Operator inequalities involving real convex functions

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Introduction

Let $\mathcal{B}(H)$ be the algebra of all bounded linear operators on a complex Hilbert space $H$ and $I$ denote the identity operator. We denote by $\mathcal{B}_h(H)$ the real subspace of all self-adjoint operators on $H$. If $\dim H = n$, we identify $\mathcal{B}(H)$ with the algebra $\mathcal{M}_n(\mathbb{C})$ of all $n \times n$ matrices with entries in the field $\mathbb{C}$ of the complex numbers. An operator $A$ is said to be positive (denoted by $A \geq 0$) if $\langle Ax, x \rangle \geq 0$ for all $x \in H$. If, in addition, $A$ is invertible, then it is called strictly positive (denoted by $A > 0$). By $A \geq B$ we mean that $A - B$ is positive, while $A > B$ means that $A - B$ is strictly positive. A mapping $\Phi$ on $\mathcal{B}(H)$ is said to be positive if $\Phi(A) \geq 0$ for each $A \geq 0$ and is called unital if $\Phi(I) = I$.

A continuous real valued function $f$ defined on an interval $J$ is said to be operator convex if $f(\lambda A + (1 - \lambda)B) \leq \lambda f(A) + (1 - \lambda)f(B)$ for all self-adjoint operators $A, B$ with spectra contained in $J$ and all $\lambda \in [0, 1]$. 
Preliminary results

If the function $f$ is operator convex, then the so-called Jensen operator inequality

$$f(\Phi(A)) \leq \Phi(f(A))$$

holds for any unital positive linear mapping $\Phi$ on $\mathcal{B}(H)$ and any $A \in \mathcal{B}_h(H)$ with spectrum contained in $J$. Many other versions of the Jensen operator inequality can be found in:


Among them, the Jensen–Mercer operator inequality reads as follows:

\[ f \left( M + m - \sum_{i=1}^{n} \Phi_i(A_i) \right) \leq f(M) + f(m) - \sum_{i=1}^{n} \Phi_i(f(A_i)), \]

where \( f \) is a convex function on an interval \([m, M]\), \( \Phi_1, \ldots, \Phi_n \) are positive linear mappings on \( \mathcal{B}(H) \) with \( \sum_{i=1}^{n} \Phi_i(I) = I \) and \( A_1, \ldots, A_n \) are self-adjoint operators with spectra contained in \([m, M]\).

If \( f : [0, \infty) \to \mathbb{R} \) is a convex function and \( f(0) \leq 0 \), then

\[
f(a) + f(b) \leq f(a + b)
\]  

(1)

for all scalars \( a, b \geq 0 \). However, if the scalars \( a, b \) are replaced by two positive operators, this inequality may not hold. There have been many interesting works devoted to obtain operator extensions of inequality (1).

- T. Kosem, *Inequalities between \( \|f(A + B)\| \) and \( \|f(A) + f(B)\| \)*, Linear Algebra Appl. 418 (2006), 153–160.
A version of Jensen’s operator inequality without operator convexity:


We can rewrite these results as follows.

Let \((A_1, \ldots, A_n)\) be an \(n\)-tuple of operators \(A_i \in \mathbb{B}_h(H)\). Let \((\Phi_1, \ldots, \Phi_n)\) be an \(n\)-tuple of positive linear mappings \(\Phi_i\) on \(\mathbb{B}(H)\) such that \(\sum_{i=1}^{n} \Phi_i(I) = I\).

If \(A_i \leq m, \ i = 1, \ldots, n_1, \ M \leq A_i, \ i = n_1 + 1, \ldots, n\), for some \(m, M \in \mathbb{R}\), \(m \leq M\) and \(m \leq A = \sum_{i=1}^{n} \Phi_i(A_i) \leq M\), then

\[
f \left( \sum_{i=1}^{n} \Phi_i(A_i) \right) \leq \sum_{i=1}^{n} \Phi_i(f(A_i))
\]

holds for every continuous convex function \(f\) on an interval \(J\) provided that the interval \(J\) contains all \(m_i, M_i\).

