \bigoplus_{\substack{\mathrm{n} \ge 0 \\ l_1 \le l_2 \le \ldots \le l_{\mathrm{n}},\, l_{\mathrm{j}} \in \mathcal{N}}} \mathcal{B}_{l_1 l_2 \ldots l_{\mathrm{n}}}\,,$$

pri čemu je svaki težinski potprostor $\mathcal{B}_{\mathrm{Q}} = \mathcal{B}_{l_1 l_2 \ldots l_{\mathrm{n}}} = span_{\mathbb{C}}\left\{\mathrm{e}_{\mathrm{j}_1 \mathrm{j}_2 \ldots \mathrm{j}_{\mathrm{n}}} \mid \mathrm{j}_1 \mathrm{j}_2 \ldots \mathrm{j}_{\mathrm{n}} \in \widehat{\mathrm{Q}}\right\}$, $\mathrm{n} \ge 1$ pridružen multiskupu $\mathrm{Q} = \{l_1 \le l_2 \le \ldots \le l_{\mathrm{n}}\}$, $l_{\mathrm{j}} \in \mathcal{N}$.

Pritom je $\widehat{\mathrm{Q}} = \mathrm{S}_{\mathrm{n}}\mathrm{Q} = \left\{\sigma\left(l_1 l_2 \ldots l_{\mathrm{n}}\right) \mid \sigma \in \mathrm{S}_{\mathrm{n}}\right\}$ skup svih permutacija multiskupa Q.

Specijalno, ako je $l_1 < l_2 < \ldots < l_{\mathrm{n}}$, $l_{\mathrm{j}} \in \mathcal{N}$, onda je $\mathrm{Q} = \{l_1, l_2, \ldots, l_{\mathrm{n}}\}$ skup pa je njemu pridružen generički težinski potprostor $\mathcal{B}_{\mathrm{Q}}$. U protivnom, multiskupu Q, koji nije skup, je pridružen degenerirani težinski potprostor $\mathcal{B}_{\mathrm{Q}}$.

Na algebri $\mathcal{B}$ uvodi se operator stupnja $\partial : \mathcal{B} \to \mathcal{B}$, $\partial = \sum_{s=1}^{N} \mathrm{e}_{\mathrm{i}_s} \partial_{\mathrm{i}_s}$, gdje su $\mathrm{e}_{\mathrm{i}_s} \partial_{\mathrm{i}_s} : \mathcal{B} \to \mathcal{B}$, $1 \le \mathrm{s} \le \mathrm{N}$ linearni operatori, stoga se konstanta u algebri $\mathcal{B}$ definira kao bilo koji element $C \in \mathcal{B}$ takav da je $\partial\, C = 0$ ili $C \in Ker\,\partial$ ($Ker\,\partial$ označava jezgru operatora stupnja ∂).

Koristeći činjenicu da operator stupnja čuva rastav algebre $\mathcal{B}$ na težinske potprostore (tj. da je $\partial\left(\mathcal{B}_{\mathrm{Q}}\right) \subset \mathcal{B}_{\mathrm{Q}}$ za svaki multiskup $\mathrm{Q} = \{l_1 \le l_2 \le \ldots \le l_{\mathrm{n}}\}$ nad $\mathcal{N}$) imamo da se problem određivanja konstanti u algebri $\mathcal{B}$ svodi na problem određivanja konstanti u pojedinačnim težinskim potprostorima.

S druge strane, u disertaciji se pokazalo da se konstante u bilo kojem degeneriranom težinskom potprostoru $\mathcal{B}_{Q'}$ (pridruženog multiskupu $Q' = \{l_1 \leq l_2 \leq ... \leq l_n\}$ nad $\mathcal{N}$) mogu konstruirati iz konstanti nekog generičkog težinskog potprostora $\mathcal{B}_Q$ (pridruženog skupu $Q = \{l_1, l_2, ..., l_n\} \subseteq \mathcal{N}$) u određenoj identifikaciji elemenata u skupu Q, što dovodi do zaključka da je dovoljno izračunati (netrivijalne) konstante u generičkim težinskim potprostorima.

Iz tog se razloga problem određivanja konstanti u algebri $\mathcal{B}$ može svesti na problem određivanja konstanti u pojedinačnim generičkim težinskim potprostorima $\mathcal{B}_Q \subset \mathcal{B}$, što je ekvivalentno traženju jezgre operatora $\partial|_{\mathcal{B}_Q} : \mathcal{B}_Q \to \mathcal{B}_Q$.

