Higher-order linked interpolation in triangular thick plate finite elements

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Abstract

Purpose – In this work, the authors aim to employ the so-called linked-interpolation concept already tested on beam and quadrilateral plate finite elements in the design of displacement-based higher-order triangular plate finite elements and test their performance.

Design/methodology/approach – Starting from the analogy between the Timoshenko beam theory and the Mindlin plate theory, a family of triangular linked-interpolation plate finite elements of arbitrary order are designed. The elements are tested on the standard set of examples.

Findings – The derived elements pass the standard patch tests and also the higher-order patch tests of an order directly related to the order of the element. The lowest-order member of the family of developed elements still suffers from shear locking for very coarse meshes, but the higher-order elements turn out to be successful when compared to the elements from literature for the problems with the same total number of the degrees of freedom.

Research limitations/implications – The elements designed perform well for a number of standard benchmark tests, but the well-known Morley’s skewed plate example turns out to be rather demanding, i.e. the proposed design principle cannot compete with the mixed-type approach for this test. Work is under way to improve the proposed displacement-based elements by adding a number of internal bubble functions in the displacement and rotation fields, specifically chosen to satisfy the basic patch test and enable a softer response in the bench-mark test examples.

Originality/value – A new family of displacement-based higher-order triangular Mindlin plate finite elements has been derived. The higher-order elements perform very well, whereas the lowest-order element requires improvement.

Keywords Higher-order linked interpolation, Mindlin plate theory, Triangular plate finite elements

Paper type Research paper

1. Introduction

Owing to its close relationship with the Timoshenko theory of thick beams, the idea of linking the displacement field to the rotations of the cross-sections has been often studied and thoroughly investigated and exploited in finite-element applications of the Mindlin moderately thick plate theory (Auricchio and Taylor, 1993, 1994; Ibrahimbegović, 1993; Chinosi and Lovadina, 1995; Auricchio and Lovadina, 2001; Taylor and Govindjee, 2002; Zienkiewicz and Taylor, 2005; Liu and Riggs, 2005; de Miranda and Ubertini, 2006; Crisfield, 1986; Zienkiewicz et al., 1993; Taylor and Auricchio, 1993; Xu et al., 1994; Chen and Cheung, 2000, 2001, 2005). It has been found out that the idea on its own cannot eliminate the problem of shear locking even though this result may be achieved for the Timoshenko beam elements (Zienkiewicz and
Taylor, 2005; Tessler and Dong, 1981; Jelenić and Papa, 2011; Przemieniecki, 1968; Rakowski, 1990; Yunhua, 1998; Reddy, 1997; Mukherjee et al., 2001). Different improvements have been proposed by different authors, which involve adjusted material parameters (Tessler and Hughes, 1985) or the assumed or enhanced strain concepts (Ibrahimbegović, 1993; Chen and Cheung, 2000, 2001; Simo and Rifai, 1990; Bathe et al., 1989; Lee and Bathe, 2004; Kim and Bathe, 2009) or are based on the use of mixed and hybrid approaches (Auricchio and Taylor, 1993, 1994; Taylor and Govindjee, 2002; de Miranda and Ubertini, 2006; Zienkiewicz et al., 1993; Taylor and Auricchio, 1993).

In this paper, we build on the ideas given in Ribarić and Jelenić (2012) where we have re-visited this classic topic and, remaining firmly in the framework of the standard displacement-based design technique, derived a family of quadrilateral thick plate elements by extending higher-order linked interpolation functions developed for the Timoshenko beams. Here, we apply the methodology to the popular class of triangular elements and contrast it to an alternative methodology of devising higher-order linked interpolation for this class of elements (Liu and Riggs, 2005).

In Section 2, we present the family of interpolation functions for the Timoshenko beam elements which provide the exact solution for arbitrary polynomial loadings (Jelenić and Papa, 2011). Even though the Mindlin theory of thick plates may be regarded as a 2D generalisation of the Timoshenko theory of thick beams, the differential equations of equilibrium for thick plates cannot be solved in terms of a finite number of parameters and so, in contrast to beams, there does not exist an exact finite-element interpolation. Nonetheless, in Auricchio and Taylor (1993, 1994, 1995) such interpolation has been used to formulate three-node triangular and four-node quadrilateral thick plate elements, while in Ibrahimbegović (1993) and Taylor and Govindjee (2002) a six-node triangular and an eight-node quadrilateral elements have been proposed. A family of triangular elements designed in this way has been proposed in Liu and Riggs (2005).

In Section 3, we outline the Mindlin plate theory and continue by considering a triangular three-node element, for which the constant shear strain condition imposed on the element edges is known to lead to an interpolation for the displacement field which is dependent not only on the nodal displacements, but also on the nodal rotations around the in-plane normal directions to the element edges. The same result may be obtained by generalising the linked interpolation for beams (Jelenić and Papa, 2011) to 2D situations. This approach enables a straightforward generalisation of the linked-interpolation beam concept to higher-order triangular plate elements leading to additional internal degrees of freedom which do not exist in the beam elements. A similar goal may be achieved following the approach presented in Liu and Riggs (2005) where a family of displacement-based linked-interpolation triangular elements has been derived by prescribing the order of the shear distribution over the element which, in contrast, does not involve internal degrees of freedom. Generalising either of these ideas to arbitrary curvilinear triangular shapes (on higher-order elements) is non-trivial and special care needs to be taken for such elements to satisfy the standard patch tests.

In Section 4, we compare the two approaches and in Section 5 summarise the finite-element results. Finally, in Section 6 we conduct numerical tests and in Section 7 draw the conclusions.
2. Solution of the Timoshenko beam problem for polynomial loading using linked interpolation of appropriate order

In contrast to the Bernoulli beam theory, in the Timoshenko beam theory the cross section of a beam remains planar after the deformation, but not necessarily orthogonal to the beam centroidal axis. This departure from orthogonality is the shear angle:

\[ \gamma = \frac{dw}{dx} + \theta = w' + \theta \]

where \( w \) is the lateral displacement of the beam shown in Figure 1, the dash (') indicates a differentiation with respect to the co-ordinate \( x \), and \( \theta \) is the rotation of a cross section.

Let, the cross-sectional stress-couple and shear stress resultants \( M \) and \( S \) be linearly dependent on curvature (change of cross-sectional rotation) and shear angle via \( M = EI \theta' \) and \( S = GA_s \gamma \), where \( EI \) and \( GA_s \) are the bending and shear stiffness, respectively. As the equilibrium equations are \( M' = S \) and \( S' = -q \), where \( q \) is the distributed loading per unit of length of the beam, this results in the following differential equations:

\[ EI \theta''' = -q, \quad GA_s (w'' + \theta') = -q, \]

with the following closed-form solution for a polynomial loading \( q \) of order \( n - 4 \) (Jelenić and Papa, 2011):

\[ \theta = \sum_{i=1}^{n} I_i \theta_i, \quad w = \sum_{i=1}^{n} I_i w_i - \frac{L}{n} \prod_{j=1}^{n} N_j \sum_{i=1}^{n} (-1)^{i-1} \binom{n-1}{i-1} \theta_i, \]

where \( L \) is the beam length, \( \theta_i \) and \( w_i \) are the values of the displacements and the rotations at the \( n \) nodes equidistantly spaced between the beam ends, \( I_i \) are the...
3. Overview of the Mindlin plate theory and a family of triangular linked-interpolation elements

The Mindlin plate theory is closely related to the Timoshenko beam theory and may be regarded as its generalisation to two-dimensional problems. The plate is assumed to be of a uniform thickness $h$ with a mid-surface lying in the horizontal co-ordinate plane and a distributed loading $q$ assumed to act on the plate mid-surface in the direction perpendicular to it. The changes of the angles which the vertical fibres close with the mid-surface are the shear angles:

$$G = g_{xz} - g_{yz} = u_y + \frac{\partial w}{\partial x} - u_x + \frac{\partial w}{\partial y}.$$

while the curvatures (the fibre's changes of rotations) are:

$$\kappa = \begin{cases} \kappa_x = \begin{bmatrix} \frac{\partial \theta_y}{\partial x} - \frac{\partial \theta_x}{\partial y} \\ \frac{\partial \theta_y}{\partial x} \end{bmatrix} = \begin{bmatrix} 0 & \frac{\partial \theta_x}{\partial y} \\ -\frac{\partial \theta_x}{\partial y} & 0 \end{bmatrix} \end{cases} \begin{bmatrix} \theta_x \\ \theta_y \end{bmatrix} = L \theta$$

where $\theta$ is the rotation vector with components $\theta_x$ and $\theta_y$ around the respective horizontal global co-ordinate axes, $w$ is the transverse displacement field, $\Gamma$ is the shear strain vector and $\nabla w$ is a gradient on the displacement field, $L$ is a differential operator on the rotation field (Auricchio and Taylor, 1994). Let us consider a linear elastic material with:

$$M = \begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \frac{Eh^3}{12(1 - \nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1 - \nu}{2} \end{bmatrix} \begin{bmatrix} \frac{\partial \theta_y}{\partial x} \\ \frac{\partial \theta_x}{\partial y} \end{bmatrix} = D_b \kappa = D_b L \theta$$

$$S = \begin{bmatrix} S_x \\ S_y \end{bmatrix} = kGh \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} = D_s \Gamma = D_s(e\theta + \nabla w),$$

where $M_x$, $M_y$, and $M_{xy}$ are the bending and twisting moments around the respective co-ordinate axes, $S_x$ and $S_y$ are the shear-stress resultants, $E$ and $G$ are the Young and shear moduli, while $\nu$ and $k$ are Poisson’s coefficient and the shear correction factor usually set to 5/6. The differential equations of equilibrium are:
\[
\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} = S_x, \quad \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} = S_y, \quad \frac{\partial S_x}{\partial x} + \frac{\partial S_y}{\partial y} = -q. \tag{5}
\]

Substituting equation (4) in equation (5), results in the differential equations which now cannot be solved in terms of a finite number of parameters as before. Still, we shall attempt to extend the results from Section 2 in order to derive more accurate Mindlin plate elements. To do so, we shall need the functional of the total potential energy:

\[
\Pi(w, \theta_x, \theta_y) = \frac{1}{2} \int (M^T\kappa) dA + \frac{1}{2} \int (S^T\Gamma) dA + \Pi_{\text{ext}} \\
= \frac{1}{2} \int (\kappa^T D_b \kappa) dA + \frac{1}{2} \int (\Gamma^T D_s \Gamma) dA + \Pi_{\text{ext}}, \tag{6}
\]

where the last term describes the potential energy of the distributed and boundary loading.