Especially, for \(n = 2\): If \(A \leq m \leq A + B \leq M \leq B\), then

\[
f(\Phi_1(A) + \Phi_2(B)) \leq \Phi_1(f(A)) + \Phi_2(f(B))
\]
Main results

**Theorem 1.**
Let $f$ be a continuous convex function on an interval $J$. Let $A, B, C, D \in \mathcal{B}_h(H)$ with spectra contained in $J$ such that

$$A + D = B + C \quad \text{and} \quad A \leq m \leq B, C \leq M \leq D$$

for two real numbers $m \leq M$. If $\Phi$ is an unital positive linear mapping on $\mathcal{B}(H)$, then

$$f(\Phi(B)) + f(\Phi(C)) \leq \Phi(f(A)) + \Phi(f(D)). \quad (2)$$

If $f$ is concave on $J$, then the inequality (2) is reversed.
Proof

We will prove only the convex case.

a) Let $m < M$. Since $A \leq m$ and $D \geq M$, then

$$f(A) \geq \frac{M-A}{M-m} f(m) + \frac{A-m}{M-m} f(M), \quad f(D) \geq \frac{M-D}{M-m} f(m) + \frac{D-m}{M-m} f(M).$$

It follows

$$\Phi(f(A)) + \Phi(f(D)) \geq \frac{2M-(\Phi(A)+\Phi(D))}{M-m} f(m) + \frac{\Phi(A)+\Phi(D)-2m}{M-m} f(M).$$

(♣)

Similarly, taking into account that $m \leq \Phi(B), \Phi(C) \leq M$, then

$$f(\Phi(B)) + f(\Phi(C)) \leq \frac{2M-(\Phi(B)+\Phi(C))}{M-m} f(m) + \frac{\Phi(B)+\Phi(C)-2m}{M-m} f(M).$$

(♠)

The inequality (2) follows from (♣) and (♠) by using $A + D = B + C$.

b) Let $m = M$. The proof is similar to the above by using the subdifferential of $f$.  

□
We give an example to clarify the situation in Theorem 1.

**Example**

Let the function \( f \) be defined on \([0, \infty)\) by \( f(t) = t^3 \) and the unital positive linear mapping \( \Phi : M_2(\mathbb{C}) \rightarrow M_2(\mathbb{C}) \) be defined by \( \Phi(A) = (\frac{1}{2} tr(A))I \) for all Hermitian matrix \( A \in M_2(\mathbb{C}) \). If 

\[
A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 6 & -1 \\ -1 & 7 \end{pmatrix}, \quad D = \begin{pmatrix} 9 & 1 \\ 1 & 10 \end{pmatrix},
\]

then

\[
0 \leq A < 3I \leq B \leq C \leq 8I < D \quad \text{and} \quad A + D = B + C,
\]

whence

\[
f(\Phi(B)) + f(\Phi(C)) = 338.625I \leq 897I = \Phi(f(A)) + \Phi(f(D)).
\]

This shows that inequality (2) can be strict.
More generally, the next corollary gives other versions of inequality (2).

**Corollary 1.**

Let \( f \) be a convex function on an interval \( J \). Let \( A_i, B_i, C_i, D_i \in \mathcal{B}(H)_h, i = 1, \ldots, n \), with spectra contained in \( J \) such that

\[
A_i + D_i = B_i + C_i \quad \text{and} \quad A_i \leq m \leq B_i, C_i \leq M \leq D_i, \ i = 1, \ldots, n
\]

for two real numbers \( m \leq M \). Let \( \Phi_1, \ldots, \Phi_n \) be positive linear mappings on \( \mathcal{B}(H) \) with \( \sum_{i=1}^{n} \Phi_i(I) = I \). Then