Pritom je djelovanje operatora $\partial|_{\mathcal{B}_Q} : \mathcal{B}_Q \to \mathcal{B}_Q$ na elementima baze generičkog težinskog potprostora $\mathcal{B}_Q$, $Q = \{l_1, l_2, ..., l_n\} \subseteq \mathcal{N}$ dano sa

$$\partial|_{\mathcal{B}_Q}\left(e_{j_1 j_2 \cdots j_n}\right) = \sum_{p=1}^{n} \left(\prod_{r=1}^{p-1} q_{j_p j_r} \right) e_{j_p j_1 j_2 \cdots j_{p-1} j_{p+1} \cdots j_n}, \quad j_1 j_2 ... j_n \in \widehat{Q} = S_n Q .$$

U disertaciji su detaljno razrađena dva pristupa za određivanje konstante u težinskim potprostorima $\mathcal{B}_Q$ algebre $\mathcal{B}$.

U prvom pristupu (drugo poglavlje) se uzima da je $\det B_Q = 0$ nužan i dovoljan uvjet egzistencije netrivijalne bazične konstante u (generičkom) težinskom potprostoru $\mathcal{B}_Q$, što je ekvivalentno uvjetu da B_Q bude singularna matrica.

Pritom je u monomijalnoj bazi težinskog potprostora $\mathcal{B}_Q$ operatoru $\partial|_{\mathcal{B}_Q}$ pridružena kvadratna matrica B_Q, stoga su retci, tj. stupci matrice B_Q indeksirani elementima baze potprostora $\mathcal{B}_Q$.

Red matrice B_Q je jednak $\dim \mathcal{B}_Q = \mathrm{Card}\,\widehat{Q}$, gdje je $\mathrm{Card}\,\widehat{Q} = \#$ permutacija multiskupa Q.

Iz jednadžbe $\det B_Q = 0$ proizlaze određeni uvjeti na parametre q_{ij}, $i, j \in \mathcal{N}$ za koje postoji netrivijalna konstanta u $\mathcal{B}_Q$, ali isto tako i u $\mathcal{B}_{\overline{Q}}$ za neki pravi podskup $\overline{Q} \subset Q$.

Pri izračunavanju konstanti u $\mathcal{B}_Q = \mathcal{B}_{l_1 l_2 \cdots l_n}$ od posebnog je interesa tzv. uvjet kocikličnosti, tj. uvjet na parametre q_{ij}, $i, j \in \mathcal{N}$ za koje postoji netrivijalna konstanta u $\mathcal{B}_Q$, ali ne postoji u $\mathcal{B}_{\overline{Q}}$ za svaki pravi podskup $\overline{Q} \subset Q$.

Ako pretpostavimo da je uvjet kocikličnosti zadovoljen, tj.

$$\sigma_{l_1 l_2 \cdots l_n} = 1, \quad \sigma_{j_1 j_2 \cdots j_i} \neq 1 \ \text{ za svaki } \ 2 \le i < n,$$

gdje je $\sigma_{j_1 j_2 \cdots j_i} = \prod_{1 \le p < r \le i} \sigma_{j_p j_r} = \prod_{1 \le p \neq r \le i} q_{j_p j_r}$, onda u potprostoru $\mathcal{B}_Q$ postoji do na faktor netrivijalna bazična konstanta C_Q.

Drugim riječima, za sve preostale vrijednosti parametara q_{ij}, $i, j \in \mathcal{N}$ za koje je $\det B_Q = 0$, a koje ne zadovoljavaju uvjet kocikličnosti, dobivaju se bazične konstante takve da proizlaze iz C_Q.

Nedostatak ovog pristupa je u nemogućnosti generalizacije formule za izračunavanje netrivijalne bazične konstante C_Q u potprostoru $\mathcal{B}_Q$, $\operatorname{Card} Q = n$, $n \ge 2$. Naime, ako je za neki fiksni $n \ge 2$, $n = \operatorname{Card} Q$ poznata bazična konstanta u potprostoru $\mathcal{B}_Q$, onda je moguće dobiti bazične konstante u svim odgovarajućim potprostorima $\mathcal{B}_{\overline{Q}}$ takvim da je $\overline{Q} \subset Q$ (tj. $\operatorname{Card} \overline{Q} = k$, $2 \le k \le n$). Obrat nije moguć, tj. iz bazične konstante u potprostoru $\mathcal{B}_{\overline{Q}}$ nije moguće dobiti bazičnu konstantu u potprostoru $\mathcal{B}_Q$, gdje je $Q \supset \overline{Q}$.

