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THE GOLDEN RATIO

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ABSTRACT: Today the golden ratio fascinates and intrigues not only mathematicians, but also artists, architects, biologists, philosophers and musicians. The interesting thing is to reveal what mathematical concepts of the golden ratio were known throughout the history. This paper gives a historical review and presents many different examples of how the golden ratio is used in architecture. Furthermore, the paper introduces some interesting constructions of the golden section in various ways, found and proved in the works of various mathematicians Borsia, Hofstetter, Huntley, Lemoine and Odom. We present construction of a regular dodecahedron by H. Serras.

Keywords: Golden Ratio, Golden Triangle, Regular pentagon, Fibonacci, Regular dodecahedron

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1. INTRODUCTION

Mario Livio:"*The Golden Ratio: The Story of Phi, the World's most astonishing Number*".
The golden section is a line segment divided according to the golden ratio: the smaller part is to the larger part as the larger part is to the whole.

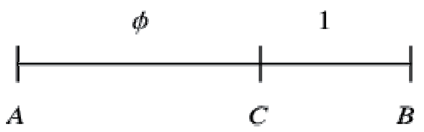

Figure 1: The golden section [13]

$$\phi = \frac{AC}{CB} = \frac{AB}{AC} \tag{1}$$

From this definition the value of the golden ratio can easily be calculated:

$$\frac{\phi + 1}{\phi} = \frac{\phi}{1} \Leftrightarrow \phi^2 - \phi - 1 = 0 \tag{2}$$

From (1) follows

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.618 \tag{3}$$

From the quadratic equation (1) the formula $\phi = 1 + \frac{1}{\phi}$ can also be derived, and then it can be expanded recursively to obtain a continued fraction for the golden ratio:

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{\cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}} \tag{4}$$

By (3) we can see that the golden number ϕ is an irrational number. There are many interesting mathematical formulas and properties of the golden number, but in searching for the golden ratio in geometry and architecture the connection to the Fibonacci numbers is important. The Fibonacci sequence is
0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144…
It can be shown that the powers of the golden ratio satisfy the formula $\phi^n = F_n\phi + F_{n-1}$, where F_n is a Fibonacci number.

2. THE GOLDEN RATIO THROUGH HISTORY

2400 years at least is a long period of the fascination with the golden ratio not only by mathematicians but also by biologists, philosophers, artists, architects, musicians and even mystics.
The golden ratio was first studied by the ancient Greek because of its frequent appearance in geometry.

The development of the idea of the golden ratio is usually attributed to Pythagoras (580-497 BC) and his students. The symbol of Pythagoras school was the regular pentagram.
Plato (428-347 BC) saw the world in terms of perfect geometric proportions and symmetry. His ideas were based on Platonic Solids: a cube for earth, a tetrahedron for fire, an octahedron for air and icosahedrons for water, some of them are related to the golden ratio.
Euclid (c. 325-c. 265 BC) in his *Elements* gives the first known definition of "extreme and mean ratio", i.e. the golden ratio, as we call it today. In several propositions in *Elements* and their proofs the golden ratio is used.
After the ancient times there was a long period of silence.
About 1500s Luca Pacioli (1445-1517) published his book "De divina proportione" (which is his name for the golden ratio), it contains drawings made by Leonardo da Vinci of the five Platonic solids and other imaginations of artist, architects, and scientists of the golden ratio.
Cardan and Bombelli (16. century) looked for the golden ratio using quadratic equations.
The first known approximation of the golden ratio as a decimal was given in a letter written in 1597 by M. Maestlin to his former student Kepler (1571-1630). But Kepler was the first who explicitly banded together Fibonacci sequence to the golden ratio in 1609.
Scottish mathematician R. Simson in 1753 first proved the next important result $\phi = \lim_{n\to\infty} \frac{F_n}{F_{n-1}}$.
At the beginning of the past century R. Penrose (1931-) discovered a symmetrical pattern that uses the golden ratio in the field of periodic tilings which led to new discoveries about quasicrystals.

