THE GENERATING CONDITION FOR THE EXTENSION OF THE CLASSICAL GAUSS SERIES-PRODUCT IDENTITY

Tomislav Šikić
University of Zagreb, Croatia

Abstract. In this paper a condition is presented on parameters (n_1, n_2, Λ_k), for arbitrary partition $\underline{n} = \{n_1, n_2\}$ $(n_1 \leq n_2)$ and $k = 1, ..., n-1$, which guarantees that two different interpretations of characters of fundamental modules $L(\Lambda_k)$ for the affine Kac-Moody Lie algebra $\widehat{\mathfrak{sl}}_n$ generate extended classical Gauss series-product identities.

1. Introduction

The classical Gauss series-product identity (see [1]) is given by

$$\sum_{n\in\mathbb{Z}} q^{2n^2+n} = \frac{\varphi(q^2)^2}{\varphi(q)} , \tag{1.1}$$

where $\varphi(q) = \prod_{j\geq 1}(1-q^j)$ is the Euler product function. The classical Gauss identity (1.1) appears in representation theory of infinite dimensional Lie algebras from the time of the first concrete computations of characters as in [3] to more recent results as in [2] and [9]. The main result of the paper [9] are two infinite families of series-product identities which are based on a classical Gauss identity and two different interpretations of characters of fundamental modules for the affine Kac-Moody Lie algebra $\widehat{\mathfrak{sl}}_n$ i.e. for the affine Lie algebras of type $A_\ell^{(1)}$ for $\ell = n-1$.
The first interpretation is based on the character formula

$$ch_{L(\Lambda)} = e^{\frac{1}{2}|\Lambda|^2\delta} \frac{\sum_{\gamma\in\overline{Q}+\overline{\Lambda}} e^{\Lambda_0+\gamma-\frac{1}{2}|\gamma|^2\delta}}{\prod_{j\geq 1}(1-e^{-j\delta})^{mult\ j\delta}} \tag{1.2}$$

2000 *Mathematics Subject Classification.* 17B67.
Key words and phrases. Affine Lie algebras, characters of fundamental modules, series-product identities, classical Gauss identity.

for level 1 dominant integral weights Λ of affine Lie algebras of type $A_\ell^{(1)}$, $D_\ell^{(1)}$, $E_\ell^{(1)}$ (see [5] and [6] or [4] Sect.12.13).
Another interpretation is based on a bosonic and fermionic construction of fundamental representations $L(\Lambda_k)$ of affine Lie algebra $\widehat{\mathfrak{gl}}_n$ (see [8]) which is special case of the more general construction [7]. The realization [8] is parameterized by partitions $\underline{n} = \{n_1, \cdots, n_r\}$ $(n_1 \leq \cdots \leq n_r)$ and the corresponding "q-dimension" trace formula of $L(\Lambda_k)$ $k = 0, 1, \dots, \ell$ for the affine Lie algebra $\widehat{\mathfrak{sl}}_n$ is

$$Trace_{L(\Lambda_k)}(q) = q^{const} \frac{\varphi(q)}{\prod_{i=1}^r \varphi(q^{1/n_i})} \sum_{k_1+\cdots+k_r=k} q^{\frac{1}{2}(\frac{k_1^2}{n_1}+\cdots+\frac{k_r^2}{n_r})} . \tag{1.3}$$

The above trace formula (1.3) is an expression for a particular specialization $\mathcal{F}_{\mathbf{s}}$ of the character $ch_{L(\Lambda_k)}$. Therefore for every fundamental module we obtain a nontrivial identity by equating (1.3) with the properly specialized character given by (1.2). Following the mentioned bosonic and fermionic construction for $\widehat{\mathfrak{gl}}_n$ we have explicit equation for arbitrary partition $\underline{n} = \{n_1, \cdots, n_r\}$ (see Propositon 4.1 in [9])

$$\mathcal{F}_{\mathbf{s}}(ch_{L(\Lambda_k)}) = q^{const} \prod_{j\geq 1}(1-q^{jN}) \frac{\sum_{k_1+\cdots+k_r=k} q^{\frac{N}{2}(\frac{k_1^2}{n_1}+\cdots+\frac{k_r^2}{n_r})}}{\prod_{i=1}^r \prod_{j\geq 1}(1-q^{\frac{jN}{n_i}})} \tag{1.4}$$

