Detecting multifractality under heavy tails

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Introduction

$\{X(t)\}$ is said to be multifractal if it has stationary increments and there exist functions $c(q)$ and $\tau(q)$ such that

$$E|X(t)|^q = c(q)t^{\tau(q)}, \quad \text{for all } t \in [0,T], q \in [q_-, q_+],$$

for some $T > 0$ and $q_-, q_+ \in \mathbb{R}$. $\tau(q)$ is called the scaling function. τ is always concave and if $\{X(t)\}$ is H-s.s., then $\tau(q) = Hq$. Legendre transform of τ is called the multifractal spectrum

$$d(h) = \inf_q (hq - \tau(q) + 1),$$

Statistical methods for detecting multifractal behavior

Partition function (or empirical structure function) - dividing the interval $[0, T]$ into N blocks of length Δt

$$S_q(T, \Delta t) = \frac{1}{N} \sum_{i=1}^{N} |X(i\Delta t) - X((i-1)\Delta t)|^q .$$

If $\{X(t)\}$ is multifractal, then from the definition

$$\ln ES_q(T, \Delta t) = \tau(q) \ln \Delta t + \ln c(q). \quad (1)$$

Suppose $X_1, \dots, X_T$ is a sample observed at discrete equally spaced time instants from a stochastic process $\{X(t), t \geq 0\}$. It is enough to consider only sampling at time instants $1, 2, \dots n$.

Empirical scaling function - for each value $q > 0$ estimate $\tau(q)$ as the slope in the simple linear regression of $\ln S_q(T, \Delta t)$ on $\ln \Delta t$

$$\hat{\tau}_{N,T}(q) = \frac{\sum_{i=1}^{N} \ln \Delta t_i \ln S_q(n, \Delta t_i) - \frac{1}{N} \sum_{i=1}^{N} \ln \Delta t_i \sum_{j=1}^{N} \ln S_q(n, \Delta t_j)}{\sum_{i=1}^{N} (\ln \Delta t_i)^2 - \frac{1}{N} \left(\sum_{i=1}^{N} \ln \Delta t_i \right)^2},$$

where $1 \leq \Delta t_i \leq T$ for $i = 1, \dots, N$. If $\hat{\tau}$ is nonlinear, then one can suspect the existence of multifractal scaling.

Estimated spectrum - estimated as Legendre transform of $\hat{\tau}$.

Assumptions

We say process $\{X(t)\}$ is of type $\mathfrak{L}$ if it satisfies

- $Y_t = X(t) - X(t-1), t \in \mathbb{N}$ is a strictly stationary sequence having a heavy-tailed marginal distribution with index α, i.e.

$$P(|Y_1| > x) = \frac{L(x)}{x^\alpha},$$

  where $L(t), t > 0$ is a slowly varying function $L(tx)/L(x) \to 1$ as $x \to \infty$, for every $t > 0$.
- (Y_t) satisfies strong mixing property with an exponentially decaying rate

$$a(\tau) = \sup_{t \geq 0} \sup_{A \in \mathcal{F}_t, B \in \mathcal{F}^{t+\tau}} |P(A \cap B) - P(A)P(B)| = O(e^{-b\tau}), \text{ as } \tau \to \infty,$$

  where $\mathcal{F}_t = \sigma\{Y_s, s \leq t\}$, $\mathcal{F}^{t+\tau} = \sigma\{Y_s, s \geq t + \tau\}$
- $EY_t = 0$ when $\alpha > 1$

- Examples: all Lévy processes with $X(1)$ heavy-tailed (e.g. α-stable Lévy processes, Student Lévy process), cumulative sum of stationary processes like Ornstein-Uhlenbeck (OU) type processes or diffusions with heavy-tailed marginal distributions

We are interested in the rate of growth of the partition function $\frac{\ln S_q(T, T^s)}{\ln T}$

Theorem (Asymptotic behaviour of the partition function)

