Electron wave functions in a magnetic field

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The problem of a single electron in a magnetic field is revisited from first principles. It is shown that the standard quantization, used by Landau, is inconsistent for this problem, with the consequence that Landau's wave functions spontaneously break the gauge symmetry of translations in the plane. The corresponding gauge current is a constant linear motion of the guiding center of the circular orbit. The one-body wave function of the physical orbit, with static orbit center, is derived here, and expressed as a superposition of Landau's wave functions. Conversely, it is shown that Landau's wave functions are a limiting case of physical solutions of a different problem. The present approach accounts for the loss of translation invariance, associated with choosing the circular orbit, without breaking translation-gauge symmetry, so that the degeneracy coming from the choice of orbit center does not appear in the one-body problem. Previous research which made explicit use of Landau's wave functions is not automatically invalidated, but may need to be reviewed, in some cases, in the light of this result. In particular, Landau's own estimate of the one-particle density of states of electrons in a magnetic field remains standing.

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I. INTRODUCTION

The motion of a free electron in a homogeneous magnetic field is a classic problem in elementary quantum mechanics. It was first treated by Landau [1] in 1930, and remains a subject of active research to this day [2]. The wave functions derived by Landau are routinely used as building blocks of many-body states in a magnetic field [3].

Canonical quantization, as established by Dirac [4], is the only impeccable way to quantize a classical system, against which all other approaches are measured. The method consists in promoting Poisson brackets of classical mechanics into commutators by the prescription $\{x, p_x\} = 1 \to [\hat{x}, \hat{p}_x] = i\hbar$, where I have taken the standard example of position and momentum dynamical variables. This step establishes the structure of the quantum theory, but leaves the freedom to concretely realize the Hilbert space on which the operators $\hat{x}$ and $\hat{p}_x$ act. The standard choice is $\hat{x} = x$, a real number, and $\hat{p}_x = -i\hbar\partial_x$, turning them into operators on the Hilbert space $\mathbf{L}^2(\mathbb{R})$ of square-integrable functions on the real line. The canonical quantization, when taken together with this particular realization of Hilbert space, is usually called "first quantization," and the corresponding wave functions are "Schrödinger wave functions." Before the work of Dirac, quantization was simply understood to be the replacement $p_x \to -i\hbar\partial_x$ in the classical Hamiltonian, irrespective of the Poisson bracket structure, a prescription called "naive quantization."

Canonical and naive quantization should in principle give the same result, because whichever canonical transformation can be made at the level of Poisson brackets, can also be made at the level of commutators, after quantization. Exceptionally, when there is a hidden gauge degree of freedom at the classical level, naive quantization overcounts the physical degrees of freedom in the system. The purpose of the present paper is to show that this is the case with a free electron in a magnetic field, with translations as the relevant gauge group. The situation is closely analogous to the ground state of the hydrogen atom, which may similarly be considered to have a huge degeneracy, coming from the different positions of the proton in space, but which is easily separated in the calculation from the one-body electron states within the atom. For an electron in a magnetic field, the separation is more difficult. The physical reason for the difficulty is that, unlike the case of the atom, the free electron chooses its own orbit-center, and the calculation has to be set up in a way invariant to that choice. The formal reason is that naive quantization is inconsistent in the presence of a gauge degree of freedom at the classical level, as first noticed by Dirac [5]. Naive quantization subtly introduces the positional degeneracy of the orbit-center into the calculation, which appears as a spurious constant of motion (momentum) in the one-body wave function. An explicit integration over the spurious momentum, Eq. (10) below, is then needed to restore the translation-gauge symmetry and give the true one-body wave function, which is non-degenerate. Canonical quantization clearly separates the one-body and many-body aspects of the problem: there is no degeneracy in the former, while in the latter, it appears manifestly as translation-gauge fixing, associated

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with choosing explicitly the position of the orbit center. It is shown below that the well-established coherent-state formalism is nothing but the most natural implementation of such translation-gauge fixing.

The Hilbert space established by first quantization is not the only one possible. Any particular choice is justified by canonical quantization if it is formally realized as a transform of the first-quantized Hilbert space. An important example, needed in the following, is the Fock space $\mathbf{F}(\mathbb{C})$, spanned by entire analytic functions $f(z)$, $z = x + iy \in \mathbb{C}$, with the property $\int dx\, dy |f(z)|^2 e^{-|z|^2} < \infty$. It is obtained from 1D first-quantized space by the Bargmann transform [6],

$$f(z) = \frac{1}{\pi^{3/4}} \int_{-\infty}^{+\infty} \exp\left(-\frac{z^2 + x^2}{2} + xz\sqrt{2}\right) \Psi(x) dx, \tag{1}$$

where $\Psi(x) \in \mathbf{L}^2(\mathbb{R})$ is a Schrödinger wave function. In particular, if $\Psi(x)$ is the Hermite function $\psi_n(x)$, the well-known Schrödinger eigenfunction of the oscillator, then $f(z) \sim z^n$, while the oscillator Hamiltonian becomes

$$\left(z\partial_z + \frac{1}{2}\right)\hbar\omega \tag{2}$$

under the same transform. Clearly, the entire functions z^n are its eigenfunctions. Notably, if the real variable x in the integral transform (1) is a point in real space, then $z = x + iy$ cannot be interpreted to refer to real space.

