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OUR FIRST INSIGHT IN SANGAKU PROBLEMS

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Abstract. During self-imposed isolation from the West in 1639-1854 Japanese mathematicians built up original math world. We can consider *sangaku* problems as an unique cultural creation with eternal beauty where art and religious aspect meet within mathematics. The advantage of *sangaku* is that it can be simple for younger students, while some problems are still a challenge for others because they are still unsolved.

Keywords*:* *sangaku* problem, Kowa Seki, *enri, soroban,* Japanese Theorem, Soddy's Hexlet

1. Introduction and Historical Review

From the 14th century onward, the dark ages of Japanese science began. During the Japanese "Edo Period", which began in 1603 and lasted until 1867, the Tokugawa shogunate isolated Japan from the rest of the world for nearly 300 years, until the 19th century. But, this was also a period of peace and significant progress in Japanese culture and art. Although there were no colleges or universities in Japan, teaching was carried out in private schools *juku* and more often in Shinto shrines and Buddhist temples. The samurai were transformed within a few generations into a highly educated class.

At the turn of the 17th and the beginning of the 18th century, math in Japan also experienced significant advancement under the leadership of Kowa Seki (1642-1718), the most important Japanese mathematician of his time. He is often called the "Newton" or "Leibnitz" of Japan. In 1683, Seki was the first mathematician who studied determinants, ten years before Leibnitz; furthermore Seki's version was the more general. In 1685, Seki solved the cubic equation $-x^3 - 5x^2 + 14x + 30 = 0$, applying the same method as Horner would a hundred years later. One of the most important Seki's contributions to Japanese mathematic *wasan* was a method named *enri*. Enri divided a circle into *n* rectangles to approximate a circle, against Europe's method of exhaustion which used *n*-sides polygons. The given enri method is rough version of integral calculus. Fukagawa and Rothman in his book [1] claim that it was nearly impossible for anyone in Japan to know about integral calculus by Newton and Leibnitz during this period.

Fig. 1: Tablet [9]

Sangaku appeared during this Edo period. Sangaku, or more precisely San Gaku literally means mathematical tablet, and those were colorful mathematical puzzles, i.e. geometrical problems written and drawn on wooden tablets. They were usually hung under the eaves in Shinto shrines and Buddhist temples as offerings to the gods, but also as challenges to all. *Sangaku* look like the works of art. Each wooden tablet often includes many illustrated problems or theorems which were frequently based on Euclidian geometry. In Fig. 1 is presented one of the few problems from the tablet in Fukushima prefecture, dealing with circles and ellipses. It is interesting that most *sangaku* contain only problems, without the solutions. The majority of these problems are not so difficult to solve, they are at the high school or college level. But, some of them are extremely difficult, because the proof of these theorems requires, for example, integral calculus which Japanese did not know during the period when the tablets were created. Some theorems remain unproven even by the application of the modern mathematics. Sangaku problems at that time on areas and volumes were solved by expansions in infinite series and term-by-term calculation.

Every tablet contains the name of creator and the date. Hence, today we know that sangaku was done by devotees of mathematics, from people in the poor castes up to highly educated samurai, women and children (aged 12-14). Sangaku were probably made by groups of students in schools. Sangaku were acts of homage, thanks to a guiding spirit, or may have been challenges to others: Solve this if you can! [2].

Fig. 2: Sangaku in Japanese [3]

The most of *sangaku* was written on Kambun, an archaic Japanese dialect related to Chinese. Kambun was language like Latin in Europe, learned and used by educated castes. It can be concluding that authors of tablets written on Kambun are samurais or some highly educated persons.

In Edo period traveling was very popular; so many mathematicians traveled and made sangaku all over the country. Kazu Yamaguchi traveled across Japan six times between 1817. and 1828. He collected data about 87 sangaku and their creators ranged them equally in rural and urban districts. Unfortunately, only two of these tablets are preserved.

Through several centuries there were made thousands of tablets, today still exist 880 sangaku. Clearly, sangaku had done their job, successfully propagated knowledge and interest for mathematics, [1], [9].

