Programming the Force Density Method

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Summary

Based on the similarity of the equation systems of the displacement method and the force density method, computer code for solving frame structures by displacement method is modified to implement force density method. Namely, force density in the homonymous method is analogous to stiffness coefficient of the frame element in the displacement method. In both methods the connections between nodes can be resolved in the same way. Therefore the stiffness matrix in the displacement method has the same structure as the system matrix in the force density method. The unknowns in displacement method are displacement components of nodes while the unknowns in force density method are nodal coordinates. Examples of the application of the written programme include single step solutions which are compared with similar physical model and minimal nets.

Keywords: force density method, displacement method, cable net, stiffness coefficient, system matrix, minimal net.

1. Introduction

The solution of equilibrium equations, while the geometry of the surface is unknown, is the task of the classical form finding methods. These methods were developed to avoid numerical problems arising in the computerization of inverse problems. Hanging chain and its inverse is one of the oldest methods known for determining the form of an arc without bending, under the influence of compressive axial forces only. It was also used by A. Gaudi. To determine the shape of shells, arc is expanded into another dimension.

Determining the form of prestressed cable nets is defined as the process of finding the equilibrium shape to meet the architect and its functional and aesthetic concept, but it will also satisfy the engineer in terms of load transfer capabilities and performance. Form finding usually consist of two phases. First a physical model is made for the given boundary conditions by using soap, stretchy fabric or elastic threads. When the shape that satisfies the desired aesthetic terms is achieved, its numerical model is formed in second stage.

The complexity of form finding process arises from the fact that there are multiple solutions that satisfy the given boundary conditions. In fact, the final form of the net is determined by three coordinates: *x, y, z* of each node in which cables (elements) intersect. If there are *n* free nodes, the number of unknowns is *3n*. For each node one can write three conditions of equilibrium in which tensile force in corresponding cables (elements) occur also as unknown. Finally, a system of *3n* equations with *3n + m* unknowns (*m* is the number of net elements) is formed. For a single node (Fig. 1) where two cables intersect (or four elements of the net), three equilibrium equations are (no external load and self-weight is ignored):

$$\begin{aligned}
\sum_{j=1}^{4} \frac{(x_i - x_j)}{\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2}} \cdot S_{ij} &= 0, \\
\sum_{j=1}^{4} \frac{(y_i - y_j)}{\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2}} \cdot S_{ij} &= 0, \\
\sum_{j=1}^{4} \frac{(z_i - z_j)}{\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2}} \cdot S_{ij} &= 0
\end{aligned} \qquad (1)$$

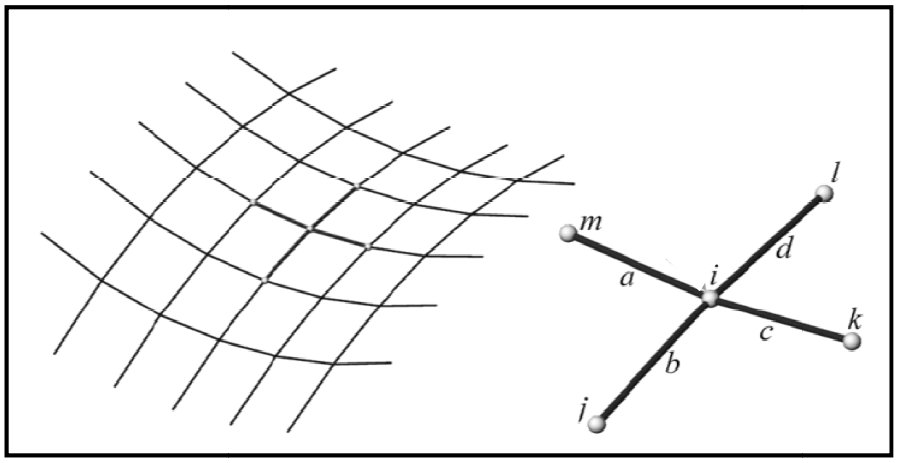

Figure1: Part of the prestressed cable net and one node secluded [1]

If the same three equations are written for every free node of the net, a system of equation is formed. The system is nonlinear, so the solution can not be obtained directly. To make the system become solvable, it is necessary to introduce some additional assumptions and limitations that will lead to a solution, but this will also conditional form of constructions. Assuming a uniform tensile force in all elements, the system becomes solvable, and construction takes the form of minimal net.

