Some Identities in the Twisted Group Algebra of Symmetric Groups

Milena Sošić

Department of Mathematics

University of Rijeka

Radmile Matejčić 2, Rijeka 51000, Croatia

msosic@math.uniri.hr

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- S_n denote the symmetric group on n letters

  i.e S_n is the set of all permutations of a set $M = \{1, 2, \ldots, n\}$ equiped with a composition as the binary operation on S_n
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- $R_n := \mathbb{C}[X_{a\,b} \mid 1 \leq a, b \leq n]$ denote the polynomial ring

  i.e the commutative ring of all polynomials in n^2 variables $X_{a\,b}$ over the set $\mathbb{C}$ with $1 \in R_n$ as a unit element of R_n.
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- S_n acts on the set X as follows: $g.X_{a\,b} = X_{g(a)\,g(b)}$.

- This action of S_n on X induces the action of S_n on R_n given by

$$g.p(\dots, X_{a\,b}, \dots) = p(\dots, X_{g(a)\,g(b)}, \dots)$$

for every $g \in S_n$ and any $p \in R_n$.

$$\mathbb{C}[S_n] = \left\{ \sum_{\sigma \in S_n} c_\sigma \sigma \mid c_\sigma \in \mathbb{C} \right\}$$

of the symmetric group S_n is a free vector space (generated with the set S_n), where the multiplication is given by

$$\left(\sum_{\sigma \in S_n} c_\sigma \sigma \right) \cdot \left(\sum_{\tau \in S_n} d_\tau \tau \right) = \sum_{\sigma, \tau \in S_n} (c_\sigma d_\tau)\, \sigma\tau.$$

Here we have used the simplified notation: $\sigma\tau = \sigma \circ \tau$

for the composition $\sigma \circ \tau$ i.e the product of σ and τ in S_n.

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- The multiplication in $\mathcal{A}(S_n)$ is given by

$$(p_1 g_1) \cdot (p_2 g_2) := (p_1 \cdot (g_1 . p_2)) \, g_1 g_2$$

where $g_1 . p_2$ is defined by: $\quad g.p(\ldots, X_{a\,b}, \ldots) = p(\ldots, X_{g(a)\,g(b)}, \ldots)$

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$$(p_1 g_1) \cdot (p_2 g_2) := (p_1 \cdot (g_1.p_2))\, g_1 g_2$$

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- The algebra $\mathcal{A}(S_n)$ is associative but not commutative.

$$I(g) = \{(a,b) \mid 1 \leq a < b \leq n,\ g(a) > g(b)\}$$

denote the set of inversions of $g \in S_n$.

Then to every $g \in S_n$ we associate a monomial in the ring R_n defined by

$$X_g := \prod_{(a,b)\in I(g^{-1})} X_{a\,b} \left(= \prod_{a<b,\ g^{-1}(a)>g^{-1}(b)} X_{a\,b} \right),$$

which encodes all inversions of g^{-1} (and of g too).

More generally, for any subset $A \subseteq \{1, 2, \ldots, n\}$ we will use the notation

$$X_A := \prod_{(a,b)\in A\times A,\ a<b} X_{a\,b} \cdot X_{b\,a} = \prod_{(a,b)\in A\times A,\ a<b} X_{\{a,\,b\}},$$

because

$$X_{\{a,\,b\}} := X_{a\,b} \cdot X_{b\,a}.$$

Definition

To each $g \in S_n$ we assign a unique element $g^* \in \mathcal{A}(S_n)$ defined by

$$g^* := X_g \, g.$$

Theorem

For every $g_1^*, g_2^* \in \mathcal{A}(S_n)$ *we have*

$$g_1^* \cdot g_2^* = X(g_1, g_2)\,(g_1 g_2)^*,$$

where the multiplication factor is given by

$$X(g_1, g_2) = \prod_{(a,b) \in I(g_1^{-1}) \setminus I((g_1 g_2)^{-1})} X_{\{a,b\}}$$

.

Recall,

$$X(g_1, g_2) = \prod_{(a,b)\in I(g_1^{-1})\setminus I((g_1 g_2)^{-1})} X_{\{a,\, b\}}$$

- Note that

$$X(g_1, g_2) = 1$$

if $\quad l(g_1 g_2) = l(g_1) + l(g_2)$.

