Box dimension and Minkowski content of generalized Euler spirals

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Introduction

By $d(x, A)$ we denote the Euclidean distance from x to a given subset A in $\mathbb{R}^m$. Let A_ε be the open ε-neighbourhood of A, called also the Minkowski sausage of radius ε around A. The upper s-dimensional Minkowski content of a bounded subset A in $\mathbb{R}^m$, $s \geq 0$, is defined as follows

$$\mathcal{M}^{*s}(A) := \limsup_{\varepsilon\to 0} \frac{|A_\varepsilon|}{\varepsilon^{m-s}},$$

where $|A_\varepsilon|$ denotes the m-dimensional Lebesgue measure of A_ε. The lower s-dimensional Minkowski content of A is defined by

$$\mathcal{M}^{s}_{*}(A) := \liminf_{\varepsilon\to 0} \frac{|A_\varepsilon|}{\varepsilon^{m-s}}.$$

If $\mathcal{M}^{*s}(A) = \mathcal{M}^{s}_{*}(A)$, the common value is denoted by $\mathcal{M}^{s}(A)$. The upper box dimension of A is defined by

$$\overline{\dim}_B A = \inf\{s \geq 0 : \mathcal{M}^{*s}(A) = 0\} = \sup\{s \geq 0 : \mathcal{M}^{*s}(A) = \infty\}.$$

Similary for the lower box dimension of A, denoted by $d = \underline{\dim}_B A$. If both of them are equal, the common value is called the box dimension of A, and denoted by $d = \dim_B A$.
If $0 < \mathcal{M}^{d}_{*}(A) \leq \mathcal{M}^{*d}(A) < \infty$, we say that A is Minkowski nondegenerate. If there exists $\mathcal{M}^{d}(A)$ for some d and $\mathcal{M}^{d}(A) \in (0, \infty)$, we say that A is Minkowski measurable.
Given any two functions f and $g : (a, \infty) \to (0, \infty)$ we write $f \sim g$ as $t \to \infty$ if $\lim_{t\to\infty} f(t)/g(t) = 1$.

p-clothoid

By p-clothoid, $p > 1$, we mean a planar curve defined parametrically by

$$\Gamma_p \cdots \begin{cases} x(t) = \int_0^t \cos(s^p)\, ds, \\ y(t) = \int_0^t \sin(s^p)\, ds, \end{cases} \qquad (1)$$

where $t \geq 0$. If we replace s^p by $|s|^p$ in (1), then we may allow $t \in \mathbb{R}$.
For $p = 2$ we obtain the standard clothoid, or the Euler spiral.
For the classical, 2-clothoid, we know that its box dimension is equal to $4/3$, see [1, Theorem 1].