S motivacijom uklanjanja navedenog nedostatka prvog pristupa, tj. s motivacijom dobivanja formule za izračunavanje netrivijalne bazične konstante C_Q u bilo kojem generičkom težinskom potprostoru $\mathcal{B}_Q$ ($n = \operatorname{Card} Q$, $n \ge 2$), u disertaciji je detaljno razrađeno treće i četvrto poglavlje, tj.

3. Zakrenuta grupovna algebra $\mathcal{A}_n$,
4. Reprezentacija zakrenute grupovne algebre $\mathcal{A}_n$ na težinskim potprostorima $\mathcal{B}_{l_1 l_2 \cdots l_n}$ algebre $\mathcal{B}$.

Ta se poglavlja temelje na radovima Svrtan-Meljanac, Krob i drugih i ujedno čine glavni dio ove disertacije.

U ovom drugom pristupu promatra se regularna reprezentacija $\mathfrak{R}:\boldsymbol{\mathcal{A}}_{\mathrm{n}}\to \mathrm{End}\left(\boldsymbol{\mathcal{B}}_{\mathrm{Q}}\right)$ zakrenute grupovne algebre $\boldsymbol{\mathcal{A}}_{\mathrm{n}}$ na bilo kojem generičkom težinskom potprostoru $\boldsymbol{\mathcal{B}}_{\mathrm{Q}}=\boldsymbol{\mathcal{B}}_{l_1 l_2 \cdots l_{\mathrm{n}}}$ algebre $\boldsymbol{\mathcal{B}}$.

Posebno se proučavaju vrijednosti regularne reprezentacije elemenata $\widetilde{\alpha_{\mathrm{n}}}$, $\widetilde{\beta_{\mathrm{n-k+1}}}$, $1\le \mathrm{k}\le \mathrm{n}$ algebre $\boldsymbol{\mathcal{A}}_{\mathrm{n}}$ na općenitom generičkom potprostoru $\boldsymbol{\mathcal{B}}_{\mathrm{Q}}$, $\mathrm{Q}\subset \mathcal{N}$, koje smo označili sa $\mathcal{A}_{\mathrm{Q}}=\mathfrak{R}\left(\widetilde{\alpha_{\mathrm{n}}}\right)\in \mathrm{End}\left(\boldsymbol{\mathcal{B}}_{\mathrm{Q}}\right)$, tj. $\mathrm{B}_{\mathrm{Q,\,n-k+1}}=\mathfrak{R}\left(\widetilde{\beta_{\mathrm{n-k+1}}}\right)\in \mathrm{End}\left(\boldsymbol{\mathcal{B}}_{\mathrm{Q}}\right)$. Regularnom reprezentacijom $\widetilde{\alpha_{\mathrm{n}}}$ na $\boldsymbol{\mathcal{B}}_{\mathrm{Q}}$ dobiva se matrica $\mathcal{A}_{\mathrm{Q}}$, koja je u direktnoj vezi s Varchenkovom matricom. Nadalje, regularnom reprezentacijom elementa $\widetilde{\beta_{\mathrm{n-k+1}}}\in \boldsymbol{\mathcal{A}}_{\mathrm{n}}$ na $\boldsymbol{\mathcal{B}}_{\mathrm{Q}}$ dobiva se matrica $\mathrm{B}_{\mathrm{Q,\,n-k+1}}$ za svaki $1\le \mathrm{k}\le \mathrm{n}-1$ (jer za $\mathrm{k}=\mathrm{n}$ imamo da je $\mathrm{B}_{\mathrm{Q,\,1}}=I$ jedinična matrica).