So we have

$$g_1^* \cdot g_2^* = (g_1 g_2)^*$$

where $\; l(g) := Card\, I(g) \;$ is the lenght of $\; g \in S_n$.

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  where $\; l(g) := Card\, I(g) \;$ is the lenght of $\; g \in S_n$.

- The factor $X(g_1, g_2)$ takes care of the reduced number of inversions in the group product of $g_1, g_2 \in S_n$.

Example

Let $g_1 = 132, \quad g_2 = 312 \in S_3$.

Then $g_1 g_2 = 213, \quad l(g_1) = 1, \quad l(g_2) = 2, \quad l(g_1 g_2) = 1$.

Note that $g_1^{-1} = 132, \quad g_2^{-1} = 231$, so

$$g_1^* \cdot g_2^* = (X_{2\,3}\, g_1) \cdot (X_{1\,3} X_{2\,3}\, g_2) = X_{2\,3} X_{1\,2} X_{3\,2}\, g_1 g_2 = X_{\{2,3\}} X_{1\,2}\, g_1 g_2.$$

On the other hand we have: $(g_1 g_2)^* = X_{1\,2}\, g_1 g_2,$

since $(g_1 g_2)^{-1} = 213$.

Thus we get $g_1^* \cdot g_2^* = X_{\{2,3\}}\,(g_1 g_2)^*$

and $X(g_1, g_2) = X_{\{2,3\}}$.

Recall, $g_1^* \cdot g_2^* = X(g_1, g_2)\,(g_1 g_2)^*$ for every $g_1^*, g_2^* \in \mathcal{A}(S_n)$

Example

For $g_1 = 132,\quad g_2 = 231$

we have $g_1 g_2 = 321,\quad l(g_1) = 1,\quad l(g_2) = 2,\quad l(g_1 g_2) = 3.$

Further $g_1^{-1} = 132,\quad g_2^{-1} = 312$ and $(g_1 g_2)^{-1} = 321,$

so we get:

$$g_1^* \cdot g_2^* = (X_{2\,3}\, g_1) \cdot (X_{1\,2} X_{1\,3}\, g_2) = X_{2\,3} X_{1\,3} X_{1\,2}\, g_1 g_2,$$

$$(g_1 g_2)^* = X_{1\,2} X_{1\,3} X_{2\,3}\, g_1 g_2.$$

Thus we get $g_1^* \cdot g_2^* = (g_1 g_2)^*$

and $X(g_1, g_2) = 1.$

We denote by

- $t_{a,b}, \quad 1 \leq a \leq b \leq n$ the following cyclic permutation in S_n:

$$t_{a,b}(k) := \begin{cases} k & 1 \leq k \leq a-1 \quad \text{or} \quad b+1 \leq k \leq n \\ b & k = a \\ k-1 & a+1 \leq k \leq b \end{cases}$$

which maps b to $b-1$ to $b-2 \cdots$ to a to b and fixes all $1 \leq k \leq a-1$ and $b+1 \leq k \leq n$ i.e

$$t_{a,b} = \begin{pmatrix} 1 & \dots & a-1 & a & a+1 & \dots & b-1 & b & b+1 & \dots & n \\ 1 & \dots & a-1 & b & a & \dots & b-2 & b-1 & b+1 & \dots & n \end{pmatrix}$$

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- $t_{b,a} := t_{a,b}^{-1}$ i.e

$$t_{b,a}(k) = \begin{cases} k & 1 \leq k \leq a-1 \quad \text{or} \quad b+1 \leq k \leq n \\ k+1 & a \leq k \leq b-1 \\ a & k = b \end{cases}$$

Then the sets of inversions are given by

$$I(t_{a,b}) = \{(a,j) \mid a+1 \le j \le b\},$$

$$I(t_{b,a}) = \{(i,b) \mid a \le i \le b-1\}.$$

so the corresponding elements in $\mathcal{A}(S_n)$ have the form:

$$t^*_{a,b} = \left(\prod_{a \le i \le b-1} X_{i\,b} \right) t_{a,b} \qquad t^*_{b,a} = \left(\prod_{a+1 \le j \le b} X_{a\,j} \right) t_{b,a}.$$

Observe: $t^*_{a,a} = id$, where $I(t_{a,a}) = \emptyset$.