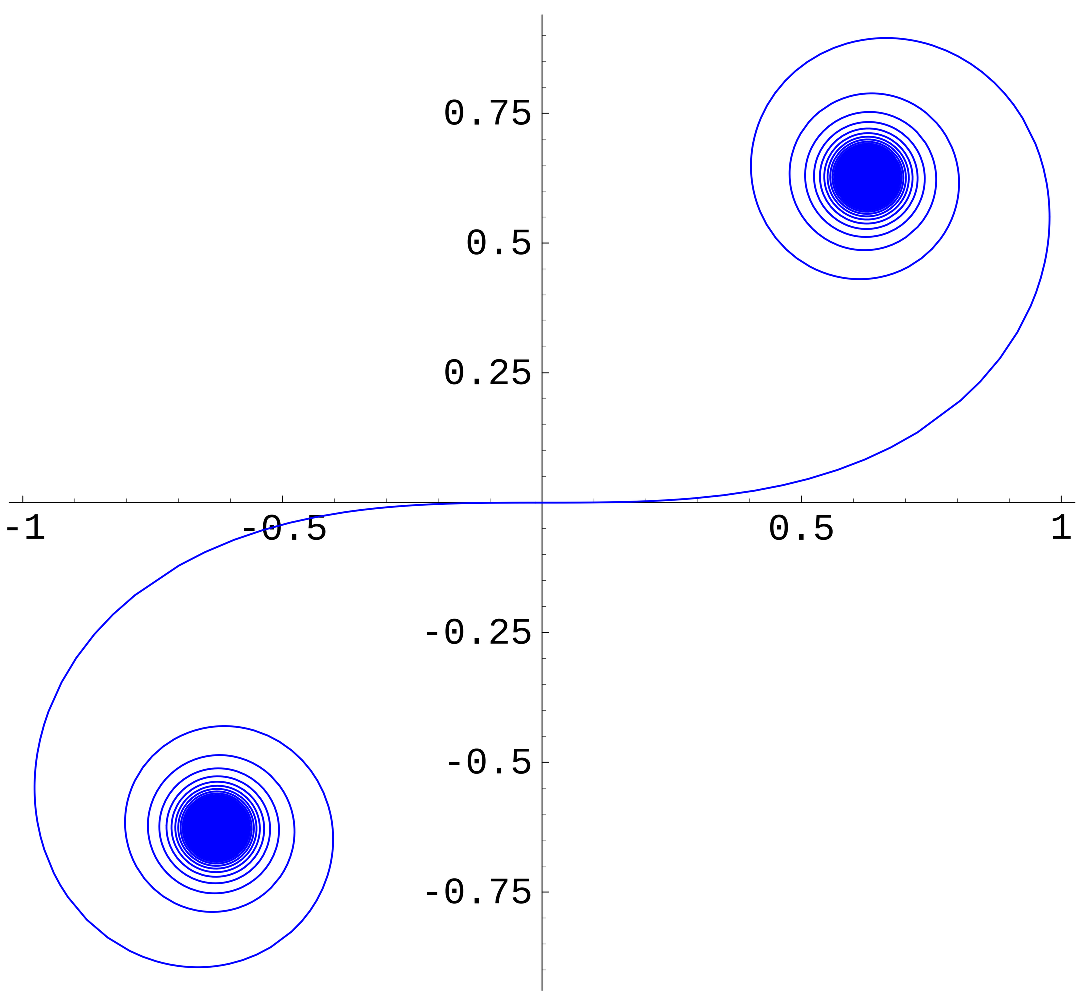

Figure 1: Euler spiral

This is a special case of the following result dealing with generalized Euler spirals.

Theorem 1 *Let Γ_p be the p-clothoid defined by (1), $p > 1$. Then $d = \dim_B \Gamma_p = 2p/(2p-1)$. Furthermore, Γ_p is Minkowski measurable and*

$$\mathcal{M}^d(\Gamma_p) = (2p-1)\left(p(p-1)^{p-1}\right)^{-2/(2p-1)} \pi^{1/(2p-1)}. \qquad (2)$$

In the proof of this theorem we use the following asymptotic expansion of Fresnel integrals associated to generalized Euler spirals. The result can be obtained using known expansions of Fresnel integrals based on complex variables and the gamma function. We propose a new, very short and elementary proof.

Theorem 2 *Let $x(t)$ and $y(t)$ be generalized Fresnel integrals defined by (1), $p > 1$, and $a = \lim_{t\to\infty} x(t)$, $b = \lim_{t\to\infty} y(t)$. Then for any nonnegative integer N we have*

$$\begin{cases} x(t) = a + A_N(t)\sin(t^p) - B_N(t)\cos(t^p) + O(t^{-(2N+3)p+1}) \\ y(t) = b - B_N(t)\sin(t^p) - A_N(t)\cos(t^p) + O(t^{-(2N+3)p+1}), \end{cases}$$

when $t \to \infty$, where

$$\begin{cases} A_N(t) = \sum_{k=0}^{N} (-1)^k a_{2k} t^{-(2k+1)p+1} \\ B_N(t) = \sum_{k=0}^{N} (-1)^k a_{2k+1} t^{-(2k+2)p+1} \\ \quad a_n = p^{-n-1}(p-1)(2p-1)\dots(np-1), \quad n \geq 1 \text{ and } a_0 = p^{-1}. \end{cases}$$

Together with the asymptotic expansion of Fresnel integrals, to prove Theorem 1, we will exploit a result from [3] which we cite here in a simplified, but equivalent form.