II. THE FOCK PROBLEM

In order to establish the paradigm of canonical quantization, needed to solve the Landau problem [1], I invoke the related Fock problem [7], of a particle of charge $q = -e < 0$ in a harmonic oscillator potential, subject to a perpendicular magnetic field. The classical Hamiltonian of Fock's problem may be written

$$H_F = \frac{\omega_+}{2}\left(v_x^2 + v_y^2\right) + \frac{\omega_-}{2}\left(x_0^2 + y_0^2\right), \tag{3}$$

where the symmetric gauge is assumed throughout, and

$$\begin{pmatrix} x_0 \\ y_0 \\ v_x \\ v_y \end{pmatrix} = \begin{pmatrix} \alpha/2 & 0 & 0 & -1/\alpha \\ 0 & \alpha/2 & 1/\alpha & 0 \\ 0 & -\alpha/2 & 1/\alpha & 0 \\ \alpha/2 & 0 & 0 & 1/\alpha \end{pmatrix} \begin{pmatrix} x \\ y \\ p_x \\ p_y \end{pmatrix}, \tag{4}$$

with $\alpha = \sqrt{M\Omega}$, is a canonical transformation of the original dynamical variables. Here $\{x_0, y_0\} = \{v_y, v_x\} = 1$, and all other Poisson brackets are zero. The frequencies are $\omega_\pm = (\sqrt{\omega_c^2 + 4\omega_0^2} \pm \omega_c)/2 \equiv (\Omega \pm \omega_c)/2$, where ω_0 is the intrinsic frequency of the oscillator, and $\omega_c = eB_0/Mc$ is the cyclotron frequency of the particle.

The classical Fock problem is easy to solve. The trajectory is epicyclic. It consists of a "fast" counterclockwise rotation, with angular frequency ω_+, on a circle, combined with a "slow" clockwise rotation of the circle center, with frequency ω_-, around the origin of the harmonic potential. To quantize it, first rescale v_y to v, given by

$$\frac{v_y}{\sqrt{\hbar}} \longrightarrow \frac{x + k_y}{\sqrt{2}} \equiv v, \tag{5}$$

where the arrow means rescaling by $r_0 = l_0\sqrt{2} = \sqrt{2\hbar c/eB_0}$, in particular $k_y = r_0 p_y/\hbar$. The conjugate variable in the Poisson bracket is now replaced by canonical prescription, $v_x/\sqrt{\hbar} \to -i\partial_v$, so the first-quantized Hamiltonian is

$$\widehat{H}_F = \frac{\hbar\omega_+}{2}\left(-\partial_{vv} + v^2\right) + \frac{\hbar\omega_-}{2}\left(-\partial_{ww} + w^2\right), \tag{6}$$

where $w = (x - k_y)/\sqrt{2}$ is the similarly rescaled x_0. Evidently its solution (the Schrödinger wave function) is

$$\psi_{n_+}(v)\psi_{n_-}(w), \tag{7}$$

where the ψ_n are Hermite functions. The spectrum is $E = (n_+ + 1/2)\hbar\omega_+ + (n_- + 1/2)\hbar\omega_-$, as obtained by Fock. However, Fock used naive quantization, so that his wave function was directly obtained in real space. Here the canonical quantization has imposed a mixed first-quantized space, real space in one direction and momentum space

in the other. In order to compare it with Fock's solution, one needs to transform this solution into real space, by a Fourier transform:

$$\int dk_y e^{iyk_y}\psi_{n_+}(\frac{x+k_y}{\sqrt{2}})\psi_{n_-}(\frac{x-k_y}{\sqrt{2}}) \sim (x+iy)^{n_+-n_-}L_{n_-}^{n_+-n_-}(x^2+y^2)e^{-(x^2+y^2)/2}, \tag{8}$$

which is precisely the wave function expressed in Laguerre functions, found by Fock [7]. The mathematical interpretation of Laguerre functions as Wigner transforms of Hermite functions is well known in the representation theory of the Heisenberg group [8]. Here a physical interpretation is obtained, that they are Fourier transforms of the Schrödinger wave functions, Eq. (7), which describe the two connected rotations of Fock's problem in first-quantized space.

For Fock's problem, naive quantization is justified by the canonical prescription. The advantage of choosing the first-quantized space in accord with the Poisson-bracket structure of the classical problem is better physical insight. In Fock's case, the natural independent variables are the velocity component v_y and orbit center component x_0, both of which oscillate harmonically, because they are 1D projections of the corresponding circular motions. This is clear from the Schrödinger wave functions (7), which are elements of $\mathbf{L}^2(\mathbb{R}^2)$ in the plane (v,w). The wave function space $\mathbf{L}^2(\mathbb{R}^2)$ in the physical plane (x,y) is an alternative choice of Hilbert space. The Hamiltonian, obtained by Fourier-transforming Eq. (6), is the same as obtained from the classical Hamiltonian by naive quantization, but, as we shall see now, that is not by itself a justification of naive quantization.