**3.1 Linked interpolation for a three-node triangular plate element**

We shall first apply the result given in equation (1) to a triangular element with three nodal points at the element vertices as in Liu and Riggs (2005), Taylor and Auricchio (1993), Chen and Cheung (2001), Tessler and Hughes (1985), Auricchio and Taylor (1995) and Zhu (1992) (Figure 2). The displacements and rotations are expressed in the so-called area coordinates which, for any interior point, make the ratio of the respective interior area to the area of the whole triangle 1-2-3 as shown in Figure 2.

Because in two dimensions any point is uniquely defined by only two coordinates, the three coordinates, \(\xi_1, \xi_2\) and \(\xi_3\) are not independent of each other and for any point within the domain they are related by the expression:

\[\xi_1 + \xi_2 + \xi_3 = 1.\]
The area coordinates of any point within the domain are transformed into the Cartesian coordinates as:

\[ x = \xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3 \]
\[ y = \xi_1 y_1 + \xi_2 y_2 + \xi_3 y_3 \]

and vice versa:

\[ \xi_1 = \frac{(x - x_3)b_1 + (y - y_3)a_1}{a_2b_1 - a_1b_2} = \frac{2A_1}{2A} = \frac{A_1}{A} \]
\[ \xi_2 = \frac{(x - x_1)b_2 + (y - y_1)a_2}{a_3b_2 - a_2b_3} = \frac{A_2}{A} \]
\[ \xi_3 = \frac{(x - x_2)b_3 + (y - y_2)a_3}{a_1b_3 - a_3b_1} = \frac{A_3}{A} \]

where \( a_i = x_k - x_i \) and \( b_i = y_j - y_i \) are the directed side-length projections along the coordinate axes and the indices \( i, j, \) and \( k \) denoting the triangle vertices are cyclic permutations of 1, 2 and 3. The area \( A_i = (1/2)(x_i y_j - y_k x_j) \) denotes the area of the interior triangle whose one vertex is at the point \((x, y)\) while the other two are the vertices \( j \) and \( k \), while \( A = (1/2)(a_1 b_1 - a_1 b_2) \) is the area of the whole triangle (Figure 2). Note that the area co-ordinates \( \xi_1, \xi_2, \xi_3 \) are in fact the standard linear Lagrangian shape functions over a triangular domain.

If any triangle side \( k \) of length \( s_k \) (where \( s_k^2 = a_k^2 + b_k^2 \)) is taken as a beam element (Figure 3), the expressions for the displacement \( w \) and the rotation around the in-plane normal \( \theta_n \) can be derived in linked form (1):

\[ w = \xi_i w_i + \xi_j w_j - \xi_k \frac{s_k}{2} (\theta_{ni} - \theta_{nj}) \]
\[ = \xi_i w_i + \xi_j w_j - \xi_k \frac{s_k}{2} [(\theta_{xi} - \theta_{xj})\cos \alpha_k - (\theta_{yi} - \theta_{yj})\sin \alpha_k] \]
\[ = \xi_i w_i + \xi_j w_j - \frac{1}{2} \xi_k [(\theta_{xi} - \theta_{xj})b_k + (\theta_{yi} - \theta_{yj})a_k], \]
\[ \theta_n = \xi_i \theta_{ni} + \xi_j \theta_{nj}, \]

while \( \xi_k = 0 \). Such interpolation provides constant moments and constant shear along the element side.

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**Figure 3.**
Triangular element's side and its rotation degrees of freedom

\( a_k = x_j - x_i \)
\( b_k = y_j - y_i \)
At each nodal point $i$ ($i = 1, 2, 3$) of an element there exist three degrees of freedom (displacement $w_i$ and rotations $\theta_xi$, and $\theta_yi$ in the global coordinate directions). The linked interpolation for the displacement and rotation field over the whole triangular domain may be now proposed as:

$$
w = \xi_1 w_1 + \xi_2 w_2 + \xi_3 w_3 - \frac{1}{2} \xi_1 \xi_2 [(\theta x_1 - \theta x_2)b_3 + (\theta y_1 - \theta y_2)a_3]$$

$$- \frac{1}{2} \xi_2 \xi_3 [(\theta x_2 - \theta x_3)b_1 + (\theta y_2 - \theta y_3)a_1]$$

$$- \frac{1}{2} \xi_3 \xi_1 [(\theta x_3 - \theta x_1)b_2 + (\theta y_3 - \theta y_1)a_2]$$

$$\theta_x = \xi_1 \theta x_1 + \xi_2 \theta x_2 + \xi_3 \theta x_3$$

$$\theta_y = \xi_1 \theta y_1 + \xi_2 \theta y_2 + \xi_3 \theta y_3.$$  

Here, $\theta x_i$, $\theta y_i$ are the rotation components at the element vertices (Figure 4). Terms in brackets are rotational projections of respective rotation components to the normal on each element side times the side length. Therefore, the interpolation is isoparametric for the rotations, while for the displacement function it includes an additional linking part schematically presented in Figure 5:

![Figure 4. Three-node triangular plate element and its nodal rotations](image-url)

**Note:** The nodal displacements are perpendicular to the element plane.
The linked interpolation as employed in a two-node Timoshenko beam element can exactly reproduce the quadratic displacement function, and the same should be expected for the 2D interpolation considered here. The finite element developed on this basis will be named $T_{3-U2}$, denoting the three-node element with the second-order displacement distribution.

3.2 Linked interpolation for a six-node triangular plate element

A six-node linked-interpolation triangular element (Figure 6) may be defined correspondingly.

Again, if any triangle side $k$ is taken as a beam element, the expressions for the displacement $w$ and the rotation around the in-plane normal $\theta_n$ can be derived in the linked form (1), with $i + 3$ node located at the middle of the side as in Figure 7:
The linked interpolation for the displacement field over the whole triangle domain may be now given as:

\[ w = \xi_1(2\xi_1 - 1)w_1 + \xi_2(2\xi_2 - 1)w_2 + \xi_3(2\xi_3 - 1)w_3 + 4\xi_1\xi_2w_4 + 4\xi_2\xi_3w_5 
\]

\[ + 4\xi_3\xi_1w_6 - \xi_1\xi_2(\xi_2 - \xi_1)\frac{1}{3}[-(\theta_{x1} + 2\theta_{x4} - \theta_{x2})b_3 
\]

\[ + \theta_{y1} + 2\theta_{y4} - \theta_{y2})a_3] 
\]

\[ - \xi_2\xi_3(\xi_3 - \xi_2)\frac{1}{3}[-(\theta_{x2} + 2\theta_{x5} - \theta_{x3})b_1 
\]

\[ + \theta_{y2} + 2\theta_{y5} - \theta_{y3})a_1] 
\]

\[ - \xi_3\xi_1(\xi_1 - \xi_3)\frac{1}{3}[-(\theta_{x3} + 2\theta_{x6} - \theta_{x1})b_2 
\]

\[ + \theta_{y3} + 2\theta_{y6} - \theta_{y1})a_2] \]

\[ w^* = \xi_1(2\xi_1 - 1)w^*_1 + \xi_2(2\xi_2 - 1)w^*_2 + \xi_3(2\xi_3 - 1)w^*_3 + 4\xi_1\xi_2w^*_4 + 4\xi_2\xi_3w^*_5 
\]

while the interpolation for the rotations takes the standard Lagrangian form:

\[ \theta_x = \xi_1(2\xi_1 - 1)\theta_{x1} + \xi_2(2\xi_2 - 1)\theta_{x2} + \xi_3(2\xi_3 - 1)\theta_{x3} + 4\xi_1\xi_2\theta_{x4} 
\]

\[ + 4\xi_2\xi_3\theta_{x5} + 4\xi_3\xi_1\theta_{x6} \]

\[ \theta_y = \xi_1(2\xi_1 - 1)\theta_{y1} + \xi_2(2\xi_2 - 1)\theta_{y2} + \xi_3(2\xi_3 - 1)\theta_{y3} + 4\xi_1\xi_2\theta_{y4} 
\]

\[ + 4\xi_2\xi_3\theta_{y5} + 4\xi_3\xi_1\theta_{y6} \]
where $\theta_{xi}$, $\theta_{yi}$ are the nodal rotation components at the element vertices and midpoints. As before, the displacement and rotation fields are interpolated using the same interpolation functions, but the displacement field has an additional linking part expressed in terms of the rotational components on each element side.

The rotations in equations (11) and (12) have a full quadratic polynomial form, but the displacement field does not have a full cubic polynomial form since expression (10) misses the tenth item in Pascal’s triangle with the function that has zero values along all the element sides which cannot be associated with any nodal degree of freedom. To provide the full cubic expansion we need to expand the result from equation (10) with an independent bubble degree of freedom $w_b$, i.e:

$$w = w^* + \xi_1 \xi_2 \xi_3 w_b$$

The finite element developed on this basis will be named $T6$-$U3$, denoting the six-node element with the third-order displacement distribution. The same interpolation for the displacements has been applied to the mixed-type six-node triangular plate element in Taylor and Govindjee (2002).

### 3.3 Linked interpolation for a ten-node triangular plate element

A ten-node linked-interpolation triangular element (Figure 8) follows analogously from the linked interpolation for the four-node Timoshenko beam element.