2. Force Density Method

The force density method has been published for the first time the 1971st in the article "*Einige Bemerkungen von zur Berechtung vorgespannten Seilnetzkonstruktionen*" by H.-J. Schek and K. Linkwitz [2]. Schek expanded the method in 1974. and described the procedure in the article "*The force density method for form finding and computation of general networks*" [3]. The method is based on a mathematical trick that turns a system of nonlinear equations into linear. The solution is prestressed cable net in equilibrium, that is, coordinates of nodes in that net. The linear system of equations is easily solved using the method of conjugate gradients.
Schek's force density method was originally developed only for equidistant square cable nets. Today's methods for form finding of tensile structures are mostly based on his method. For example, method with triangular mesh surface elements is developed and is called the surface stress density method [4].
The force density method is proved to be a powerful tool for assembling and solving the equilibrium equations of prestressed cable nets and the membrane, without the need to determine the initial coordinates. The elementary equations are based on achieving equilibrium in nodes (Fig. 1): internal, prestressed forces (s) and external forces (p) are in balance:

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where::

s_a, s_b, s_c, s_d.. are prestressed forces in elements (cables) *a, b, c, d*,
a, b, c, d … are nonlinear functions of the nodes coordinates:

_____

_____

_____

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Prestressed forces are dependent on the length of the undeformed cables and Hooke's law. The form finding process tends to find shape with the required prestress and surface or net discretization. These two data form the so-called force density: $q=s_d/a$. If the force densities are chosen before solving equationes (2), these nonlinear equationes are transformed into linear:

$$\begin{aligned} q_a(x_m - x_i) + q_b(x_j - x_i) + q_c(x_k - x_i) + q_d(x_i - x_i) &= p_x, \\ q_a(y_m - y_i) + q_b(y_j - y_i) + q_c(y_k - y_i) + q_d(y_i - y_i) &= p_y, \\ q_a(z_m - z_i) + q_b(z_j - z_i) + q_c(z_k - z_i) + q_d(z_i - z_i) &= p_z. \end{aligned} \qquad (4)$$

The system of equations (4) is easily solved using conjugate gradient or the Gaussian elimination. Input data for the process of form finding are the connections between the elements, the force density of each element and the boundary conditions (the coordinates of boundary nodes). Several computer programs were written that implement the force density method to the form finding procedure. The following describes one possible algorithm.

2.1 Typical matrix record of the force density method

Even trivial force density values such as 1, give a satisfactory form of prestressed net. The method can satisfy the requirement of equal force in all net elements (equal prestress in all cables), as well as the requirement to preserve the rectangular or square grid. The method has developed today to so-called nonlinear force density method that includes additional theoretical formulations and iterative procedures [5]. However, even the basic formulation of this method represents an important step in the design of tensile structures.

The force density method assumes a straight cable and hinge between the cables, as well as those on the edge (connection to the supporting structure). The first step is to determine the net and nodes numbered from *1* to n_s, and elements from *1* to *m*. Edge nodes (n_f) are numbered last. The remaining *n* nodes are free, which gives the total number of nodes: $n_s = n + n_f$.

Numbered nodes are entered into the matrix of nodes connection, **Cs**. Connection matrix can be divided into two new matrices:

$$[\mathbf{C_s}] = [\mathbf{C}\,\mathbf{C_f}\,], \qquad (5)$$

where:

C… is matrix of free nodes connection,

$\mathbf{C}_f$ …is matrix of edge nodes connection.

Let us recall that the force densities are defined by the ratio of force in the cable and its length. Also vectors of projected elements in all three coordinate directions can be written using the matrix of nodes connection, $\mathbf{C_s}$. Thus:

$$\begin{aligned} \mathbf{Q}_{mx} &= \mathbf{Q}\mathbf{L}_{mx}, \\ \mathbf{Q}_{my} &= \mathbf{Q}\mathbf{L}_{my}, \\ \mathbf{Q}_{mz} &= \mathbf{Q}\mathbf{L}_{mz}, \end{aligned} \qquad (6) \qquad\qquad \begin{aligned} \mathbf{L}_{mx} &= \mathbf{C}_s\mathbf{x}_s, \\ \mathbf{L}_{my} &= \mathbf{C}_s\mathbf{y}_s, \\ \mathbf{L}_{mz} &= \mathbf{C}_s\mathbf{z}_s, \end{aligned} \qquad (7)$$

where:

$\mathbf{L}_{mx}$, $\mathbf{L}_{my}$, $\mathbf{L}_{mz}$....are vectors of the projected length of the elements on the axis *x, y, z*,

Q .... is the force density matrix,

$\mathbf{Q}_{mx}$, $\mathbf{Q}_{my}$, $\mathbf{Q}_{mz}$ ... diagonal matrix of *x, y* and *z* components of force in the elements,

$\mathbf{x_s}$, $\mathbf{y_s}$, $\mathbf{z_s}$ .....vectors containing nodes coordinates.