1. \[
f(\sum_{i=1}^{n} \Phi_i(B_i)) + f(\sum_{i=1}^{n} \Phi_i(C_i)) \leq \sum_{i=1}^{n} \Phi_i(f(A_i)) + \sum_{i=1}^{n} \Phi_i(f(D_i)),
\]
2. \[
\sum_{i=1}^{n} \Phi_i(f(B_i)) + \sum_{i=1}^{n} \Phi_i(f(C_i)) \leq f(\sum_{i=1}^{n} \Phi_i(A_i)) + f(\sum_{i=1}^{n} \Phi_i(D_i)),
\]
3. \[
\sum_{i=1}^{n} \Phi_i(f(B_i)) + f(\sum_{i=1}^{n} \Phi_i(C_i)) \leq f(\sum_{i=1}^{n} \Phi_i(D_i)) + \sum_{i=1}^{n} \Phi_i(f(A_i)).
\]

If \( f \) is concave on \( J \), then the above inequalities are reversed.
Moreover, we can present another version of Corollary 1.

**Corollary 2.**

Let $f$ be a convex function on an interval $J$. If $A_i, B_i, C_i, D_i \in \mathbb{B}_h(H)$, $i = 1, \cdots, n$, with spectra contained in $J$ such that

$$A_i + D_i = B_i + C_i \quad \text{and} \quad A_i \leq m \leq B_i, \ C_i \leq M \leq D_i, \ i = 1, \cdots, n,$$

then

(1') \hspace{1em} f\left(\sum_{i=1}^{n} B_i\right) + f\left(\sum_{i=1}^{n} C_i\right) \leq \sum_{i=1}^{n} f(A_i) + \sum_{i=1}^{n} f(D_i),

(2') \hspace{1em} \sum_{i=1}^{n} f(B_i) + \sum_{i=1}^{n} f(C_i) \leq f\left(\sum_{i=1}^{n} A_i\right) + f\left(\sum_{i=1}^{n} D_i\right),

(3') \hspace{1em} \sum_{i=1}^{n} f(B_i) + f\left(\sum_{i=1}^{n} C_i\right) \leq f\left(\sum_{i=1}^{n} D_i\right) + \sum_{i=1}^{n} f(A_i).

Especially, we have the following result: If $A \leq m \leq C, \ D \leq M \leq B$ for two real numbers $m \leq M$ and $A + B = C + D$, then

$$f(C) + f(D) \leq f(A) + f(B).$$
Corollary 3.

Let $f$ be a convex function on an interval $J$. Let $A_i, B_i \in B_h(H)$, $i = 1, \ldots, n$, with spectra contained in $J$. Let $\Phi_i, i = 1, \ldots, n$, be positive linear mappings on $B(H)$ with $\sum_{i=1}^n \Phi_i(I) = I$. If $A_i \leq m \leq \frac{A_i + B_i}{2} \leq M \leq B_i$, $i = 1, \ldots, n$, for two real numbers $m \leq M$, then

\[
f \left( \sum_{i=1}^n \Phi_i \left( \frac{A_i + B_i}{2} \right) \right) \leq \sum_{i=1}^n \Phi_i \left( \frac{f(A_i) + f(B_i)}{2} \right),
\]

\[
f \left( \sum_{i=1}^n \Phi_i \left( \frac{A_i + B_i}{2} \right) \right) \leq \frac{1}{2} \sum_{i=1}^n \Phi_i (f(A_i)) + \frac{1}{2} f \left( \sum_{i=1}^n \Phi_i (B_i) \right), \tag{3}
\]

\[
f \left( \sum_{i=1}^n \Phi_i \left( \frac{A_i + B_i}{2} \right) \right) \leq \frac{1}{2} f \left( \sum_{i=1}^n \Phi_i (A_i) \right) + \frac{1}{2} \sum_{i=1}^n \Phi_i (f(B_i)),
\]

\[
f \left( \sum_{i=1}^n \Phi_i \left( \frac{A_i + B_i}{2} \right) \right) \leq \frac{1}{2} f \left( \sum_{i=1}^n \Phi_i (A_i) \right) + \frac{1}{2} f \left( \sum_{i=1}^n \Phi_i (B_i) \right).
\]