III. THE LANDAU PROBLEM

Having established the quantization method, and checked it on Fock's problem, I now apply it to Landau's problem [1] of an electron in empty space, subject to a constant magnetic field $\mathbf{B} = B_0\hat{\mathbf{z}}$. Leaving implicit the motion along the field, the quantization in the perpendicular plane can be obtained by setting $\omega_0 = 0$ in Fock's problem, whence $\Omega = \omega_+ = \omega_c$, and $\omega_- = 0$. The first-quantized Hamiltonian becomes

$$\widehat{H}_L = \frac{\hbar\omega_c}{2}\left(-\partial_{vv} + v^2\right), \tag{9}$$

because the orbit center has stopped moving. Significantly, the velocity component v_y, which appears here, has no reference to any particular position in space. The Schrödinger wave function is $\psi_n(v)$, which refers only to the "internal" motion of the electron, i.e. relative to a given center, wherever it may be. It is obviously non-degenerate, and means that the projection v_y of the velocity on the y-axis oscillates harmonically, which is, of course, true for uniform circular motion. The corresponding real-space wave function is obtained by the same Fourier transform, from velocity space to real space, as in Fock's problem, which gives (scaling by $r_0 = l_0\sqrt{2}$)

$$\int dk_y e^{iyk_y}\psi_n(\frac{x+k_y}{\sqrt{2}}) \sim e^{-ixy}\psi_n(y\sqrt{2}), \tag{10}$$

because Hermite functions are eigenfunctions of the Fourier transform [8]. This is the correct physical solution of the one-body Landau problem, and the central result of the present paper. In the Landau gauge and dimensionful variables, it is just $\psi_n(y/l_0)$. Note that Landau's original solution, $e^{iyk_y}\psi_n((x+k_y)/\sqrt{2})$, reappears as its Fourier component, so the physical solution is a superposition of Landau's formal solutions as basis vectors: the latter are "harmonics" of the physical orbit. This means that Landau's solution is not in real space: its variables are x and k_y, while y is a parameter. It only appears to be in real space (x and y variables, k_y a parameter) if one tries to solve the naively quantized Hamiltonian by direct attack, because then one is working "blindly" in the whole space $\mathbf{L}^2(\mathbb{R}^2)$ of wave functions in the plane, while the Schrödinger wave functions of Eq. (9) are in $\mathbf{L}^2(\mathbb{R})$, because the Landau Hamiltonian has only one degree of freedom. In particular, it is misleading to relabel k_y as $-x'$ (say), which is commonly done in the literature, without taking into account that it is also a momentum label. It implies that while Landau's wave functions are a basis in real space, a physical wave function of a single electron cannot be described by a single Landau vector, because an electron orbiting around a stationary center has no net linear momentum. Eq. (10) is precisely the summation over this spurious constant of motion, required to produce a physical solution, whose orbit center is motionless. The dual role of k_y will be fully explained in the next section.

Eq. (10) is not symmetric with respect to the Cartesian axes. This is a reflection of the "handedness" of the original problem, formally encoded in the signs of the Poisson brackets: $\{v_y, v_x\} = 1$, rather than -1. Indeed, for a positively charged particle the x- and y- axes exchange place. In effect, the time-reversal symmetry breaking of the magnetic field is reduced to the statement that the Hamiltonian (9) describes the oscillation of the y-component of the velocity, rather than the x-component. It is particularly gratifying in this context that the Landau gauge-fixing term e^{-ixy} appears as a result in Eq. (10), namely as the eigenvalue of the Fourier transform, rather than an assumption. Hence

the preference of one axis over the other is not due to any a priori choice to search for the solution in the Landau gauge. Because $v_y = (x + k_y)/\sqrt{2}$, one can absorb the gauge-fixing term into the velocity, so the same Eq. (10) can be written

$$\psi_n(y\sqrt{2}) \sim \int dv_y e^{iv_y y\sqrt{2}} \psi_n(v_y), \tag{11}$$

showing that the Landau-gauge solution in real space is nothing but a Fourier transform directly from velocity space. Physically, this equation means that harmonic oscillation in the y-component of the velocity is equivalent to harmonic oscillation in the y-coordinate of the position. The gauge-fixing term appears as an independent factor only if one insists on writing the Fourier transform in the wave number k_y, in spite of the velocity being the canonical variable of the problem: $iy \cdot v_y \sim iy \cdot (x + k_y) \sim iy \cdot x + iy \cdot k_y$. A Landau "solution" (basis vector) in the Landau gauge is just a single term under the integral in Eq. (11), as can be seen by rewriting it in terms of k_y instead.

Notably, if the Hamiltonian (9) is Fourier-transformed to real space, one does obtain, as in Fock's case, the same Hamiltonian as by naive quantization. However its solutions are limited by the same transform to the highly restricted class of Eq. (10), meaning that the transform to real space is also a prescription to embed $\mathbf{L}^2(\mathbb{R})$ into $\mathbf{L}^2(\mathbb{R}^2)$. Physically, the 1D oscillation of the projection of a circular orbit is being expressed in the larger wave-function space, capable of carrying true 2D oscillation.

