USING MATHEMATICA IN ALTERNATIVE DERIVATION OF FUSS' RELATION FOR BICENTRIC QUADRILATERAL

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Abstract. The bicentric n-gon is polygon with n sides that are tangential for incircle and chordal for circumcircle. Different methods were used to prove the condition that n-gon is bicentric one. In this article we present the alternative way for the construction of bicentric quadrilateral. Using computer program Mathematica, we show that radii of circles and their mutual distance satisfy well known Fuss' relation for here presented quadrilateral. Our Mathematica code is in appendix. It can be notice that such construction is not successfull for bicentric polygons with n > 4 sides.

Key words and phrases. circle, bicentric polygon, Fuss’ relations.

1 The construction of bicentric quadrilateral

Inside of the cirle $C_1$ with radius $r$ and center in origin, we choose the point $m$ on the x-axis. Through that point we draw two orthogonal straight lines. In the intersections of those lines with given circle we withdraw four tangent lines. Those lines define tangential quadrilateral with vertices in there intersections, as is shown in Fig 1.

We show that all vertices $A_i$, $i = 1, 2, 3, 4$ lie on the circle $C_2$ which is circumcircle for now bicentric quadrilateral. Moreover, for given circles, we derive relation between the corresponding radii and their mutual distance which is known as Fuss’ relation for quadrilateral.

2 Derivation of Fuss’ relation

In our derivation we have used computer program Mathematica (code is given in appendix). The points $T_1(x_1, y_1)$, $T_3(x_3, y_3)$ and $T_2(x_2, y_2)$, $T_4(x_4, y_4)$ are obtained by solving following system

\[ x^2 + y^2 = r^2, \quad k(x - m) = y \]  \hfill (1a)

and

\[ x^2 + y^2 = r^2, \quad \frac{1}{k}(x - m) = y \]  \hfill (1b)
Figure 1: The circle $C_1 \ldots x^2 + y^2 = r^2$ and point $M(m, 0)$ inside $C_1$ are given. Two orthogonal lines through $M$ are $p_1$ and $p_2$. Intersections of lines with circle are points $T_i$, $i = 1, 2, 3, 4$. The tangent lines in points $T_i$ are $t_i$. Intersections of tangents are points $A_i = t_i \cap t_{i+1}$. Points $A_i$ define tangential quadrilateral for incircle $C_1$. The point $I(d, 0)$ is center of circumcircle $C_2$ with radius $R$. Distance between centers of circles is denoted by $d$.

respectively.

The equation of tangent in point $T(x_T, y_T)$ is

$$x_T x + y_T y = r^2. \quad (2)$$

Inserting coordinates of points $T_i$ we get four tangents $t_i$. Then, we found intersections $A_i(x_i, y_i)$ where $i = 1, 2, 3, 4$ of these tangents. Points $A_i$ are vertices of tangential quadrilateral.

Our claim is that all four vertices of obtained quadrilateral lie on the same circle.

Using three of four vertices in the following system of equations

$$(x_i - x_0)^2 + (y_i - y_0)^2 = R^2, \quad i = 1, 2, 3 \quad (3)$$

we find the center and radius of circumcircle $C_2$

$$x_0 = \frac{mr^2}{m^2 - r^2}, \quad y_0 = 0, \quad R = \frac{r^2 \sqrt{2r^2 - m^2}}{r^2 - m^2}. \quad (4)$$

We can notice that, according to symmetry reason, circle $C_2$ has the center on the $x$-axis.

Now, we have to show that vertex $A_4(x_4, y_4)$ lie on the circumcircle $C_2$. After substitution coordinates $(x_4, y_4)$ and values of $x_0, y_0$ in expression

$$(x_4 - x_0)^2 + (y_4 - y_0)^2 \quad (5)$$
we obtain
\[
\frac{r^4}{\left( m - k^2 \right) m + k \sqrt{-m^2 + (1 + k^2) r^2} + \sqrt{r^2 + k^2 (-m^2 + r^2)} }^2 \\
\left( k m^2 + \sqrt{-m^2 + (1 + k^2) r^2} \sqrt{r^2 + k^2 (-m^2 + r^2)} \right)^2 \\
+ \left( - \frac{m r^2}{m^2 - r^2} \right) \\
\frac{r^2}{k m^2 + \sqrt{-m^2 + (1 + k^2) r^2} \sqrt{r^2 + k^2 (-m^2 + r^2)} }^2 \\
= \frac{r^4 (2 r^2 - m^2)}{(r^2 - m^2)^2}.
\]
(6)

The last expression is value $R^2$ given in (4). We can conclude that vertex $A_4$ is on the circumcircle $C_2$ what we claimed.