Any triangle side can be taken as a beam element and expressions for $w$ and $\theta_n$ can be expressed in the linked form (1). Completed over the whole triangle, the interpolations for the displacement and the rotations follow as:
\[ w = \xi_1(3\xi_1 - 2)(3\xi_1 - 1)\frac{1}{2}w_1 + \xi_1 \xi_2(3\xi_1 - 1)\frac{9}{2}w_4 + \xi_1 \xi_2(3\xi_2 - 1)\frac{9}{2}w_5 + \xi_2(3\xi_2 - 2)(3\xi_2 - 1)\frac{1}{2}w_2 + \xi_2 \xi_3(3\xi_2 - 1)\frac{9}{2}w_6 + \xi_2 \xi_3(3\xi_3 - 1)\frac{9}{2}w_7 + \xi_3(3\xi_3 - 2)(3\xi_3 - 1)\frac{1}{2}w_3 + \xi_3 \xi_1(3\xi_3 - 1)\frac{9}{2}w_8 + \xi_3 \xi_1(3\xi_1 - 1)\frac{9}{2}w_9 + 27\xi_1 \xi_2 \xi_3 w_{10} - \xi_1 \xi_2(3\xi_1 - 1)(3\xi_2 - 1)\frac{1}{8}[(\theta_{x1} + 3\theta_{x4} - 3\theta_{x5} + \theta_{x2})b_3 + (-\theta_{y1} + 3\theta_{y4} - 3\theta_{y5} + \theta_{y2})a_3] - \xi_2 \xi_3(3\xi_2 - 1)(3\xi_3 - 1)\frac{1}{8}[(\theta_{x2} + 3\theta_{x6} - 3\theta_{x7} + \theta_{x3})b_1 + (-\theta_{y2} + 3\theta_{y6} - 3\theta_{y7} + \theta_{y3})a_1] - \xi_3 \xi_1(3\xi_3 - 1)(3\xi_1 - 1)\frac{1}{8}[(\theta_{x3} + 3\theta_{x8} - 3\theta_{x9} + \theta_{x1})b_2 + (-\theta_{y3} + 3\theta_{y8} - 3\theta_{y9} + \theta_{y1})a_2] \]

\[ \theta_x = \xi_1(3\xi_1 - 2)(3\xi_1 - 1)\frac{1}{2} \theta_{x1} + \xi_1 \xi_2(3\xi_1 - 1)\frac{9}{2} \theta_{x4} + \xi_1 \xi_2(3\xi_2 - 1)\frac{9}{2} \theta_{x5} + \xi_2(3\xi_2 - 2)(3\xi_2 - 1)\frac{1}{2} \theta_{x2} + \xi_2 \xi_3(3\xi_2 - 1)\frac{9}{2} \theta_{x6} + \xi_2 \xi_3(3\xi_3 - 1)\frac{9}{2} \theta_{x7} + \xi_3(3\xi_3 - 2)(3\xi_3 - 1)\frac{1}{2} \theta_{x3} + \xi_3 \xi_1(3\xi_3 - 1)\frac{9}{2} \theta_{x8} + \xi_3 \xi_1(3\xi_1 - 1)\frac{9}{2} \theta_{x9} + 27\xi_1 \xi_2 \xi_3 \theta_{x10} \]

\[ \theta_y = \xi_1(3\xi_1 - 2)(3\xi_1 - 1)\frac{1}{2} \theta_{y1} + \xi_1 \xi_2(3\xi_1 - 1)\frac{9}{2} \theta_{y4} + \xi_1 \xi_2(3\xi_2 - 1)\frac{9}{2} \theta_{y5} + \xi_2(3\xi_2 - 2)(3\xi_2 - 1)\frac{1}{2} \theta_{y2} + \xi_2 \xi_3(3\xi_2 - 1)\frac{9}{2} \theta_{y6} + \xi_2 \xi_3(3\xi_3 - 1)\frac{9}{2} \theta_{y7} + \xi_3(3\xi_3 - 2)(3\xi_3 - 1)\frac{1}{2} \theta_{y3} + \xi_3 \xi_1(3\xi_3 - 1)\frac{9}{2} \theta_{y8} + \xi_3 \xi_1(3\xi_1 - 1)\frac{9}{2} \theta_{y9} + 27\xi_1 \xi_2 \xi_3 \theta_{y10} \]

where \( \theta_{x1}, \theta_{y1} \) are the nodal rotation components at the element vertices and the mid-side points.

The rotations are expressed as a full cubic polynomial, but the displacement field does not have a full quartic polynomial forms. To extend expression (14) to a full quartic form (with all 15 items in Pascal’s triangle), two more parameters are needed and they are related with the functions that have zero values along any element side and at the central point (node with index 10). Those parameters are some internal bubbles so finally the displacement field may be completed as:

\[ \bar{w} = w^* + \xi_1 \xi_2 \xi_3 (\xi_1 - \xi_2) w_{b1} + \xi_1 \xi_2 \xi_3 (\xi_2 - \xi_3) w_{b2} \]

The third term that appears to be missing in expression (17) to complete the cyclic triangle symmetry, namely:
is actually linearly dependent on the two other added terms and the tenth term in equation (14). The finite element developed on the basis of this interpolation will be named $T10-U4$, denoting the ten-node element with the fourth-order displacement distribution.

4. Comparison with Liu-Riggs family of purely displacement-based triangular elements

If arbitrary direction $s$ crossing the triangle element is chosen (Figure 9), the shear strain can be expressed in terms of the shear strains along the directions of the co-ordinate axes $x$ and $y$ as:

$$\gamma_s = \gamma_x \cos \alpha + \gamma_y \sin \alpha,$$

where $\alpha$ is the angle between the $s$-direction and the $x$-axis.

In the linked interpolation formulation presented in this work, the expression for the shear along an element side is a polynomial which is two orders lower than the displacement interpolation polynomial. This is also valid for any direction parallel to an element side.

Liu and Riggs (2005) have derived the family of triangle elements likewise based purely on displacement interpolations, which eventually, turn out to be of the linked type in the sense that the displacement distribution also depends on the nodal rotations. In contrast to the approach presented here, however, the requirement that the authors set is that the shear strain along arbitrary direction $s$, and not only those parallel to the element sides, should satisfy the above condition (Figure 9), thus imposing the $p$th derivative of equation (18) to be zero for arbitrary $\alpha$, while the displacement interpolation is of the order $p + 2$ and the interpolation for the rotations is of the order $p + 1$.
4.1 Liu-Riggs interpolation for a three-node triangular plate element
Liu and Riggs (2005) and Tessler and Hughes (1985) before them, have derived interpolation functions for the triangular element named MIN3 with three nodes and nine degrees of freedom (the same degrees as the T3-U2 element presented in Section 3.1) from the condition that the shear strain must be constant along any direction within the element (Tessler and Hughes have additionally introduced a shear-relaxation factor in order to improve the element performance). Their interpolations for the displacement and the rotation fields are:

\[ w = N_i w_i + L_i \theta_{x_i} + M_i \theta_{y_i}, \quad \theta_x = N_i \theta_{x_i} \quad \text{and} \quad \theta_y = N_i \theta_{y_i} \]  \hfill (19)

where the interpolation functions are:

\[ N_i = \xi_i, \quad L_i = \frac{1}{2} (b_k \xi_i \xi_j - b_j \xi_k \xi_i) \quad \text{and} \quad M_i = -\frac{1}{2} (a_j \xi_k \xi_i - a_k \xi_i \xi_j) \]  \hfill (20)

with \( i, j \) and \( k \) again being the cyclic permutations of 1, 2 and 3. It can be verified that the rigid body conditions are satisfied because:

\[ \sum_{i=1}^{3} N_i = 1, \quad \sum_{i=1}^{3} L_i = 0 \quad \text{and} \quad \sum_{i=1}^{3} M_i = 0, \]  \hfill (21)

and it can be also verified by direct calculation that the Liu-Riggs interpolation is the same as the linked interpolation given in Section 3.1. Therefore, for the triangular element with three nodes, in the present formulation the shear strain is also constant in any direction and not only along the directions parallel to the sides of the triangle.

4.2 Liu-Riggs interpolation for a six-node triangular plate element – MIN6
Liu and Riggs (2005) have next derived a family of elements based on upgrading the criteria for the shear strain along an arbitrary direction over the element. The second member of the family is the so-called MIN6 element with six nodal points. The interpolations are derived to provide linear shear in any direction crossing the element. Interpolations in MIN6 for the displacement and rotations are again:

\[ w = N_i w_i + L_i \theta_{x_i} + M_i \theta_{y_i}, \quad \theta_x = N_i \cdot \theta_{x_i} \quad \text{and} \quad \theta_y = N_i \cdot \theta_{y_i} \quad \text{for} \quad i = 1, 2, \ldots, 6 \]  \hfill (22)

where the actual interpolation functions are:

\[ N_i = \xi_i (2 \xi_i - 1), \quad N_{i+3} = 4 \xi_i \xi_j \quad \text{for} \quad i = 1, 2, 3 \]  \hfill (23a)

\[ L_i = -\xi_i (2 \xi_i - 1) \frac{1}{3} (b_k \xi_j - b_j \xi_k), \quad L_{i+3} = -4 \xi_i \xi_j \frac{1}{3} \left[ b_j \left( \xi_i - \frac{1}{2} \right) - b_i \left( \xi_j - \frac{1}{2} \right) \right] \]  \hfill (23b)
and:

\[ M_i = -\xi_i (2\xi_i - 1) \frac{1}{3} (a_k \xi_j - a_j \xi_k), \quad M_{i+3} = -4 \xi_i \xi_j \frac{1}{3} [a_j (\xi_i - \frac{1}{2}) - a_i (\xi_j - \frac{1}{2})] \]  

(23c)

It should be stressed that in the expressions for \( L_{i+3} \) and \( M_{i+3} \) given here a typographic error in the Liu-Riggs original (Liu and Riggs, 2005) is corrected to satisfy the rigid body conditions:

\[ \sum_{i=1}^{6} N_i = 1, \quad \sum_{i=1}^{6} L_i = 0 \quad \text{and} \quad \sum_{i=1}^{6} M_i = 0. \]  

(24)