2. Some Typical Sangaku Problems

In *sangaku* problems we can usually find multitudes of circles or of spheres within others, ellipses, cylinders and other figures.

2.1. Circles in a Square

In a square there are two circles touching the bottom side of the square. At the same time, there are two small circles tangent to the same side of the square and the incircle of the square. Let two tangent of smaller circles meet each other at midpoint of the top side of square, they make an isosceles triangle in square. Find the radius of the incircle of that triangle if we know the radius of incircle of the square.

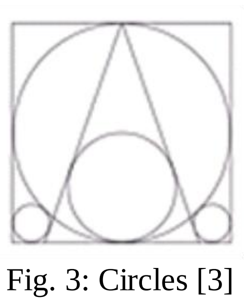


Fig. 3: Circles [3]

Solution: The medium circle is half the size of the largest circle.

2.2. Sangaku with Pentagons: Golden Ratio

Fig. 4: Pentagon [3]

This is sangaku from Miyage prefecture, 1912. Six congruent right triangles fan out along the sides of a regular pentagon of side a. Find the length of the hypotenuse t of these triangles in terms of a.

Solution: The diagonal with a side of pentagon in golden ratio, using similar triangles and the fact $\cos 36^\circ = \frac{\varphi}{2}$ leads to solution $t = 2\varphi a$.

Two congruent regular pentagons with a common side are inscribed in a large circle. A circle of radius r touches the large circle and the sides of pentagons, and the incircle of the triangle ABH has radius t. Show that $r = 2t$. *Solution:* Similar triangles EPR and ABH .

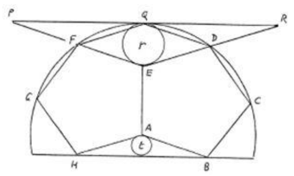


Fig. 5: Two pentagons [4]

2.3. Balls Problem Restored by H. Fukagawa

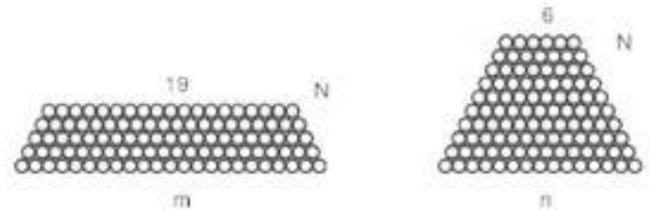


Fig. 6: Sangaku with Balls [1]

There are N balls. First stack them with 19 balls on the top and m balls on the bottom, as in the figure left. Then stack them with 6 balls on the top and n on the bottom, as in the figure right. Find N, m, and n.

Solutions: 105, 23, 15; 12075, 156, 155.

2.4. Sangaku Proposed by Okuda Tsume

In circle of $AB = 2R$, draw 2 arcs of radius R with centers A and B respectively and ten inscribed circles, two of diameter R (light), 4 dark of radius t, and 4 lighter of radius t'. Show that $t = t' = \frac{R}{6}$. For solution apply Pythagorean Theorem and the law of cosine.
This is a rare problem proposed by woman.

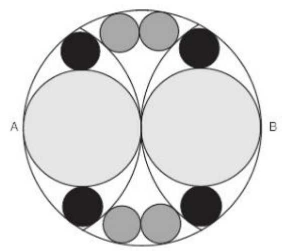


Fig. 7: Circles and Arcs [1]

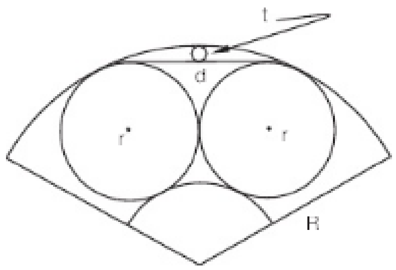


Fig. 8: Sangaku in Sector of Annulus [1]

2.5. *Sangaku* Problems upon a Sector of Annulus

Here is example of *sangaku* that could be drawn upon folding fan such as were popular in later Edo period. This is also an example of how in Japanese problems the parameters were sometimes strange, but the aim was to have nice result or to make calculation easier.