If $\{X(t)\}$ is of type $\mathfrak{L}$, then for $q > 0$ and every $s \in (0, 1)$

$$\frac{\ln S_q(T, T^s)}{\ln T} \xrightarrow{P} R_\alpha(q, s) := \begin{cases} \frac{sq}{\alpha}, & \text{if } q \leq \alpha \text{ and } \alpha \leq 2, \\ s + \frac{q}{\alpha} - 1, & \text{if } q > \alpha \text{ and } \alpha \leq 2, \\ \frac{sq}{2}, & \text{if } q \leq \alpha \text{ and } \alpha > 2, \\ \max\left\{s + \frac{q}{\alpha} - 1, \frac{sq}{2}\right\}, & \text{if } q > \alpha \text{ and } \alpha > 2, \end{cases}$$

as $T \to \infty$, where $\xrightarrow{P}$ stands for convergence in probability.

Relation (1) holds approximately.

Theorem (Asymptotic behaviour of the scaling function)

Suppose Δt_i is of the form $T^{\frac{i}{N}}$ for $i = 1, \dots, N$. Then, for every $q > 0$,

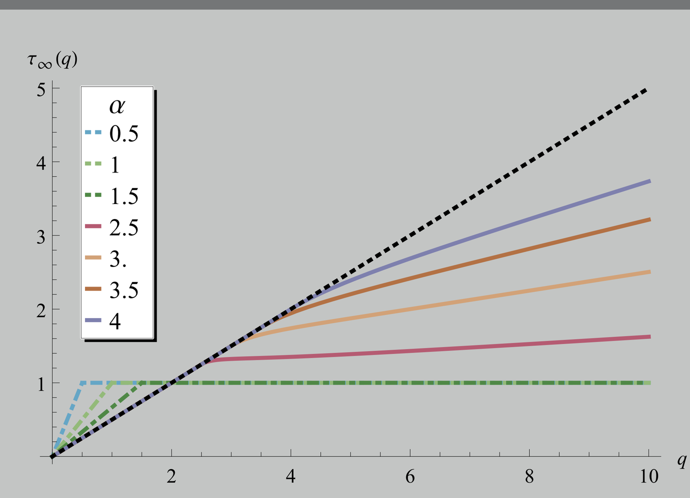
$$\lim_{N \to \infty} \operatorname{plim}_{T \to \infty} \hat{\tau}_{N,T}(q) = \tau_\infty(q),$$

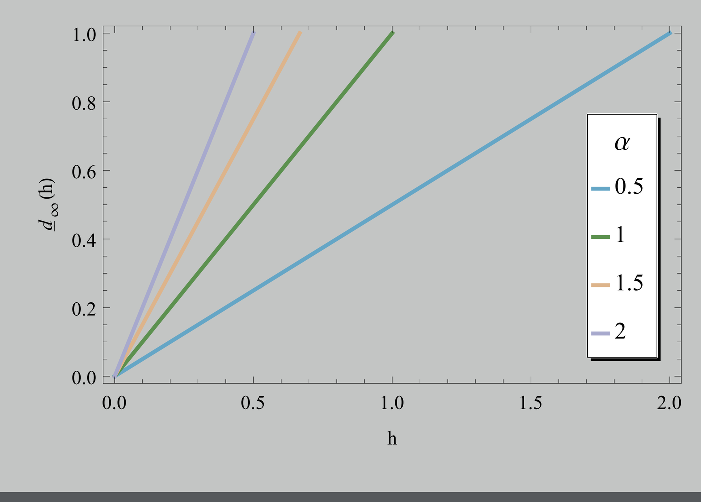
where plim stands for limit in probability and