The element based on interpolation (equation (23)) – denoted as MIN6 – has been coded in the finite-element programme environment FEAP (Zienkiewicz and Taylor, 2005) along with the earlier elements T3-U2, T6-U3 and T10-U4. In contrast to MIN3 (without shear relaxation), which corresponds exactly to the T3-U2 presented in Section 3.1, the MIN6 element is different from the T6-U3 presented in Section 3.2, which has an additional bubble degree of freedom. It can be verified by direct calculation that the Liu–Riggs interpolation for MIN6 should coincide with the T6-U3 interpolation given in Section 3.2 if the bubble degree of freedom were constrained to:

\[
\begin{align*}
\frac{1}{3} [(b_3 - b_2) \theta_{x1} + (a_3 - a_2) \theta_{x1} + (b_1 - b_3) \theta_{x2} + (a_1 - a_3) \theta_{x2} + (b_2 - b_1) \theta_{x3} \\
+ (a_2 - a_1) \theta_{x3}] + \frac{2}{3} [(b_2 - b_1) \theta_{x4} + (a_2 - a_1) \theta_{x4} + (b_3 - b_2) \theta_{x5} + (a_3 - a_2) \theta_{x5} \\
+ (b_1 - b_3) \theta_{x6} + (a_1 - a_3) \theta_{x6}]
\end{align*}
\]

(25)

It should be made clear that the shear strain distribution of a certain order along an arbitrary direction is the basic underlying condition from which the MINn family of elements has been derived, while the shear strain distribution of a certain order along a direction parallel to the element sides is a consequence, rather than the origin of the family of elements presented in Section 3. The presented methodology generates the linked interpolation from the underlying interpolation functions developed for the beam elements and may be consistently and straightforwardly applied to triangular plate elements of arbitrary order. In contrast, the MINn methodology requires a symbolic manipulation of algebraic expressions which get progressively more complicated as the order of the element is raised.

5. Finite element stiffness matrix and load vector

The earlier interpolations may be written in matrix form as:

\[ w = I_{ww} w + N_{wb} \theta_{x,y} + N_{wb} w_b, \]  

(26)

\[ \begin{bmatrix} \theta_x \\ \theta_y \end{bmatrix} = I_{\theta \theta} \theta_{x,y}, \]  

(27)
where $\mathbf{I}_{\text{w}}$ is a matrix of all interpolation functions concerning the nodal displacement parameters with the dimension $1 \times N_{nd}$, where $N_{nd} = n(n+1)/2$ and $n$ is the number of nodes per element side. Also, $\mathbf{w}$ is the vector of nodal displacement parameters with the dimension $N_{nd}$. $N_{nd} = n(n+1)/2$ and $n$ is the number of nodes per element side. Also, $\mathbf{w}$ is the vector of nodal displacement parameters with the dimension $N_{nd}$.

$\mathbf{T} = \langle w_1, \ldots, w_n \rangle$, $N_{w\theta}$ is the matrix of all linked interpolation functions with the dimension $1 \times 2N_{nd}$ and $\mathbf{\theta}_{x,y}$ is the vector of nodal rotations in global coordinate directions with the dimension $2N_{nd}$: $\mathbf{\theta}_{x,y} = \langle \theta_{x1}, \theta_{x2}, \ldots, \theta_{xn}, \theta_{yn} \rangle$. Further, $N_{wb}$ is the matrix of bubble interpolation functions given in equation (13) or equation (17) with the dimension $1 \times N_{wb}$ and $\mathbf{w}_b$ is the bubble parameter vector with the dimension $N_{wb} = n^2 - 2$: $\mathbf{w}_b = \langle w_{b1}, \ldots, w_{b,n-2}, 1 \rangle$, only for $n \geq 2$. $\mathbf{I}_{\theta\theta}$ is again the matrix of all rotational parameters described in equations (8), (9), (11) and (12) or equations (15) and (16) and has the dimension $2 \times 2N_{nd}$.

The formation of the element stiffness matrix and the external load vector for the interpolation functions defined in this way follows the standard finite-element procedure described in text-books (Bathe, 1989; Hughes, 2000; Zienkiewicz et al., 2005). A functional of the total energy of the system is given in equation (6) and from the stationarity condition for the total potential energy of an element, a system of algebraic equations is derived:

$$
\begin{bmatrix}
\mathbf{K}_{\text{sww}} & \mathbf{K}_{\text{sw} \theta}^T & \mathbf{K}_{\text{swb}}^T \\
\mathbf{K}_{\text{sw} \theta} & \mathbf{K}_{\text{B} \theta} + \mathbf{K}_{\text{S} \theta} & \mathbf{K}_{\text{S} \theta}^T \\
\mathbf{K}_{\text{swb}} & \mathbf{K}_{\text{S} \theta} & \mathbf{K}_{\text{S} \theta}^T
\end{bmatrix}
\begin{bmatrix}
\mathbf{w} \\
\mathbf{\theta}_{x,y} \\
\mathbf{w}_b
\end{bmatrix}
= 
\begin{bmatrix}
\mathbf{f}_w \\
\mathbf{f}_\theta \\
\mathbf{f}_b
\end{bmatrix},
$$

where vectors $\mathbf{f}_w$, $\mathbf{f}_\theta$ and $\mathbf{f}_b$ are the terms due to external loading. The submatrices in the stiffness matrix follow:

$$
\begin{align*}
\mathbf{K}_{\text{B} \theta} &= \int_A (\mathbf{L} \mathbf{I}_{\theta\theta})^T \mathbf{D}_b (\mathbf{L} \mathbf{I}_{\theta\theta}) dA \\
\mathbf{K}_{\text{sww}} &= \int_A (\nabla \mathbf{I}_{\text{sw}})^T \mathbf{D}_s (\nabla \mathbf{I}_{\text{sw}}) dA \\
\mathbf{K}_{\text{S} \theta} &= \int_A (\mathbf{e} \mathbf{I}_{\theta\theta} + \nabla \mathbf{N}_{\text{w\theta}})^T \mathbf{D}_s (\mathbf{e} \mathbf{I}_{\theta\theta} + \nabla \mathbf{N}_{\text{w\theta}}) dA \\
\mathbf{K}_{\text{swb}} &= \int_A (\nabla \mathbf{N}_{\text{wb}})^T \mathbf{D}_s (\nabla \mathbf{I}_{\text{wb}}) dA \\
\mathbf{K}_{\text{sw} \theta} &= \int_A (\nabla \mathbf{N}_{\text{w\theta}})^T \mathbf{D}_s (\mathbf{e} \mathbf{I}_{\theta\theta} + \nabla \mathbf{N}_{\text{w\theta}}) dA \\
\mathbf{K}_{\text{S} \theta} &= \int_A (\mathbf{e} \mathbf{I}_{\theta\theta} + \nabla \mathbf{N}_{\text{w\theta}})^T \mathbf{D}_s (\mathbf{e} \mathbf{I}_{\theta\theta} + \nabla \mathbf{N}_{\text{w\theta}}) dA \\
\mathbf{K}_{\text{S} \theta} &= \int_A (\nabla \mathbf{N}_{\text{w\theta}})^T \mathbf{D}_s (\mathbf{e} \mathbf{I}_{\theta\theta} + \nabla \mathbf{N}_{\text{w\theta}}) dA \\
\mathbf{K}_{\text{S} \theta} &= \int_A (\nabla \mathbf{N}_{\text{w\theta}})^T \mathbf{D}_s (\mathbf{e} \mathbf{I}_{\theta\theta} + \nabla \mathbf{N}_{\text{w\theta}}) dA
\end{align*}
$$

where $\mathbf{L}$ and $\nabla$ are the differential operators from equations (2) and (3) acting on the interpolation functions in equations (27) and (26), respectively, while $\mathbf{e}$ is a transformation matrix given in equation (2).

### 6. Test examples

In all the examples the results for the elements $T3-U2$, $T6-U3$ and $T10-U4$ are compared to the mixed-type element of Auricchio and Taylor (1995) denoted as $T3-LIM$ and integrated in FEAP (a finite element analysis program) by the same authors, or to the $T3BL$ element (Taylor and Auricchio, 1993). Also, comparison is
made to the MIN6 element (Liu and Riggs, 2005) and the hybrid-type element 9βQ4 (de Miranda and Ubertini, 2006) as well as the linked-interpolation quadrilateral elements $Q4-U2$, $Q9-U3$ and $Q16-U4$ (Ribarić and Jelenić, 2012).

### 6.1 Patch test and eigenanalysis of the stiffness matrix

Consistency of the developed elements is tested for the constant strain conditions on the patch example with ten elements, covering a rectangular domain of a plate as shown in Figure 10. The displacements and rotations for the four internal nodes within the patch are checked for the specific displacements and rotations given at the four external nodes (Chen and Cheung, 2000, 2001; Chen, 2006; Chen et al., 2009). The plate properties are $E = 10^5$, $\nu = 0.25$, $k = 5/6$, while two different thicknesses corresponding to a thick and a thin plate extremes are considered: $h = 1.0$ and $h = 0.01$.

Two strain-stress states are analysed (Chen et al., 2009):

1. **Constant bending state**
   Displacements and rotations are expressed, respectively, by:
   $$ w = \frac{(1 + x + 2y + x^2 + xy + y^2)}{2}, \quad \theta_x = \frac{(2 + x + 2y)}{2}, \quad \theta_y = -\frac{(1 + 2x + y)}{2}. $$
   The exact displacements and rotations at the internal nodes and the exact strains and stress resultants at every integration point are expected. The moments are constant $M_x = M_y = -11,111.11$ $h^3$, $M_{xy} = -33,333.33$ $h^3$ and the shear forces vanish ($S_x = S_y = 0$).