In Landau's approach, n and k_y are quantum numbers, so the above canonical approach removes the infinite degeneracy in the second quantum number. This is appropriate, because the two degrees of freedom in Fock's problem are reduced to one in Landau's problem. It may similarly be noted that when $\omega_- = 0$, the solution to Fock's problem, Eq. (8), becomes infinitely degenerate in the quantum number n_-, and may also serve as a formal solution (basis) for Landau's problem. However, it cannot be the physical solution, formally because a problem with one degree of freedom cannot have two quantum numbers, and physically because it describes a circle whose center also moves on a circle, while in the Landau problem the center is motionless.

The last observation points to a way to guess the subclass of solutions to Fock's problem which are also solutions of Landau's problem: simply put $n_- = 0$ in Eq. (8). This immediately gives ($n_+ = n$)

$$(x + iy)^n e^{-(x^2+y^2)/2}, \tag{12}$$

which agrees with the Fock-space solution of a single harmonic oscillator, given in the Introduction. This solution is not in Fock space, but still in real space, as indicated by the Gaussian factor (absent in Fock space). Alternatively, the Bargmann transform of the Hamiltonian (9) from the variable v to the variable z gives the Fock-space solution z^n directly from $\psi_n(v)$. This justifies the guess (12), showing also that in this case the Fock-space $z = x + iy$ does refer to a position in real space, which is possible because the original variable v was not itself in real space.

Both the real-space solution (10) and the Fock-space $(x + iy)^n$ are just different images, under the Fourier and Bargmann transforms respectively, of the original solution $\psi_n(v)$ of the one-particle Hamiltonian (9) in velocity space, so they naturally inherit its non-degeneracy. To put these one-particle results in the many-body context, and make contact with well-known results, note that there is still expected a *parametric* infinite degeneracy of the problem in real space, because the orbit center can be anywhere in the plane. This degeneracy is physically different than degeneracy in a quantum number, and in particular the parameter denoting the orbit center must be a continuous c-number, with a sharp value. To identify it in the real-space representation, write an *ansatz* for the wave function as

$$\psi(z, \bar{z}) = u(z, \bar{z}) e^{-|z|^2/2} \tag{13}$$

where $z, \bar{z} = x \pm iy$ is just a coordinate transformation here, so x and y do refer to a real-space position. The real-space eigenvalue equation for u becomes

$$-u_{z\bar{z}} + z u_z = n u, \tag{14}$$

where $n = E/(\hbar\omega_c) - 1/2$. Solving it by separation of variables, $u(z, \bar{z}) = f(z)g(\bar{z})$, one obtains quite generally

$$u(z, \bar{z}) = (z - z_0)^n e^{z_0 \bar{z}}, \tag{15}$$

where z_0 is the separation constant, a complex number. Notably, this separation of variables yields a physically different set of solutions than the usual $u(z, \bar{z}) = f(|z|)g(\arg z)$, which leads to Fock's solution (8). Eq. (15) is a particular resummation of an infinite series in the 'moving-center' solutions (8) over the quantum number n_-, which produces an electron orbit with a stationary center at z_0. Such a construction is analogous to the expression (10). When $z_0 = 0$, the series collapses to a single term, Fock's solution (8) with $n_- = 0$, as guessed above. These eigenstates are also coherent states, and were originally introduced for that reason, in the context of many-body path

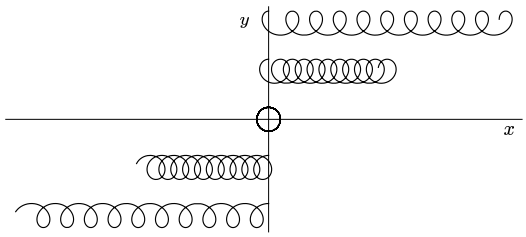

FIG. 1: Several trajectories of the class (17), evolved for the same length of time. Here $\omega_c/\omega_0 = 5$, or $\omega_c/\omega \approx 0.98$.

integrals [9]. As coherent states, they are complete precisely when the density of the orbit-center grid z_0 is one orbit per flux quantum [9, 10], which is an independent confirmation of Landau's estimate of the degeneracy of Landau levels. This discretization of the separation constant z_0 plays the role of a spectrum.

Parenthetically, it is unexpected for coherent states to be simultaneously the eigenstates of the Hamiltonian. Usually, they are only eigenstates of some lowering operator. The states (15) achieve this surprising duality by being eigenstates of the first term in Eq. (6), and coherent in the second term, which is zero in the Landau Hamiltonian. The corresponding lowering operator is $\hat{x}_0 + i\hat{y}_0$, physically the orbit-center position, with eigenvalue z_0.