Theorem 3 (Minkowski measurable spirals) *Assume that $f : [\varphi_1, \infty) \to (0, \infty)$ is a decreasing, C^2 function converging to zero, and $\varphi_1 > 0$. Assume that there exists the limit*

$$m := \lim_{\varphi\to\infty} \frac{f'(\varphi)}{(\varphi^{-\alpha})'}.$$

Let there be a positive constant C such that $|f''(\varphi)| \leq C\varphi^{-\alpha}$ for all $\varphi \geq \varphi_1$. Let Γ be the graph of the spiral $\rho = f(\varphi)$ with $\alpha \in (0, 1)$, and define $d := 2/(1+\alpha)$. Then $\dim_B \Gamma = d$, the spiral is Minkowski measurable, and moreover,

$$\mathcal{M}^d(\Gamma) = m^d \pi(\pi\alpha)^{-2\alpha/(1+\alpha)} \frac{1+\alpha}{1-\alpha}.$$

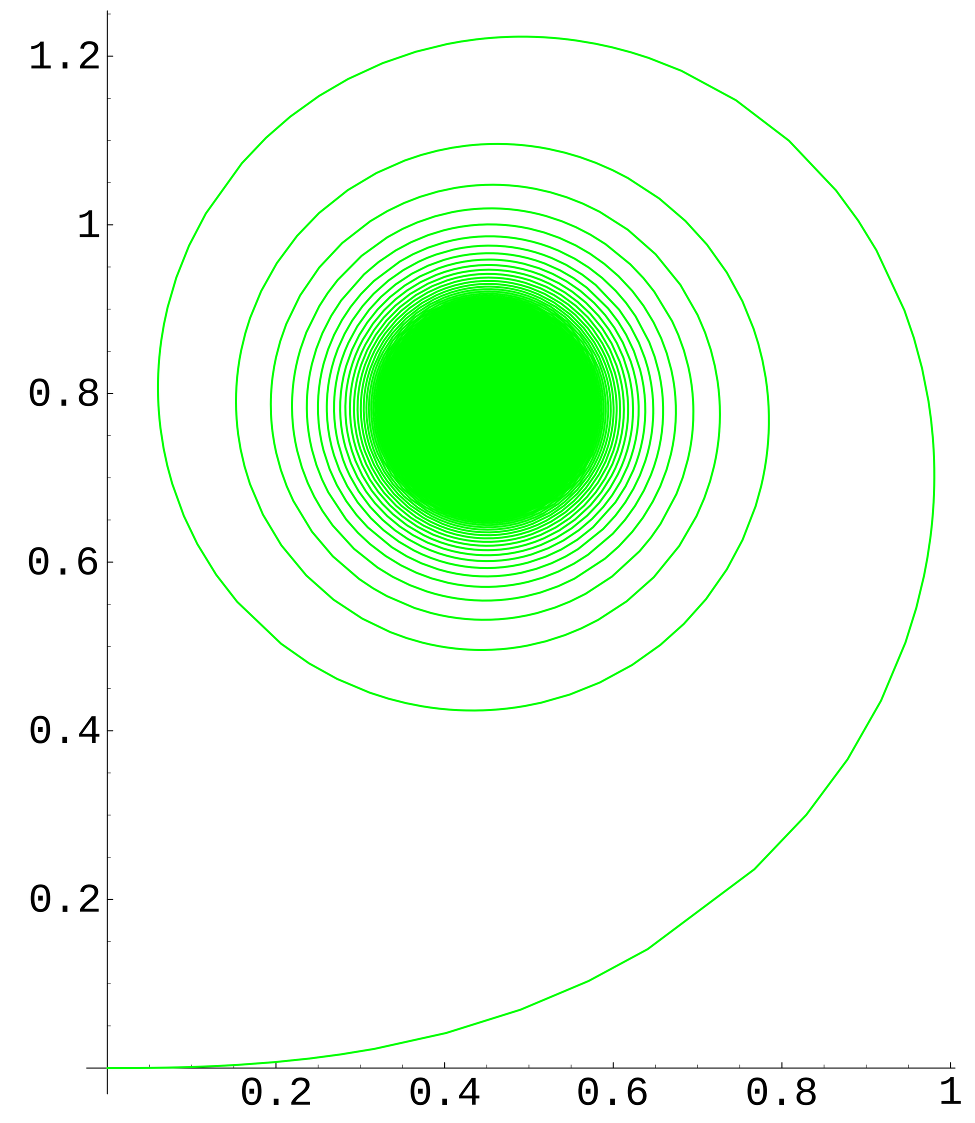

Figure 2: p-clothoid for $p = 3/2$

q-clothoid

Let $q : (0, \infty) \to \mathbb{R}$ be a given function such that $q(t) \sim t^p$, $p > 1$, when $t \to \infty$. By the clothoid generated by control function q, or q-clothoid Γ_q, we mean a planar curve defined parametrically by

$$\Gamma_q \cdots \begin{cases} x(t) = \int_0^t \cos(q(s))\, ds \\ y(t) = \int_0^t \sin(q(s))\, ds. \end{cases} \qquad (3)$$

Now we have even more general situation than in case od p-clothoid.