Vectors $\mathbf{x_s}$, $\mathbf{y_s}$, $\mathbf{z_s}$ can be decomposed into vectors of edge nodes coordinates, $\mathbf{x_f}$, $\mathbf{y_f}$, $\mathbf{z_f}$, and vectors of free nodes coordinates, **x, y, z.**

With some transformation and by using expressions (6) and (7), one can get a matrix notation of equilibrium equations in any free node and for all three coordinate directions:

$$\mathbf{C}^{\mathrm{T}}\mathbf{Q}\mathbf{C}_{\mathbf{s}}\mathbf{x}_{\mathbf{s}} = \mathbf{F}_{\mathrm{x}}, \qquad \mathbf{C}^{\mathrm{T}}\mathbf{Q}\mathbf{C}\mathbf{x} + \mathbf{C}^{\mathrm{T}}\mathbf{Q}\mathbf{C}_{\mathbf{f}}\mathbf{x}_{\mathrm{f}} = \mathbf{F}_{\mathrm{x}}, \qquad \mathbf{x} = \mathbf{D}^{-1}(\mathbf{F}_{\mathrm{x}} - \mathbf{D}_{\mathrm{f}}\mathbf{x}_{\mathrm{f}}),$$

$$\mathbf{C}^{\mathrm{T}}\mathbf{Q}\mathbf{C}_{\mathbf{s}}\mathbf{y}_{\mathbf{s}} = \mathbf{F}_{\mathrm{y}}, \quad (8) \qquad \mathbf{C}^{\mathrm{T}}\mathbf{Q}\mathbf{C}\mathbf{y} + \mathbf{C}^{\mathrm{T}}\mathbf{Q}\mathbf{C}_{\mathbf{f}}\mathbf{y}_{\mathrm{f}} = \mathbf{F}_{\mathrm{y}}, \quad (9) \qquad \mathbf{y} = \mathbf{D}^{-1}\left(\mathbf{F}_{y} - \mathbf{D}_{\mathrm{f}}\mathbf{y}_{\mathrm{f}}\right), \quad (10)$$

$$\mathbf{C}^{\mathrm{T}}\mathbf{Q}\mathbf{C}_{\mathbf{s}}\mathbf{z}_{\mathbf{s}} = \mathbf{F}_{\mathrm{z}}. \qquad \mathbf{C}^{\mathrm{T}}\mathbf{Q}\mathbf{C}\mathbf{z} + \mathbf{C}^{\mathrm{T}}\mathbf{Q}\mathbf{C}_{\mathbf{f}}\mathbf{z}_{\mathrm{f}} = \mathbf{F}_{\mathrm{z}}. \qquad \mathbf{z} = \mathbf{D}^{-1}(\mathbf{F}_{z} - \mathbf{D}_{\mathrm{f}}\mathbf{z}_{\mathrm{f}}),$$

where:

$\mathbf{D} = \mathbf{C}^{\mathrm{T}}\mathbf{Q}\mathbf{C}$,

$\mathbf{D_f} = \mathbf{C}^{\mathrm{T}}\mathbf{Q}\mathbf{C_f}$.

$\mathbf{F_x}, \mathbf{F_y}, \mathbf{F_z}$ ..... are components of external load in the nodes, in direction of coordinate axis.

Having obtained the equilibrium state, it is possible to calculate the forces in cables (Fi):

$$\mathrm{F_i} = \mathrm{q_i} \cdot \mathrm{L_i} \qquad (13)$$

where:

q_i ..... is the force density of the cable,

L_i ..... length of the element in the resulting equilibrium position.

We can also calculate the stress in the cable: ($\sigma_i = F_i/A_i$).