Finally, we denote distance between circles with $d = x_0$, and eliminate $m$ from equations (4). Thus we get a condition that tangential quadrilateral is also a chordal one
\[
d^4 + (\frac{-2 r^2 - 2 R^2}{R^2 (2 r^2 - R^2)}) = R^2 (2 r^2 - R^2)
\]
(7)
in accordance with Fuss' relation obtained by other methods.

3 Mathematica code for construction of bicentric quadrilateral

Here we give mathematica code for construction of bicentric quadrilateral:

(* First we define two perpendicular lines with slope k and -1/k through the point (m,0) and circle with center in origin (incircle) *)

\[\{p1,p2\} = \{y == k(x-m), y == \frac{-1}{k}(x-m)\};\]
\[\text{incircle} = x^2 + y^2 == r^2;\]

(* Intersections of lines with circles are $T[i], i = 1,2,3,4$ *)

\[T = \text{Table}\{-999, -888, \{4\}\} \text{ (*reserve space for list*)}\]
\[\text{Solve}\{\{p1, \text{incircle}\}, \{x, y\}\};\]
\[x, y/.%;\]
\[T[[1]] == \%[[1]]; T[[3]] == \%[[2]];\]
\[\text{Solve}\{\{p2, \text{incircle}\}, \{x, y\}\};\]
\[x, y/.%;\]
\[T[[2]] == \%[[1]]; T[[4]] == \%[[2]];\]

(* Tangents in intersections points $T_i$ *)

\[t = \text{Table}\{T[[i,2]]y + T[[i,1]]x == r^2\}, i, 4\];
(* Mutual intersection of tangents are vertex \( A[[i]] = t[[i]] \cap t[[i+1]] \) of quadrilateral *)

\[
xs = \text{Table}[-222, 4], \; ys = \text{Table}[-333, 4]
\]

For\( i = 1; a = \{-11, -22\}, i \leq 4, i ++, \)
\[
j = \text{If}[i < 4, i + 1, 1]; tt1 = t[[i, 1]], tt2 = t[[j, 1]];
\]
\[
rez = \text{Solve}[[tt1, tt2], \{x, y\}];
\]
\[
a = \text{FullSimplify}[[x, y] \;/ rez];
\]
\[
xs[[i]] = a[[1, 1]]; ys[[i]] = a[[1, 2]]
\]

(* Find center \((x_0, y_0)\) and radius of circle which is defined by three not yet defined points \((xs_i, ys_i), \; i = 1, 2, 3\) *)

\[
eq = ((xs1 - x0)^2 + (ys1 - y0)^2) == (xs2 - x0)^2 + (ys2 - y0)^2,
\]
\[
(xs3 - x0)^2 + (ys3 - y0)^2 == (xs2 - x0)^2 + (ys2 - y0)^2
\]
\[
\text{Solve}[\text{eq}, \{x0, y0\}];
\]
\[
\{x0, y0\}/.%; \text{center} = \%[[1]]
\]

(* Insert concrete values of vertices, and calculate \( r^2 \) *)

\[
\text{substitution} = \{xs1 \rightarrow xs[[1]], xs2 \rightarrow xs[[2]], xs3 \rightarrow xs[[3]], xs4 \rightarrow xs[[4]],
\]
\[
ys1 \rightarrow ys[[1]], ys2 \rightarrow ys[[2]], ys3 \rightarrow ys[[3]], ys4 \rightarrow ys[[4]]\}
\]
\[
x0s = \text{FullSimplify}[\text{center}[[1]]/\text{substitution}]
\]
\[
y0s = \text{FullSimplify}[\text{center}[[2]]/\text{substitution}]
\]
\[
R2 = (xs[[1]] - x0s)^2 + (ys[[1]] - y0s)^2
\]

(* From the symmetry of the problem it must be \( y0s = 0 \) *)

(* \( d = x0s \) is distance of centers, and \( R = \sqrt{R2} \) is radius of circumcircle. *)

(* We eliminate \( m \) to get relation for quadrilateral *)

\[
\text{Eliminate}[[d == x0s, R^2 == R2], m];
\]

(* That relation is in accordance with well known Fuss' relation *)

(* Proof that \( A4 \) is on the circumcircle, expression must be zero *)

\[
(xs[[1]] - x0s)^2 + (ys[[1]] - y0s)^2
\]
\[
\text{Simplify}[%]
\]
\[
R2 - %
\]

(* rezult is 0, QED *)