Multiple Conclusion Deductions in Classical Logic

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LAP 2013-09

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1. Introduction

Natural deductions

Natural deductions are calculi with assumptions

- Lukasiewicz, Jaśkowski 1926.–1930.
- Gentzen 1933.–1935.
  - "ein Kalkül des natürlichen Schliessen"
  - "Untersuchungen über das logische Schliessen"

$$\dfrac{\dfrac{A \qquad A \rightarrow B}{B} \qquad \dfrac{A \wedge C}{C}}{B \wedge C}$$

Linear Jaskowski-type (Fitch)

$$-A \to -B$$

$$B$$

$$-A \to -B$$

$$-A$$

$$-A \to -B$$

$$-B$$

$$B$$

$$A$$

$$B \to A$$

$$(-A \to -B) \to (B \to A)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A}\,(1) \qquad A \to B}{B} \qquad \dfrac{}{-B}\,(2)}{\bot}}{-A}\,(1)$$

$$\dfrac{-A}{-B \to -A}\,(2)$$

Deduction of $-B \to -A$ from $A \to B$.

$$(\mathsf{I}\wedge)\ \frac{A \quad B}{A \wedge B} \qquad\qquad (\mathsf{E}\wedge)\ \frac{A \wedge B}{A} \qquad \frac{A \wedge B}{B}$$

$$(\mathsf{I}\vee)\ \frac{A}{A \vee B} \qquad \frac{B}{A \vee B} \qquad\qquad (\mathsf{E}\vee)\ \frac{A \vee B \qquad \begin{array}{c} \overline{A}\ \text{(i)} \\ \vdots \\ C \end{array} \qquad \begin{array}{c} \overline{B}\ \text{(i)} \\ \vdots \\ C \end{array}}{C}\ \text{(i)}$$

$$(\mathsf{I}\rightarrow)\;\dfrac{\begin{matrix}\dfrac{}{A}\;(\mathrm{i})\\ \vdots\\ B\end{matrix}}{A \rightarrow B}\;(\mathrm{i})$$

$$(\mathsf{E}\rightarrow)\;\dfrac{A \rightarrow B \qquad A}{B}$$

$$(\mathsf{I}-)\;\dfrac{\begin{matrix}\dfrac{}{A}\;(\mathrm{i})\\ \vdots\\ \bot\end{matrix}}{\overline{A}}\;(\mathrm{i})$$

$$(\mathsf{E}-)\;\dfrac{\overline{A} \qquad A}{\bot}$$

$$(\mathsf{TND})\;\dfrac{}{A \vee \overline{A}}$$

Features of NK

"Local" and "global" inference rules

$$(\mathsf{I}\wedge)\ \frac{A \qquad B}{A \wedge B}$$

vs.

$$(\mathsf{E}\vee)\ \frac{A \vee B \qquad \begin{array}{c} \overline{\ A\ }^{\,(i)} \\ \vdots \\ C \end{array} \qquad \begin{array}{c} \overline{\ B\ }^{\,(i)} \\ \vdots \\ C \end{array}}{C}\ (i)$$

Premisses $(\mathsf{E}\vee)$ are *deductions*, not formulae!

$$\dfrac{\dfrac{A \wedge B}{A} \qquad B}{A \wedge B} \qquad\qquad \dfrac{\dfrac{B}{A \vee B} \qquad \begin{matrix}\overline{A}\\ \vdots \\ C\end{matrix} \qquad \begin{matrix}\overline{B}\\ \vdots \\ C\end{matrix}}{C}$$

Principle of inversion (Prawitz)
About elimination following an introduction:

> *one essentially restores what had already been established*

Equivalently:

> *No formula-node is a major formula of two consecutive inferences.*

Normal form theorem for NK

Theorem 1.
If $\Gamma \vdash A$ *in NK then there is a natural deduction of* A *from* Γ *that is in normal formal.*

Gentzen's unpublished proof for NJ was found much later.

LK — Logistischer Klasischer Kalkül

Inference rules

$$(\neg\vdash)\ \frac{\Gamma \vdash A, \Delta}{\Gamma, \overline{A} \vdash \Delta} \qquad (\vdash\neg)\ \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \overline{A}, \Delta}$$

$$(\wedge\vdash)\ \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \qquad (\vdash\wedge)\ \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}$$

$$(\vee\vdash)\ \frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta} \qquad (\vdash\vee)\ \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}$$

$$(\rightarrow\vdash)\ \frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \rightarrow B \vdash \Delta} \qquad (\vdash\rightarrow)\ \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \rightarrow B, \Delta}$$

Hauptsatz

Cut elimination theorem; or "normal form theorem for LK"

Theorem 2 (Hauptsatz). *Cut rule is eliminable in LK.*

Nice consequences:

- subformula property
- consistency
- Craig's lemma
- bottom-up proof search
- LK proof calculus is **analytic**

Consistency

Derivation of empty sequent $\vdash$:

$$\dfrac{\vdash A \wedge -A \qquad \dfrac{\dfrac{\overline{A \vdash A}}{A, -A \vdash}}{A \wedge -A \vdash}}{\vdash}$$

But there is no cut-free derivation of empty sequent
(*cut is the only simplyfing rule*)

2. Multiple Conclusion Deductions

Duality of $\wedge$, $\vee$ presents itself in symmetry of the pair:

$$(\mathsf{I}\vee)\ \frac{A}{A \vee B} \qquad \frac{B}{A \vee B} \qquad\qquad (\mathsf{E}\wedge)\ \frac{A \wedge B}{A} \qquad \frac{A \wedge B}{B}$$