3. DiM vs FDM

Existing computer program named *DiM* which analyzes static two-dimensional beam systems will be revised for the analysis of prestressed cable nets in the form finding phase. To implement such analysis *DiM* should be extended into the third dimension. And for the form finding the essential information are the interconnection of net elements, force density of each element and location of edge (boundary) nodes. The value and position of the external loads are not required input data for the form finding phase (as well as the self weight of the cables).

Program *DiM* uses the displacement method in which one system of equations is assembled and the unknowns are the displacements and rotations of nodes. Certain displacement component of one node affects the component of the force that is not in its direction. On the other hand, the system of equations in the force density method can be broken down into three smaller systems because of the independence of the equilibrium equations in three perpendicular directions. Elements of the equilibrium equations in the displacement method are analogous to the members of the equilibrium equations in the force density method. Stiffness matrix in the displacement method has the same structure as the matrix $\mathbf{D} = \mathbf{C}^{\mathrm{T}}\mathbf{Q}\mathbf{C}$ from expression (10). Elements on the diagonal of matrix **D** are the sum of the force densities of connecting cables, and outside the diagonal are the negative values of the force densities in case of link between two nodes. If the nodes are not connected at this point in the matrix **D** zero is entered. On the other hand, on the diagonal of the stiffness matrix is the sum of the stiffness coefficients of the connected elements corresponding with the displacement of the specific node and outside the diagonal coefficients of stiffness or zero depending on whether the corresponding displacement affects the individual static value. Thus, the force density in the force density method is analogous to the stiffness coefficient in the displacement method, and connection between the nodes is defined in the matrix **D** as in the stiffness matrix, with elements that connect nodes.

We can also draw a parallel between the unknowns of these two methods. The unknowns in the displacement method are the displacements of nodes in the static system, while in the force density method we are looking for the coordinates of free nodes in the net. The coordinates of boundary nodes are included in the equation as part of the free members of the system ("right side" of the system of equations, although actually in the code "right side" remains on the left).

Displayed matrix formulation of the force density method is a formal description of the method. However, the procedure includes complex matrix operations such as multiplication of multiple matrices, finding inverse matrix, all of which considerably slows down the computer program. The existing program *DiM*, on the other hand, when numbering nodes does not require a special division of free and boundary nodes, and defining the elements using correlation matrix is not used. Connection between nodes is determined by the list of elements. Computer code that describes the

force density method is seamlessly integrated into the *DiM* code, which represents a shift from formal description of the method. This is possible because of the analogy between the matrices of equilibrium equations (in the displacement method and the force density method) and the similarity of unknowns.

3.1 Description of computer code (package *FDM*)

```
vectorOfNodalCoords [nds_List, els_List, supps_List, fds_List] :=
Module [
        { n, dof, mD, mDf, xyz },
        { n, dof } = tableOfNodeDsOF [Length[nds], supps];
        mD = matrixD [n, dof, els, fds];
        prn = False;
        If [prn, Print [MatrixForm [mD]]];
        mDf = xyzDf [n, dof, nds, els, fds];
        If [prn, Print [MatrixForm [mDf]]];
        xyz = LinearSolve [mD, -mDf];
        If [prn, Print [MatrixForm [xyz]]];
        insertNodeCoords [dof, nds, xyz]
  ]
```

If we draw a flow chart corresponding to the written computer program named *FDM*, the above function describes this flow. The function first defines the degrees of freedom and shapes matrix **D**. Then defines a matrix of free members and eventually solves the system. The solution (the coordinates of free nodes) is entered in the list of coordinates of all nodes.

The order of coordinates input defines the numbering sequence of network nodes. The coordinates of the free nodes are unknowns, and their coordinates can be left empty. Thus the program distinguishes free nodes and one at the edge. Since the formation of matrix of the system (as well as the stiffness matrix in the displacement method) goes by the elements, each element contains indices of unknowns in its nodes. When defining matrix **D** we use the analogy of the matrix and the stiffness matrix in displacement method. From this analogy the interpretation of force density as the stiffness coefficient is deriving, so that the process of forming the matrix **D** is identical to the process forming the stiffness matrix.
Free member ("right side") is a product of the force density of the element that connects the free node with the edge node and the edge node coordinate. Free member can be compared with the fix boundary forces caused by the forced displacement in the displacement method, which is the product of the stiffness coefficient and forced displacement.
The coordinates of the free nodes are solution of equilibrium equations, and from those coordinates the new length of the element is determined. Iterative procedure can obtain a solution which approaches the minimum net. Iteration begins by determining the mean prestressed force of all the elements, and the known length of the element determines the new force density of each element. The system is solved again and new equilibrium position is determined. The process is repeated as many times as desired, or until we are satisfied with the values of prestressed forces.