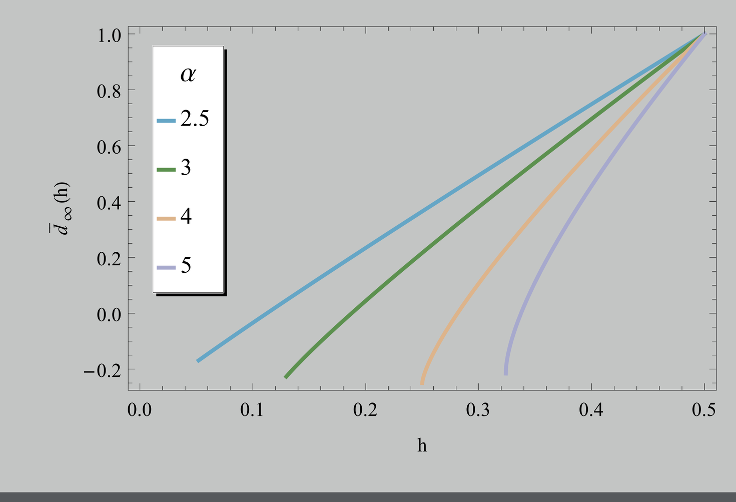
$$\tau_\infty(q) = \begin{cases} \frac{q}{\alpha}, & \text{if } 0 < q \leq \alpha \ \& \ \alpha \leq 2, \\ 1, & \text{if } q > \alpha \ \& \ \alpha \leq 2, \\ \frac{q}{2}, & \text{if } 0 < q \leq \alpha \ \& \ \alpha > 2, \\ \frac{q}{2} + \frac{2(\alpha - q)^2 (2\alpha + 4q - 3\alpha q)}{\alpha^3 (2-q)^2}, & \text{if } q > \alpha \ \& \ \alpha > 2. \end{cases}$$

Under heavy-tails τ is estimated as non-linear function.

Asymptotic scaling function and spectrum







Example 1

Reproducing analysis from Fisher, Calvet, Mandelbrot [1997] and Calvet and Fisher [2002] of DM/USD exchange rate data with Student Lévy process

Figure: Scaling function of the data - Figure 6. from Calvet and Fisher [2002]

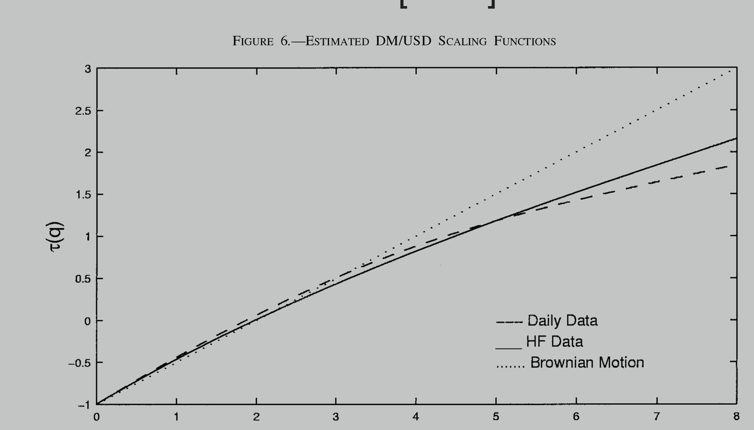


Figure: Estimated scaling function of generated Student Lévy process

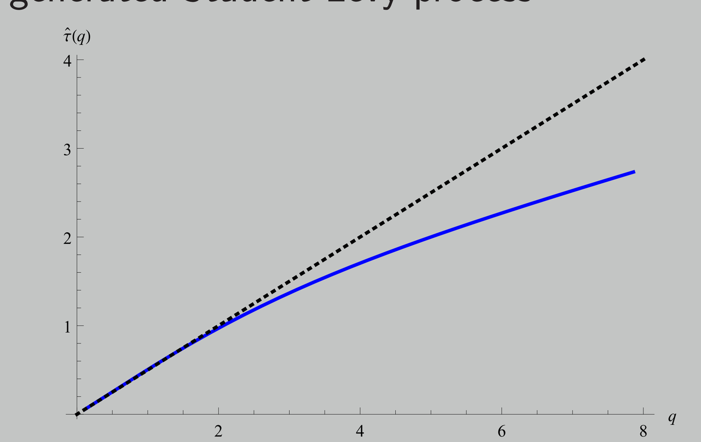


Figure: Estimated spectrum of the data - Figure 7. from Calvet and Fisher [2002]