2. **Constant shear state**
   Displacements and rotations are expressed, respectively, by:
   $$ w = -\frac{h^2}{5(1 - \nu)}(14x + 18y) + x^3 + 2y^3 + 3x^2y + 4xy^2, \quad \theta_x = 3x^2 + 8xy + 6y^2, \quad \theta_y = -(3x^2 + 6xy + 4y^2). $$

---

**Figure 10.** Element patch for consistency assessment of three-node elements

- **(0.00, 0.00)** prescribed d.o.f
- **(0.00, 0.12)** checked d.o.f
- **(0.04, 0.02)**
- **(0.08, 0.08)**
- **(0.12, 0.04)**
- **(0.16, 0.08)**
- **(0.20, 0.04)**
- **(0.24, 0.00)**
- **(0.24, 0.12)**

- **(0.18, 0.03)**
- **(0.04, 0.02)**
- **(0.16, 0.08)**
- **(0.21, 0.12)**
The exact displacements and rotations at the internal nodes and the exact strains and stress resultants at every integration point are expected again. The shear forces are constant \( S_x = -124,400.0 \, h^3 \) and \( S_y = -160,000.0 \, h^3 \) in every Gauss point and the moments are linearly distributed according to:

\[
M_x = -\frac{Eh^3}{12(1-\nu^2)} [x(6 + 8\nu) + y(6 + 12\nu)],
\]
\[
M_y = -\frac{Eh^3}{12(1-\nu^2)} [x(8 + 6\nu) + y(12 + 6\nu)] \quad \text{and}
\]
\[
M_{xy} = -\frac{Eh^3}{12(1-\nu^2)} \frac{1 - \nu}{2} (12x + 16y).
\]

The three-node triangle element \( T3-U2 \) is tested on the patch given in Figure 10. For the given values for the displacements and rotations at the external nodes calculated from the above data, the displacements and rotations at the internal nodes as well as the bending and torsional moments and the shear forces at the integration points are calculated and found out to correspond exactly to the analytical results given above for the constant bending test, but not for the constant shear test. It should be noted that the constant shear test performed here is related to a linear change in curvature (third-order cylindrical bending) and in fact by definition requires an element to enable cubic distribution of the displacement field and the quadratic distribution of the rotation fields, for which the analysed element is not designed. This test should not be mistaken for the constant shear test with no curvature as a consequence of a suitable choice of distributed moment loadings (de Miranda and Ubertini, 2006), which the analysed element also passes.

The six-node triangle element \( T6-U3 \) is tested on the similar patch example (the mesh is given in Figure 11). Again, only the displacements and rotations at the boundary nodes are given (eight displacements and 16 rotations), while all the internal nodal displacements and rotations are to be calculated by the finite-element

Figure 11. Element patch for consistency assessment of six-node elements

85
solution procedure. In fact, they are calculated exactly for both the constant bending and the constant shear test. The moments and shear forces at the integration points are also exact.

The same patch tests are also successfully performed by the ten-node elements $T10-U4$, where 48 parameters for the degrees of freedom are prescribed and 108 others are checked (Figure 12).

The results of the patch tests for all three proposed elements are given in Table I for the displacement at node 1 with co-ordinates $(0.04, 0.02)$. The results are not altered if any of the internal nodes changes its position in the mesh (for example node 2 in Figure 10) for $T3-U2$ element’s constant curvature test or for $T6-U3$ and $T10-U4$ elements in both tests. The results of the patch tests are not sensitive to mesh distortion.

Furthermore, the quartic interpolation for the displacement field would enable the ten-node element $T10-U4$ to exactly reproduce even the cylindrical bending of the fourth order.

For stability assessment of the elements, a patch test must be obviously satisfied, but the eigenanalysis on the single element should be also checked out (Auricchio and

![Element patch for consistency assessment of ten-node elements](image)

<table>
<thead>
<tr>
<th>Elements</th>
<th>Patch test for constant curvature $h = 1.0$</th>
<th>$h = 0.01$</th>
<th>Result</th>
<th>Patch test for constant shear $h = 1.0$</th>
<th>$h = 0.01$</th>
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<td>-0.2450933</td>
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</tbody>
</table>

Note: Displacement at point 1: $w_1$
Taylor, 1994, 1995). Several cases of span-to-thickness ratio are considered: \( L/h = 10, \ L/h = 1,000 \) and \( L/h = 100,000 \) (Figure 13). In eigenanalysis the bending stiffness is kept constant by scaling the Young modulus proportionally to \((1/h)^3\). The elements have always the correct number of zero eigenvalues that correspond to rigid body modes. The \( T3-U2 \) element has three eigenvalues that are associated with shear, which experience considerable growth as the element thickness is reduced indicating a propensity of the element to lock. The other three eigenvalues are bending dependent and they remain constant. The results are given in Table II. In \( T6-U3 \) and \( T10-U4 \) elements there also exist the growing shear-related eigenvalues, but there is also an increased number of the eigenvalues which remain constant (Tables III and IV). It will be shown in Sections 6.2-6.5 that, in spite of the growing eigenvalues, these elements considerably reduce or completely eliminate the locking effect.

6.2 Clamped square plate
In this example, a square plate with clamped edges is considered. Only one quarter of the plate is modeled with symmetric boundary conditions imposed on the symmetry lines. Two ratios of span versus thickness are analysed, \( L/h = 10 \) representing a relatively thick plate and \( L/h = 1,000 \) representing its thin counterpart. The loading on the plate is uniformly distributed of magnitude \( q = 1 \). The plate material properties are \( E = 10.92 \) and \( v = 0.3 \).

The numerical results for the mesh pattern in Figure 14 are given in Tables V and VII and compared to the elements presented in Taylor and Auricchio (1993) and Auricchio and Taylor (1995) based on the mixed approach. The dimensionless results \( w = w/(qL^4/100D) \) and \( M = M/(qL^2/100) \), where \( D = Eh^3/(12(1 - v^2)) \) and \( L \) is the plate span, given in these tables are related to the central displacement of the plate and the bending moment at the integration point nearest to the centre of the plate. The number of elements per mesh in these tables is given for one quarter of the structure as shown in Figure 14 for the \( 4 \times 4 \) mesh consisting of 32 elements.

Clearly, all the new elements converge towards the same solution as the elements from the literature, and the higher-order elements exhibit an expected faster convergence rate. Still, the lowest-order element \( T3-U2 \) is somewhat inferior to \( T3BL \), which is not surprising knowing that that element is actually based on the linked interpolation as in \( T3-U2 \) on top of which additional improvements are made.

![Figure 13.](image-url)
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Table III.
Right angle triangle element $T_6-U_3$.
Eigenvalues of the element stiffness matrix.
Table IV.

Right angle triangle

Element T6-U3.

Eigenvalues of the
element stiffness matrix

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<td>5.0913 × 10^{-01}</td>
<td>3.7071 × 10^{-01}</td>
<td>3.4339 × 10^{-01}</td>
<td>2.1503 × 10^{-01}</td>
<td>1.5811 × 10^{-01}</td>
<td>8.4796 × 10^{-02}</td>
</tr>
<tr>
<td>29</td>
<td>6.1038 × 10^{-05}</td>
<td>5.2861 × 10^{-05}</td>
<td>5.1867 × 10^{-05}</td>
<td>4.5341 × 10^{-05}</td>
<td>4.4273 × 10^{-05}</td>
<td>3.7483 × 10^{-05}</td>
<td>1.3092 × 10^{-05}</td>
<td>1.3092 × 10^{-05}</td>
</tr>
<tr>
<td>30</td>
<td>7.7227 × 10^{-04}</td>
<td>7.2023 × 10^{-04}</td>
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<td>5.2870 × 10^{-04}</td>
<td>5.0940 × 10^{-04}</td>
<td>4.9274 × 10^{-04}</td>
<td>2.1609 × 10^{-04}</td>
<td>1.6841 × 10^{-04}</td>
</tr>
<tr>
<td>78</td>
<td>1.7292 × 10^{-00}</td>
<td>1.5489 × 10^{-00}</td>
<td>5.4943 × 10^{-01}</td>
<td>4.1616 × 10^{-01}</td>
<td>3.8821 × 10^{-01}</td>
<td>2.3809 × 10^{-01}</td>
<td>1.6841 × 10^{-01}</td>
<td>8.8825 × 10^{-02}</td>
</tr>
<tr>
<td>90</td>
<td>1.6798 × 10^{-11}</td>
<td>-1.5475 × 10^{-11}</td>
<td>1.0763 × 10^{-13}</td>
<td>7.8089 × 10^{-17}</td>
<td>7.7479 × 10^{-17}</td>
<td>6.8300 × 10^{-17}</td>
<td>5.7991 × 10^{-17}</td>
<td>4.8800 × 10^{-17}</td>
</tr>
</tbody>
</table>
The minute differences between the results of $T6-U3$ and $MIN6$ are attributed to the slight difference in these elements as explained in Section 4.2.

Comparing the present triangular linked-interpolation elements to their quadrilateral counterparts (Ribarić and Jelenić, 2012) shows that for the same number of the degrees of freedom the latter, converge a little faster (Table VI).

Convergence of the central displacement for the thick-plate case is presented in Figure 15, with respect to the number of degrees of freedom (in logarithmic scale).
Best convergence with respect to the number of degrees of freedom can be observed in elements with higher-order linked interpolation and it may be concluded that for the thick clamped plate the present elements converge competitively for a comparable number of degrees of freedom.
The $M_x$ moment distribution along the $x$-axis is computed at the Gauss points closest to the axis and, beginning from the centre point of the plate, shown in Figure 16. These results are the same as the results for the moment $M_y$ along the $y$-direction. For $T3-U2$ element, the moment is constant across the element, owing to its dependence on the derivatives of rotations (8) and (9) in both directions. For elements $T6-U3$ and $T10-U4$, the moment distribution is accordingly linear or quadratic, respectively, and the results converge towards the exact distribution fast. Similar observations may be made for the distribution of the shear-stress resultants.

For the thin plate case shown in Table VII, the elements $T3-U2$ and $T6-U3$ suffer from some shear locking when the meshes are coarse, but as expected they converge to the correct result. The higher-order elements exhibit an expected faster convergence rate.