where

$$\begin{aligned} \mathbf{s} \;=\; & N\Big(\frac{n_1+n_r}{2n_1n_r}, \frac{1}{n_1}, \dots, \frac{1}{n_1}, \frac{n_1+n_2}{2n_1n_2} - 1, \frac{1}{n_2}, \dots, \frac{1}{n_2}, \\ & \frac{n_2+n_3}{2n_2n_3} - 1, \dots, \frac{1}{n_{r-1}}, \dots, \frac{1}{n_{r-1}}, \frac{n_{r-1}+n_r}{2n_{r-1}n_r} - 1, \frac{1}{n_r}, \dots, \frac{1}{n_r}\Big) . \end{aligned} \tag{1.5}$$

and

$$N = \begin{cases} N' & if\ N'(\frac{1}{n_i}+\frac{1}{n_j}) \in 2\mathbf{Z} \quad \forall\, i,j \in \{1,\dots,r\} \\ 2N' & if\ N'(\frac{1}{n_i}+\frac{1}{n_j}) \notin 2\mathbf{Z} \quad for\ a\ pair\ (i,j) \end{cases} \tag{1.6}$$

(N' is least common multiple of $n_1, \cdots, n_r$).
By using the Gauss identity (1.1) for two special choices of partitions

$$\underline{n} = \{n_1, n_2\} \ (n_1 \leq n_2)$$

and the corresponding fundamental weights Λ_k we can transform the right-hand side of equation (1.4) into infinite products and obtain two infinite families of series-product identities (see again [9]).

In this paper a condition is presented on parameters (n_1, n_2, Λ_k) which guarantees that equation (1.4) generate new series-product identities based on the classical Gauss identity (1.1) following the methodology as in the paper [9].

It is very important to accentuate that this generating condition discover infinitely many new examples of extended classical Gauss identities which are not presented in the paper [9].
Certainly, the parameters (n_1, n_2, Λ_k) for two mentioned infinite families of series-product identities from paper [9] definitely satisfies this generating condition. For instance, for affine Lie algebra $\widehat{\mathfrak{sl}}_{16}$ the triples $(n_1, n_2, \Lambda_k) = (1, 15, \Lambda_{12})$ and $(4, 12, \Lambda_{15})$ are corresponded to mentioned two families from paper [9]. But, from condition which will be explain later in this paper, we can discover triple $(n_1, n_2, \Lambda_k) = (2, 14, \Lambda_{14})$ which create a new example of extended classical Gauss identity. This first sporadic new example was presented on $4^{th}CMC$, [10]. As one may expect, Example 3.5 confirms that the number of new examples of extended classical Gauss identity for affine Lie algebra $\widehat{\mathfrak{sl}}_n$ will increases with n.

2. The basic notation

Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{sl}_n$, the simple Lie algebra of type A_ℓ for $n = \ell+1$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and $\mathcal{R}$ the corresponding root system. We may identify $\mathfrak{h} \cong \mathfrak{h}^*$ via a normalized Killing form $(\, . \mid . \,)$ of the Lie algebra $\mathfrak{sl}_n$ such that $(\theta \mid \theta \,) = 2$ where $\theta = \alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_\ell$ is maximal root for a fixed set of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$.
For a root α, by $\alpha^\vee$ we denote the dual root. Let

$$\mathfrak{sl}_n = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathcal{R}} \mathfrak{g}_\alpha$$

be a root space decomposition. Let

$$\widehat{\mathfrak{sl}}_n = \mathfrak{sl}_n \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d \ .$$