…but not in the other one:

$$(\mathsf{I}\wedge)\ \frac{A \quad B}{A \wedge B} \qquad\qquad (\mathsf{E}\vee)\ \frac{A \vee B \qquad \begin{matrix} \overline{A}^{\,(i)} \\ \vdots \\ C \end{matrix} \qquad \begin{matrix} \overline{B}^{\,(i)} \\ \vdots \\ C \end{matrix}}{C}\,(i)$$

Hypothetical rules (I$\to$), (I$-$), (E$\vee$) are complicated:

$$(\mathsf{I}\!\to)\ \dfrac{\begin{array}{c}\dfrac{}{A}\ (\mathrm{i})\\ \vdots\\ B\end{array}}{A \to B}\ (\mathrm{i}) \qquad\qquad (\mathsf{I}-)\ \dfrac{\begin{array}{c}\dfrac{}{A}\ (\mathrm{i})\\ \vdots\\ \bot\end{array}}{\overline{A}}\ (\mathrm{i})$$

$$(\mathsf{E}\vee)\ \dfrac{A \vee B \qquad \begin{array}{c}\dfrac{}{A}\ (\mathrm{i})\\ \vdots\\ C\end{array} \qquad \begin{array}{c}\dfrac{}{B}\ (\mathrm{i})\\ \vdots\\ C\end{array}}{C}\ (\mathrm{i})$$

Kneale's Remedy

Multiple Conclusion Inference Rules

$$(\mathsf{I}\wedge)\ \frac{A \qquad B}{A \wedge B} \qquad\qquad (\mathsf{E}\wedge)\ \frac{A \wedge B}{A} \quad \frac{A \wedge B}{B}$$

$$(\mathsf{I}\vee)\ \frac{A}{A \vee B} \quad \frac{B}{A \vee B} \qquad\qquad (\mathsf{E}\vee)\ \frac{A \vee B}{A \qquad B}$$

$$(\mathsf{I}\rightarrow)\ \frac{B}{A \rightarrow B} \quad \frac{}{A \qquad A \rightarrow B} \qquad\qquad (\mathsf{E}\rightarrow)\ \frac{A \qquad A \rightarrow B}{B}$$

$$(\mathsf{I}-)\ \frac{}{A \qquad -A} \qquad\qquad (\mathsf{E}-)\ \frac{A \qquad -A}{}$$

Kneale's Remedy

Multiple Conclusion Inference Rules

(I∧) $\dfrac{A \qquad B}{A \wedge B}$

(E∧) $\dfrac{A \wedge B}{A} \qquad \dfrac{A \wedge B}{B}$

(I∨) $\dfrac{A}{A \vee B} \qquad \dfrac{B}{A \vee B}$

(E∨) $\dfrac{A \vee B}{A \qquad B}$

(I→) $\dfrac{B}{A \to B} \qquad \dfrac{}{A \qquad A \to B}$

(E→) $\dfrac{A \qquad A \to B}{B}$

(I−) $\dfrac{}{A \qquad -A}$

(E−) $\dfrac{A \qquad -A}{}$

Note: major formulas are conclusions of introductions and premisses eliminations.

Example
"development"

Proofs are formula trees. Instances of inference rules are building blocks of proofs.

$$-A \vee B$$

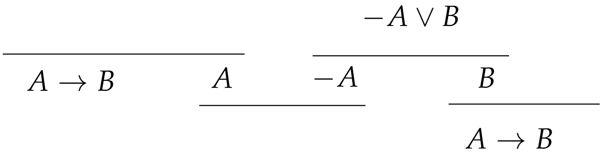
$$\overline{\quad A \rightarrow B \quad} \qquad A \qquad \overline{-A \qquad\qquad B}$$

$$\overline{A \qquad -A} \qquad \overline{\quad B \quad}$$

$$A \rightarrow B$$

Proofs are branching up and down.

Kneale's "development"



proves $-A \lor B \vdash_{\mathsf{KN}} A \to B$.

Building blocks are:

$$\frac{}{A \to B \qquad A} \qquad \frac{A \qquad -A}{} \qquad \frac{-A \lor B}{-A \qquad B} \qquad \frac{B}{A \to B}$$

Trees don't have cycles

Joining

$$\frac{A \vee B}{A \qquad B} \qquad + \qquad \frac{A \qquad B}{A \wedge B} \qquad ?$$

Certainly not

$$\cfrac{\cfrac{A \vee B}{A \qquad B}}{A \wedge B}$$

What do we get with KN?

Symmetry of dual $\wedge$ and $\vee$:

$$\text{(I}\wedge\text{)}\ \frac{A \qquad B}{A \wedge B} \qquad\qquad \text{(E}\vee\text{)}\ \frac{A \vee B}{A \qquad B}$$

What do we get with KN?

$$
(\mathsf{E}\lor)\;\dfrac{A \lor B \qquad \begin{array}{c}\overline{\;A\;}\ (i)\\ \vdots \\ C\end{array} \qquad \begin{array}{c}\overline{\;B\;}\ (i)\\ \vdots \\ C\end{array}}{C}\ (i)
$$

$$
\xrightarrow{\text{becomes}} \quad \begin{array}{c} A \lor B \\ \hline \begin{array}{cc} A & B \\ \vdots & \vdots \\ C & C \end{array} \end{array}
$$

No discharging. All inferences are local.