As for the case of the thick plate, the lowest-order element $T3-U2$ is still somewhat inferior to $T3BL$, while $T6-U3$ is marginally better than $MIN6$. Likewise, comparing the triangular linked-interpolation elements to their quadrilateral counterparts (Ribaric and Jelenic, 2012) shows that for the same number of the degrees of freedom the latter, converge a little faster (Table VIII).

If the same example were run with a different orientation of the triangular elements (with the longest element side orthogonal to the diagonal passing through the centre of the plate) the results would turn out to be slightly worse even though for the higher-order elements the trend gets reversed as the mesh is refined. This is shown in Table IX for the thin plate case.

The differences in the results given for the two orientations (Tables VII and IX) drop below 3.3 per cent for the $T3-U2$ displacement with a 16x16 mesh already.

![Figure 16. Moment $M_x$ distribution along element’s Gauss points closest to the $x$ axis on the $4 \times 4$ regular mesh for the clamped plate with $L/h = 10$](image_location)
6.3 Simply supported square plate

In this example the square plate as before is considered, but this time with the simply supported edges of the type SS2 (displacements and rotations around the normal to the edge set to zero) as shown in Figure 17. The same elements as before are tested and the results are given in Tables X and XII for the thick and the thin plate, respectively, compared against the elements presented in Taylor and Auricchio (1993) and Auricchio and Taylor (1995) (Tables X-XIII).

The dimensionless results
\[ w^* = \frac{w}{(qL^4/100D)} \quad M^* = \frac{M}{(qL^2/100)} \]
given in these tables are related to the central displacement of the plate and the bending moment at the integration point nearest to the centre of the plate. The number of elements per mesh in these tables relates to one quarter of the plate.

For the thick plate case, it can be concluded that elements \( T6-U3 \) and \( T10-U4 \) converge considerably faster than elements based on the mixed approach and no locking can be observed on coarse meshes, even for the three-node element \( T3-U2 \). In the thin plate example, locking on the coarse meshes can be observed for \( T3-U2 \), but higher-order elements, again, show very good convergence rate.

In contrast to the clamped plate problem, the mesh pattern used here has slightly better convergence than the mesh pattern used in that example.

6.4 Simply supported skew plate

In this example the rhombic plate is considered with the simply supported edges (this time, of the so-called soft type SS1 (Babuška and Scapolla, 1989) to test performance of the rhombic elements. The problem geometry and material properties are given in Figure 18, where an example of a \( 8 \times 8 \)-mesh is shown (128 triangular elements).

The same three elements as before are tested and the results are given in Tables XIV and XVIII for the thick and the thin plate, respectively. The dimensionless results
<table>
<thead>
<tr>
<th>Element mesh</th>
<th>$9\beta Q4$ (de Miranda and Ubertini, 2006)</th>
<th>$Q4-U2$</th>
<th>$Q9-U3$</th>
<th>$Q16-U4$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$w^*$</td>
<td>$M^*$</td>
<td>$w^*$</td>
<td>$M^*$</td>
</tr>
<tr>
<td>$1 \times 1$</td>
<td>0.0000027</td>
<td>0.00027</td>
<td>0.00746</td>
<td>0.13241</td>
</tr>
<tr>
<td>$2 \times 2$</td>
<td>0.13766768</td>
<td>2.745544</td>
<td>0.00013</td>
<td>$-0.0001$</td>
</tr>
<tr>
<td>$4 \times 4$</td>
<td>0.12938531</td>
<td>2.423885</td>
<td>0.00469</td>
<td>0.10731</td>
</tr>
<tr>
<td>$8 \times 8$</td>
<td>0.12725036</td>
<td>2.323949</td>
<td>0.05988</td>
<td>1.18496</td>
</tr>
<tr>
<td>$16 \times 16$</td>
<td>0.12671406</td>
<td>2.298876</td>
<td>0.11899</td>
<td>2.17415</td>
</tr>
<tr>
<td>$32 \times 32$</td>
<td>0.12657946</td>
<td>2.292605</td>
<td>0.12600</td>
<td>2.28275</td>
</tr>
<tr>
<td>$64 \times 64$</td>
<td>0.12648</td>
<td>2.28988</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ref. sol. (Zienkiewicz et al., 1993)</td>
<td>0.126532</td>
<td>2.29051</td>
<td>0.126532</td>
<td>2.29051</td>
</tr>
</tbody>
</table>

**Note:** Displacement and moment at the centre using a quadrilateral hybrid element (de Miranda and Ubertini, 2006) and linked-interpolation elements (Ribarić and Jelenić, 2012), $L/h = 1,000$
The tested example has two orthogonal axes of symmetry, A-C-B and D-C-E, and only one triangular quarter may be taken for analysis (Hughes, 2000; Hughes and Tezduyar, 1981). Since there is a singularity in the moment field at the obtuse vertex, this test example is a difficult one. Even more, the analytical solution (Morley, 1962) reveals that moments in the principal directions near the obtuse vertex have opposite signs.

In contrast to the earlier examples, it must be noted that here the new displacement-based elements perform worse than the elements given in Taylor and Auricchio (1993) and Auricchio and Taylor (1995), both for the thick and the thin plate examples. Tables XIV and XVIII now reveal slightly more pronounced differences in the results obtained using elements $T6-U3$ and $MIN6$, where the latter are somewhat worse, apparently owing to the absence of the internal bubble parameter present in $T6-U3$ (Section 4.2). Also, from Tables XV and XIX it is apparent that for this test

$$w^* = w/(qL^4/10^4D), \ M^*_{11} = M_{11}/(qL^2/100) \text{ and } M^*_{22} = M_{22}/(qL^2/100)$$

are related to the central displacement of the plate and the principal bending moments in diagonal directions at the integration point nearest to the centre of the plate.

The tested example has two orthogonal axes of symmetry, A-C-B and D-C-E, and only one triangular quarter may be taken for analysis (Hughes, 2000; Hughes and Tezduyar, 1981). Since there is a singularity in the moment field at the obtuse vertex, this test example is a difficult one. Even more, the analytical solution (Morley, 1962) reveals that moments in the principal directions near the obtuse vertex have opposite signs.

In contrast to the earlier examples, it must be noted that here the new displacement-based elements perform worse than the elements given in Taylor and Auricchio (1993) and Auricchio and Taylor (1995), both for the thick and the thin plate examples. Tables XIV and XVIII now reveal slightly more pronounced differences in the results obtained using elements $T6-U3$ and $MIN6$, where the latter are somewhat worse, apparently owing to the absence of the internal bubble parameter present in $T6-U3$ (Section 4.2). Also, from Tables XV and XIX it is apparent that for this test

Table IX.
Clamped square plate

<table>
<thead>
<tr>
<th>Element mesh</th>
<th>$T3-U2 \ M^*$</th>
<th>$T6-U3 \ M^*$</th>
<th>$T10-U4 \ M^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 × 1</td>
<td>0.000001</td>
<td>0.00057</td>
<td>0.00195</td>
</tr>
<tr>
<td>2 × 2</td>
<td>0.000052</td>
<td>0.00112</td>
<td>1.09704</td>
</tr>
<tr>
<td>4 × 4</td>
<td>0.001531</td>
<td>0.02764</td>
<td>1.87195</td>
</tr>
<tr>
<td>8 × 8</td>
<td>0.025112</td>
<td>0.38406</td>
<td>2.18328</td>
</tr>
<tr>
<td>16 × 16</td>
<td>0.100985</td>
<td>1.64321</td>
<td>2.27223</td>
</tr>
<tr>
<td>32 × 32</td>
<td>0.124358</td>
<td>2.20020</td>
<td>2.28838</td>
</tr>
<tr>
<td>64 × 64</td>
<td>0.126356</td>
<td>2.28191</td>
<td>2.29051</td>
</tr>
</tbody>
</table>

Ref. sol. (Zienkiewicz et al., 1993) 0.126532 2.29051 0.126532 2.29051

Note: Displacement and moment at the centre using opposite orientation of triangular elements in the mesh, $L/h = 1,000$
example the new triangular family of linked-interpolation elements is in fact superior to the family of quadrilateral linked-interpolation elements presented in Ribaric and Jelenic (2012).

The mesh pattern chosen for computing the results in Table XIV is the best among the uniformly distributed meshes. For example, meshes (b) and (c) in Figure 19 give less good results for the similar numbers of degrees of freedom as can be noticed in Tables XVI and XVII.
The distribution of the principal moments between the obtuse angle at A and the centre-point C is very complex owing to the presence of singularity at A and worth particular consideration. The principal moment $M_{11}$ acting around the in-plane normal to the shorter diagonal converges towards the exact solution satisfactorily, but for the principal moment $M_{22}$ acting around the shorter diagonal it is obvious that this family of elements finds it difficult to follow the exact moment distribution near the