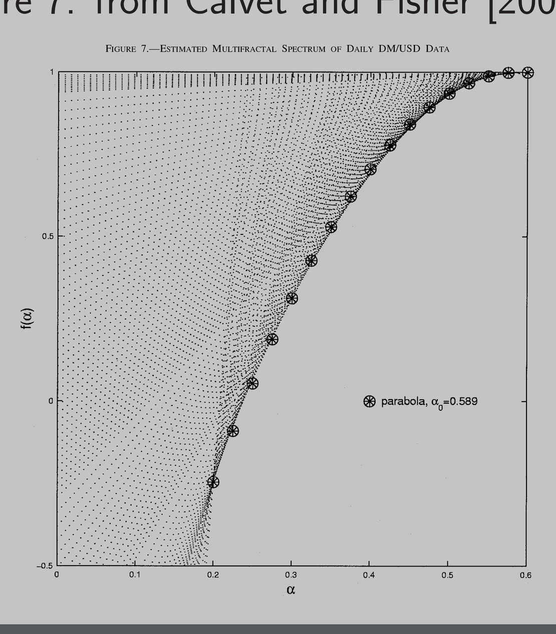
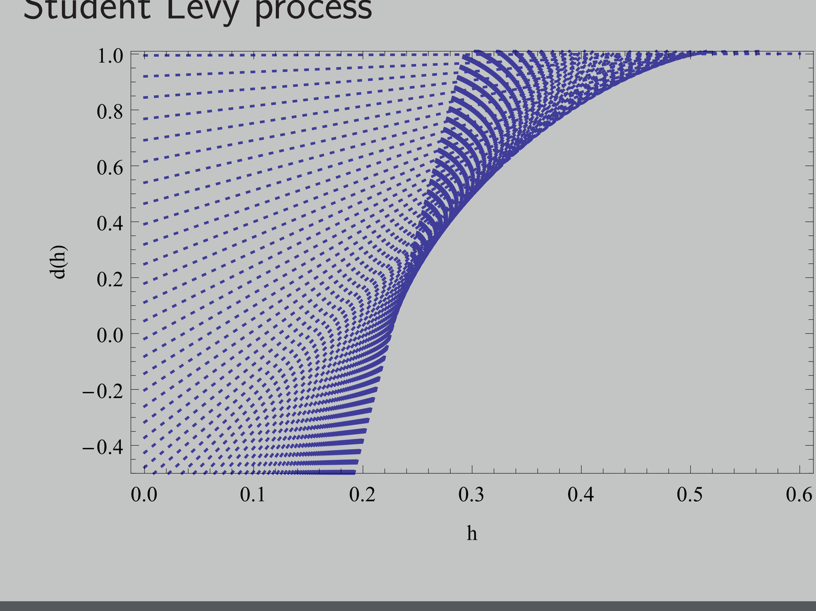


Figure: Estimated spectrum of generated Student Lévy process



Example 2

Comparison: 5307 daily closing values of S&P500 stock market index collected in the period from January 1, 1980 until December 31, 2000 vs. same length sample path of Student Lévy process with $X(1) \stackrel{d}{=} T(2.5, 0.0072, 0)$