The term $e^{z_0\bar{z}}$ appears above as the extra factor in the wave function, needed to compensate the action of translating the physical orbit in the plane, $z \to z - z_0$. In other words, it is a gauge-fixing term, with translations as the underlying gauge group. The shift of the orbit center by z_0 comes after the choice of origin of coordinates has been fixed by the zero of the vector potential, which is where the Gaussian envelope in Eqs. (10) and (13) is centered. Shifting the origin of coordinates shifts the potential by a constant, which is an electromagnetic gauge transformation. Thus translation and electromagnetic gauge fixing appear independently in the wave function. In gauge-theory idiom, motion in z is physical, while motion in $\bar{z}$ is "pure gauge." It is possible to write the same states in the form of shifted Gaussians with an extra phase factor [10], which however hides the natural structure of the two distinct gauge groups. The separation of physical and gauge motion plays a striking role in the demonstration that the Landau-level quantum number n is just the physical angular momentum around the arbitrary orbit center, as shown in the Appendix. Because of this, the preference of z over $\bar{z}$ in the holomorphic coordinates is physically more intuitive than the preference of y over x in the Cartesian coordinates, although both are ultimately due to the time-reversal symmetry breaking. The reduction to one coordinate reflects the loss of the orbit-center degree of freedom in both cases.

The translation gauge reappeared in Eq. (15) only because the wave equation in real space was being solved by direct attack. That is precisely what one should not do with a transformed one-body equation. Instead, all of its physically distinct solutions are to be generated from the first-quantized solutions by the self-same transform. Technically, this correctly embeds the space $\mathbf{L}^2(\mathbb{R})$, of Schrödinger wave functions which solve the Landau problem, into the space $\mathbf{L}^2(\mathbb{R}^2)$, of all possible solutions of the two-dimensional real-space Hamiltonian. Physically, the resummation in Eq. (10) may be considered as restoration of the broken translation-gauge symmetry.

The translation-gauge degree of freedom becomes physical, however, when one considers the many-body problem. The localized electrons then fill the translation-gauge spectrum, given by the discretized orbit-center grid z_0. The energy spectrum, on the other hand, is given by the first-quantized Hamiltonian (9), free of the translation gauge.

IV. A PROBLEM SOLVED BY LANDAU-LIKE WAVE FUNCTIONS

As shown in Eq. (10) above, Landau's wave functions are eigenfunctions of the Landau problem, but not, individually, solutions. Similarly, Fock's wave functions can be a basis for Landau's problem, but they only solve Fock's problem: a single electron can be in a single Fock basis state only if there is an isotropic 2D harmonic potential present, otherwise the circular motion of the orbit center is unphysical. In Landau's wave functions, the orbit center moves linearly, so the question is, is there a related problem for which that motion is physical. Intuition suggests to add a 1D harmonic potential to the magnetic field in Landau's problem, and I prove here that it is basically correct, up to a slight generalization.

Orienting the 1D oscillator along the x-axis, the classical Hamiltonian is

$$\begin{aligned} H &= \frac{1}{2}M(\dot{x}^2 + \dot{y}^2) + \frac{M\omega_0^2}{2}y^2 \\ &= \frac{1}{2M}(p_x^2 + p_y^2) + \frac{\omega_c}{2}(xp_y - yp_x) + \frac{M\omega_c^2}{8}x^2 + \frac{M\Omega^2}{8}y^2, \end{aligned} \tag{16}$$

where $\Omega^2 = \omega_c^2 + 4\omega_0^2$. A representative class of trajectories starts on the y-axis with velocity horizontal; they are

$$x(t) = V_x t - R\frac{\omega_c}{\omega}\sin\omega t, \qquad y(t) = Y_0 + R\cos\omega t, \tag{17}$$

where $\omega^2 = \omega_c^2 + \omega_0^2$, and, importantly,

$$\omega_c V_x = \omega_0^2 Y_0. \tag{18}$$

This constraint already indicates the dual role played by the momentum in Landau's wave function, as it relates the distance from the oscillator axis to the velocity of the orbit center. Some trajectories are shown in Fig. 1. Note that they describe an *ellipse* being translated in the x direction, because the confining potential is anisotropic. Nevertheless, the motion of both projections on the ellipse axes is harmonic, which indicates the problem will still give rise to a quantum oscillator. Accounting for the ellipse requires a generalization of Landau's wave function.

Inserting the solution into the Hamiltonian, the latter becomes

$$\begin{aligned} H &= \frac{1}{2}M\omega^2 R^2 + \frac{1}{2}MV_x^2 + \frac{1}{2}M\omega_0^2 Y_0^2 \\ &= \frac{1}{2}M\omega^2\left(\frac{\omega^2}{\omega_c^2}X^2 + Y^2\right) + \frac{1}{2}M\omega^2\frac{\omega_0^2}{\omega_c^2}Y_0^2, \end{aligned} \tag{19}$$

where, in the second step, it has been prepared for quantization, by collapsing the last two terms with the help of the constraint (18), and expanding the radius in terms of the components along the axes. Explicitly, the new dynamical variables are

$$X_0 \equiv V_x t = \frac{2\omega_0^2 + \omega_c^2}{2\omega^2}x - \frac{\omega_c}{M\omega^2}p_y, \tag{20}$$

$$Y_0 = \frac{\omega_c^2}{2\omega^2}y + \frac{\omega_c}{M\omega^2}p_x, \tag{21}$$