<table>
<thead>
<tr>
<th>Element mesh</th>
<th>$w^*$</th>
<th>$M^*$</th>
<th>$w^*$</th>
<th>$M^*$</th>
<th>$w^*$</th>
<th>$M^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T3-U2$</td>
<td>0.325522</td>
<td>3.38542</td>
<td>0.325543</td>
<td>2.51970</td>
<td>0.405992</td>
<td>4.53393</td>
</tr>
<tr>
<td>$T6-U3$</td>
<td>0.325541</td>
<td>3.07155</td>
<td>0.386721</td>
<td>3.56892</td>
<td>0.406198</td>
<td>4.78210</td>
</tr>
<tr>
<td>$T10-U4$</td>
<td>0.326331</td>
<td>2.67668</td>
<td>0.402526</td>
<td>4.25254</td>
<td>0.406234</td>
<td>4.78844</td>
</tr>
<tr>
<td>$T3BL$ (Taylor and Auricchio, 1993)</td>
<td>0.339027</td>
<td>2.80820</td>
<td>0.406211</td>
<td>4.76246</td>
<td>0.4062374</td>
<td>4.78860</td>
</tr>
<tr>
<td>$MIN6$</td>
<td>0.403889</td>
<td>4.67235</td>
<td>0.406237</td>
<td>4.78543</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T3-LIM$ (Auricchio and Taylor, 1995)</td>
<td>0.406062</td>
<td>4.77806</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Element mesh</th>
<th>$w^*$</th>
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<th>$w^*$</th>
<th>$M^*$</th>
<th>$w^*$</th>
<th>$M^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$9BQ4$ (de Miranda and Ubertini, 2006)</td>
<td>0.0000008</td>
<td>0.00093</td>
<td>0.35527</td>
<td>3.67992</td>
<td>0.41220</td>
<td>5.49186</td>
</tr>
<tr>
<td>$Q4-U2$</td>
<td>0.4063653</td>
<td>5.241963</td>
<td>0.0031093</td>
<td>0.03842</td>
<td>0.39807</td>
<td>4.61311</td>
</tr>
<tr>
<td>$Q9-U3$</td>
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<td>4.901453</td>
<td>0.053621</td>
<td>0.68507</td>
<td>0.40475</td>
<td>4.79293</td>
</tr>
<tr>
<td>$Q16-U4$</td>
<td>0.4062559</td>
<td>4.817513</td>
<td>0.29658</td>
<td>3.54861</td>
<td>0.40615</td>
<td>4.80990</td>
</tr>
<tr>
<td>$Q24-U4$</td>
<td>0.4062241</td>
<td>4.768785</td>
<td>0.39706</td>
<td>4.68543</td>
<td>0.40624</td>
<td>4.78943</td>
</tr>
<tr>
<td>$Q32-U4$</td>
<td>0.4062386</td>
<td>4.790442</td>
<td>0.40626</td>
<td>4.78188</td>
<td>0.40624</td>
<td>4.78884</td>
</tr>
<tr>
<td>$Q64-U4$</td>
<td>0.406216</td>
<td>4.78844</td>
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<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table XII. Simply supported square plate (SS2) under uniformly distributed load

<table>
<thead>
<tr>
<th>Element mesh</th>
<th>$w^*$</th>
<th>$M^*$</th>
<th>$w^*$</th>
<th>$M^*$</th>
<th>$w^*$</th>
<th>$M^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w^*$</td>
<td>0.326331</td>
<td>2.67668</td>
<td>0.402526</td>
<td>4.25254</td>
<td>0.406234</td>
<td>4.78844</td>
</tr>
<tr>
<td>$M^*$</td>
<td>0.39027</td>
<td>2.80820</td>
<td>0.406211</td>
<td>4.76246</td>
<td>0.4062374</td>
<td>4.78860</td>
</tr>
<tr>
<td>$T3BL$ (Taylor and Auricchio, 1993)</td>
<td>0.339027</td>
<td>2.80820</td>
<td>0.406211</td>
<td>4.76246</td>
<td>0.4062374</td>
<td>4.78860</td>
</tr>
<tr>
<td>$MIN6$</td>
<td>0.403889</td>
<td>4.67235</td>
<td>0.406237</td>
<td>4.78543</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T3-LIM$ (Auricchio and Taylor, 1995)</td>
<td>0.406062</td>
<td>4.77806</td>
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</tr>
</tbody>
</table>

Note: Displacement and moment at the centre, $L/h = 1,000$

Table XIII. Simply supported square plate (SS2) under uniformly distributed load

<table>
<thead>
<tr>
<th>Element mesh</th>
<th>$w^*$</th>
<th>$M^*$</th>
<th>$w^*$</th>
<th>$M^*$</th>
<th>$w^*$</th>
<th>$M^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w^*$</td>
<td>0.0000008</td>
<td>0.00093</td>
<td>0.35527</td>
<td>3.67992</td>
<td>0.41220</td>
<td>5.49186</td>
</tr>
<tr>
<td>$M^*$</td>
<td>0.4063653</td>
<td>5.241963</td>
<td>0.0031093</td>
<td>0.03842</td>
<td>0.39807</td>
<td>4.61311</td>
</tr>
<tr>
<td>$Q4-U2$</td>
<td>0.4063062</td>
<td>4.901453</td>
<td>0.053621</td>
<td>0.68507</td>
<td>0.40475</td>
<td>4.79293</td>
</tr>
<tr>
<td>$Q9-U3$</td>
<td>0.4062559</td>
<td>4.817513</td>
<td>0.29658</td>
<td>3.54861</td>
<td>0.40615</td>
<td>4.80990</td>
</tr>
<tr>
<td>$Q16-U4$</td>
<td>0.4062241</td>
<td>4.768785</td>
<td>0.39706</td>
<td>4.68543</td>
<td>0.40624</td>
<td>4.78943</td>
</tr>
<tr>
<td>$Q24-U4$</td>
<td>0.4062386</td>
<td>4.790442</td>
<td>0.40626</td>
<td>4.78188</td>
<td>0.40624</td>
<td>4.78884</td>
</tr>
<tr>
<td>$Q32-U4$</td>
<td>0.406216</td>
<td>4.78844</td>
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</tr>
<tr>
<td>$Q64-U4$</td>
<td>0.406219</td>
<td>4.78764</td>
<td></td>
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<td></td>
<td></td>
</tr>
</tbody>
</table>

Note: Displacement and moment at the centre using a quadrilateral hybrid element (de Miranda and Ubertini, 2006) and linked-interpolation elements (Ribaric and Jelenic, 2012), $L/h = 1,000$
singularity point. These results are shown in Figure 20. It should be noted that near the singularity point the moments are getting high values of opposite signs, and there even a small relative difference between the exact result and the finite-element solution in one of the principal moments may strongly influence the other principal moment since they are related via Poisson’s coefficient ($v = 0.3$ in this case) as shown in equation (4). Specifically, even though the finite-element solutions for the principal moment $M_{11}$ recognise the monotonous trend of the exact solution, the fact that, as absolute values, these moments are overestimated makes it difficult for the element to provide a solution.
for $M_{22}$ which would recognise the change in sign, slope and curvature evident in the exact solution. Of course, there exist techniques to reduce the error in $M_{11}$ which, as a result, would also correct this anomaly in $M_{22}$, e.g. the shear correction factor concept (Figure 14 in Tessler and Hughes (1985)), an idea that has not been followed up in this paper (see Liu and Riggs (2005) for evidence and Tessler (1989) for explanation why this concept has a diminishing effect as the interpolation order is increased) (Tables XVIII and XIX).

### Table XV.

<table>
<thead>
<tr>
<th>Element mesh</th>
<th>$W^*$</th>
<th>$M_{22}^*$</th>
<th>$M_{11}^*$</th>
<th>$w^*$</th>
<th>$M_{22}^*$</th>
<th>$M_{11}^*$</th>
<th>$w^*$</th>
<th>$M_{22}^*$</th>
<th>$M_{11}^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2 \times 2$</td>
<td>0.06153</td>
<td>0.1349</td>
<td>0.3763</td>
<td>0.21493</td>
<td>0.4850</td>
<td>1.0536</td>
<td>0.28406</td>
<td>0.8404</td>
<td>1.5335</td>
</tr>
<tr>
<td>$4 \times 4$</td>
<td>0.16287</td>
<td>0.3659</td>
<td>0.9226</td>
<td>0.32974</td>
<td>0.8227</td>
<td>1.6486</td>
<td>0.37778</td>
<td>0.9642</td>
<td>1.8310</td>
</tr>
<tr>
<td>$8 \times 8$</td>
<td>0.29165</td>
<td>0.6870</td>
<td>1.4858</td>
<td>0.38719</td>
<td>1.0083</td>
<td>1.8508</td>
<td>0.40497</td>
<td>1.0681</td>
<td>1.8962</td>
</tr>
<tr>
<td>$16 \times 16$</td>
<td>0.37449</td>
<td>0.9470</td>
<td>1.7959</td>
<td>0.40904</td>
<td>1.0890</td>
<td>1.9094</td>
<td>0.41774</td>
<td>1.1172</td>
<td>1.9345</td>
</tr>
<tr>
<td>$24 \times 24$</td>
<td>0.39633</td>
<td>1.0348</td>
<td>1.8690</td>
<td>0.41554</td>
<td>1.1115</td>
<td>1.9279</td>
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<td></td>
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</tr>
<tr>
<td>$32 \times 32$</td>
<td>0.40536</td>
<td>1.0696</td>
<td>1.8890</td>
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<td></td>
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</tr>
<tr>
<td>$48 \times 48$</td>
<td>0.41326</td>
<td>1.0928</td>
<td>1.9213</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ref. (Zhu, 1992)</td>
<td>0.423</td>
<td>0.423</td>
<td>0.423</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note: Displacement and moment at the centre using a quadrilateral hybrid element (de Miranda and Ubertini, 2006) and linked-interpolation elements (Ribarić and Jelenić, 2012), $L/h = 100$

6.5 **Simply supported circular plate**

The circular plate with the simply supported edges is analysed next. The element mesh is here irregular and the influence of such irregularity is studied on the element family in consideration.