What do we get with KN?

Kneale version of another hypothetical deduction:

$$\dfrac{\dfrac{\dfrac{}{A}\,(i)}{\begin{array}{c}\vdots\\ B\end{array}}}{A \to B}\,(\mathsf{I}{\to})\ (i)$$

is built from

$$\begin{array}{c} A \\ \vdots \\ B \end{array} \qquad \dfrac{}{A \qquad A \to B} \qquad \dfrac{B}{A \to B}$$

…

$$
\begin{array}{l}
\overline{\;A \qquad\qquad A \to B\;} \\
\quad\vdots \\
\quad B \\
\hline
A \to B
\end{array}
$$

Theorem 3.
KN is sound, but incomplete.

KN-unprovable tautologies:

- $(A \to A) \wedge (A \vee (A \to A))$
- distributive law:

$$A \vee (B \wedge C) \vdash (A \vee B) \wedge (A \vee C)$$

Contractions

Some (simple) cycles have to be allowed:

$$\frac{A \qquad A}{A} \qquad \text{and} \qquad \frac{A}{A \qquad A} \;.$$

For example:

$$\cfrac{\cfrac{A \vee A}{A \qquad A}}{A} \;.$$

Contractions

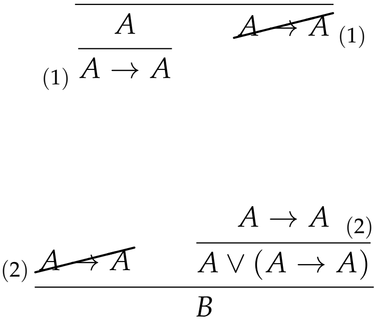
We "borrow" NK-notation for discharging hypothesis. We’ll "discharge" duplicate premisses (conclusions).

Ex:

$$\dfrac{(A \wedge B) \vee (A \wedge C)}{\dfrac{A \wedge B}{A_{(1)}} \qquad \dfrac{A \wedge C}{\cancel{A}_{(1)}}}$$

MCD deduction of a formula unprovable in KN

with contractions



MCD deduction of a formula unprovable in KN

with contractions

$$\dfrac{\cancel{A \to A} \qquad \dfrac{\dfrac{\dfrac{}{A \qquad\qquad \cancel{A \to A}}}{A \to A}}{A \lor (A \to A)}}{(A \to A) \land (A \lor (A \to A))}$$

3. Proof search

I. Analysis ✓

We can search for proofs in normal form.
Top and bottom nodes are obvious pieces of the puzzle.

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II. Synthesis

We can search for proofs in normal form.
Top and bottom nodes are obvious pieces of the puzzle.

proof search (sketch)

Proof search sketch $\Gamma \vdash \Delta$

I. Analysis
- find all analytic deductions of Γ formulas as premisses
- find all analytic deductions of Δ formulas as conclusions

II. Synthesis
- try to match/build $\Gamma \vdash \Delta$ from analytic parts

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We'll present such a procedure and prove it to be equivalent to tableaux method.

Proof search for $A \wedge B \vdash A \wedge B$:

Proof search for $A \wedge B \vdash A \wedge B$:

$$\frac{A \wedge B}{A} \qquad \frac{A \wedge B}{B} \qquad \frac{A \qquad B}{A \wedge B}$$

Motivating Example

$A \wedge B \vdash A \wedge B$

Proof search for $A \wedge B \vdash A \wedge B$:

$$\frac{A \wedge B}{A} \qquad \frac{A \wedge B}{B} \qquad \frac{A \qquad B}{A \wedge B}$$

$$\frac{\dfrac{A \wedge B}{A} \qquad \dfrac{A \wedge B}{B}}{A \wedge B}$$

Algorithm 1: Analysis of F going down

input : non-atomic F

$$\begin{array}{c}\boxed{F}\\ \downarrow\end{array}$$

output : appropriate inference(s) with appropriate major formula F.

$$\begin{array}{c}\boxed{F}\\ \downarrow\end{array} \quad\mapsto\quad \frac{\boxed{F}}{F_1 \qquad F_2} \qquad \text{or} \qquad \begin{array}{c}\boxed{F}\\ \downarrow\end{array} \quad\mapsto\quad \frac{\boxed{F}}{F_1}\,,\ \frac{\boxed{F}}{F_2}$$

Algorithm returns all analytical deductions of F as premiss.
(normal form, one major formula)

Example

$$\overset{\uparrow}{\boxed{X \to (Y \wedge Z)}}$$

$$\frac{\overset{\uparrow}{\boxed{Y \wedge Z}}}{X \to (Y \wedge Z)} \quad , \qquad \frac{}{X \qquad X \to (Y \wedge Z)}$$

$$\frac{\dfrac{Y \qquad Z}{Y \wedge Z}}{X \to (Y \wedge Z)} \quad , \qquad \frac{}{X \qquad X \to (Y \wedge Z)}$$

Matching (sparivanje)

ad hoc, for now

Matching – series of connections that yields π za $\Gamma \vdash \Delta$

(premisses are in Γ, conclusions are in Δ)

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N.B. We don't get a necessarily minimal deduction, but we get one in normal form.
Connections are Intro-Elim.

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N.B. We don't get a necessarily minimal deduction, but we get one in normal form.
Connections are Intro-Elim.