$$X \equiv x - V_x t = \frac{\omega_c^2}{2\omega^2}x + \frac{\omega_c}{M\omega^2}p_y, \tag{22}$$

$$Y \equiv Y_0 - y = -\frac{2\omega_0^2 + \omega_c^2}{2\omega^2}y + \frac{\omega_c}{M\omega^2}p_x, \tag{23}$$

with the only non-zero Poisson brackets

$$\{X, Y\} = \{X_0, Y_0\} = \frac{\omega_c}{M\omega^2}. \tag{24}$$

Standard canonical quantization in the new variables introduces

$$\sqrt{\frac{M\omega^2}{\hbar\omega_c}}X \to \widetilde{u}, \quad \sqrt{\frac{M\omega^2}{\hbar\omega_c}}Y \to -i\frac{\partial}{\partial\widetilde{u}} \tag{25}$$

$$\sqrt{\frac{M\omega^2}{\hbar\omega_c}}X_0 \to \widetilde{s}, \quad \sqrt{\frac{M\omega^2}{\hbar\omega_c}}Y_0 \to -i\frac{\partial}{\partial\widetilde{s}} \tag{26}$$

in terms of which the quantum Hamiltonian reads

$$\widehat{H} = \frac{\hbar\omega}{2}\left(-\partial_{uu} + u^2\right) + \frac{\hbar\omega}{2}\frac{\omega_0^2}{\omega_c^2}(-\partial_{ss}), \tag{27}$$

where the rescaling

$$u = \sqrt{\frac{\omega}{\omega_c}}\widetilde{u}, \quad s = \sqrt{\frac{\omega}{\omega_c}}\widetilde{s}, \tag{28}$$

was found convenient after quantization. The solution is evidently

$$e^{i\nu s}\psi_n(u), \tag{29}$$

where ν is the momentum conjugate to s. In order to compare it with Landau's wave function, it must be Fourier-transformed to real space. To facilitate the comparison, all lengths are now rescaled by $\lambda_0 = \sqrt{\hbar/(M\omega)}$. Then the Fourier transform gives

$$\int dk_y e^{iyk_y} \exp\left[i\nu\left(\frac{2\omega_0^2+\omega_c^2}{2\omega\omega_c}x - k_y\right)\right]\psi_n\left(\frac{\omega_c}{2\omega}x + k_y\right) \sim \exp\left(-i\frac{\omega_c}{2\omega}xy\right)\exp\left(i\frac{\omega}{\omega_c}\nu x\right)\psi_n(y-\nu) \equiv \Psi_L, \qquad (30)$$

which is indeed Landau's wave function in the limit $\omega = \omega_c$, if one can identify $\nu = Y_0$. That identification is correct, because the corresponding contribution to the eigenenergy

$$E_n = \frac{\hbar\omega}{2}(2n+1) + \frac{\hbar\omega}{2}\frac{\omega_0^2}{\omega_c^2}Y_0^2 \qquad (31)$$

is then identical to the classical expression (19), given the scaling by λ_0. This completes the derivation. One can also show directly that the wave function (30) is the eigenfunction of the naively quantized Hamiltonian (16), with the same eigenvalue, as expected.

The above derivation shows that Landau's wave function, strictly speaking, is not a solution of the one-body problem with a 1D potential well. It is the limiting case of such solutions, when the well potential disappears. The classical solution physically constrains the orbit center to stop moving when $\omega_0 = 0$. The wave function (30) does not follow suit: the quantum version of the constraint (18) is the eigenvalue equation (scaling by λ_0)

$$-i\omega_c\frac{\partial}{\partial X_0}\Psi_L = \omega Y_0 \Psi_L, \qquad (32)$$

which survives the limit $\omega_0 \to 0$ ($\omega = \omega_c$): the coordinate Y_0 of the orbit center is the eigenvalue of the translation operator for its coordinate X_0. The wave function merely reduces the moving ellipse to a moving circle, albeit with the energy of the linear motion vanishing, as it becomes "pure gauge."

The situation is closely analogous to the 2D oscillator, where Fock's wave function similarly does not stop the orbit center when $\omega_0 \to 0$. This is particularly obvious in the latter case, because the frequency does not appear in Fock's wave function at all: the moving orbit is also a circle, because the potential is isotropic. In both cases, the wave function raises a persistent gauge current, instead of physically stopping the orbit center, when $\omega_0 \to 0$. The persistent current reflects a spontaneous breaking of translation-gauge symmetry, which is a remnant of the breaking, induced by the external potential, after the latter vanished.

V. DISCUSSION

It has been shown in the present work how the problem of a single electron in a magnetic field can be quantized, and solved, without encountering the infinite degeneracy, related to the choice of orbit-center. It remains to understand how this degeneracy crept into Landau's now-standard approach, where one solves the naively quantized Schrödinger equation by direct attack in real space. Why did Landau obtain a basis with moving orbit centers as the solution, instead of a solution with fixed center, say at the origin? The analogy with the hydrogen atom in the Introduction shows that something is indeed amiss: if one puts the proton at rest at the beginning, one does not expect it to start moving by the end of the calculation.