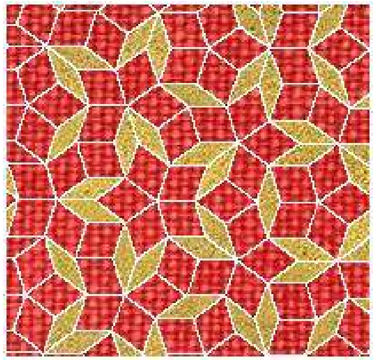

Figure 2: Penrose tiling [7]

2.1 Terms and labels for the golden ratio

As stated before, Euclid used the term "extreme and mean ratio". Euclid's term for the golden ratio was used until about the 16th century. Then Pacioli introduced the term "divine proportion" and some writers adopted it. "Proportionally divided", "continuous proportion" and similar expression were also used. In 1835 Martin Ohm in his book introduced a new term "golden section", the name which is now used. Modern terms are also "the golden ratio" and "the golden number".
The golden ratio is often represented by the Greek letter τ ("tau"), which means "the cut" or "the section" in Greek.
But, Mark Barr (early 18th century) represented the golden ratio as ϕ ("phi"), because the ϕ ("phi") is the first letter in the name of Greek architect and sculptor Phidias who's work often symbolized the golden ratio.

3. GOLDEN SECTION IN ARCHITECTURE

3.1 Ancient Egypt

Today there are many theories about the Great Pyramid of Giza (c. 2570 BC). Many of them claim that Egyptian designers used the golden proportion in the construction of the Great Pyramid. It can be measured that the ratio of the slant height of the pyramid to half the length of the base as 612.01 (feet)/377.9 (feet) = 1.6195 is the golden ratio. But, this is not proof that the Egyptians intentionally used the golden ratio. We don't know the real measures of the pyramid; it was partially ruined by taking off the paneling. And also there is no written evidence about the design and the building of the pyramid.

3.2 Ancient Greece

Interesting data is that the golden ratio was used in some Greek theatres. The Epidaurus Theatre which was designed by Polykleitos the Younger in the 4th century BC used the golden ratio. The auditorium was divided into two parts; one had 34 rows and the other 21 rows (Fibonacci numbers). The angle between the theatre and the stage divides a circumference of the basis of an amphitheatre in the ratio: 137°.5: 222°.5 = 0.618 (the golden proportion).

Figure 3: The Epidaurus Theater [8]

Similarly, the Theatre of Dionis in Athens has three circles, first with 13 rows, second with 21, and last with 34, again Fibonacci numbers.

The famous Greek temple, the Parthenon (c.430 BC), in the Acropolis in Athens includes golden rectangles in many proportions.

Figure 4: Parthenon [7]

But, it is necessary to underline the fact that there are no original plans for the Parthenon, and there is no documentary evidence that this was deliberately designed. The temple is damaged and all the measures are only approximate.

3.3 Gothic

One of the most popular monuments from this period is Notre Dame Cathedral in Paris (built 1163-1346). F. Macody Lung in his book Ad Quadratum (1919) claims that this church, Cathedral of Chartres (early 12th century), the Notre-Dame of Laon (12th-13th century) were designed according to the golden ratio.

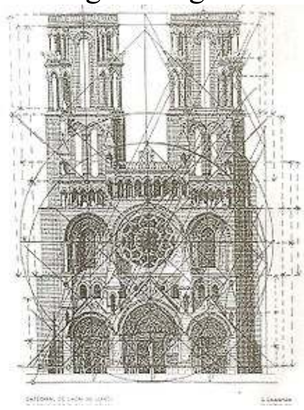

Figure 5: Notre-Dame of Laon [6]

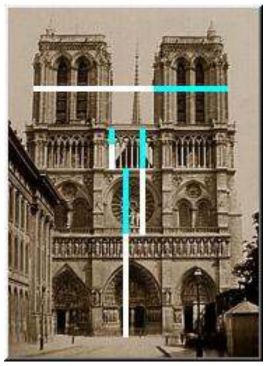

Figure 6: Notre-Dame of Paris [7]