4. Semantic version of analysis

TF semantic analysis

$\boxed{-\top}\quad \dfrac{(-A)\top}{A\bot}$

$\boxed{-\bot}\quad \dfrac{(-A)\bot}{A\top}$

$\boxed{\wedge\top}\quad \dfrac{(A \wedge B)\top}{\begin{array}{c}A\top\\ B\top\end{array}}$

$\boxed{\wedge\bot}\quad \dfrac{(A \wedge B)\bot}{A\bot \qquad B\bot}$

$\boxed{\vee\top}\quad \dfrac{(A \vee B)\top}{A\top \qquad B\top}$

$\boxed{\vee\bot}\quad \dfrac{(A \vee B)\bot}{\begin{array}{c}A\bot\\ B\bot\end{array}}$

$\boxed{\rightarrow\top}\quad \dfrac{(A \rightarrow B)\top}{A\bot \qquad B\top}$

$\boxed{\rightarrow\bot}\quad \dfrac{(A \rightarrow B)\bot}{\begin{array}{c}A\top\\ B\bot\end{array}}$

Analytic deductions and clauses

Let π be a.d. of F as premiss:

$$F,\ A_1, .., A_n \vdash B_1, .., B_m$$

(A_i and B_j are atoms)

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$$F\top \models -A_1, .., -A_n, B_1, .., B_m \ .$$

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$$\{-A_1, .., -A_n,\ B_1, \ldots, B_m\}\ .$$

Atoms on top are $\bot$-marked. Atoms on the bottom are $\top$-marked.

$$\frac{A \qquad B}{A \wedge B} \quad \xrightarrow{\text{read as}} \quad \frac{(A \wedge B)\bot}{A\bot \qquad B\bot}$$

"$(A \wedge B)\bot$ implies $A\bot$ or $B\bot$."

Semantic Analysis with MCDs

$$(\wedge\bot)\ \frac{A \qquad B}{A \wedge B}$$

$$(\wedge\top)\ \frac{A \wedge B}{A} \quad \text{and} \quad \frac{A \wedge B}{B}$$

$$(\vee\bot)\ \frac{A}{A \vee B} \quad \text{and} \quad \frac{B}{A \vee B}$$

$$(\vee\top)\ \frac{A \vee B}{A \qquad B}$$

$$(\rightarrow\bot)\ \frac{B}{A \rightarrow B} \quad \text{and} \quad \frac{}{A \qquad A \rightarrow B}$$

$$(\rightarrow\top)\ \frac{A \qquad A \rightarrow B}{B}$$

$$(-\bot)\ \frac{}{A \qquad -A}$$

$$(-\top)\ \frac{A \qquad -A}{}$$

Introductions

Eliminations

Sequence of analytic deductions...

is conjunction of signed formulas...

Theorem 4.
All analytic deductions of $F\top$ determine logically equivalent ***clausal form*** *of F.*

By analogy $F\bot$ gets clausal form for $-F$.

5. Synthesis

What about the connection?
(of two analytic deductions)

Let $\mathcal{L}$ and $\mathcal{M}$ denote clauses belonging to deductions that can be matched through A.

$$Z_1 \models \mathcal{L}, A$$
$$Z_2 \models \mathcal{M}, -A$$

After connecting

$$Z_1, Z_2 \models \mathcal{L}, \mathcal{M} \, .$$

(A is eliminated ("resolved"))

Resolution:

$$\frac{\mathcal{M}, A \qquad \mathcal{N}, \overline{A}}{\mathcal{M}, \mathcal{N}}$$

(Clause $\mathcal{M}, \mathcal{N} = \mathcal{M} \cup \mathcal{N}$ is **resolvent** of clauses $\mathcal{M}, A = \mathcal{M} \cup \{A\}$ i $\mathcal{N}, \overline{A} = \mathcal{N} \cup \{\overline{A}\}$.)

Theorem 5.
Clausal form is not satisfiable if we can derive an empty clause from it by resolution.

Resolution:

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Theorem 5.
Clausal form is not satisfiable if we can derive an empty clause from it by resolution.

Successful matching corresponds with resolution of empty clause.

Theorem 6.
Resolution is decision procedure for propositional classical logic.

Example

Analytic deductions of

$$A \wedge -C, -B \vee C \models A \wedge -B, -A \vee C$$

are:

$$\frac{A \wedge -C}{A} \qquad \frac{A \wedge -C}{-C} \qquad \frac{-B}{-B \vee C} \qquad \frac{C}{-B \vee C}$$

$$\frac{A \qquad -B}{A \wedge -B} \qquad \frac{-A}{-A \vee C} \qquad \frac{C}{-A \vee C}$$

Example

$$\dfrac{\{-C\} \qquad \dfrac{\dfrac{\{A\} \qquad \{-A, B\}}{\{B\}}\ (1) \qquad \{-B, C\}}{\{C\}}\ (2)}{\emptyset}\ (3)$$

$$\frac{A \wedge -C}{A} \qquad \frac{A \wedge -C}{-C} \qquad \frac{-B}{-B \vee C} \qquad \frac{C}{-B \vee C}$$

$$\frac{A \qquad -B}{A \wedge -B} \qquad \frac{-A}{-A \vee C} \qquad \frac{C}{-A \vee C}$$

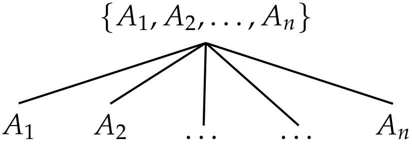
$$\dfrac{\dfrac{A \wedge -C}{A} \qquad \dfrac{}{-B}}{A \wedge -B} \qquad \dfrac{\dfrac{}{B} \qquad \dfrac{-B \vee C}{-B}}{} \qquad \dfrac{C}{-A \vee C}$$

Clausal Satisfiability Tree — (motivation are semantic trees): formula tree with open and closed branches, nodes are literals – open branches represent models.