The classical formulations of Fock's and Landau's problems already show why naive quantization suffices for the former, but not for the latter. Specific linear combinations of positions and momenta, x_0 and y_0 in Eq. (4), have the physical meaning of orbit center coordinates. They can thus be arbitrarily set to a constant value, say zero, in Landau's problem, but not in Fock's, where the orbit center itself moves under Hamiltonian action. At the same time, the Poisson bracket $\{x_0, y_0\} \neq 0$. This leads to a contradiction in the naive quantization of Landau's problem, which "blindly" promotes Poisson brackets $\{x, p_x\}$ and $\{y, p_y\}$ to commutators with a constant value, because then one can construct the linear combination $[0, 0] = i\hbar$. Such contradictions were first investigated by Dirac himself [5], who developed special methods of "constrained quantization" to eliminate them. In the Landau case, constrained quantization would redefine the Poisson brackets $\{x, p_x\}$ and $\{y, p_y\}$, so as to set their particular combination $\{x_0, y_0\}$ to zero. The present work is a pedagogical example of an alternative procedure, which is available whenever a canonical transformation is known, in which the gauge variables appear as conjugate pairs in their own right. Such is the transformation of Eq. (4), which isolates the translation-gauge variables x_0 and y_0. Then one can simply use standard canonical quantization in the new variables, and observe that some conjugate pairs do not appear in the Hamiltonian. In technical language, the gauge Hamiltonian (which is zero!) has been uncoupled from the physical Hamiltonian. More intuitively, the problem has been dimensionally reduced, to the subspace of only those dynamical variables

which do appear in the Hamiltonian. The contradiction $[0,0] = i\hbar$ can no longer be constructed, because y_0 has been replaced by $-i\hbar\partial_{x_0}$. The only price is that non-intuitive independent variables can appear in first quantization, such as the velocity component v_y in Eq. (9), making an integral transform necessary to obtain the variables of choice.

Historically, Landau [1] did notice that after naive quantization, the commutator between velocity components was not zero, but used this only to argue, correctly, that the spectrum must be that of the harmonic oscillator. His own use of his wave functions in the calculation of the orbital magnetic response was limited to an estimate of the one-particle density of states, for which he had to count their degeneracy in the quantum number k_y for each oscillator state n, given by Eq. (10). Because each of Landau's wave functions is exactly one Fourier component of the solution in Eq. (10), his approach amounts to counting the number of terms in a resolution of unity, so that his subsequent reasoning is not affected by the present development. Remarkably, Landau's counting of the degeneracy in terms of x and k_y reveals the physical insight, that these are the true independent variables of his wave function, not x and y, despite its ostensibly real-space derivation. This insight is formally proven correct by Eq. (10) above.

The solution of a differential equation is physical if it satisfies the physical symmetries and boundary conditions of the problem. In Landau's problem, the relevant condition is that the orbit center be motionless. It is satisfied by the coherent eigenstates of Eq. (13), as well as by the solution (10). Notably, the latter was obtained without introducing this condition *a priori*. Canonical quantization of the one-body problem is thus naturally free of the degeneracy associated with breaking of translational invariance.

In the next step, when considering the many-body problem, states are constructed by putting whole physical electrons into individual degenerate basis states. Then a physical choice must be made, how to describe the degeneracy. On the one side are bases which break the translation-gauge symmetry by a persistent orbit-center current, such as Landau's original wave functions [1], or general ($n_- \neq 0$) solutions of the Fock problem, Eq. (8). These are delocalized in real space, because the orbit center moves, as signalled by the appearance of a spurious conserved momentum, linear or angular, respectively. On the other side is the basis of localized coherent states (13), where the orbit center is static, and each basis state is by itself a physical solution of the Landau problem. If the first choice is made, in a many-body perturbation setting, it is to be expected on general grounds that the symmetry broken at zeroth order is not subsequently restored. One should check in such cases, that the spurious orbit-center current in the basis vectors disappears from the final results, or, in more technical language, that these results are properly projected onto the physical subspace, which is invariant to both the translation and electromagnetic gauges.

An alternative canonical quantization of Landau's problem, shown in the Appendix, is to decompose the motion into the coordinates of the orbit center, as above, and the coordinates relative to the center, instead of the velocities above. Nothing new is obtained, but it is most obvious in that formulation, that the oscillators in Landau's and Fock's problems are simply a quantum-physical realization of the elementary observation, that uniform motion on a circle projects onto oscillatory motion on any axis through the center. On the other hand, it is more immediately clear in the velocity formulation, that the first (Landau) part of Fock's Hamiltonian (6) has no reference point in space, while the second, in which the orbit-center coordinate x_0 oscillates, obviously refers to the origin of the harmonic potential. Notably, the time-reversal symmetry breaking inherent in the magnetic field is not manifest in the canonically quantized Hamiltonian (9), which is real, but is innocuously hidden in the choice, to which of the two components of velocity its variable v refers: v_y (v_x) for a negatively (positively) charged particle.