3.4 Modern architecture

The most popular modern architect connected to the golden section is the Swiss architect Le Corbusier (1887-1965). Le Corbusier explicitly used the golden ratio in his *Modulor* system for the scale of architecture proportion. He wanted to improve the function of architecture using the proportions of the human body. The best example of this idea is Villa Stein (1927) in Garches. The villa's ground plan and inner structure including elevation are approximate golden rectangles.

Another Swiss architect, Mario Botta (1943), is famous for using the golden section in his work. For example in the house *Origlio* the golden ratio is the proportion between the central section and the side sections of the house.
The CN Tower in Toronto (1974) is interesting. The height of once the world's tallest tower is 553.33 m, and the height of the glass floor is 342 m. The proportion of these two values is 1.617924- the golden ratio.

Figure 7: CN Tower [7]

Furthermore, just a few years old Education Centre called The Core in SW England has been designed using the Fibonacci numbers as an example of plant spirals which symbolize nature and plants from all over the world.

Figure 8: The Core [12]

Another very new example is Engineering Plaza in California. Its designer J. Gordon Smith (former student of this University) was also guided by Fibonacci series spiral.

Figure 9: Cal Poly Engineering Plaza [12]

3.5 Few examples in Croatia

So far we have found just two examples of the exploitation of the golden ratio. The first is the oldest Croatian artifact (c.1100) Bascanska ploca.

Figure 10: Bascanska ploca [15]

Furthermore, the windows at the front of a secondary school (19^{th} century) are golden rectangles.

Figure 11: School "Gornjogradska gimnazija" [10]

4. CONSTRUCTION OF THE GOLDEN RATIO

4.1 The Golden Ratio by Huntley

In a right triangle ABC with sides BC =3, $AC = 4$, and AB = 5, the point O is the foot of the angle bisector at B. If we draw a circle with the center O and the radius CO and extend BO to meet the circle at P and Q, then the golden ratio appears as $PQ : BP = \phi$.

Proof: First notice an angle bisector BO divides AC in the ratio of the sides $AB: BC$ as we see $\frac{AO}{CO} = \frac{AB}{AC} = \frac{5}{3}$.

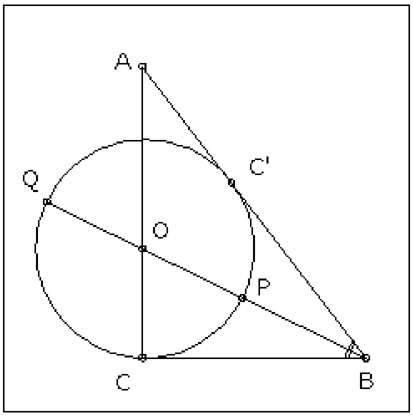

Figure 12: The Golden Ratio in right triangle

These results with $AO = \frac{5}{2}$ and the circle's radius $CO = \frac{3}{2} = r$.

By the Power of a Point Theorem $BP \cdot BQ = BC^2$ and after short research we find $BO = \frac{3\sqrt{5}}{2}$ and $BP = BO - r = \frac{3(\sqrt{5}-1)}{2}$.

Finally, $\frac{PQ}{BP} = \frac{2r}{BP} = \phi$.

4.2 The Golden Ratio by Gabries Borsia

Gabriel Borsia has discovered another interesting way to construct the golden ratio. He associated the right triangle $1:2:\sqrt{5}$ with the right triangle 3–4–5 as on Figure 12.