Then $\widehat{\mathfrak{sl}}_n$ is the affine Lie algebra with

$$[x(i), y(j)] = [x, y](i+j) + i\delta_{i+j,0}(x \mid y)c \ ,$$

c being a central element and d a scaling element with $[d, x(i)] = ix(i)$. The affine Lie algebra $\widehat{\mathfrak{sl}}_n$ is a Kac-Moody Lie algebra of type $A_\ell^{(1)}$ (see [4]), and

$$\hat{\mathfrak{h}} = \mathfrak{h} \oplus (\mathbb{C}c + \mathbb{C}d)$$

is its Cartan subalgebra. We identify $\mathfrak{h}^* \subset \hat{\mathfrak{h}}^*$ using $\mathfrak{h}^*|_{(\mathbb{C}c+\mathbb{C}d)} = 0$ and define δ by $\delta|_{\mathfrak{h}\oplus\mathbb{C}c} = 0$, $\delta(d) = 1$. The root system $\hat{\mathcal{R}}$ of the affine Lie algebra $\widehat{\mathfrak{sl}}_n$ is composed of the real and imaginary roots

$$\hat{\mathcal{R}} = \hat{\mathcal{R}}^{Re} \cup \hat{\mathcal{R}}^{Im} = \{\alpha + n\delta | \, \alpha \in \mathcal{R}, n \in \mathbb{Z}\} \cup \{n\delta | \, n \in \mathbb{Z} \setminus \{0\}\} \ .$$

If we denote by α_0 the root $\alpha_0 = -\theta + \delta$, then $\hat{\Delta} = \{\alpha_0, \alpha_1, \ldots, \alpha_\ell\}$ forms a base of the root system $\hat{\mathcal{R}}$. The corresponding root lattice is $Q = \sum_{i=0}^{\ell} \mathbb{Z}\alpha_i$, and

$$\delta = \sum_{i=0}^{\ell} \alpha_i \ . \tag{2.1}$$

If we denote $\alpha_0^\vee = -\theta^\vee + c$, then $\{\alpha_i | \, i = 0, 1, \dots, \ell\}$ is a set of simple roots and $\{\alpha_i^\vee | \, i = 0, 1, \dots, \ell\}$ is a set of simple coroots of Kac-Moody Lie algebra $\widehat{\mathfrak{sl}}_n$ (cf. [4]). The fundamental weights Λ_k, for $k = 0, 1, \dots, \ell$, are defined by

$$\Lambda_k(\alpha_j^\vee) = \delta_{jk} \ , \qquad j = 0, 1, \dots, \ell \ \ \textit{and} \ \ \Lambda_k(d) = 0 \ .$$

For a subset $S \subseteq \hat{\mathfrak{h}}^*$ by $\overline{S}$ is denoted the orthogonal projection of S on $\mathfrak{h}^*$. In the case of the affine Lie algebra $\widehat{\mathfrak{sl}}_n$ (i.e. $A_\ell^{(1)}$) (see [4]) we have

$$\begin{aligned} \overline{\hat{\Delta}} &= \Delta \ , \\ \overline{Q} &= \sum_{i=1}^{\ell} \mathbb{Z}\alpha_i \ . \end{aligned} \tag{2.2}$$

3. The generating condition

As it was already said in the introduction, in this paper a condition will be presented on parameters (n_1, n_2, Λ_k), where $\underline{n} = \{n_1, n_2\}$ $(n_1 \leq n_2)$ is an arbitrary partition and Λ_k, for $k = 1, ..., n-1$, a fundamental weight, which guarantees that equation (1.4) generates series-product identities based on the classical Gauss identity (1.1) as in the paper [9]. Since the following equation holds

$$\sum_{n \in \mathbb{Z}} q^{2n^2+n} = \sum_{n \in \mathbb{Z}} q^{2n^2-n} \ , \tag{3.1}$$

we have the following definition of the similarity to the classical Gauss identity.