Pokazuje se da se matrica $\mathcal{A}_{\mathrm{Q}}$ može faktorizirati matricama $\mathrm{B}_{\mathrm{Q,\,n-k+1}}$, $1\le \mathrm{k}\le \mathrm{n}-1$ po formuli $\mathcal{A}_{\mathrm{Q}}=\overleftarrow{\prod_{1\le \mathrm{k}\le \mathrm{n}-1}}\mathrm{B}_{\mathrm{Q,\,n-k+1}}=\mathrm{B}_{\mathrm{Q,\,2}}\cdot \mathrm{B}_{\mathrm{Q,\,3}}\cdots \mathrm{B}_{\mathrm{Q,\,n-1}}\cdot \mathrm{B}_{\mathrm{Q,\,n}}$.

Pritom je od posebnog interesa matrica $\mathrm{B}_{\mathrm{Q,\,n}}$ za koju se pokazuje da je jednaka matrici B_{Q}, gdje se B_{Q} dobiva pri izračunavanju konstanti u potprostoru $\boldsymbol{\mathcal{B}}_{\mathrm{Q}}$.

Koristeći činjenicu da se problem nalaženja prostora konstanti u algebri $\boldsymbol{\mathcal{B}}$ svodi na traženje jezgre operatora $\mathbf{B}_{\mathrm{Q}}=\partial|_{\boldsymbol{\mathcal{B}}_{\mathrm{Q}}}:\boldsymbol{\mathcal{B}}_{\mathrm{Q}}\to \boldsymbol{\mathcal{B}}_{\mathrm{Q}}$ (kojemu je u monomijalnoj bazi potprostora $\boldsymbol{\mathcal{B}}_{\mathrm{Q}}$ pridružena matrica B_{Q} takva da je $\mathrm{B}_{\mathrm{Q}}=\mathrm{B}_{\mathrm{Q,\,n}}$) došlo se je do sljedećih rezultata:

Ako je uvjet kocikličnosti zadovoljen, tj. ako vrijedi

$$\sigma_{l_1 l_2 \cdots l_{\mathrm{n}}}=1,\quad \sigma_{\mathrm{j}_1 \mathrm{j}_2 \cdots \mathrm{j}_{\mathrm{i}}}\neq 1 \quad \text{za svaki } 2\le \mathrm{i}<\mathrm{n},$$

onda u potprostoru $\boldsymbol{\mathcal{B}}_{\mathrm{Q}}$ (pridruženog skupu $\mathrm{Q}=\{l_1,l_2,l_3,\ldots,l_{\mathrm{n}}\}$) postoji $(\mathrm{n}-2)!$ netrivijalnih baričnih konstanti, koje označavamo s $C_{l_1 l_2 \mathrm{j}_3 \mathrm{j}_4 \cdots \mathrm{j}_{\mathrm{n}}}$. Pritom su $l_1,l_2\in \mathrm{Q}$ fiksirani, a $\mathrm{j}_3\mathrm{j}_4\ldots\mathrm{j}_{\mathrm{n}}\in \mathrm{S}_{\mathrm{n}-2}\mathrm{Q}'$, gdje je $\mathrm{S}_{\mathrm{n}-2}\mathrm{Q}'=\left\{\sigma\left(l_3 l_4\ldots l_{\mathrm{n}}\right)\mid \sigma\in \mathrm{S}_{\mathrm{n}-2}\right\}$ skup svih permutacija skupa $\mathrm{Q}'=\mathrm{Q}\setminus\{l_1,l_2\}=\{l_3,l_4,\ldots,l_{\mathrm{n}}\}$.

Bazične konstante $C_{l_1 l_2 \mathrm{j}_3 \mathrm{j}_4 \cdots \mathrm{j}_{\mathrm{n}}}$ opisane su u teoremu 4.2.10 formulom (135), tj. u raspisanom obliku formulom (136).

Tim formulama imamo da se svaka bazična konstanta $C_{l_1 l_2 j_3 j_4 \cdots j_n}$ može prikazati kao linearna kombinacija od $(n-1)!$ iteriranih komutatora oblika $Y_{l_1\xi}$, $\xi \in \widehat{Q''}$, gdje je $\widehat{Q''} = S_{n-1}Q'' = \{\tau(l_2 l_3 \ldots l_n) \mid \tau \in S_{n-1}\}$ skup svih permutacija skupa $Q'' = Q \setminus \{l_1\}$.