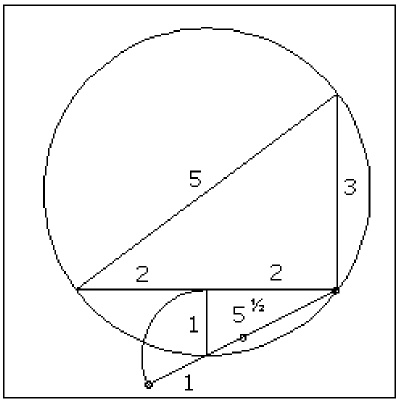

Figure 13: GR in right triangle 1:2:$\sqrt{5}$

4.3 The Golden Ratio by George Odom

Let ABC be an equilateral triangle with L and M the midpoints of its sides AB and AC. Let X and Y be the intersections of LM extended with the circumcircle of the triangle ABC. Then $LM : MY = \phi$.

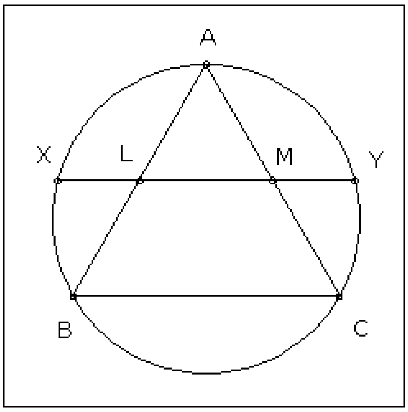

Figure 14: GR in equilateral triangle

Proof: If we take 2a as the side length of ABC, then follows $AM = MC = LM = a$. Let us assume $XL = MY = b$. By Intersecting Chord's theorem $MX \cdot MY = AM \cdot MC$ and $(a+b) \cdot b = a \cdot a$. If we denote $\frac{a}{b} = x$ it appears equation $1 + x = x^2$, which results in $x = \phi$.

4.4 Hofstetter's Construction with Compass

An elegant construction of the golden ratio can be found in the work of K. Hofstetter in Forum Geomericorum, Vol 2 (2002),

p.65-66.[9]
Let us mark *S (P)* the circle with the centre *S* through point *P*. Let us take *A* and *B* to be the two points. Circles *A(B)* and *B(A)* intersect in *C* and *D* and cross the line *AB* in points *E* and *F*. Circle *A(F)* and *B(E)* intersect in *X* and *Y* as in the diagram. Points *X*, *D*, *C* and *Y* are collinear (due symmetry). Then $\phi = CX : CD$.

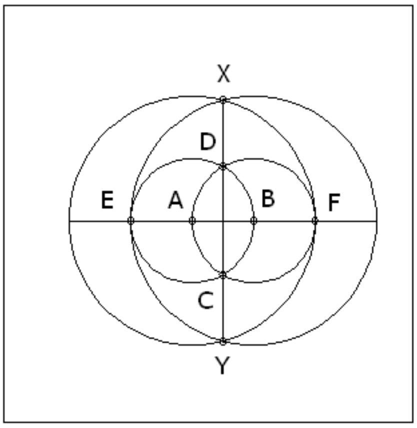

Figure 15: GR from four circles

Proof: Let us assume $AB = 2$. Than $CD = 2\sqrt{3},\ \ CX = \sqrt{15} + \sqrt{3}$ and we obtain $\dfrac{CX}{CD} = \dfrac{\sqrt{15}+\sqrt{3}}{2\sqrt{3}} = \phi$ which means *D* is gold point of the segment *CX*. Note the points *E* and *F* lie on the circle *C (D)*. That is why the construction can be accomplished with compass only.

4.5 Lemoine's Construction

This construction and proof was given by Lemoine in 1902 and rediscovered by Hofstetter as 5-step construction [9].
Two circles $A(B)$ and $B(A)$ and C, D their points of intersection. The third circle $C(A)$ intersects $A(B)$ in *E* and the line *CD* in *F*.
The fourth circle *E* (*F*) intersects line *AB* in points *G* and *G*'.
So, $\dfrac{AB}{AG} = \phi$ and $\dfrac{AG}{AB} = \phi$.