Definition 3.1. *Let*

$$\sum_{n \in \mathbb{Z}} q^{a_2 n^2 + a_1 n + a_0} \tag{3.2}$$

is a formal series, where $a_2, a_1, a_0 \in \mathbb{R}$. The formal series (3.2) is similar to the classical Gauss identity if there exists a pair (λ, c) from $\mathbb{R}^\star \times \mathbb{Z}$ such that

$$\sum_{n \in \mathbb{Z}} q^{a_2 n^2 + a_1 n + a_0} = q^{const} \sum_{n \in \mathbb{Z}} q^{\lambda(2(n-c)^2 \pm (n-c))} \ .$$

Remark 3.2. Let $\mathfrak{g} = \widehat{\mathfrak{sl}}_n$ and let $\underline{n} = \{n_1, n_2\}$ be an arbitrary partition of the positive integer n. For the fundamental weight $\Lambda = \Lambda_k$ the sum

$$\sum_{k_1 + k_2 = k} q^{\frac{N}{2}(\frac{k_1^2}{n_1} + \frac{k_2^2}{n_2})} \tag{3.3}$$

from the right-hand side of the equation (1.4), after substitution $k_2 = k - k_1$, could be transformed to formal series

$$\sum_{k_1 \in \mathbb{Z}} q^{a_2 k_1^2 + a_1 k_1 + a_0}$$

for a triple

$$(a_2, a_1, a_0) = (\frac{N}{2n_1 \cdot n_2} \cdot n, -\frac{N}{n_2} \cdot k, \frac{k^2 \cdot N}{2n_2}) \ . \tag{3.4}$$

Since the condition $n_1 \leq n_2$ holds it is not necessary to consider the dual substitution $k_1 = k - k_2$ for the polynomial $\frac{k_1^2}{n_1} + \frac{k_2^2}{n_2}$.

As a result of remark above we can reformulate Definition 3.1 in the special case of the formal series (3.3).

Definition 3.3. *Let $\underline{n} = \{n_1, n_2\}$ be an arbitrary partition of the positive integer n. For $k = 0, 1, ..., n-1$ and N defined in (1.6), the triple (n_1, n_2, Λ_k) satisfies the generating condition for the classical Gauss identity if there exists a pair (λ, c) from $\mathbb{R}^\star \times \mathbb{Z}$ such that*

$$\sum_{k_1+k_2=k} q^{\frac{N}{2}(\frac{k_1^2}{n_1}+\frac{k_2^2}{n_2})} = \left[_{k_2=k-k_1}\right] = q^{const} \sum_{k_1 \in \mathbb{Z}} q^{\lambda(2(k_1-c)^2 \pm (k_1-c))} \ . \tag{3.5}$$

It is interesting to notice that the triple (n_1, n_2, Λ_0) does not satisfy the generating condition (3.5) at all. Now, the following lemma holds.

Lemma 3.4. *Let $\underline{n} = \{n_1, n_2\}$ be an arbitrary partition of the positive integer n and $k = 1, \cdots, n-1$. The triple (n_1, n_2, Λ_k) satisfies the generating conditions for the classical Gauss identity (3.5) if and only if the following equation holds*

$$4k \cdot n_1 = (4c \pm 1)n \tag{3.6}$$

for some nonnegative integer c.

Proof. First of all, as in (3.4), the substitution $k_2 = k - k_1$ implies an equation

$$\frac{N}{2} \ (\frac{k_1^2}{n_1} + \frac{k_2^2}{n_2}) = \frac{N}{2n_1 \cdot n_2} \ (nk_1^2 - 2n_1 k \cdot k_1) + const \ \ . \tag{3.7}$$

Suppose that the generating condition (3.5) holds for the triple (n_1, n_2, Λ_k). From the comparison between polynomial in the variable k_1

$$\lambda(2(k_1-c)^2 \pm (k_1-c)) = 2\lambda k_1^2 - \lambda(4c \mp 1)k_1 + \lambda(2c^2 \mp c)$$

and polynomial (3.7) we have the following system equations

$$
\begin{aligned}
\frac{N}{2n_1n_2}\cdot n &= 2\lambda \\
-\frac{N}{n_2}\cdot k &= -(4c\pm 1)\lambda\ .
\end{aligned}
\tag{3.8}
$$