Drugim riječima, dobivamo da se svaka bazična konstanta $C_{l_1 l_2 j_3 j_4 \cdots j_n}$ može prikazati u bazi iteriranih komutatora $Y_{l_1\xi} = Y_{l_1 j_2 j_3 j_4 \cdots j_n}$ takvih da im je prvi indeks $l_1 \in Q$ fiksiran, a preostalih $n-1$ indeksa $j_2, j_3, j_4, \ldots, j_n \in Q'' = Q \setminus \{l_1\} = \{l_2, l_3, \ldots, l_n\}$ permutiraju.

Pritom je $\xi := j_2 j_3 j_4 \ldots j_n \in \widehat{Q''} = S_{n-1}Q'' = \{\tau(l_2 l_3 \ldots l_n) \mid \tau \in S_{n-1}\}$.

Prednost ovog drugog pristupa je da se uz zadovoljen uvjet kocikličnosti dobiva elegantna formula za izračunavanje bazičnih konstanti u bilo kojem generičkom težinskom potprostoru $\mathcal{B}_Q$, $\text{Card } Q = n$, $n \geq 2$.

Treba naglasiti da je do sličnog rezultata po pitanju izračunavanja konstanti u potprostoru $\mathcal{B}_Q$ također došao i Correll u svom radu [C], gdje je opisao konstante u specijalnom slučaju kada su svi kompleksni parametri q_{ij} jednaki kompleksnom parametru q, tj. $q_{ij} := q$ za svaki $i, j \in \mathcal{N} = \{i_1, i_2, \ldots, i_N\}$.

Lako se može pokazati ako je $q_{ij} := q$ za svaki $i, j \in \mathcal{N}$, onda iz formule (135) u teoremu 4.2.10 proizlaze konstante koje je opisao Correll u [C].

Na kraju napomenimo sljedeće:

umjesto rastava permutacije $g \in S_n$ u produkt cikličkih permutacija navedenog u teoremu 3.15, permutaciju $g \in S_n$ mogli smo također rastaviti na načine kako je dano u komentaru 3.16.

Rastav (31) je odabran ciljano, jer uporabom bilo kojeg od tri rastava navedena u komentaru 3.16 dobivamo $B_{Q,n} \neq B_Q$, tj. ne vrijedi jednakost matrica $B_{Q,n}$ i B_Q, iako su njihove determinante jednake.

Dakle, identitet (31) je korišćen s ciljem da se dobije jednakost matrica $B_{Q,n}$ i B_Q.

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Sažetak

Promatramo slobodnu $\mathbb{C}$-algebru $\mathcal{B} = \mathbb{C}\langle e_{i_1}, e_{i_2}, ..., e_{i_N} \rangle$ s jedinicom (gdje je stupanj svakog generatora e_{i_s} jednak jedan) u kojoj je uvedena **q**-diferencijalna struktura familijom linearnih **q**-diferencijalnih operatora $\partial_{i_1}, \partial_{i_2}, ..., \partial_{i_N} : \mathcal{B} \to \mathcal{B}$ zadanih rekruzivno sa:

$$\partial_i(1) = 0, \ \partial_i(e_j) = \delta_{ij}, \ \partial_i(e_j x) = \delta_{ij} x + q_{ij} e_j \partial_i x \quad \text{za svaki } x \in \mathcal{B}, \ i, j \in \mathcal{N}.$$

Pritom je $\mathcal{N} = \{i_1, i_2, ..., i_N\}$ fiksirani podskup skupa nenegativnih cijelih brojeva.

Općenito je rastav algebre $\mathcal{B}$ po multihomogenim komponentama dan sa

$$\mathcal{B} = \bigoplus_{\substack{n \geq 0 \\ l_1 \leq l_2 \leq ... \leq l_n, l_j \in \mathcal{N}}} \mathcal{B}_{l_1 l_2 \dots l_n},$$

pri čemu je svaki težinski potprostor $\mathcal{B}_Q = \mathcal{B}_{l_1 l_2 \dots l_n} = span_{\mathbb{C}} \left\{ e_{j_1 j_2 \cdots j_n} \mid j_1 j_2 ... j_n \in \widehat{Q} = S_n Q \right\}$, $n \geq 1$ pridružen multiskupu $Q = \{ l_1 \leq l_2 \leq ... \leq l_n \}$, $l_j \in \mathcal{N}$.