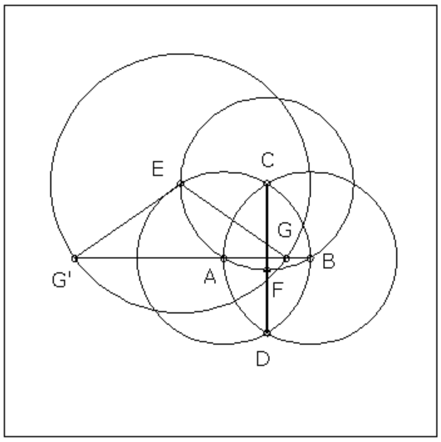

Figure 16: Lemoine's construction

Proof: Let *AB* be unit length. Then $CD = \sqrt{3}$, and $EG = EF = \sqrt{2}$. Let *H* be the orthogonal projection of *E* on the line *AB*. Since $HA = \dfrac{1}{2}$ and $HG^2 = EG^2 - EH^2 = \dfrac{5}{4}$, it results in $AG = HG - HA = \dfrac{\sqrt{5}-1}{2}$. *G* divides *AB* in the golden section.

4.6 Hofstetter's 5-step construction

Let us draw circles from points *A* and *B*, segment *AB* being unit length, let *C* and *D* be the intersections of $A(B)$ and $B(A)$. Extend *AB* beyond *A* to the intersection *E* with $A(B)$.

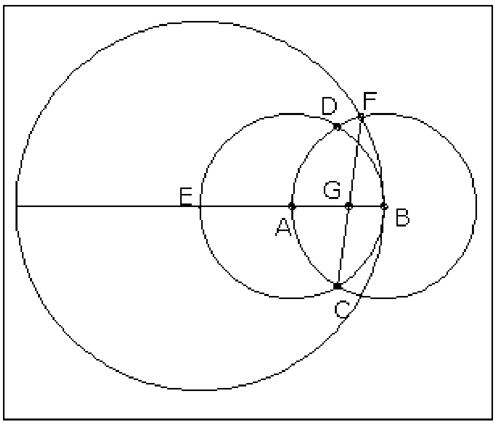

Figure 17: 5-step division of a segment in the Golden Section

Draw $E(B)$ and let *F* be the intersection of $E(B)$ and $B(A)$ further from *D*. *CF* intersects *AB* in *G*. $AG : BG = \phi$. [2]

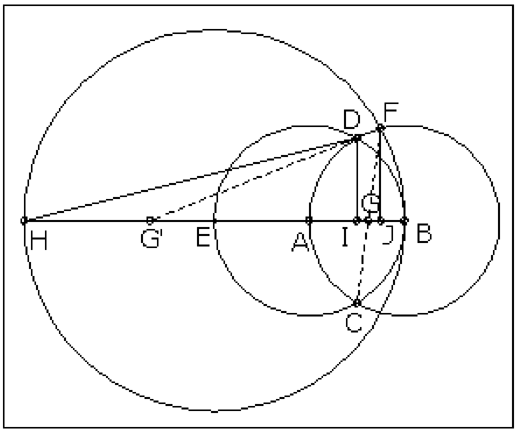

Figure 18: Proof

Proof: Extend AB to intersect the circle $E(B)$ at H. Let the point I be intersection of CD and AB and J the foot of perpendicular from F to AB. We can see $BH = 4$ and $BF = 1$ in the right triangle BFH. Since $BH \cdot BJ = BF^2$, $BJ = \frac{1}{4}$. Further, $IJ = \frac{1}{4}$, $JF = \frac{1}{4}\sqrt{15}$. Then,

$$\frac{IG}{GJ} = \frac{IC}{JF} = \frac{\frac{\sqrt{3}}{2}}{\frac{\sqrt{15}}{4}} = \frac{2}{\sqrt{15}}$$

$$IG = \frac{2IJ}{2+\sqrt{5}} = \frac{\sqrt{5}-2}{2},$$

$$AG = \frac{1}{2} + IG = \frac{\sqrt{5}-1}{2}.$$

The point G divides AB in the golden section.