Figure: Estimated scaling function of S&P 500 index with τ_∞ for $\alpha = 2.5$

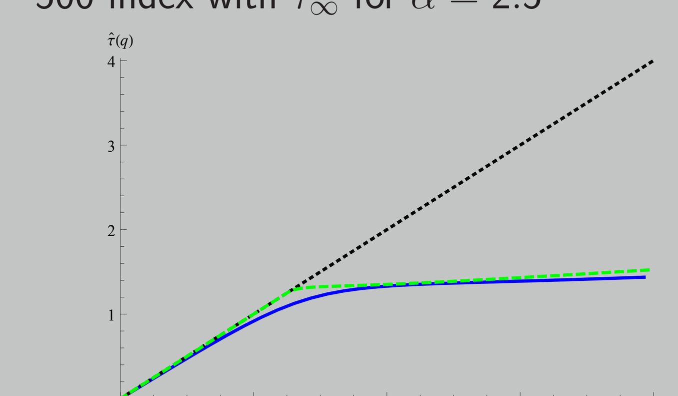


Figure: Estimated scaling function of generated Student Lévy process

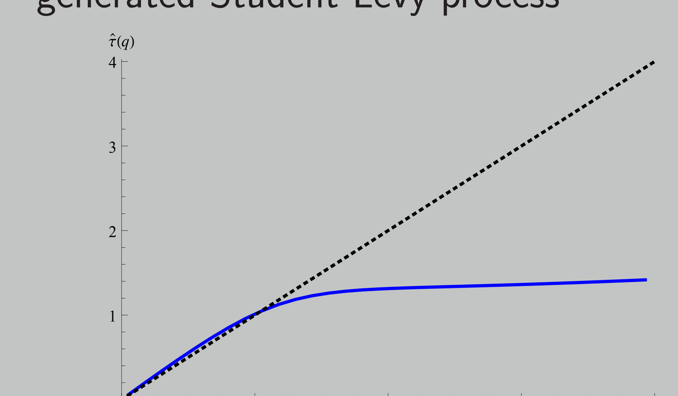


Figure: Estimated spectrum of S&P 500 index

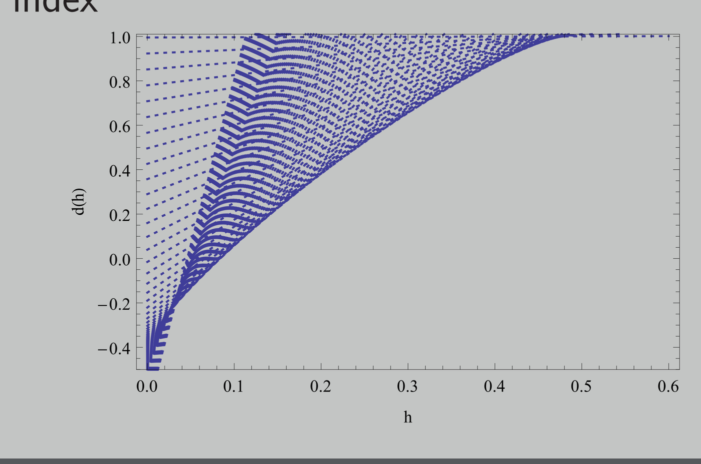
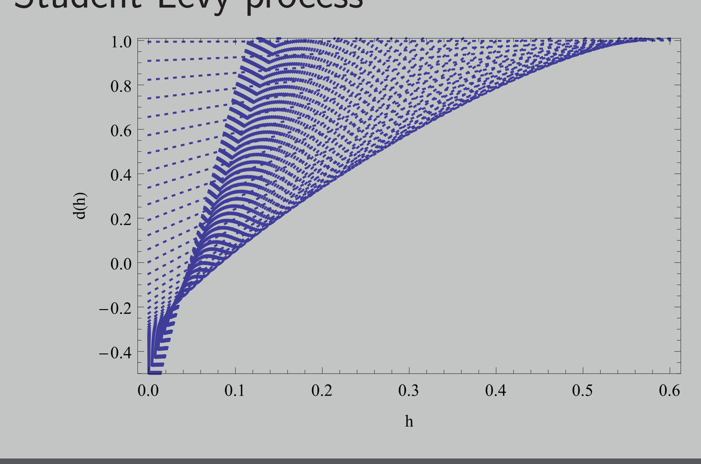


Figure: Estimated spectrum of generated Student Lévy process



Acknowledgments and references

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