Specijalno, ako je $l_1 < l_2 < ... < l_n$, $l_j \in \mathcal{N}$, onda je $Q = \{l_1, l_2, ..., l_n\}$ skup, njemu pridružen težinski potprostor $\mathcal{B}_Q$ zovemo generičkim. Multiskupu Q (koji nije skup) pridruženi težinski potprostor $\mathcal{B}_Q$ zovemo degeneriranim.

U algebri $\mathcal{B}$ pod konstantom $C \neq 0$ podrazumijeva se netrivijalni element, koji ima svojstvo da je $\partial_i C = 0$ za svaki $i \in \mathcal{N}$.

U ovoj disertaciji, pomoću eksplicitnih formula, proučavamo konstante u bilo kojem težinskom potprostoru $\mathcal{B}_Q$ ($1 \leq \text{Card } Q \leq 5$). U potprostorima $\mathcal{B}_Q$, $\text{Card } Q = 1$ nema netrivijalnih konstanti. Također se pokazuje da konstante u bilo kojem degeneriranom težinskom potprostoru možemo konstruirati iz konstanti nekog generičkog težinskog potprostora $\mathcal{B}_Q$, što nas dovodi do zaključka da je dovoljno izračunati netrivijalne bazične konstante u generičkim težinskim potprostorima.

Proučavanjem radova Svrtan-Meljanac, Krob, Fronsdal, Corell i drugih, gdje je korišten matrični pristup, u disertaciji smo razvili formalizam zakrenute grupovne algebre te smo dobili sljedeće rezultate.

Ako pretpostavimo da je $\sigma_{l_1 l_2 \cdots l_n} = 1$, $\sigma_{j_1 j_2 \cdots j_i} \neq 1$ za svaki $2 \leq i < n$ (uvjet kocikličnosti), onda u generičkom težinskom potprostoru $\mathcal{B}_Q$, $\text{Card } Q = n$, postoji $(n-2)!$ netrivijalnih baznih konstanti, koje označavamo s $C_{l_1 l_2 j_3 j_4 \cdots j_n}$, gdje su $l_1, l_2 \in Q$ fiksirani, a $j_3 j_4 \ldots j_n \in S_{n-2} Q'$.

Pritom $S_{n-2} Q' = \{\sigma(l_3 l_4 \ldots l_n) \mid \sigma \in S_{n-2}\}$ označava skup svih permutacija skupa $Q' = Q \setminus \{l_1, l_2\} = \{l_3, l_4, \ldots, l_n\}$.

Pokazujemo da formule (135), (136) u Teoremu 4.2.10 opisuju bazične konstante $C_{l_1 l_2 j_3 j_4 \cdots j_n}$. Primjenom tih formula nalazimo da se svaka bazična konstanta $C_{l_1 l_2 j_3 j_4 \cdots j_n}$ može prikazati pomoću iteriranih komutatora $Y_{l_1 \xi} = Y_{l_1 j_2 j_3 j_4 \cdots j_n}$ kojima je prvi indeks $l_1 \in Q$ fiksiran, a preostalih $n-1$ indeksa $j_2, j_3, j_4, \ldots, j_n \in Q'' = Q \setminus \{l_1\} = \{l_2, l_3, \ldots, l_n\}$ variraju.

Također opći Teorem 4.2.10 iz ove disertacije ilustriramo eksplicitnije za sve generičke težinske potprostore $\mathcal{B}_Q$ u slučaju da je $2 \leq \text{Card } Q \leq 5$ (vidi primjere 4.2.11 - 4.2.14).

Specijalno ako je $q_{ij} := q$ za svaki $i, j \in \mathcal{N}$, onda proizlaze konstante koje je Correll opisao u svojoj disertaciji [C] iz formule (135) u Teoremu 4.2.10.

Summary

We consider a free unital $\mathbb{C}$-algebra $\mathcal{B} = \mathbb{C}\langle e_{i_1}, e_{i_2}, \ldots, e_{i_N} \rangle$ (where degree of each generators e_{i_s} is equal one) in which a **q**-differential structure is introduced by a set of linear **q**-differential operators $\partial_{i_1}, \partial_{i_2}, \ldots, \partial_{i_N} : \mathcal{B} \to \mathcal{B}$ defined recursively by:

$$\partial_i(1) = 0, \quad \partial_i(e_j) = \delta_{ij}, \quad \partial_i(e_j x) = \delta_{ij} x + q_{ij} e_j \partial_i x \quad \text{for each } x \in \mathcal{B}, \ i, j \in \mathcal{N}.$$

Here $\mathcal{N} = \{i_1, i_2, \ldots, i_N\}$ is a fixed subset of the set of nonnegative integers.