Since λ and $\frac{N}{2n_1n_2}$ are nonzero real numbers the system equations (3.8) is equivalent with the following one

$$
\begin{aligned}
\frac{n}{2} &= \lambda\,\frac{2n_1n_2}{N} \\
2n_1k &= (4c\pm 1)\cdot\lambda\,\frac{2n_1n_2}{N}\quad ,
\end{aligned}
$$

which implies the existence of integer c in the equation (3.6). Moreover, from (3.6) it is obvious that c is a nonnegative integer.
Suppose that the equation (3.6) holds for some nonnegative integer c. Since the integer $4c\pm 1$ is an odd number (i.e. relatively prime with 2) it is obvious to notice that n is divisible by 4 and $\frac{4n_1k}{n}$ is positive odd number. Then from (3.7) the following calculation

$$
\begin{aligned}
\frac{N}{2}\left(\frac{k_1^2}{n_1}+\frac{k_2^2}{n_2}\right) &= \frac{N}{2n_1\cdot n_2}(nk_1^2-2n_1k\cdot k_1)+const \\
&= \frac{N}{2n_1\cdot n_2}\cdot\frac{n}{2}\left(2k_1^2-\frac{4n_1k}{n}\cdot k_1\right)+const \\
\stackrel{(3.6)}{=}\ & \frac{N}{2n_1\cdot n_2}\cdot\frac{n}{2}(2k_1^2-(4c\pm 1)\cdot k_1)+const \\
&= \frac{N}{2n_1\cdot n_2}\cdot\frac{n}{2}(2(k_1-c)^2\mp(k_1-c))+const
\end{aligned}
\tag{3.9}
$$

implies the generating condition (3.5) for the pair $(\lambda,c)=(\frac{N\cdot n}{4n_1\cdot n_2},c)$. □

Due to Lemma 3.4 we can interpret the condition (3.5) as the condition (3.6). Therefore in the rest of the paper we always use the following notation "the generating condition for the classical Gauss identity (3.6)" instead of the condition (3.5).

EXAMPLE 3.5. Let $\mathfrak{g}=\mathfrak{sl}_{24}$. In paper [9] only two triples are presented in the case of $n=24$:

$$
\begin{aligned}
&(1,23,\Lambda_{18})\\
&(6,18,\Lambda_{23})\quad .
\end{aligned}
$$

Using the generating condition for the classical Gauss identity (3.6) we can list all triples as follows

(n_1, n_2, Λ_k)	c	(n_1, n_2, Λ_k)	c	(n_1, n_2, Λ_k)	c	(n_1, n_2, Λ_k)	c
$(6, 18, \Lambda_1)$	0	$(7, 17, \Lambda_6)$	2	$(6, 18, \Lambda_{13})$	3	$(7, 17, V\Lambda_{18})$	5
$(3, 21, \Lambda_2)$	0	$(9, 15, \Lambda_6)$	2	$(3, 21, \Lambda_{14})$	2	$(9, 15, \Lambda_{18})$	7
$(9, 15, \Lambda_2)$	1	$(11, 13, \Lambda_6)$	3	$(9, 15, \Lambda_{14})$	5	$(11, 13, \Lambda_{18})$	8
$(2, 22, \Lambda_3)$	0	$(6, 18, \Lambda_7)$	2	$(2, 22, \Lambda_{15})$	1	$(6, 18, \Lambda_{19})$	5
$(6, 18, \Lambda_3)$	1	$(2, 22, \Lambda_9)$	1	$(6, 18, \Lambda_{15})$	4	$(2, 22, \Lambda_{21})$	2
$(10, 14, \Lambda_3)$	1	$(6, 18, \Lambda_9)$	2	$(10, 14, \Lambda_{15})$	6	$(6, 18, \Lambda_{21})$	5
$(6, 18, \Lambda_5)$	1	$(10, 14, \Lambda_9)$	4	$(6, 18, \Lambda_{17})$	4	$(10, 14, \Lambda_{21})$	9
$(1, 23, \Lambda_6)$	0	$(3, 21, \Lambda_{10})$	1	$(1, 23, \Lambda_{18})$	1	$(3, 21, \Lambda_{22})$	3
$(3, 21, \Lambda_6)$	1	$(9, 15, \Lambda_{10})$	4	$(3, 21, \Lambda_{18})$	2	$(9, 15, \Lambda_{22})$	8
$(5, 19, \Lambda_6)$	1	$(6, 18, \Lambda_{11})$	3	$(5, 19, \Lambda_{18})$	4	$(6, 18, \Lambda_{23})$	6

REMARK 3.6. It is interesting to notice that when the triple (n_1, n_2, Λ_k) satisfies the generating condition for the classical Gauss identity (3.6) for integer c, then the triple $(n_1, n_2, \Lambda_{n-k})$ also satisfies the generating condition for $c' = n_1 - c$. This conclusion is based on the following simple calculation

$$4n_1(n-k) = (3.6) = 4n_1 n - (4c \pm 1)n = (4(n_1 - c) \mp 1)n \ .$$

Denote by C the Cartan matrix of $\mathfrak{sl}_n$. Now we have the following result.