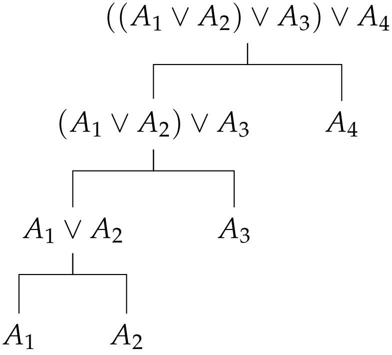
τ_A and τ_B are equivalent if every model of τ_A is a model of τ_B and vice versa.

CNF adjustment

we'll allow clausal trees – clause analysis in one step

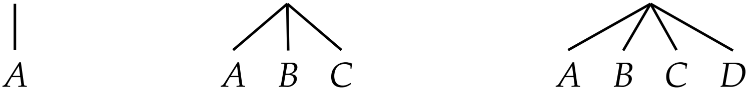


So we don't have to deal with



Clausal Satisfiability Tree

2° build it directly



Clausal Satisfiability Tree

2° build it directly from clauses

Clausal form

$$A \wedge (B \vee \overline{C}) \wedge (\overline{A} \vee B \vee C)$$

Clausal Satisfiability Tree

2° build it directly from clauses

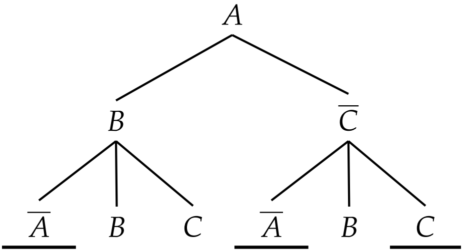
Clausal form

$$A \wedge (B \vee \overline{C}) \wedge (\overline{A} \vee B \vee C)$$

yields analytic deductions whose clauses are

$$\{A\}, \{B, \overline{C}\} \quad \text{i} \quad \{\overline{A}, B, C\},$$

which finally yields:



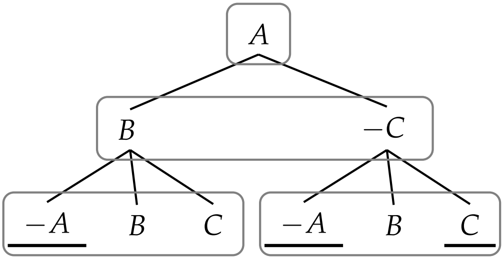


Fig. : Clauses are visible in the tree

Remember:

an. deductions → clauses → clausal tree

We'll use clausal tree as a map to find matching.

Unbalanced c.s. trees

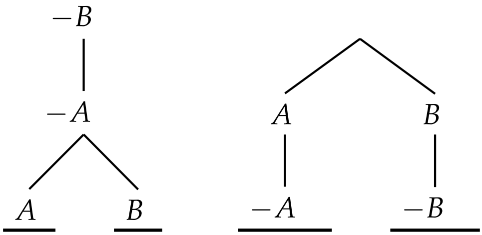


Fig. : Simple (left) and unbalanced (right) clausal sat. tree for $\{A, B\}$, $\{-A\}$, $\{-B\}$

As long as they are closed, or open branches "go through" all clauses, they are equivalent.

Distributive law:

$$A \vee (B \wedge C) \vdash (A \vee B) \wedge (A \vee C)$$

Analysis

I. Analysis yields a sequence of analytic deductions

$$\dfrac{A \lor (B \land C)}{A \qquad \dfrac{B \land C}{B}} \qquad\qquad \dfrac{A \lor (B \land C)}{A \qquad \dfrac{B \land C}{C}}$$

$\top$-analysis

$$\dfrac{\dfrac{A}{A \lor B} \qquad \dfrac{A}{A \lor C}}{(A \lor B) \land (A \lor C)} \qquad\qquad \dfrac{\dfrac{A}{A \lor B} \qquad \dfrac{C}{A \lor C}}{(A \lor B) \land (A \lor C)}$$

$$\dfrac{\dfrac{B}{A \lor B} \qquad \dfrac{A}{A \lor C}}{(A \lor B) \land (A \lor C)} \qquad\qquad \dfrac{\dfrac{B}{A \lor B} \qquad \dfrac{C}{A \lor C}}{(A \lor B) \land (A \lor C)}$$

$\bot$-analysis

For given set of clauses $\{\overline{A}\}$, $\{A, B\}$, $\{A, C\}$, $\{\overline{A}, \overline{C}\}$, $\{\overline{A}, \overline{B}\}$, $\{\overline{B}, \overline{C}\}$.

form a (closed) clausal satisfiability tree:

For given set of clauses $\{\overline{A}\}$, $\{A, B\}$, $\{A, C\}$, $\{\overline{A}, \overline{C}\}$, $\{\overline{A}, \overline{B}\}$, $\{\overline{B}, \overline{C}\}$.