4. Examples

For all examples a 24x16m grid is chosen with cables every four meters (Fig. 2a). If all the boundary nodes lie in the same plane, then free nodes also belong to the same plane. Such solution of the equilibrium equation is called trivial (Fig. 3a).

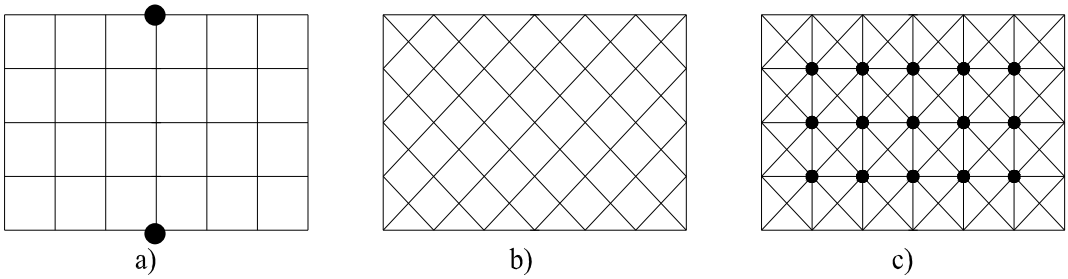

Figure 2: a) Nodes and elements for examples 1, 2 and 3; b) Nodes and elements for example 4; c) Matching nodes in example 3 and 4

Second example uses the same numeration and node connections as the privies. The only differences are coordinates of boundary nodes. Node 4 and 32 (indicated on the Fig. 2a) are lifted 4m up in z axis direction. All the other edge nodes connected with 4 and 32 changed their position as well. After fifty-three iteration using *FDM,* tensile force in all elements coincide on the third decimal place. We can say that this equilibrium position approximates minimal net (Fig. 3b).

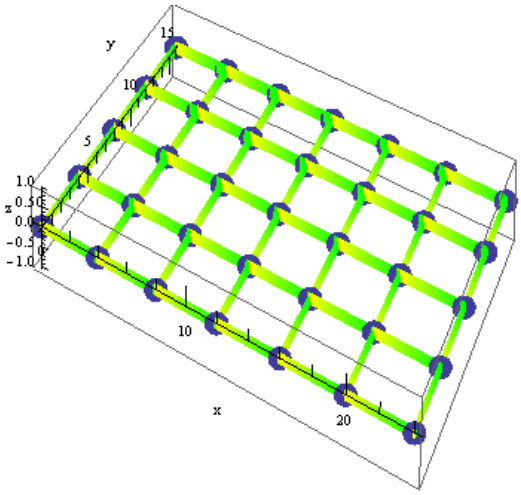

Figure 3a: trivial solution

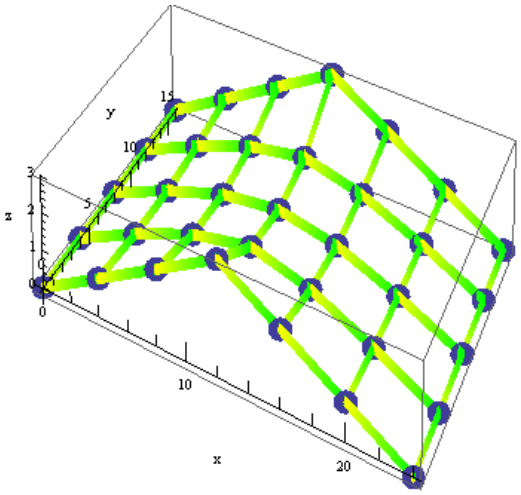

Figure 3b: Example 2 – approximately minimal net after 53. iterations

If node 4 and 32 change vertical position again, now being lifted 9m in z axis direction, *FDM* gives new equilibrium position. After 55 iteration minimal net is again approximated (Fig. 4a). Even in this simple graphic as in Fig. 3b and 4a, we can see difference in curvature of nets from example 2 and 3. With increasing number of nodes (and elements as well) net becomes smoother. This is done in next example. Fig. 4b shows solution for example 4 after 71 iterations. Number of nodes in example 3 and 4 differ as their connections as well. Fig. 2b shows position of nodes and elements for example 4. It is obvious that this new net has more nodes then example 3 and that all of the nodes from that example are also nodes in example 4 (Fig. 2c)

The numerical solution from *FDM* was compared with coordinates from physical model of example 3 [6]. It was noticed that higher correspondence is between physical model and equilibrium after first iteration (Fig. 5b). The same comparison was made for numerical model in example 4 (Fig 6a and 6b).