THEOREM 3.7. *Let $\underline{n} = \{n_1, n_2\}$, $(n_1 \leq n_2)$ be a partition of a positive integer n and the corresponding N is defined in (1.6). Let $k = 1, \ldots, n-1$. If the triple (n_1, n_2, Λ_k) satisfies the generating condition for the classical Gauss identity (3.6) then the equation (1.4) generates a series-product identity in the following form*

$$\sum_{\xi \in \mathbb{Z}^{n-1}} q^{\frac{N}{2}\xi C \xi^t + lin(\xi)} = q^{const} \varphi(q^N)^n \frac{\varphi(q^{\frac{nN}{2n_1 n_2}})^2}{\varphi(q^{\frac{N}{n_1}})\varphi(q^{\frac{N}{n_2}})\varphi(q^{\frac{nN}{4n_1 n_2}})} \tag{3.10}$$

for

$$lin(\xi) = N\xi_k - \sum_{i=1}^{n-1} s_i \xi_i \tag{3.11}$$

where

$$\begin{aligned} \mathbf{s} &= (s_0, s_1, \ldots, s_{n-1}) \\ &= N(\frac{n_1+n_2}{2n_1 n_2}, \underbrace{\frac{1}{n_1}, \ldots, \frac{1}{n_1}}_{(n_1-1)\times}, \frac{n_1+n_2}{2n_1 n_2} - 1, \underbrace{\frac{1}{n_2}, \ldots, \frac{1}{n_2}}_{(n_2-1)\times}) \ . \end{aligned} \tag{3.12}$$

PROOF. Since $n-1$ is equal to ℓ and $\Lambda_k = \Lambda_0 + \overline{\Lambda}_k$ from (2.2) we can write

$$\overline{\Lambda}_k = \lambda_{k,1}\alpha_1 + \lambda_{k,2}\alpha_2 + \cdots , \lambda_{k,l}\alpha_l$$

for a fundamental weight of the simple Lie algebra $\mathfrak{sl}_n$ (i.e. type A_ℓ). Furthermore, the n-tuple $\mathbf{s}$ (3.12) is special case of the n-tuple (1.5) for partition $\underline{n} = \{n_1, n_2\}$ (i.e. $r = 2$). So for affine Lie algebra $\widehat{\mathfrak{sl}}_n$ (i.e. type $A_\ell^{(1)}$) the numerator of the formula (1.2) has the form

$$e^{\Lambda_0 + \frac{1}{2}|\Lambda_k|^2\delta} \sum_{\gamma \in \overline{Q} + \overline{\Lambda}_k} e^{\gamma - \frac{1}{2}|\gamma|^2\delta}$$

for

$$\gamma = \xi_1\alpha_1 + \xi_2\alpha_2 + \cdots + \xi_\ell\alpha_\ell + \overline{\Lambda}_k \ . \tag{3.13}$$

Since the Cartan matrix of type A_ℓ is given by

$$C = [(\alpha_i|\alpha_j)]_{i,j\in\{1,\dots,\ell\}} = \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & \cdots & 0 & 0 & -1 & 2 \end{bmatrix}$$

we have the following formula

$$|\gamma|^2 = (\gamma \mid \gamma) = \xi C \xi^t + 2\xi_k + (\overline{\Lambda}_k|\overline{\Lambda}_k) \ . \tag{3.14}$$

Now (2.1) implies

$$\mathcal{F}_{\mathbf{s}}(e^{-\delta}) = q^N \tag{3.15}$$

for the specialization defined by

$$\mathcal{F}_{\mathbf{s}}(e^{-\alpha_i}) = q^{s_i}, \ i = 0, 1, \cdots, l \ . \tag{3.16}$$