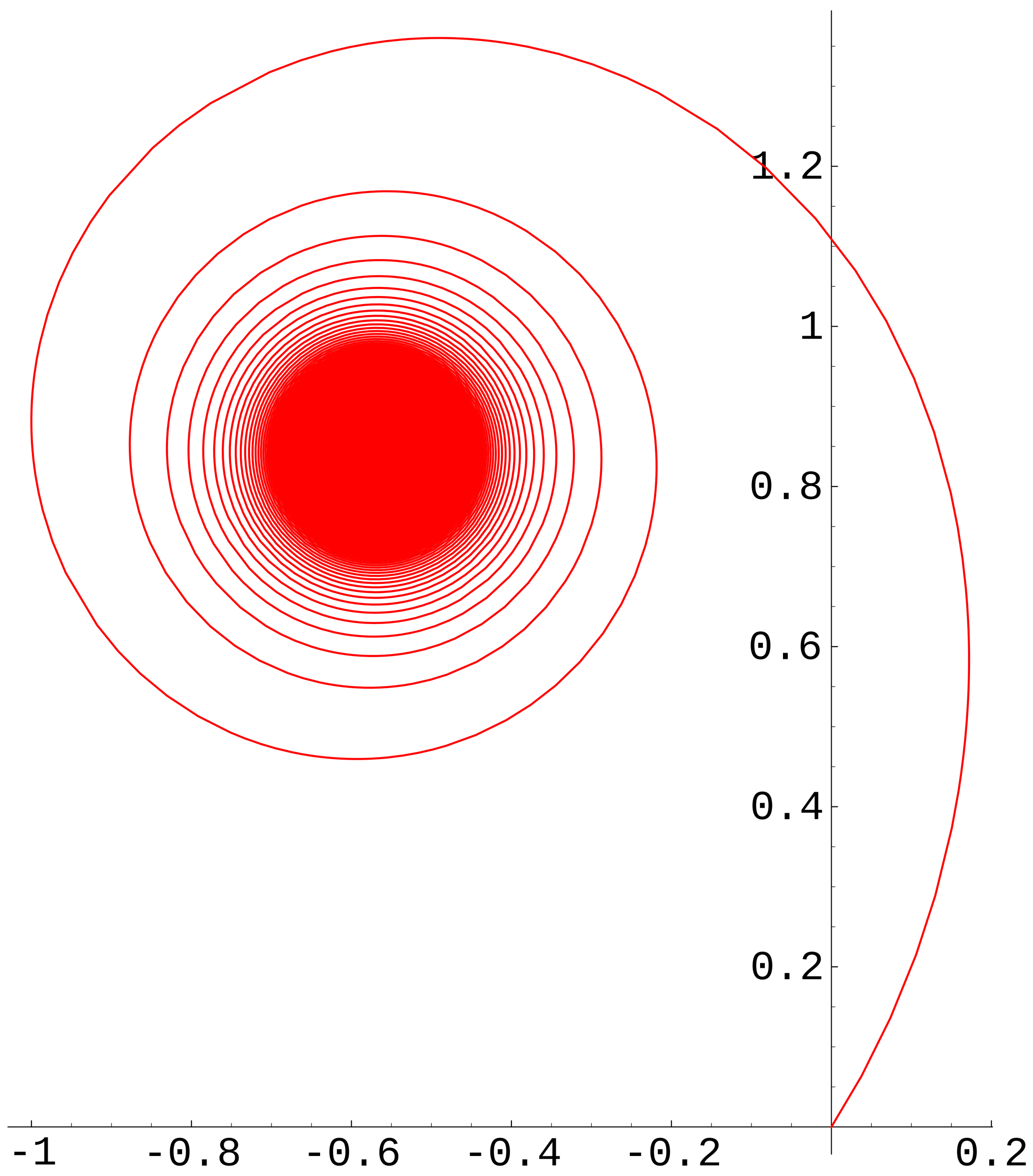

Figure 3: q-clothoid for $q(s) = \sqrt{s^3 + 2s + 1}$

We first establish sufficient conditions needed for convergence of Generalized Fresnel integrals (3) when $t \to \infty$.

Lemma 1 *Let us assume that $q(t)$ is increasing and convex C^1 function, and $q'(t) \to \infty$ as $t \to \infty$. Then the limits*

$$\lim_{t\to\infty} x(t) = a, \quad \lim_{t\to\infty} y(t) = b$$

of functions defined by (3) exist.

Now we describe an algorithm for the asymptotic expansion of Generalized Fresnel integrals. First, assuming that $q(t)$ is sufficiently smooth, we introduce a sequence of auxiliary functions $D_n(t)$, $n \geq 0$, as follows:

$$D_0(t) = \frac{1}{q'(t)}, \quad D_{n+1}(t) = \frac{(-1)^{n+1}}{q'(t)} D_n'(t), \; n \geq 0.$$

Let us define remainder terms $R_N^{(i)}(t)$, $i = 1, 2$, by