In a sector of radius R two circles of radius r are tangent to each other and touch the sector internally. At the same time a small circle of radius t touches both the sector and a chord of length d. If $d = 3.62438$ and $2t = 0.34\,2$, find $2r$.
Often they offer wrong answer on tablets. Here, the correct answer is $2r = 3.025$, but the answer on the tablet states that $2r = 3.0076$.

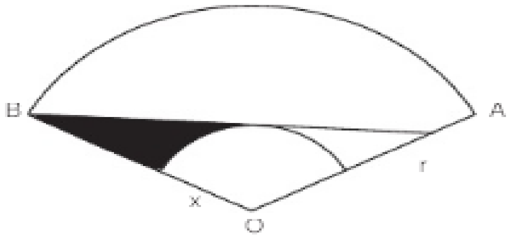


Fig. 9: Fan Sangaku [1]

2.6. Problem Maxime

In a sector AOB of radius r draw a small circle of radius x with center O. Draw the tangent to the small circle from the vertex B as shown. As x is varied, the area $S(x)$ of the black part of the figure will also vary. Shaw that $S(x)$ is a maximum when $x = \frac{293}{744} r$.

2.7. *Sangaku* from Yamaguchi Kazan's Diary

Hung by Seki Terutoshi in Zenkoji temple in 1804 this sangaku is known as Japanese theorem. Yamaguchi Kanzan did not record the answer in his *Travel Diary* because it was too complicated.

In a regular *n*-sided polygon, with length side *a*, we draw the diagonals from the vertex *A*, so we get $(n-2)$ triangles. Find the sum of radii of the inscribed circles in terms of *a* and prove that it does not depend on the way we triangulate the polygon.

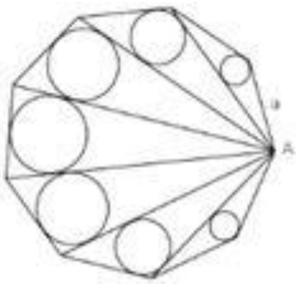
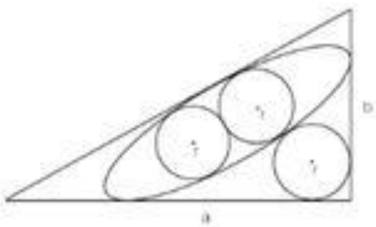


Fig. 10: Japanese Theorem [1]

2.8. Still Unsolved Problem with Ellipse by Sawa Masayoshi



In the right triangle an ellipse is inscribed with its major axes parallel to the hypotenuse. Two circles of radius *r* are inscribed within ellipse and the third circle with the same radius *r* touches the ellipse and two shorter sides of the triangle, *a* and *b*. Find *r* in terms of *a* and *b*.

Fig. 11: Unsolved Problem [1]

3. Solids in Sangaku Tablets

3.1. A Special Case Tangent Spheres

This problem is from an 1822 tablet in Kanagawa Prefecture, which predates by more than a century a theorem of F. Soddy, the British chemist [1], [3], [5].

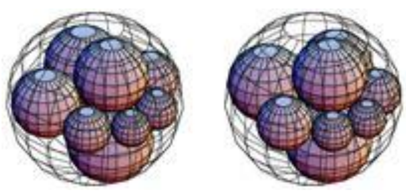


Fig. 12: Soddy's Hexlet [3]

Two spheres touch each other and also touch the inside of the largest sphere. A chain of smaller, different-size spheres circles the "neck" between the spheres. Each small sphere touches its neighbours and given three spheres. Find how many spheres must be there? How are the radii of the spheres related?

$$\frac{1}{r_1}+\frac{1}{r_4}=\frac{1}{r_2}+\frac{1}{r_5}=\frac{1}{r_3}+\frac{1}{r_1}$$

Easy way to solve this problem is by the method of inversion. Traditional Japanese mathematicians did not know about inversion. This problem is connected with the envelope of Soddy's hexlets is a Dupin cyclide, an inversion of the torus.