Additionally, not only the vertex nodes, but also the side nodes of the higher-order elements (six-node $T6-U3$ and ten-node $T10-U4$) are now placed on the circular boundary. The edge elements are not following the straight line rule, so they must behave as curvilinear transformed triangles for which linked interpolation does not solve the patch test exactly (unless the elements become infinitesimally small). Only the transverse displacements of the nodes on the circular plate boundary are restrained and the rotations remain free (SS1 boundary condition). The results are given in Tables XX and XXI for the thick and the thin plate, respectively. The problem geometry and material properties are given in Figure 21 (only one quarter of the plate is analysed), where examples of the three-node element mesh is shown. A comparison with the linked-interpolation quadrilateral elements (Ribarić and Jelenić, 2012) is given in Table XXII:
Figure 19.
A simply supported (SS1) skew plate with three different mesh patterns

Table XVI.
Simply supported skew plate (SS1)
\[
\begin{align*}
w_c^* &= \frac{w_c}{q} \frac{100D}{(2R)^4}, \\
D &= \frac{Eh^3}{12(1 - \nu^2)}, \\
M_c^* &= \frac{M_c}{q} \frac{100}{(2R)^2} \quad \text{(at closest Gauss point)}
\end{align*}
\]

\[M_x = M_y = M_c - \text{moment at the central point}\]

In Table XXIII the results for the simply supported circular plate subject to a concentrated load at the center point are given. The geometry and material properties

<table>
<thead>
<tr>
<th>Element mesh</th>
<th>d.o.f.</th>
<th>(T3-U2) (w^<em>) (M_{22}^</em>) (M_{11}^*)</th>
<th>(w^*)</th>
<th>(T6-U3) (M_{22}^<em>) (M_{11}^</em>)</th>
<th>(w^*)</th>
<th>(MIN6) (Liu-Riggs) (M_{22}^<em>) (M_{11}^</em>)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4 \times 4</td>
<td>107</td>
<td>0.394314 1.767770 0.442714 1.57560</td>
<td>2.42260</td>
<td>0.443030 1.58489 2.442300</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8 \times 8</td>
<td>403</td>
<td>0.380630 1.09990 1.85088 0.398472</td>
<td>1.99636</td>
<td>0.394705 1.28551 1.995570</td>
<td></td>
<td></td>
</tr>
<tr>
<td>16 \times 16</td>
<td>1,571</td>
<td>0.405027 1.09959 1.90959</td>
<td>0.412030</td>
<td>1.16231 1.93387 0.410138</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Table XVII.** Simply supported skew plate (SS1)

**Note:** Displacement and moment at the centre, \(L/r = 100\) – mesh pattern (c)

Figure 20.
Simply supported skew plate under uniform load

Notes: (a) Principal moment \(M_{11}\) distribution between points A and C; (b) principal moment \(M_{22}\) distribution between points A and C
### Table XVIII.

<table>
<thead>
<tr>
<th>Element mesh</th>
<th>( W^* )</th>
<th>( M_{22}^* )</th>
<th>( M_{11}^* )</th>
<th>( w^* )</th>
<th>( M_{22}^* )</th>
<th>( M_{11}^* )</th>
<th>( w^* )</th>
<th>( M_{22}^* )</th>
<th>( M_{11}^* )</th>
<th>( w^* )</th>
<th>( M_{22}^* )</th>
<th>( M_{11}^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 x 2</td>
<td>0.421115</td>
<td>0.64790</td>
<td>1.33819</td>
<td>0.443104</td>
<td>1.62431</td>
<td>2.52722</td>
<td>0.246108</td>
<td>0.60215</td>
<td>1.16324</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4 x 4</td>
<td>0.393909</td>
<td>1.01757</td>
<td>1.67013</td>
<td>0.348698</td>
<td>1.30423</td>
<td>1.98429</td>
<td>0.356469</td>
<td>1.05704</td>
<td>1.76663</td>
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<td></td>
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</tr>
<tr>
<td>8 x 8</td>
<td>0.305318</td>
<td>1.11745</td>
<td>1.58532</td>
<td>0.326564</td>
<td>0.86335</td>
<td>1.71994</td>
<td>0.365434</td>
<td>0.95047</td>
<td>1.78325</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>16 x 16</td>
<td>0.285607</td>
<td>1.05519</td>
<td>1.61422</td>
<td>0.358165</td>
<td>0.97179</td>
<td>1.81092</td>
<td>0.390331</td>
<td>1.02111</td>
<td>1.85543</td>
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</tr>
<tr>
<td>24 x 24</td>
<td>0.309866</td>
<td>1.05211</td>
<td>1.68162</td>
<td>0.376441</td>
<td>0.99480</td>
<td>1.82813</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>32 x 32</td>
<td>0.330657</td>
<td>1.00984</td>
<td>1.72183</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>48 x 48</td>
<td>0.360440</td>
<td>0.96062</td>
<td>1.77377</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ref. (Morley, 1962)</td>
<td>0.4080</td>
<td>1.08</td>
<td>1.91</td>
<td>0.4080</td>
<td>1.08</td>
<td>1.91</td>
<td>0.4080</td>
<td>1.08</td>
<td>1.91</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Note:** Displacement and moment at the centre with regular meshes, \( L/h = 1,000 \)

### Table XIX.

<table>
<thead>
<tr>
<th>Element mesh</th>
<th>( w^* )</th>
<th>( M_{22}^* )</th>
<th>( M_{11}^* )</th>
<th>( w^* )</th>
<th>( M_{22}^* )</th>
<th>( M_{11}^* )</th>
<th>( w^* )</th>
<th>( M_{22}^* )</th>
<th>( M_{11}^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 x 2</td>
<td>0.537100</td>
<td>0.74915</td>
<td>1.44018</td>
<td>0.443109</td>
<td>1.62421</td>
<td>2.52671</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>4 x 4</td>
<td>0.420881</td>
<td>0.95544</td>
<td>1.72257</td>
<td>0.348478</td>
<td>1.30308</td>
<td>1.98405</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8 x 8</td>
<td>0.415286</td>
<td>1.09045</td>
<td>1.89280</td>
<td>0.324182</td>
<td>0.83373</td>
<td>1.70398</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>16 x 16</td>
<td>0.413626</td>
<td>1.10028</td>
<td>1.91234</td>
<td>0.354328</td>
<td>0.93747</td>
<td>1.78247</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>24 x 24</td>
<td>0.412734</td>
<td>1.09998</td>
<td>1.91781</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>32 x 32</td>
<td>0.412062</td>
<td>1.10022</td>
<td>1.91790</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>48 x 48</td>
<td>0.4080</td>
<td>1.08</td>
<td>1.91</td>
<td>0.4080</td>
<td>1.08</td>
<td>1.91</td>
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<tr>
<td>64 x 64</td>
<td>0.412062</td>
<td>1.10022</td>
<td>1.91790</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Note:** Displacement and moment at the centre using a quadrilateral hybrid element (de Miranda and Ubertini, 2006) and linked-interpolation elements (Ribaric and Jelenic, 2012), \( L/h = 1,000 \)

---

**Higher-order linked interpolation**

103
are identical to the problem from Figure 21. For the thin plate situation \((R/h = 50)\) the exact solution for the central displacement is known (de Miranda and Ubertini, 2006), while the bending moment at the centre has a singularity:

\[
\text{With: } \quad w_c^* = \frac{w_c}{P} \frac{100D}{4R^2}, \quad M_c^* = \frac{M_c}{P} \quad \text{(at closest Gauss point)}
\]

<table>
<thead>
<tr>
<th>Table XX.</th>
<th>Simply supported circular plate (SS1) with uniform load on meshes from Figure 21</th>
</tr>
</thead>
<tbody>
<tr>
<td>Element mesh</td>
<td>d.o.f.</td>
</tr>
<tr>
<td>6</td>
<td>12</td>
</tr>
<tr>
<td>24</td>
<td>42</td>
</tr>
<tr>
<td>96</td>
<td>156</td>
</tr>
<tr>
<td>384</td>
<td>600</td>
</tr>
<tr>
<td>Ref. (de Miranda and Ubertini, 2006)</td>
<td>0.415994</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table XXI.</th>
<th>Simply supported circular plate (SS1) with uniform load on meshes from Figure 21</th>
</tr>
</thead>
<tbody>
<tr>
<td>Element mesh</td>
<td>d.o.f.</td>
</tr>
<tr>
<td>6</td>
<td>12</td>
</tr>
<tr>
<td>24</td>
<td>42</td>
</tr>
<tr>
<td>96</td>
<td>156</td>
</tr>
<tr>
<td>384</td>
<td>600</td>
</tr>
<tr>
<td>Ref. (de Miranda and Ubertini, 2006)</td>
<td>0.398315</td>
</tr>
</tbody>
</table>

**Note:** Displacement and moment at the centre, \(R/h = 5\)
7. Conclusions

A displacement-based linked-interpolation concept for designing triangular Mindlin plate finite elements has been presented and numerically verified. The idea has been presented in its general form applicable to triangular elements of any order, but the numerical results have been provided for the first three elements of the family, i.e. the three-, six- and ten-node plate elements. In all the elements from the family the leading
design principle is borrowed from the underlying family of Timoshenko beam elements. Thus, the original shear strain condition of a certain order imposed along the beam has been applied to the plate element edges leading to 2D generalisation of the beam-type linked interpolation for the displacement field, whereby, both nodal rotations contribute to the element out-of-plane displacements.

It has been shown that, for the elements with more than two nodes per side, additional internal degrees of freedom are required in order to provide full polynomial expansion of certain order. The number of these degrees of freedom is linearly growing by one, beginning with no internal degrees of freedom for the lowest-order three-node member of the family. The elements developed in this way give exact result for cylindrical bending of a corresponding order, e.g. quadratic distribution of the out-of-plane displacements for the three-node elements, cubic for the six-node elements and so on. However, in contrast to beams, the developed elements still suffer from shear locking for very coarse meshes of the lowest-order element types. In particular, the results for the thin clamped square plate have shown that the lowest-order triangular linked-interpolation element ($T3-U2$) may suffer from considerable shear locking and cannot be considered as reliable, while a certain amount of locking is also present in $T6-U3$.

Performance of these elements has been numerically analysed and it has been found out, that they perform well for a number of standard benchmark tests. It has to be stressed, however, that the well-known Morley’s skewed plate example again turns out to be rather demanding confirming the earlier conclusion for the quadrilateral displacement-based linked-interpolation elements that, for this test, the proposed design principle cannot compete with the mixed-type approach. Otherwise, the higher-order members of the element family turn out to be successful when compared to the lower-order elements from literature for the problems with the same total number of the degrees of freedom.

Nevertheless, the lowest-order element with three nodes is the one with the greatest practical interest for its versatility. Work is under way to improve it by adding a number of internal bubble functions in the displacement and rotation fields associated with corresponding bubble parameters (Ribarić, 2012), which are specifically chosen to satisfy the basic patch test necessary for the element consistency and enable a softer response in the benchmark test examples.

References


About the authors
Dragan Ribarić is a Senior Lecturer of mechanics at the Faculty of Civil Engineering, University of Rijeka. He is a doctoral candidate and a Teacher for a number of undergraduate courses in engineering and structural mechanics. His research interests are plate and shell structures, structural analysis, and finite element design.

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