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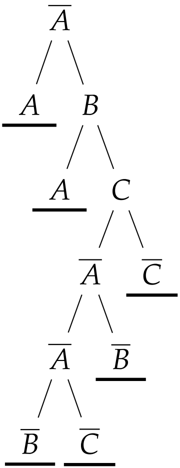
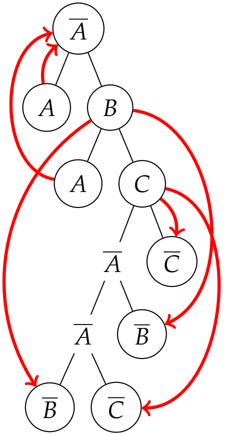
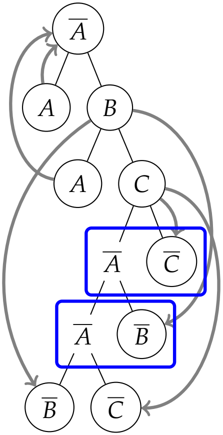
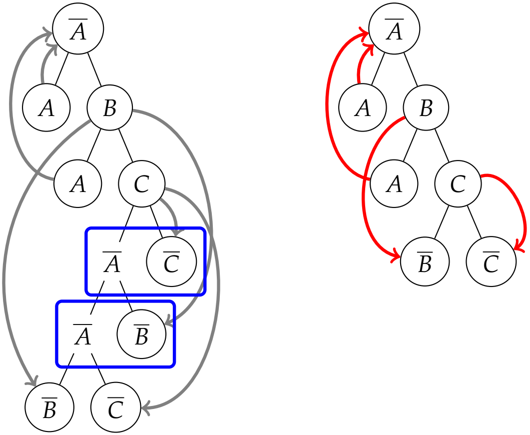




We augment the tree with all arrows that close its branches. Nodes incident with arrows are circled.



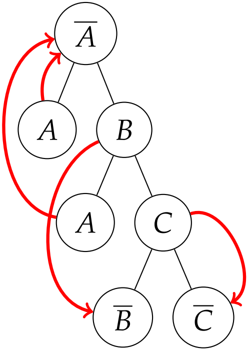
Unmatched clauses $\{\overline{A}, \overline{C}\}$ i $\{\overline{A}, \overline{B}\}$ are not necessary.



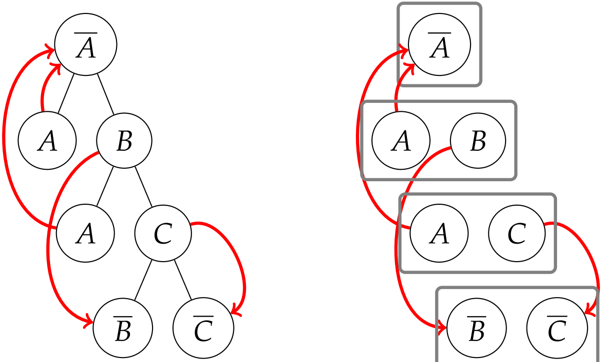
Unmatched clauses $\{\overline{A}, \overline{C}\}$ i $\{\overline{A}, \overline{B}\}$ are not necessary.

Throwing them out is enough for a matched tree.

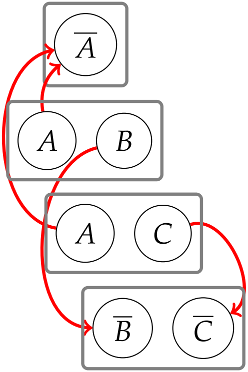
Closed & matched tree



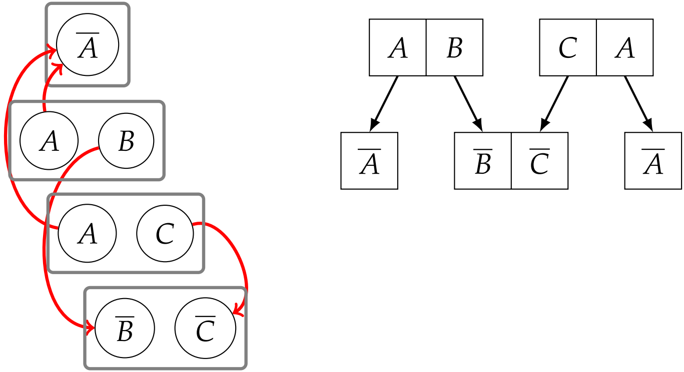
Closed & matched tree



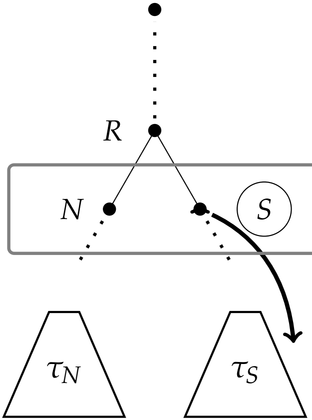
Top-sort the DAG (arrows point down)



Top-sort the DAG (arrows point down)

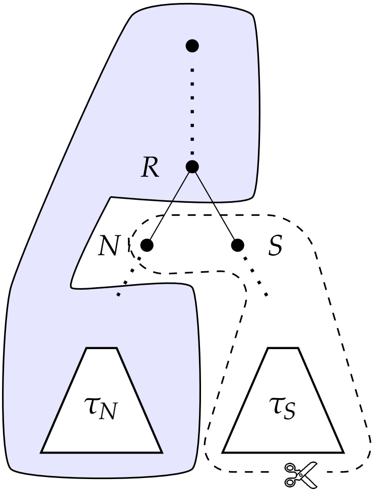


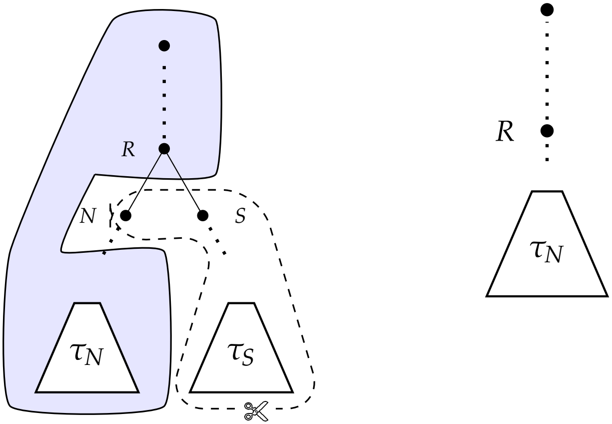
Removal of an unmatched clause in a closed tree can "break" (un-match) some other matched clause. It is not a problem, though.