Using (3.13) – (3.16) and from the fact that $mult\ j\delta = dim\mathfrak{h} = \ell$ we have the following calculation

$$\mathcal{F}_{\mathbf{s}}\Big(e^{\frac{1}{2}|\Lambda_k|^2\delta} \frac{\sum_{\gamma\in\overline{Q}+\overline{\Lambda}_k} e^{\Lambda_0+\gamma-\frac{1}{2}|\gamma|^2\delta}}{\prod_{j\geq 1}(1-e^{-j\delta})^{mult\ j\delta}}\Big) = q^{const}\frac{1}{\varphi(q^N)^\ell} \cdot \mathcal{F}_{\mathbf{s}}\Big(\sum_{\gamma\in\overline{Q}+\overline{\Lambda}_k} e^{\gamma-\frac{1}{2}|\gamma|^2\delta}\Big)$$

$$= \frac{q^{const}}{\varphi(q^N)^\ell} \sum_{\gamma\in\overline{Q}+\overline{\Lambda}_k} \mathcal{F}_{\mathbf{s}}(e^\gamma) \cdot [\mathcal{F}_{\mathbf{s}}(e^{-\delta})]^{\frac{1}{2}|\gamma|^2}$$

$$= \frac{q^{const}}{\varphi(q^N)^\ell} \sum_{\xi\in\mathbb{Z}^\ell} q^{-(\xi_1+\lambda_{k,1})s_1-\cdots-(\xi_\ell+\lambda_{k,l})s_\ell} \cdot q^{\frac{N}{2}(\xi C\xi^t + 2\xi_k + (\overline{\Lambda}_k|\overline{\Lambda}_k))} \ .$$

Finally the left-hand side of the formula (1.4) has the form

$$\mathcal{F}_{\mathbf{s}}\Big(e^{\frac{1}{2}|\Lambda_k|^2\delta}\frac{\sum_{\gamma\in\overline{Q}+\overline{\Lambda}_k} e^{\Lambda_0+\gamma-\frac{1}{2}|\gamma|^2\delta}}{\prod_{j\geq 1}(1-e^{-j\delta})^{mult\ j\delta}}\Big) = \tag{3.17}$$

$$= \frac{q^{const}}{[\varphi(q^N)]^\ell}\sum_{\xi\in\mathbb{Z}^\ell} q^{\frac{N}{2}\xi C\xi^t+N\xi_k-\xi_1 s_1-\cdots-\xi_\ell s_\ell}$$

for particular specialization (3.12).

The right-hand side of the formula (1.4) for the triple (n_1, n_2, Λ_k) is

$$q^{const}\quad \prod_{j\geq 1}(1-q^{Nj})\frac{\sum_{k_1+k_2=k} q^{\frac{N}{2}(\frac{k_1^2}{n_1}+\frac{k_2^2}{n_2})}}{\prod_{j\geq 1}(1-q^{\frac{Nj}{n_1}})\prod_{j\geq 1}(1-q^{\frac{Nj}{n_2}})} =$$

$$= \quad q^{const}\quad \frac{\varphi(q^N)}{\varphi(q^{\frac{N}{n_1}})\varphi(q^{\frac{N}{n_2}})}\sum_{k_1+k_2=k} q^{\frac{N}{2n_1n_2}(n_2k_1^2+n_1k_2^2)}\ .$$

After the substitution $k_2 = k - k_1$ and using the calculation (3.9) from Lemma 3.4 we have

$$\sum_{k_1+k_2=k} q^{\frac{N}{2n_1n_2}[n_2k_1^2+n_1k_2^2]} = q^{const}\sum_{k_1\in\mathbb{Z}} q^{\frac{N}{2n_1n_2}[n\cdot k_1^2-2n_1k\cdot k_1]} = (3.6) = \tag{3.18}$$

$$= q^{const}\cdot q^{-\frac{nN}{4n_1n_2}(2c^2\pm c)}\sum_{k_1\in\mathbb{Z}} q^{\frac{N}{2n_1n_2}\cdot\frac{n}{2}[2(k_1-c)^2\mp(k_1-c)]}\ .$$