If $f$ is concave, then the above inequalities are reversed.
Note that the existence of scalars \( m \leq M \) is essential in Corollary 3, i.e., the inequalities (3) may not hold if \( A, B \not\in \Omega \), where

\[
\Omega = \left\{ (A, B) \mid A, B \in \mathbb{B}_h(H) \text{ and } A \leq m \leq \frac{A+B}{2} \leq M \leq B, \right. \\
\left. \text{for some } m, M \in \mathbb{R} \right\}.
\]

Example

Consider the convex function \( f(t) = t^3 \) on \([0, \infty)\). Putting

\[
A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}
\]

we have \( 0 \leq A \leq B \). There is no scalar \( m \) such that \( A \leq m \leq \frac{A+B}{2} \). Now

\[
f\left(\frac{A+B}{2}\right) = \begin{pmatrix} 6 & 14 & 0 \\ 14 & 34 & 0 \\ 0 & 0 & 3.375 \end{pmatrix} \nless \begin{pmatrix} 6 & 15 & 0 \\ 15 & 43 & 0 \\ 0 & 0 & 4.5 \end{pmatrix} = \frac{f(A) + f(B)}{2}.
\]
The Jensen–Mercer operator inequality follows directly from Corollary 2:

**Corollary 4.**

Let $\Phi_i$, $i = 1, \ldots, n$, be positive linear mappings on $\mathbb{B}(H)$ with $\sum_{i=1}^{n} \Phi_i(I) = I$ and $B_i \in \mathbb{B}_h(H)$, $i = 1, \ldots, n$, with spectra contained in $[m, M]$. If $f$ is a convex function on $[m, M]$, then

$$f(m + M - \sum_{i=1}^{n} \Phi_i(B_i)) \leq f(m) + f(M) - \sum_{i=1}^{n} \Phi_i(f(B_i)).$$

**Proof**

Clearly $m \leq B_i \leq M$, $i = 1, \ldots, n$. Set $C_i = M + m - B_i$. Then $m \leq C_i \leq M$ and $B_i + C_i = m + M$, $i = 1, \ldots, n$. Applying inequality (3) of Corollary 2 when $A_i = mI$ and $D_i = M I$ we obtain

$$\sum_{i=1}^{n} \Phi_i(f(B_i)) + f(\sum_{i=1}^{n} \Phi_i(C_i)) \leq f(m) + f(M)$$

which is the desired inequality.
Operator version of the Petrović inequality

Petrović inequality see e.g. in


**First we give a generalization of this inequality.**

**Corollary 5.**

Let $A, B, C_i \in \mathcal{B}_h(H)$, $i = 1, \ldots, n$, $n > 1$, with spectra contained in an interval $J$ such that $A + B = \sum_{i=1}^{n} C_i$ and $A \leq m \leq C_i \leq M \leq B$, $i = 1, \ldots, n$, for two real numbers $0 \leq m \leq M$. Then

$$\sum_{i=1}^{n} f(C_i) \leq f(B) + (n-1)f\left(\frac{1}{n-1}A\right)$$

for every convex function $f$ on $J$. If $f$ is concave, then the inequality (4) is reversed.
As a special case of Corollary 5 we have operator version of the Petrović inequality as follows.