Unmatched clause

Removal of an unmatched clause
... and its subtree ... is not a problem.





Lemma 7. *A closed tree remains closed after removal of unmatched clauses.*

Finally, when there is nothing left to remove – we get a matching.

Theorem 8.
MCD is analytic (and complete) calculus for classical propositional logic.

Completeness follows from equivalence with semantic trees.

6. First Order Logic

NK inference rules for quantifiers

(I∃) $$\frac{A(a/x)}{\exists x A(x)}$$

(E∃) $$\frac{\exists x A(x) \qquad \begin{array}{c}\overline{A(a/x)}\\ \vdots \\ C\end{array}}{C}$$

(a doesn't appear in hypotheses, nor in C)

(I∀) $$\frac{A(a/x)}{\forall x A(x)}$$

(a doesn't appear in hypotheses)

(E∀) $$\frac{\forall x A(x)}{A(a/x)}$$

Symmetry lost, hypothesis discharging reenters the story…?

Existential Instantiation

solution for (E∃) with cumbersome stipulation

$$\frac{\exists x A(x)}{A(a/x)}$$

and dual (I∀) rule:

$$\frac{A(a/x)}{\forall x A(x)}$$

must not be *a*-connected to an (E∃) inference instance that introduces *a*.

Don't want to connect

$$\frac{\exists x Fx}{Fa} \qquad \frac{Fa}{\forall x Fx}$$

to get absurd deduction

$$\cfrac{\cfrac{\exists x Fx}{Fa}}{\forall x Fx} \; .$$

Luckily, *a* being "fresh" is enough for tableaux, and also enough for us.

Example of "new" deduction

$$\dfrac{\dfrac{\dfrac{\dfrac{\exists y \forall x Fxy}{\forall x Fxa}}{Fba}}{\exists y Fby}}{\forall x \exists y Fxy}$$

Symmetry and (apparent) locality of inference rules are preserved.

Semantic analysis for ∀, ∃

$\boxed{\forall\top}$ $$\frac{\forall x A x \ \top}{A a \ \top}$$

$\boxed{\forall\bot}$ $$\frac{\forall x A x \ \bot}{A a \ \bot}$$
(new a)

$\boxed{\exists\top}$ $$\frac{\exists x B x \ \top}{B a \ \top}$$
(new a)

$\boxed{\exists\bot}$ $$\frac{\exists x B x \ \bot}{B a \ \bot}$$

γ	γ_1	γ_2	$\ldots$
$\forall x A x \top$	$A a \top$	$A b \top$	$\ldots$
$\exists x A x \bot$	$A a \bot$	$A b \bot$	$\ldots$

γ never finishes on infinite universum $\{a, b, c, \ldots\}$ ...

δ	δ_1
$\exists x A x \top$	$A a \top$ new/fresh a
$\forall x A x \bot$	$A a \bot$ new/fresh a

Tableaux Example

$$\forall x \forall y (Rxy \to -Rxx \wedge -Ryy) \qquad \top, \textcircled{a}$$

$$\forall x - Rxx \quad \textcircled{\bot}$$

$$-Raa \quad \textcircled{\bot}$$

$$\textcircled{Raa} \quad \top$$

$$Raa \to -Raa \wedge -Raa \quad \textcircled{\top}$$

Left branch:

$$\textcircled{Raa} \quad \bot$$

Right branch:

$$-Raa \quad \textcircled{\top}$$

$$\textcircled{Raa} \quad \bot$$

Cyclic method – analyzing in cycles

Tableaux closes for a valid $\Gamma \models \Delta$ as long as every formula gets analyzed.

For example, we can queue formulas in cycles.
Then prioritize (within the cycle):

1. α, β (Truth functional)
2. δ (instantiation)
3. γ (application)

Theorem 9.
Cyclic variant of analytic tableaux is a positive decision procedure for FOL.

But, tableuax is decision procedure for monadic fragment of FOL.

Analysis

again, analytic deductions end with atoms

- Atoms $Aa, Ab, \ldots$
- Literals $Aa, -Aa, \ldots$
- Clauses, clause trees...

Only difference – we'll have to build the clausal tree and try to close it as soon as possible.