Denote: $t_a = t_{a,a+1}$ $(= t_{a+1,a})$, $1 \le a \le n-1$

(the transposition of adjacent letters a and $a+1$).

Then: $t^*_a = X_{a\,a+1}\, t_a$, with $I(t_a) = \{(a,a+1)\}$.

Recall, $g_1^* \cdot g_2^* = X(g_1, g_2)\,(g_1 g_2)^*$ for every $g_1^*, g_2^* \in \mathcal{A}(S_n)$

Corollary

For each $1 \leq a \leq n-1$ *we have* $(t_a^*)^2 = X_{\{a,\,a+1\}}\, id.$

Here we have used that $t_a t_a = id$ and $X_{\{a,\,a+1\}} = X_{a\,a+1} \cdot X_{a+1\,a}$.

Corollary

For each $g \in S_n$, $1 \leq a < b \leq n$ *we have*

$$g^* \cdot t_{b,a}^* = \left(\prod_{a<j\leq n,\, g(a)>g(j)} X_{\{g(j),\, g(a)\}} \right) (g t_{b,a})^*.$$

In the case $g \in S_j \times S_{n-j}$, $1 \leq j \leq k \leq n$ *we have* $g^* \cdot t_{k,j}^* = (g t_{k,j})^*$.

Recall, $g_1^* \cdot g_2^* = X(g_1, g_2)\,(g_1 g_2)^*$ for every $g_1^*, g_2^* \in \mathcal{A}(S_n)$

Corollary (Braid relations)

We have

(i) $t_a^* \cdot t_{a+1}^* \cdot t_a^* = t_{a+1}^* \cdot t_a^* \cdot t_{a+1}^*$ *for each* $1 \leq a \leq n-2$,

(ii) $t_a^* \cdot t_b^* = t_b^* \cdot t_a^*$ *for each* $1 \leq a, b \leq n-1$ *with* $|a-b| \geq 2$.

Corollary (Commutation rules)

We have

(i) $t_{m,k}^* \cdot t_{p,k}^* = (t_k^*)^2 \cdot t_{p,k+1}^* \cdot t_{m-1,k}^*$ *if* $1 \leq k \leq m \leq p \leq n$.

(ii) *Let* $w_n (= n\, n-1 \cdots 2\, 1)$ *be the longest permutation in* S_n. *Then for every* $g \in S_n$ *we have*

$$(g w_n)^* \cdot w_n^* = w_n^* \cdot (w_n g)^* \left(= \prod_{a<b,\, g^{-1}(a)<g^{-1}(b)} X_{\{a,\,b\}} \right) g^*.$$

- Observe first:

for $\forall g \in S_n \quad \exists\, g_1 \in S_1 \times S_{n-1}$ and $1 \leq k_1 \leq n$ such that

$$g = g_1 t_{k_1,1}$$

Then $g(k_1) = g_1(t_{k_1,1}(k_1)) = g_1(1) = 1$ implies $k_1 = g^{-1}(1)$.

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- Subsequently, the permutation $g_1 \in S_1 \times S_{n-1}$

  can be represented uniquely as $g_1 = g_2 t_{k_2,2}$

  with $g_2 \in S_1 \times S_1 \times S_{n-2}$ and $2 \leq k_2 \leq n$.

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- Subsequently, the permutation $g_1 \in S_1 \times S_{n-1}$ can be represented uniquely as $g_1 = g_2 t_{k_2,2}$ with $g_2 \in S_1 \times S_1 \times S_{n-2}$ and $2 \le k_2 \le n$.
  Then $g_1(k_2) = g_2(t_{k_2,2}(k_2)) = g_2(2) = 2$ implies $k_2 = g_1^{-1}(2)$.