**Corollary 6.**

If $f : [0, \infty) \to \mathbb{R}$ is a convex function and $C_i$, $i = 1, \ldots, n$, are positive operators such that $\sum_{i=1}^{n} C_i = MI$ for some scalar $M \geq 0$, then

$$\sum_{i=1}^{n} f(C_i) \leq f\left( \sum_{i=1}^{n} C_i \right) + (n - 1)f(0)I.$$

**Proof**

We put $B = MI = \sum_{i=1}^{n} C_i$ and $A = 0$ in Corollary 5.
Superadditivity inequalities

If $f : [0, \infty) \rightarrow \mathbb{R}$ is a convex function such that $f(0) \leq 0$, then inequality

$$f(a) + f(b) \leq f(a + b)$$

(5)

holds for all non-negative scalars $a, b$.

(see e.g.


)

Generally, the inequality (5) would be false if we replace scalars $a, b$ with two arbitrary positive operators (see the next example).
Example

Consider the convex function $f(t) = t^3$ on $[0, \infty)$. Putting

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Then $f(A + B) - f(A) - f(B) =$

$$= \begin{pmatrix} 48 & 112 & 0 \\ 112 & 272 & 0 \\ 0 & 0 & 27 \end{pmatrix} - \begin{pmatrix} 5 & 8 & 0 \\ 8 & 13 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 7 & 22 & 0 \\ 22 & 73 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

$$= \begin{pmatrix} 37 & 82 & 0 \\ 82 & 186 & 0 \\ 0 & 0 & 18 \end{pmatrix} \not\geq 0.$$

So there is no relationship between $f(A + B)$ and $f(A) + f(B)$ under the operator order.
The following corollary gives an operator version of (5). Also, we may compare this result with the inequality for unitarily invariant norms given in

T. Kosem, *Inequalities between* $\|f(A + B)\|$ *and* $\|f(A) + f(B)\|$, Linear Algebra Appl. **418** (2006), 153–160.

**Corollary 7.**

*If* $f : [0, \infty) \to \mathbb{R}$ *is a convex function with* $f(0) \leq 0$ *then*

$$\sum_{i=1}^{n} f(C_i) \leq f \left( \sum_{i=1}^{n} C_i \right)$$

*(6)*

*for all positive operators* $C_i$ *such that* $C_i \leq M \leq \sum_{i=1}^{n} C_i$, $i = 1, \ldots, n$, *for some scalar* $M \geq 0$. *If* $f$ *is concave, then the reverse inequality is valid in* (6).

*In particular:*

*if* $f$ *is convex and* $f(0) \leq 0$, *then* $f(A) + f(B) \leq f(A + B)$ *for all positive operators* $A, B$ *such that* $A, B \leq MI \leq A + B$ *for some scalar* $M \geq 0$. 
Further generalizations

**Theorem 2.**

Let $f$ be a continuous function on an interval $J$. Let $A, B, C, D \in \mathcal{B}_h(H)$ with spectra contained in $J$ such that $A \leq m \leq B$, $C \leq M \leq D$ for two real numbers $m \leq M$. Let $\Phi$ be an unital positive linear mapping on $\mathcal{B}(H)$. If $f$ is convex and one of the following conditions

(i) $B + C \leq A + D$ and $f(m) \leq f(M)$
(ii) $A + D \leq B + C$ and $f(M) \leq f(m)$

is satisfied, then

$$f(\Phi(B)) + f(\Phi(C)) \leq \Phi(f(A)) + \Phi(f(D)).$$

(7)

If $f$ is concave and one of the following conditions

(iii) $B + C \leq A + D$ and $f(M) \leq f(m)$
(iv) $A + D \leq B + C$ and $f(m) \leq f(M)$

is satisfied, then the inequality (7) is reversed.
Proof

We will prove only the case when $f$ is convex and (i) is valid.

**a)** Let $m < M$. Denote $a_f := \frac{f(M)-f(m)}{M-m}$ and $b_f := \frac{mf(M)-Mf(m)}{M-m}$.
Since $A \leq m$ and $D \geq M$, then $f(A) \geq a_f A + b_f$, $f(D) \geq a_f D + b_f$.
It follows
\[
\Phi(f(A)) + \Phi(f(D)) \geq a_f (\Phi(A) + \Phi(D)) + 2b_f. \quad (♣')
\]
Similarly, taking into account that $m \leq \Phi(B), \Phi(C) \leq M$, then
\[
f(\Phi(B)) + f(\Phi(C)) \leq a_f (\Phi(B) + \Phi(C)) + 2b_f. \quad (♠')
\]
The inequality (7) follows from (♣') and (♠') by using $B + C \leq A + D$ and $a_f \geq 0$.