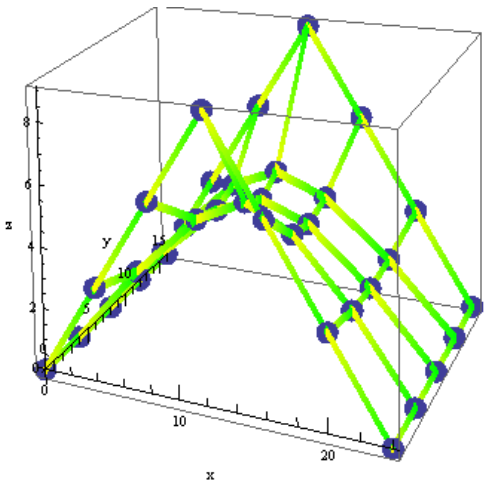

Figure 4a: Example 3 – approximately minimal net after 55. iterations

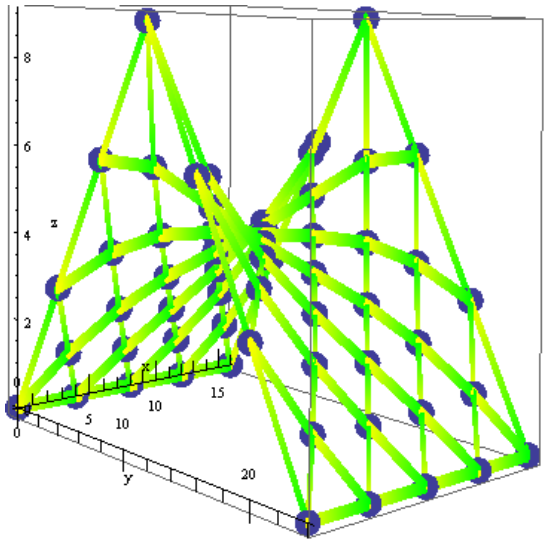

Figure 4b: Example 4 – approximately minimal net after 71. iterations

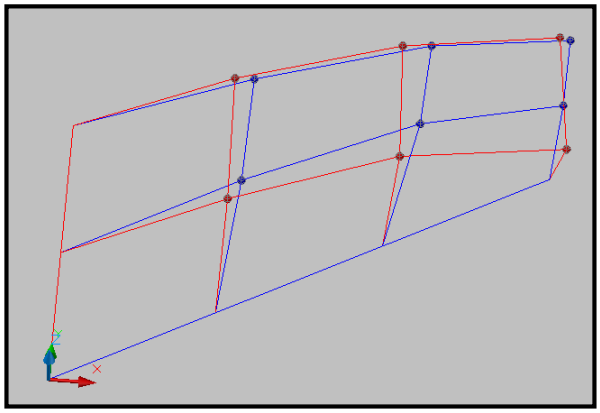

Figure 5a: Differences in numerical (minimal net; blue) and physical model (red) of example 3

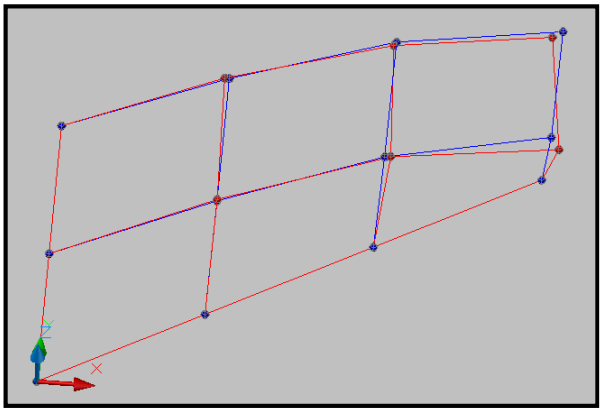

Figure 5b: Differences in numerical (after first iteration; blue) and physical model (red) of example 3