4.7 A Simple Hofstetter's construction with a rusty compass

Construction with a rusty compass means opening can be set only once.

First draw $A(B)$ and $B(A)$ and find C and D their intersection. AB intersects CD at the point M. Than construct $C(M, AB)$, a circle with centre M and radius AB. Let it intersects $B(A)$ in F and another point, F being the farthest from D. Define G as the intersection of AB and DG. G is the sought point.

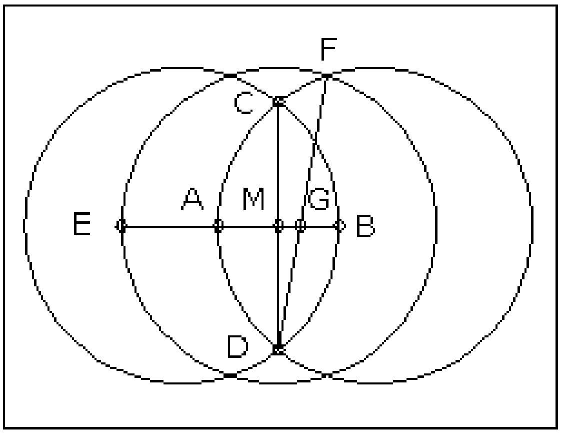

Figure 19: GR with rusty compass

Proof: $BF = FM$, the projection of F to AB the point K is the midpoint of segment AB, then $KM = BK$. Right triangles GMD and GKF are similar. Let us assume $AB = 4$, then $FM = BF = 4$, $AM = BM = 2$, $KM = BK = 1$. Solving right triangles results we conclude $AG : BG = \phi$.

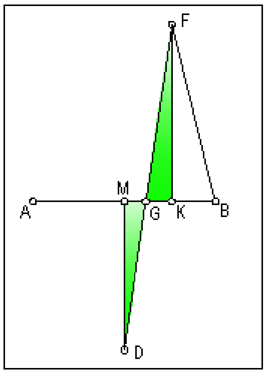

Figure 20: Proof

5. FROM ISOSCELES TRIANGLE VIA PENTAGON TO DODECAHEDRON WITH THE GOLDEN RATIO

5.1 Golden triangle

The Golden Ratio also appears in some special isosceles triangles.

An isosceles triangle ABC with a top angle measuring 36° and both base angles measure 72° we find in regular decagon, in a regular pentagon, in a pentagram as in a regular dodechaedron.

If the baseline of this triangle is unit length its sides will be ϕ.

This triangle has an interseting property that bisecting the angle in *B* by drawing *BD*, we obtain an isosceles triangle *BCD* simillar to *ABC*, and we can proof : $\frac{AD}{DC} = \phi$. That is why this tall triangle is usually named the “golden triangle”. The remaining obtuse isosceles triangle 36˚- 36˚-108˚ is named the “golden gnomon”.

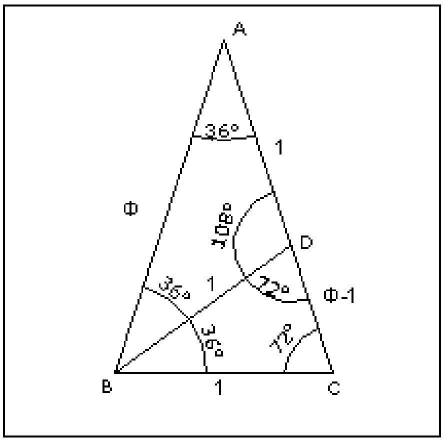

Figure 21: Golden Triangle

We can also name triangle *ABC* with angles 72˚-36˚-72˚ “sharp” triangle, and triangle *BCD* with angles 36˚-108˚-36˚“flat” triangle.
These two types of isosceles triangles are the basic building shapes of Penrose tilings, an interesting way of applying the golden section.
We can keep on drawing: e.x. bisector *D* in the small triangle. Then the golden ratio appears in every next small triangle. If we draw arcs from the vertices of the obtuseangles of the isosceles triangles from a point to point and connect them we draw Golden Spiral.[1]