**b)** Let $m = M$. The proof is similar to the above by using the subdifferential of $f$. \hfill \square
More generally, the next corollary gives other versions of inequality (7).

**Corollary 8.**

Let $f$ be a continuous function on an interval $J$. Let $A_i, B_i, C_i, D_i \in \mathbb{B}(H)_h$, $i = 1, \cdots, n$, with spectra contained in $J$ such that $A_i \leq m \leq B_i, C_i \leq M \leq D_i$ for two real numbers $m \leq M$. Let $\Phi_1, \cdots, \Phi_n$ be positive linear mappings on $\mathbb{B}(H)$ with $\sum_{i=1}^{n} \Phi_i(I) = I$. If $f$ is convex and one of the following conditions

(i) $B_i + C_i \leq A_i + D_i$, $i = 1, \cdots, n,$ and $f(m) \leq f(M)$

(ii) $A_i + D_i \leq B_i + C_i$, $i = 1, \cdots, n,$ and $f(M) \leq f(m)$

is satisfied, then

(1) $f \left( \sum_{i=1}^{n} \Phi_i(B_i) \right) + f \left( \sum_{i=1}^{n} \Phi_i(C_i) \right) \leq \sum_{i=1}^{n} \Phi_i(f(A_i)) + \sum_{i=1}^{n} \Phi_i(f(D_i))$,

(2) $\sum_{i=1}^{n} \Phi_i(f(B_i)) + \sum_{i=1}^{n} \Phi_i(f(C_i)) \leq f \left( \sum_{i=1}^{n} \Phi_i(A_i) \right) + f \left( \sum_{i=1}^{n} \Phi_i(D_i) \right)$,

(3) $\sum_{i=1}^{n} \Phi_i(f(B_i)) + f \left( \sum_{i=1}^{n} \Phi_i(C_i) \right) \leq f \left( \sum_{i=1}^{n} \Phi_i(D_i) \right) + \sum_{i=1}^{n} \Phi_i(f(A_i))$. 
As an immediate consequence of Theorem 2, we have the following corollary.

**Corollary 9.**

Let $f$, $A$, $B$, $C$, $D$, $\Phi$ and conditions (i)–(iv) be as in Theorem 2. If $f$ is convex and one of the conditions (i) or (ii) is satisfied, then

$$f(\Phi(B)) + f(\Phi(C)) \leq \Phi(g(A)) + \Phi(g(D))$$

(8)

for every continuous function $g \geq f$ on $J$ and

$$g(\Phi(B)) + g(\Phi(C)) \leq \Phi(f(A)) + \Phi(f(D))$$

(9)

for every continuous function $g \leq f$ on $J$.

If $f$ is concave and one of the conditions (iii) or (iv) is satisfied, then the inequality (8) is reversed for $g \leq f$ on $J$ and the inequality (9) is reversed for $f \leq g$ on $J$. 
Further generalizations

Applying the above corollary to the power functions we get the following corollary.

**Corollary 10.**

Let \( A, B, C, D \in \mathcal{B}_h(H) \) such that

\[
I \leq A \leq m \leq B, \ C \leq M \leq D
\]

for two real numbers \( m \leq M \). If one of the following conditions

(a) \( B + C \leq A + D \) and \( p \geq 1 \)

(b) \( A + D \leq B + C \) and \( p \leq 0 \)

is satisfied, then

\[
B^p + C^p \leq A^q + D^q
\]

for each \( q \geq p \).
Thank you very much for your attention