- By repeating the above procedure we get the following decomposition:

$$g = t_{k_n,n} \cdot t_{k_{n-1},n-1} \cdots t_{k_j,j} \cdots t_{k_2,2} \cdot t_{k_1,1} \left(= \overleftarrow{\prod_{1 \le j \le n}} t_{k_j,j} \right).$$

Recall, $g = t_{k_n,n} \cdot t_{k_{n-1},n-1} \cdots t_{k_j,j} \cdots t_{k_2,2} \cdot t_{k_1,1}.$

Example

- Let $S_3 = \{123, 132, 312, 321, 231, 213\}$ then in $\mathcal{A}(S_3)$ we have:

$$123^* = t^*_{3,3} \cdot t^*_{2,2} \cdot t^*_{1,1}, \quad 132^* = t^*_{3,3} \cdot t^*_{3,2} \cdot t^*_{1,1}, \quad 312^* = t^*_{3,3} \cdot t^*_{3,2} \cdot t^*_{2,1},$$

$$321^* = t^*_{3,3} \cdot t^*_{3,2} \cdot t^*_{3,1}, \quad 231^* = t^*_{3,3} \cdot t^*_{2,2} \cdot t^*_{3,1}, \quad 213^* = t^*_{3,3} \cdot t^*_{2,2} \cdot t^*_{2,1}.$$

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- Now, assume that

$$\alpha_3^* = \sum_{g \in S_3} g^*.$$

Then we get the following product form:

$$\alpha_3^* = \underbrace{\left(t^*_{3,3}\right)}_{\beta_1^* = (id)} \cdot \underbrace{\left(t^*_{3,2} + t^*_{2,2}\right)}_{\beta_2^*} \cdot \underbrace{\left(t^*_{3,1} + t^*_{2,1} + t^*_{1,1}\right)}_{\beta_3^*}$$

of simpler elements β_i^*, $1 \leq i \leq 3$ of the algebra $\mathcal{A}(S_3)$.

The general situation in $\mathcal{A}(S_n)$:

Definition

For every $1 \leq k \leq n$ we define

$$\beta^*_{n-k+1} := t^*_{n,k} + t^*_{n-1,k} + \cdots + t^*_{k+1,k} + t^*_{k,k} \quad \left(= \overleftarrow{\sum_{k \leq m \leq n}} t^*_{m,k} \right).$$

Theorem

Let

$$\alpha^*_n = \sum_{g \in S_n} g^*.$$

Then

$$\alpha^*_n = \beta^*_1 \cdot \beta^*_2 \cdots \beta^*_n \quad \left(= \overleftarrow{\prod_{1 \leq k \leq n-1}} \beta^*_{n-k+1} \right).$$

In what follows we are going to introduce some new elements in the algebra $\mathcal{A}(S_n)$ by which we will reduce β^*_{n-k+1}, $1 \leq k \leq n$.

The motivation is to show that the element $\alpha^*_n \in \mathcal{A}(S_n)$ can be expressed in turn as products of yet simpler elements of the algebra $\mathcal{A}(S_n)$.

Definition

For every $1 \leq k \leq n-1$ we define

$$\gamma^*_{n-k+1} := \left(id - t^*_{n,k}\right) \cdot \left(id - t^*_{n-1,k}\right) \cdots \left(id - t^*_{k+1,k}\right),$$

$$\delta^*_{n-k+1} := \left(id - (t^*_k)^2\, t^*_{n,k+1}\right) \cdot \left(id - (t^*_k)^2\, t^*_{n-1,k+1}\right) \cdots \left(id - (t^*_k)^2\, t^*_{k+1,k+1}\right)$$

Recall, $(t^*_k)^2 = X_{\{k,\, k+1\}}\, id\ (= X_{k\, k+1} \cdot X_{k+1\, k}\, id)$ and $t^*_{k+1,k+1} = id$.

Theorem

For every $1 \leq k \leq n$ *we have the following factorization*

$$\beta^*_{n-k+1} = \delta^*_{n-k+1} \cdot \left(\gamma^*_{n-k+1}\right)^{-1}.$$

Recall, $\alpha_n^* = \beta_1^* \cdot \beta_2^* \cdots \beta_n^*$ with $\beta_1^* = id$

$$\beta_{n-k+1}^* = \delta_{n-k+1}^* \cdot \left(\gamma_{n-k+1}^*\right)^{-1}$$

$$\gamma_{n-k+1}^* = \left(id - t_{n,k}^*\right) \cdot \left(id - t_{n-1,k}^*\right) \cdots \left(id - t_{k+1,k}^*\right)$$

$$\delta_{n-k+1}^* = \left(id - (t_k^*)^2\, t_{n,k+1}^*\right) \cdot \left(id - (t_k^*)^2\, t_{n-1,k+1}^*\right) \cdots \left(id - (t_k^*)^2\, t_{k+1,k+1}^*\right)$$