3.2. Sangaku Dated from 1798

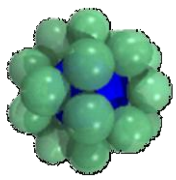


Fig. 13: Small Sphere around a Big Sphere [3]

Let a large sphere be surrounded by 30 small, identical spheres, each of which touches its four small-sphere neighbors as well as the large sphere. How is the radius of the large sphere related to that of the small spheres?

3.3. Sphere and Cylinder

If a cylinder intersects a sphere so that the outside of the cylinder is tangent to the inside of the sphere, what is the surface area of the part of the cylinder contained inside the sphere? [5], [7].

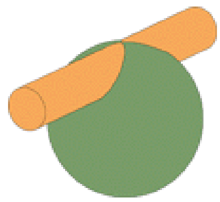


Fig. 14: Sphere Tangents Cylinder [3], [7]

3.4. Sangaku from Atsuta Tablet

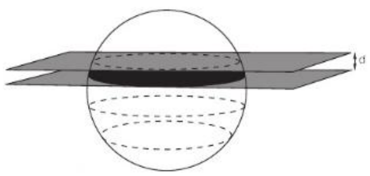


Fig. 15: Two Parallel Planes Cut Sphere [1]

Two parallel planes separated by a distance d cut a sphere of radius r. Find the surface area A (dark) of the cut-out section in terms of d and r. *Solution:* $A = 2\pi rd$.

3.5. Euler and Sangaku from Daikokutendo Temple, Tokyo

On a sphere of radius r draw three circular arcs AB, AC and CA. Let the straight line segments $AB = c$, $BC = a$ and $CA = b$. Find the area S of the spherical triangle ABC in terms of a, b, c and r. The original solution involved definite integral. We can find the same problem in L. Euler's work "*Varie Speculations Super Area Triangulorum Sphaerocorum*"[1].

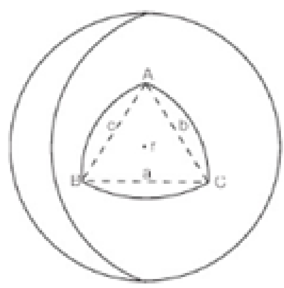


Fig. 16: Sphere and Three Circular Arcs [1]

3.6. Kyoto's Gion Shrine Problem

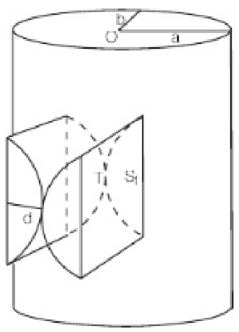


Fig. 17: Elliptic Cylinder [1]

Find the area *A* on an elliptic cylinder that intersects two sectors of a right circular cylinder, if the diameter of the cylinder is *D* and the depth of the sector is *d*. This famous problem was solved by Ajima Nanobu at the age of 42, in 1774. The original solution lead to equation of 1024 degrees in terms of the length of the chord, and Ajima reduced it to a problem on the tenth degree by the same method Laplace used in 1722 to expand determinants.

3.7. Two Cylinders cut the Sphere

(1829, Tokyo) Two identical and parallel right circular cylinders pass through the center of the sphere such that the line of contact between the cylinders passes through center of the sphere. If the radius of the sphere is *2r*, find the volume *V* cut out of the sphere. Also find the area of the surface cut out by the cylinders.

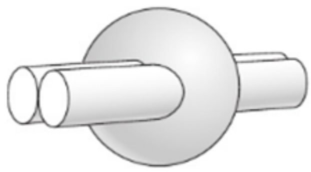


Fig. 18: Cylinders Cut Sphere [1]

$$V = \frac{16}{9}(2r)^3,\ A = 8(2r)^2$$

4. Summary

Today *sangaku* problems draws attention of students, mathematicians, artists and other people just like before 250 years ago. Galleries and scientific articles on this topic can be found on the net. Why? For mathematicians these are complex examples, including more challenging in positive way. We have to combine what we know; it has to look nice, and must have an elegant solution. They are indeed worthy of our attention. These problems "unique among the world's cultural creations" [1], are good educational examples and should not be forgotten in practice.

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