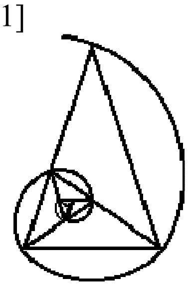

Figure 22: Golden Spiral

5.2 The Regular Pentagon

We can find the golden triangle in a regular pentagon. The base of triangular *DE* here is the side of regular pentagon, and side *BD* is the diagonal. The ratio of the diagonal and the side in the regular pentagon is $d : s = \phi$.

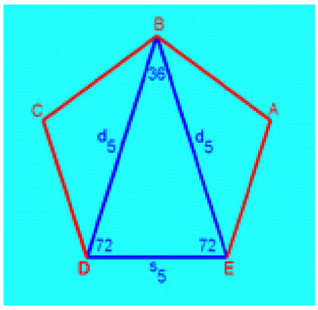

Figure 23: Golden triangle in a regular pentagon [4]

Two diagonals of the regular pentagon divide each other in the golden ratio.

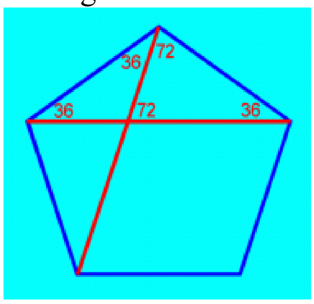

Figure 24: GR in a regular pentagon [4]

When we draw two intersecting diagonals, we get two golden triangles (sharp) and a gnomon (flat).
Let us remark that by drawing all five diagonals of a regular pentagon we define a pentagram example of a regular non – convex polygon where again the golden ratio appears.

5.3 Diagonals of a regular pentagon and Fibonacci

Even the Pythagoreans studied a series of regular pentagons. That is the way they discovered incommensurability and the golden ratio.[4]
Let a pentagon side be s_n and we take it to be diagonal d_{n-1} of the previous smaller regular pentagon, than $s_n = d_{n-1}$. In that case diagonal d_n is the sum of the side and the diagonal of the previous pentagon.

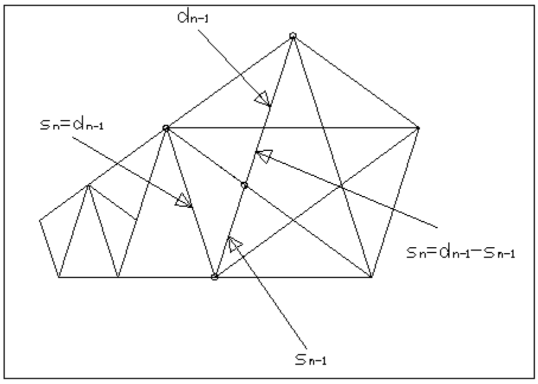

Figure 25: A serie of regular pentagons

This can be written write as reccurrence relationships $s_n = d_{n-1}$ and $d_n = d_{n+1} + s_{n-1}$. With $s_1 = 2$ and $d_1 = 3$ leads to the sequence $\frac{d_n}{s_n}$ of ratios $\frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \ldots$ which are successive Fibonacci numbers. A formal proof can be found in The Golden Ratio: The story of Phi the World's most Astonishing Number.[1]

5.4 Construction of a regular dodecahedron around a cube

This method is mentioned by Euclid in book XIII proposition 17 and is given in [4].

We can construct a regular dodecahedron by putting appropriate "roofs" on each face of the cube.