$$\begin{aligned} x(t) &= a + \sin q(t) \sum_{k=0}^{\lfloor \frac{N}{2} \rfloor} D_{2k}(t) - \cos q(t) \sum_{k=0}^{\lfloor \frac{N-1}{2} \rfloor} D_{2k+1}(t) + R_N^{(1)}(t) \\ y(t) &= b - \sin q(t) \sum_{k=0}^{\lfloor \frac{N-1}{2} \rfloor} D_{2k+1}(t) - \cos q(t) \sum_{k=0}^{\lfloor \frac{N}{2} \rfloor} D_{2k}(t) + R_N^{(2)}(t), \end{aligned}$$

where $N \geq 1$.
In the following theorem we find some sufficient conditions for $R_{N-1}^{(i)}(t)$ to be small for sufficiently large t. It extends Theorem 2.

Theorem 4 *Let $x(t)$ and $y(t)$ be defined by (3), where the control function $q : (0, \infty) \to \mathbb{R}$ is increasing, convex, of class C^{N+2}, $N \geq 1$, and $q(t) \sim t^p$ as $t \to \infty$, and $p > 1$. Assume that $D_n(t) \to 0$ as $t \to \infty$ for all $n = 0, 1, \dots, N-1$. If $D_N(t) = O(t^{-(N+1)p+1})$ and $D_N'(t) = O(t^{-(N+1)p})$, then for any γ such that $Np - 1 < \gamma < (N+1)p - 1$, we have $R_{N-1}^{(i)}(t) = o(t^{-\gamma})$ as $t \to \infty$, for $i = 1, 2$.*