Example (The factorization of $\alpha_2^* \in \mathcal{A}(S_2)$)

We have

$$\alpha_2^* = \beta_2^*$$

i.e

$$\alpha_2^* = \left(id - (t_1^*)^2\right) \cdot \left(id - t_{2,1}^*\right)^{-1}$$

Recall, $\alpha_n^* = \beta_1^* \cdot \beta_2^* \cdots \beta_n^*$ with $\beta_1^* = id$

$$\beta_{n-k+1}^* = \delta_{n-k+1}^* \cdot \left(\gamma_{n-k+1}^*\right)^{-1}$$

$$\gamma_{n-k+1}^* = \left(id - t_{n,k}^*\right) \cdot \left(id - t_{n-1,k}^*\right) \cdots \left(id - t_{k+1,k}^*\right)$$

$$\delta_{n-k+1}^* = \left(id - (t_k^*)^2\, t_{n,k+1}^*\right) \cdot \left(id - (t_k^*)^2\, t_{n-1,k+1}^*\right) \cdots \left(id - (t_k^*)^2\, t_{k+1,k+1}^*\right)$$

Example (The factorization of $\alpha_3^* \in \mathcal{A}(S_3)$)

We have

$$\alpha_3^* = \beta_2^* \cdot \beta_3^*$$

where

$$\beta_2^* = \left(id - (t_2^*)^2\right) \cdot \left(id - t_{3,2}^*\right)^{-1}$$

$$\beta_3^* = \left(id - (t_1^*)^2 \cdot t_{3,2}^*\right) \cdot \left(id - (t_1^*)^2\right) \cdot \left(id - t_{2,1^*}\right)^{-1} \cdot \left(id - t_{3,1}^*\right)^{-1}$$

Recall, $\alpha_n^* = \beta_1^* \cdot \beta_2^* \cdots \beta_n^*$ with $\beta_1^* = id$

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$$\gamma_{n-k+1}^* = \left(id - t_{n,k}^*\right) \cdot \left(id - t_{n-1,k}^*\right) \cdots \left(id - t_{k+1,k}^*\right)$$
$$\delta_{n-k+1}^* = \left(id - (t_k^*)^2\, t_{n,k+1}^*\right) \cdot \left(id - (t_k^*)^2\, t_{n-1,k+1}^*\right) \cdots \left(id - (t_k^*)^2\, t_{k+1,k+1}^*\right)$$

Example (The factorization of $\alpha_4^* \in \mathcal{A}(S_4)$)

We have

$$\alpha_4^* = \beta_2^* \cdot \beta_3^* \cdot \beta_4^*$$

where

$$\beta_2^* = \left(id - (t_3^*)^2\right) \cdot \left(id - t_{4,3}^*\right)^{-1}$$

$$\beta_3 = \left(id - (t_2^*)^2 \cdot t_{4,3}^*\right) \cdot \left(id - (t_2^*)^2\right) \cdot \left(id - t_{3,2}^*\right)^{-1} \cdot \left(id - t_{4,2}^*\right)^{-1},$$

$$\beta_4^* = \left(id - (t_1^*)^2 \cdot t_{4,2}^*\right) \cdot \left(id - (t_1^*)^2 \cdot t_{3,2}^*\right) \cdot \left(id - (t_1^*)^2\right) \cdot \left(id - t_{2,1}^*\right)^{-1} \cdot \left(id - t_{3,1}^*\right)^{-1} \cdot \left(id - t_{4,1}^*\right)^{-1}.$$

In order to replace the matrix factorizations (from the right) given in [1] by twisted algebra computation, we nead to consider similar factorizations (but from the left).
Here we used factorizations from the left, because they are more suitable for computing constants in the algebra of noncommuting polynomials (this will be elaborated in a fortcoming paper).

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S. Meljanac, D. Svrtan, Determinants and inversion of Gram matrices in Fock representation of q_{kl}-canonical commutation relations and applications to hyperplane arrangements and quantum groups. Proof of an extension of Zagier's conjecture, arXiv:math-ph/0304040vl, 26 Apr 2003.

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