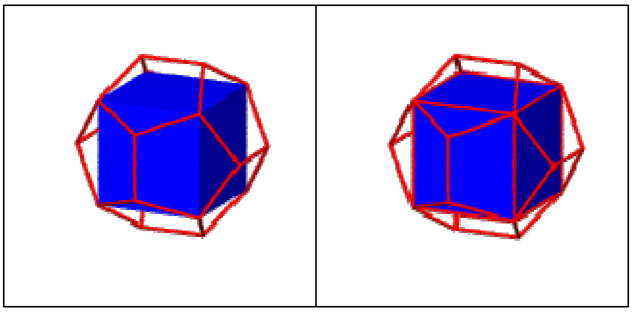

Figure 26: From a cube to the regular dodecahedron [4]

The bases of such roofs are the faces of the cube. All edges of the roofs that are not edges of the cube must have the same length. The adjacent faces of meeting roofs must form a regular plan pentagon.

The length of the diagonals of the pentagons equals the length of the sides of the cube. The length of the edges of the dodechaedron is greater of the two parts in which the edge of the cube is divided by the golden ratio.

The angles between the base and a triangular face and between the base and a trapezoidal face of the roof are complementary. It means the pentagons are planar. That we can

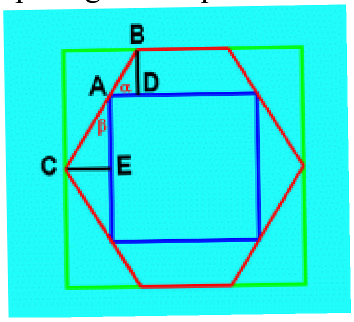

Figure 27: Frontal view [4]

see from a frontal view.

As $BD = \frac{\varphi}{2}$ and $AD = \frac{1-\varphi}{2}$, $1-\varphi = \varphi^2$, $\tan\alpha = \frac{\varphi}{1-\varphi} = \frac{1}{\varphi}$.

As $AC = \frac{\varphi}{2}$ and $AE = \frac{1}{2}$, $\tan\beta = \varphi = \frac{1}{\tan\alpha}$

$\alpha + \beta = 90^\circ$. We conclude that the pentagons are planar. If the length of the edge of the starting cube is 1, then the length of the edge of the dodecahedron is $\varphi = \frac{\sqrt{5}-1}{2}$. The length of the edge of the cube desribed (the green one) around the dodechaerdon is $\phi = \frac{\sqrt{5}+1}{2}$. The same sphere circumscribes the dodechaedron and the original cube, $r = \frac{\sqrt{3}}{2}$.

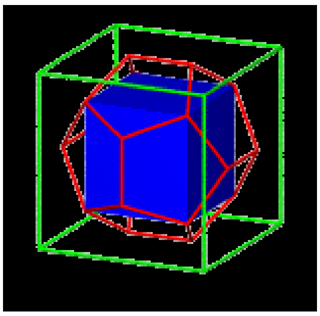

Figure 28: Dodechaedron and two cubes [4]

5.5 A cube to dodecahedron transformation

Bob Faulkner gave Herman Serras an interesting idea how to construct a regular dodechaedron starting with a cube.[4]

Define six roofs upon a given cube. Complete the roofs by their square basis. Choose one of the roofs as fixed, four adjacent roofs were hinged along their edges common with fixed roof.

The sixth roof was hinged to one of its already hinged neighbours.

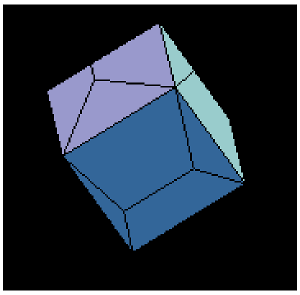

Figure 29: A Cube in Dodecahedron

It could be unfold into the regular dodechaedron and refold into the cube. This served as the basis for Serras' nice animation.[4]

It is possible to see a cube in a regular dodecahedron if we use one diagonal on each face.

The diagonals of a regular dodechaedron are ϕ times as long as its sides then the cube's sides so the dodechaedron's sides are in the golden ratio. In fact, five distinct cubes can be fitted into the dodecahedron with the vertices of the cube meeting at the vertices of the dodechaedron.

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