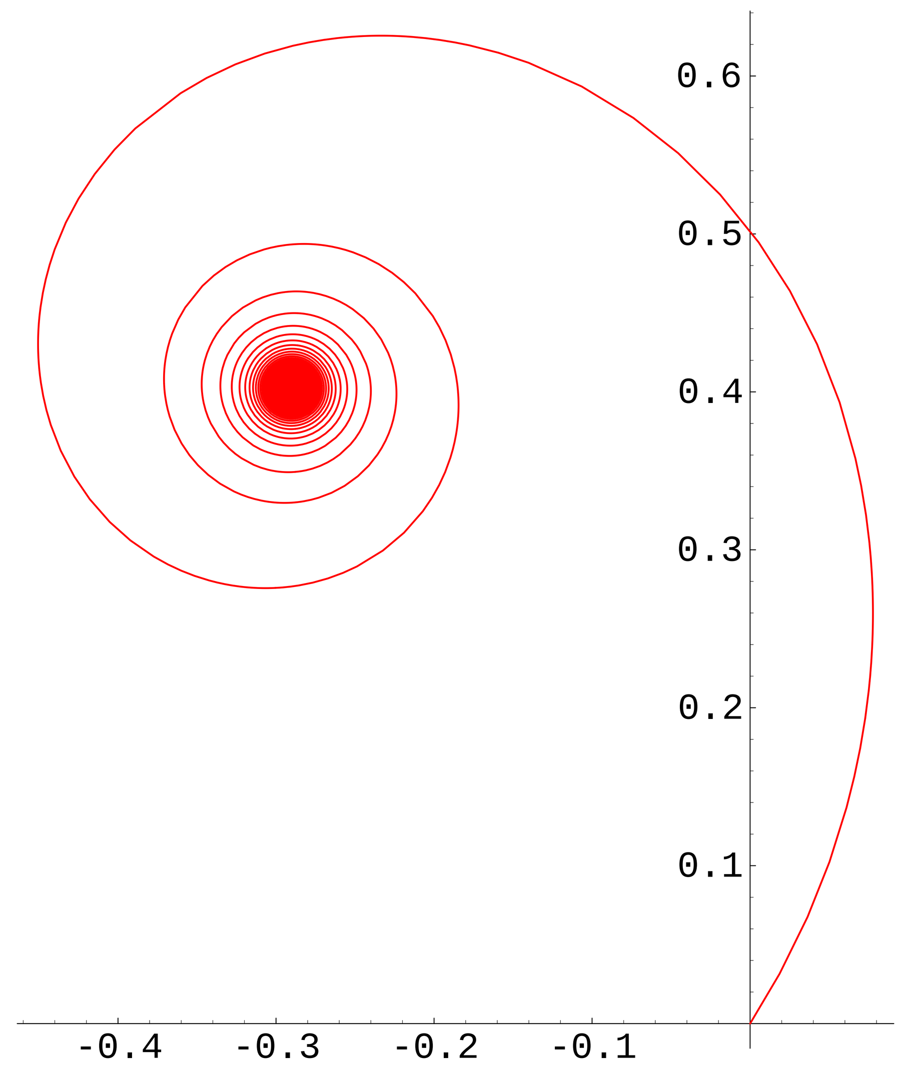

Figure 4: q-clothoid for $q(s) = s^3 + 2s + 1$

Finally we give generalization of Theorem 1 regarding control function q.

Theorem 5 *Assume that $q : (0, \infty) \to \mathbb{R}$ is* **increasing**, **convex**, *and of* **class** C^5. *Let*

$$\begin{cases} q(t) \sim t^p, \quad \dot{q}(t) \sim p\, t^{p-1} \\ \ddot{q}(t) \sim p(p-1)t^{p-2}, \quad q^{(3)}(t) \sim p(p-1)(p-2)t^{p-3} \\ q^{(4)}(t) = O(t^{p-4}), \quad q^{(5)}(t) = O(t^{p-5}), \quad \text{as } t \to \infty \end{cases}$$

be satisfied. Then $d = \dim_B \Gamma_q = 2p/(2p-1)$. Furthermore, the spiral Γ_q is Minkowski measurable, and its d-dimensional Minkowski content is equal to the value in (2).

Example. Theorem 5 applies for example to control functions $q(t) = t^3 + 2t + 1$ with $p = 3$, or to $q(t) = \sqrt{t^3 + 2t + 1}$ with $p = 3/2$, etc.

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References

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