To conclude, the problem of a single electron in a constant magnetic field has been solved by canonical quantization in velocity space. In accord with classical intuition, the electron cannot work radially against the effective potential created by the magnetic field, so all its energy is in angular motion around an arbitrary center. Its wave function has only one, Landau-level, quantum number, physically the angular momentum around that center, in which it is not degenerate. The degeneracy of Landau's and some other solutions in a second quantum number stems from an incorrect treatment of the arbitrariness of the center. Physically, it means that they contain components of spurious (gauge) motion of the orbit center, or, equivalently, that fixing the center has not been properly separated from the calculation of the internal (one-body) motion relative to a given center. One should not jump to the conclusion that all work based on such wave functions was incorrect. Much depends on how they were used, and how the result was interpreted physically. It seems appropriate to wait some time for those most familiar with particular calculations to review them in the light of the above results.

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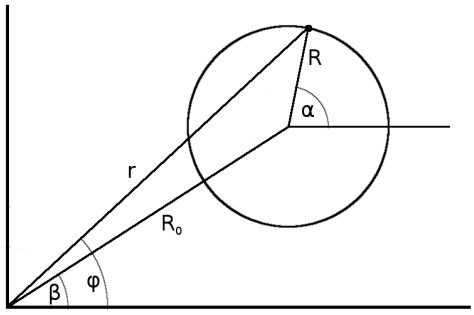

FIG. 2: Classical solution of the Landau problem. Here $\alpha = \omega_c t$.

Appendix A: The role of angular momentum

It is useful to consider the classical solution to the Landau problem in polar coordinates in light of the above discussion. The trajectory can be written by inspection, referring to Fig. 2 for the notation:

$$r(t) = \sqrt{R_0^2 + R^2 + 2R_0 R\cos(\alpha - \beta)}, \quad \alpha = \omega_c t. \tag{A1}$$

Inserting this solution into the canonical equations of motion, one easily finds the canonical angular momentum to be

$$p_\phi = \frac{M\omega_c}{2}(R^2 - R_0^2) \equiv p_\alpha - p_\beta, \tag{A2}$$

where the sign of p_ϕ has been chosen so that the Hamiltonian gives the known energy of rotational motion, $H = M\omega_c^2 R^2/2$. These classical results clearly show how the naive quantization $p_\phi \to -i\partial_\phi$ admixes the orbit's fixed distance R_0 from an arbitrary origin and its physical angular momentum p_α into the eigenvalue $m = n_+ - n_-$. A large negative p_ϕ simply means that one is far from the origin (determined by the zero of the vector potential), as measured by the spurious contribution p_β, which does not refer to anything that moves. Conversely, p_ϕ does give the physical angular momentum if $R_0 = 0$, when it achieves its maximum positive value. This is the classical analogue of setting $n_- = 0$ to obtain Eq. (12).

After the coordinate transformation $z, \bar{z} = x \pm iy$, the quantum angular momentum acquires the well-known form

$$L_z = -i\partial_\phi = z\partial_z - \bar{z}\partial_{\bar{z}}, \tag{A3}$$

which identifies the physical angular momentum p_α with the physical-motion term $z\partial_z$, while the spurious p_β corresponds to the gauge-motion term $\bar{z}\partial_{\bar{z}}$. But $z\partial_z$ is just the term in the eigenvalue equation (14), or equivalently in Eq. (2), whose eigenvalue is the quantum number n in the Landau-level spectrum. Hence all the energy of the quantum system is in the angular momentum around the physical orbit center, corresponding to the classical p_α. (The analogous classical statement is $H = \omega_c p_\alpha$.)

It is possible to quantize this problem canonically in the variables $X = R\cos\alpha$, $Y = R\sin\alpha$, $X_0 = R_0\cos\beta$, and $Y_0 = R_0\sin\beta$, in terms of which the Hamiltonian is

$$H = \frac{M\omega_c^2}{2}\left(X^2 + Y^2\right) = \frac{\omega_c}{2}\left(\widetilde{X}^2 + \widetilde{Y}^2\right), \tag{A4}$$

and indeed $\{\widetilde{Y}, \widetilde{X}\} = 1$. Similarly $\{X_0\sqrt{M\omega_c}, Y_0\sqrt{M\omega_c}\} = \{\widetilde{X}_0, \widetilde{Y}_0\} = 1$. However, nothing new is obtained, because there is a fixed relationship between the radius components (X, Y) and the velocity components (v_x, v_y), which were quantized in Eq. (3). On the other hand, it is perhaps most obvious in this formulation, that the harmonic oscillators in Fock's and Landau's problems are nothing but a quantum-physical realization of the elementary observation, that uniform motion on a circle projects to oscillatory motion on any axis through the center.

The above quantization shows that the single half-quantum of zero-point motion comes simply from the uncertainty of the projected position, which is one-dimensional. If one insists on imagining the particle on the parent circular orbit, the uncertainty smoothly oscillates between being wholly radial near the turning-points of the projection, and

wholly angular near the mid-point. Such a separation of the uncertainty into radial and angular parts is arbitrary, because the axis of projection may be chosen at will.

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