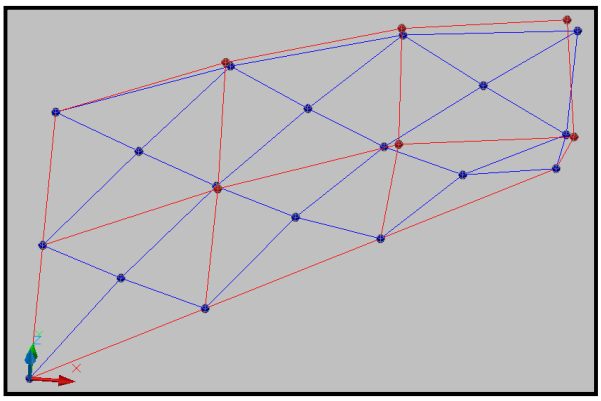

Figure 6a: Differences in numerical (minimal net; blue) and physical model (red) of example 4

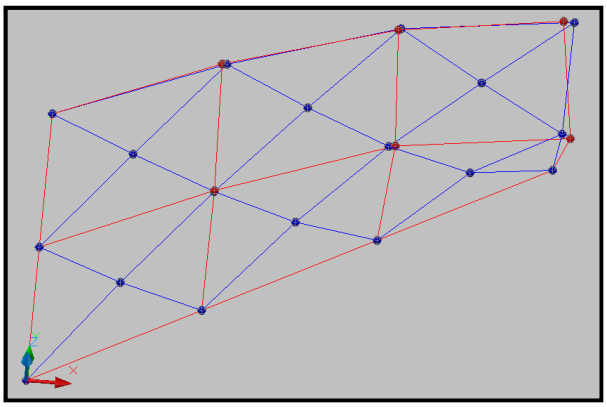

Figure 6b: Differences in numerical (after first iteration; blue) and physical model (red) of example 4

5. Conclusions

Although many years have passed since introducing the force densities method, it is still the most common method for calculation of the equilibrium state of tensile structures. In particular, this applies to prestressed cable nets for which this method was developed. Later improvements, as this work also are not considerably shifted from the initial idea of H.-J. Schek and K. Linkwitz. An interesting analogy exist between the force density method and the displacement method (and the finite element method also). Although these two methods describe the equilibrium states of completely different structural systems, the analogy between the components in the system of equilibrium equations is certainly there. Components of the system matrix are compiled in the same way, so that we can think of the force density as the stiffness coefficient.

From a comparison of the coordinates of physical and numerical models in the examples 3 and 4 the shortcomings of both methods for seeking an equilibrium state are evident. Material of this physical model does not allow the achievement of minimum net, because the distance between nodes are predefined (changing only because of the elastic deformation - extension –of threads). On the other hand, the numerical model achieved (with sufficient accuracy) the minimum net after fifty-three iterations (Example 3) and after seventy-one iterations (Example 4), and the coordinates of the nodes deviated from those of the physical model. It was also shown that the solution of the equations after the first iteration in both cases, better describe the same physical model. We conclude that by comparing the equilibrium states of physical and numerical model after fifty-third or seventy-first iteration, we are actually comparing models that describe different physical conditions. The best shapes are obtained using both the physical and numerical model simultaneously. In this way, these models serve as a corrective to one another.

Written computer code certainly is in development stage and currently does not offer much solution variations. The package *FDM* was tested with examples from this work and as such has proved to be correct. In the future, the package is to be extended to the static calculation phase of prestressed cable nets introducing the possibility of placing the load to the nodes. Also, great help would be the possibility of generating nodes and elements, and placing the various force densities to elements of the net in order to expand the range of possible solutions of equilibrium equations. The introduction of the boundary conditions other than hinge would enable definition of the symmetry conditions, if necessary.

6. References

References should be listed in the order in which they appear in the paper.Provide a reference number in square brackets [1] and use an overhanging indentation of 10 mm. In the text the citation should be made by using the appropriate reference number in square brackets. In the list of references, authors' names should be followed by the initials. The titles of books, journals and proceedings should be in italics. The year of publication and the page number(s) should be given.

[1] Lee SJ, BaeJE, and HintonE., "Shell topology optimization using the layered artificial material model", *International Journal for Numerical Methods in Engineering*, Vol. 47, No. 6, 2000, pp. 843-867.

[2] Motro R, *Tensegrity: Structural Systems for the Future*, Kogan Page Science, London, 2003, p. 43.

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