Since $\frac{n}{4}$ and $\frac{N}{n_1n_2}$ are positive integers, the calculation (3.18), equation (3.1) and the Gauss identity (1.1) imply that the right-hand side of the formula (1.4) has the form

$$\frac{\prod_{j\geq 1}(1-q^{Nj})\sum_{k_1+k_2=k} q^{\frac{N}{2}(\frac{k_1^2}{n_1}+\frac{k_2^2}{n_2})}}{\prod_{j\geq 1}(1-q^{\frac{Nj}{n_1}})\prod_{j\geq 1}(1-q^{\frac{Nj}{n_2}})} = \tag{3.19}$$

$$q^{const}\frac{\varphi(q^N)}{\varphi(q^{\frac{N}{n_1}})\varphi(q^{\frac{N}{n_2}})}\cdot\frac{\varphi(q^{\frac{nN}{2n_1n_2}})^2}{\varphi(q^{\frac{nN}{4n_1n_2}})}\ .$$

Now, from (3.17) and (3.19) it is obvious that the series-product identity (3.10) holds when $lin(\xi)$ has the form (3.11) for the n-tuple (3.12). □

Finally, the simple calculation for following triples

$$(n_1, n_2, \Lambda_k) = (1, 4m-1, \Lambda_{3m}),\ m\in\ \mathbb{Z}^+$$

$$(n_1, n_2, \Lambda_k) = (m, 3m, \Lambda_{4m-1}),\ m\in\ \mathbb{Z}^+$$

and corresponding ℓ-tuples ($\ell = 4m-1$)

$$(s_1, \cdots, s_\ell) = (1-2m, 1, \cdots, 1)$$

$$(s_1, \cdots, s_m, \cdots, s_\ell) = (3, \cdots, 3, 2-3m, 1, \cdots, 1)$$

confirms that two infinite families of series-product identities from paper [9] are just two special cases of more general formula (3.10).

References

[1] G. E. Andrews, The theory of partitions. Encyclopedia of Mathematics and its Applications, Vol. 2, Addison-Wesley, Reading MA 1976.

[2] W. J. Cook, H.-S. Li and K. C. Misra, *A recurrence relation for characters of highest weight integrable modules for affine Lie algebras,* Commun. Contemp. Math. **9** (2007), 121–133.

[3] A. J. Feingold and J. Lepowsky, *The Weyl-Kac character formula and power series identities,* Adv. Math. **29** (1978), 271–309.

[4] V. G. Kac, Infinite dimensional Lie algebras (3^{rd} edition), Cambridge University Press, Cambridge 1990.

[5] V. G. Kac, *Infinite-dimensional Lie algebras and Dedekind's η-function, classical Möbious function and very strange formula,* Adv. Math. **30** (1978), 85–136.

[6] V. G. Kac, and D. H. Peterson, *Infinite-dimensional Lie algebras, theta functions and modular forms,* Adv. Math. **53** (1984), 125–264.

[7] V. G. Kac, and D. H. Peterson, *112 construction of the basic representation of the loop group of E_8,* in: Proceedings of the conference "Anomalies, geometry, topology", World Sci., Argone 1985, 276–298.

[8] F. ten Kroode and J. van de Leur, *Bosonic and fermionic realization of the affine algebra $\widehat{\mathfrak{gl}}_n$,* Comm. Math. Phys. **137** (1991), 67–107.

[9] T. Šikić, *An extension of the classical Gauss series-product identity by boson-fermionic realization of the affine algebra $\widehat{\mathfrak{gl}}_n$,* J. Algebra Appl. **9** (2010), 123–133.

[10] T. Šikić, *Some series-product identities and boson-fermionic realization of the affine Lie algebra $\widehat{\mathfrak{gl}}_n$,* on: 4^{th} Croatian Mathematical Congres, Osijek, Croatia 2008.

Tomislav Šikić
Faculty of Electrical Engineering and Computing
University of Zagreb
Unska 3, 10000 Zagreb
Croatia
E-mail: tomislav.sikic@fer.hr