Generally, we have the direct sum decomposition

$$\mathcal{B} = \bigoplus_{\substack{n \geq 0 \\ l_1 \leq l_2 \leq \ldots \leq l_n, l_j \in \mathcal{N}}} \mathcal{B}_{l_1 l_2 \ldots l_n},$$

where each weight subspace

$$\mathcal{B}_Q = \mathcal{B}_{l_1 l_2 \ldots l_n} = span_{\mathbb{C}} \{ e_{j_1 j_2 \ldots j_n} \mid j_1 j_2 \ldots j_n = \text{rearrangement of } l_1, l_2, \ldots, l_n \}, \ n \geq 1$$

is associated to the multiset $Q = \{ l_1 \leq l_2 \leq \ldots \leq l_n \}$, $l_j \in \mathcal{N}$.

In particular, when $l_1 < l_2 < \ldots < l_n$, $l_j \in \mathcal{N}$ (i.e. $Q = \{l_1, l_2, \ldots, l_n\}$ is a set) the associated weight subspace $\mathcal{B}_Q$ is called generic. A degenerate weight subspace $\mathcal{B}_Q$ is associated to a multiset Q which is not a set.

In the algebra $\mathcal{B}$ by a constant $C \neq 0$ we shall mean a nontrivial element with the property $\partial_i C = 0$ for all $i \in \mathcal{N}$.

In this dissertation we study, by means of explicit formulas, the constants in any weight subspace $\mathcal{B}_Q$ ($1 \leq \text{Card } Q \leq 5$) were calculating in detail. In the subspaces $\mathcal{B}_Q$, $\text{Card } Q = 1$ there are no nontrivial constants. It is also shown that constants in degenerate weight subspaces can be constructed from constants in generic weight subspaces $\mathcal{B}_Q$ by certain specialization procedure. So we conclude, that is enough to calculate nontrivial basic constants in generic weight subspaces.

By studying some papers of Svrtan-Meljanac, Krob, Fronsdal, Corell′s and others, which use matrix approach we develop some twisted group algebras machinery and obtain the following results.

If we assume that $\sigma_{l_1 l_2 \cdots l_n} = 1$, $\sigma_{j_1 j_2 \cdots j_i} \neq 1$ for each $2 \leq i < n$ (cocycle condition), then in the generic weight subspace $\mathcal{B}_Q$, $\mathrm{Card}\, Q = n$ there are $(n-2)!$ nontrivial basic constants which we denote by $C_{l_1 l_2 j_3 j_4 \cdots j_n}$, where $l_1, l_2 \in Q$ are fixed and $j_3 j_4 \ldots j_n \in S_{n-2} Q'$. $S_{n-2} Q' = \{\sigma(l_3 l_4 \ldots l_n) \mid \sigma \in S_{n-2}\}$ denote the set of all permutations of $Q' = Q \setminus \{l_1, l_2\} = \{l_3, l_4, \ldots, l_n\}$.

We show that formulas (135), (136) in Theorem 4.2.10 describes the basic constants $C_{l_1 l_2 j_3 j_4 \cdots j_n}$. By applying these formulas we find that each basic constant $C_{l_1 l_2 j_3 j_4 \cdots j_n}$ can be expressed in terms of iterated commutators $Y_{l_1 \xi} = Y_{l_1 j_2 j_3 j_4 \cdots j_n}$, where first index $l_1 \in Q$ is fixed and remaining $n-1$ indices $j_2, j_3, j_4, \ldots, j_n \in Q'' = Q \setminus \{l_1\} = \{l_2, l_3, \ldots, l_n\}$ vary.

We also illustrate our general Theorem 4.2.10 of this dissertation more explicitly for all weight subspaces $\mathcal{B}_Q$ in case $2 \leq \mathrm{Card}\, Q \leq 5$ (see examples 4.2.11 - 4.2.14).

In particular, if $q_{ij} := q$ for all $i, j \in \mathcal{N}$, then the constants described in Corell′s thesis [C] come out from equation (135) in Theorem 4.2.10.

Životopis

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