Working with prenex formulas makes life easier.
(branching is postponed)

Example

Show $\exists y \forall x Fxy \models \forall x \exists y Fxy$ with analytic deduction method:

$$\begin{array}{c} \exists y \forall x Fxy \\ \hline \delta \end{array} \qquad \begin{array}{c} \delta \\ \hline \forall x \exists y Fx \end{array}$$

$$\begin{array}{c} \exists y \forall x Fxy \\ \hline \textcircled{\delta} \quad a \text{ new} \\ \hline \forall x Fay \\ \hline \gamma \quad a \end{array} \qquad \begin{array}{c} \delta \\ \hline \forall x \exists y Fx \end{array}$$

$$\begin{array}{c} \exists y \forall x Fxy \\ \hline \textcircled{\delta} \quad a \text{ new} \\ \hline \forall x Fay \\ \hline \gamma \quad a, b \end{array} \qquad \begin{array}{c} \gamma \quad a, b \\ \hline \exists y Fb \\ \hline \delta \quad b \text{ new} \\ \hline \forall x \exists y Fx \end{array}$$

$$
\begin{array}{c}
\exists y\forall x Fxy \\ \hline
\textcircled{δ} \quad a \text{ novi} \\ \hline
\forall x Fay \\ \hline
\gamma \quad a, \textcircled{b} \\ \hline
Fab
\end{array}
\qquad\qquad
\begin{array}{c}
Fab \\ \hline
\gamma \quad \textcircled{a}, b \\ \hline
\exists y Fb \\ \hline
\textcircled{δ} \quad b \text{ novi} \\ \hline
\forall x \exists y Fx
\end{array}
$$

Without the aux. notation:

$$
\begin{array}{c}
\exists y\forall x Fxy \\ \hline
\forall x Fay \\ \hline
Fab
\end{array}
\qquad\qquad
\begin{array}{c}
Fab \\ \hline
\exists y Fb \\ \hline
\forall x \exists y Fx
\end{array}
$$

Matching here is trivial: connect through Fab results in

$$
\begin{array}{c}
\exists y\forall x Fxy \\ \hline
\forall x Fxa \\ \hline
Fba \\ \hline
\exists y Fby \\ \hline
\forall x \exists y Fxy
\end{array}
$$

7. Conclusion

- $\Gamma \not\models \Delta$ denotes that $\Gamma \models \Delta$ is not valid
- Read $\Gamma \not\models \Delta$ as

$$\Gamma\top, \Delta\bot$$

- Example

$$A \not\models B, C$$

denotes $A\top, B\bot, C\bot$

Rules for $\not\models$ analysis

$\boxed{-\top}$ $$\frac{\Gamma, \overline{A} \not\models \Delta}{\Gamma \not\models A, \Delta}$$

$\boxed{-\bot}$ $$\frac{\Gamma \not\models \overline{A}, \Delta}{\Gamma, A \not\models \Delta}$$

$\boxed{\wedge\top}$ $$\frac{\Gamma, A \wedge B \not\models \Delta}{\Gamma, A, B \not\models \Delta}$$

$\boxed{\wedge\bot}$ $$\frac{\Gamma \not\models A \wedge B, \Delta}{\Gamma \not\models A, \Delta \qquad \Gamma \not\models B, \Delta}$$

$\boxed{\vee\top}$ $$\frac{\Gamma, A \vee B \not\models \Delta}{\Gamma, A \not\models \Delta \qquad \Gamma, B \not\models \Delta}$$

$\boxed{\vee\bot}$ $$\frac{\Gamma \not\models A \vee B, \Delta}{\Gamma \not\models A, B, \Delta}$$

$\boxed{\rightarrow\top}$ $$\frac{\Gamma, A \rightarrow B \not\models \Delta}{\Gamma \not\models A, \Delta \qquad \Gamma, B \not\models \Delta}$$

$\boxed{\rightarrow\bot}$ $$\frac{\Gamma \not\models A \rightarrow B, \Delta}{\Gamma, A \not\models B, \Delta}$$

Semantic $\not\models$ Tree

- Semantic tree grows downward
- α i β rules are essentially identical
- Branch closes if it its sequent is overlapping

$$\Gamma, C \not\models C, \Delta \,.$$

- Fully expanded branches

$$\dfrac{-A \vee B \not\models A \to B}{\dfrac{-A \not\models A \to B}{\dfrac{-A, A \not\models B}{\underline{A \not\models A, B}}} \qquad \dfrac{B \not\models A \to B}{\underline{B, A \not\models B}}}$$

Semantic $\not\models$ Tree

$$\dfrac{-A \text{\textcircled{$\vee$}} B \not\models A \to B}{\dfrac{-A \not\models A \text{\textcircled{$\to$}} B}{\dfrac{\text{\textcircled{$-A$}}, A \not\models B}{\underline{A \not\models A, B}}} \qquad \dfrac{B \not\models A \text{\textcircled{$\to$}} B}{\underline{B, A \not\models B}}}$$

- $\not\models$ Tree is (notationally) equivalent to analytic tableaux
- Closed $\not\models$ tableaux can be inverted upside down. Replacing $\not\models$ with $\vdash$ gives a LK proof of $\Gamma \vdash \Delta$ without cut.
- "*Sequent proofs are upside down tableaux proofs*"

Turning the tableaux upside-down
gives LK proof

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A, B}}{-A, A \vdash B}}{-A \vdash A \to B} \qquad \dfrac{\dfrac{}{B, A \vdash B}}{B \vdash A \to B}}{-A \vee B \vdash A \to B}$$

.

Conclusion

- LK and analytic tableaux are equivalent (as notational variants)
- MCD's are really not that different – proof search is analytic, relates to common techniques
- Analysis is similar to tableaux (converts to CNF, with tableaux we get DNF)
- Synthesis is simple, relates to resolution

…about multiple sequents:

"It must be admitted that this new concept of a sequent in general already constitutes a departure from 'natural' and that its introduction is primarily justified by the considerable formal advantages ..."

The End.

Thank you