ON THE SPECIAL SURFACES THROUGH THE ABSOLUTE CONIC WITH A SINGULAR POINT OF THE HIGHEST ORDER

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ABSTRACT: In this paper we observe a special class of surfaces in the Euclidean space E^3 which touch the plane at infinity through the absolute conic and have a singular point of the highest order. We study their properties and visualize them with the program *Mathematica*.

Keywords: surfaces, absolute conic, tangent cone, singular point

1. INTRODUCTION

In this paper we will study one special class of surfaces in the Euclidean space E^3 so let us start by recalling some definitions and facts about the surfaces. In the real three-dimensional projective space $P^3(\mathbb{R})$, in homogeneous Cartesian coordinates $(x:y:z:w)$, $(x,y,z \in \mathbb{R}, w \in \{0,1\}$, $(x:y:z:w) \neq (0:0:0:0))$, the equation

$$F_n(x,y,z,w) = 0,$$

where F_n is a homogeneous algebraic polynomial of degree n, defines an n^{th} order surface S_n. This equation can be written as

$$f_n(x,y,z) + wf_{n-1}(x,y,z) + \dots \\ \dots + w^{n-1}f_1(x,y,z) + w^n f_0(x,y,z) = 0,$$

where f_i, $i = 0,1,\dots,n$, are homogeneous algebraic polynomials of degree i.

Any straight line, not lying on S_n, intersects S_n in n points and any plane intersects S_n in the n^{th} order plane curve.

A point T of the surface S_n for which at least one partial derivation of F_n is not equal to zero is called the regular point of S_n. All tangents to the surface at that point lie in one plane - the tangent plane of S_n at T.

A point T of the surface S_n for which all partial derivations of F_n are equal to zero is called the singular point of S_n. The tangents to S_n at this point form an algebraic cone with vertex in T. If the tangent cone is of order k, the point T is the k-fold point of the surface S_n. Every plane through T intersects S_n in the n^{th} order plane curve with the k-fold point in T.

If the origin $O(0:0:0:1)$ is the k-fold point of S_n, then S_n has the equation

$$f_n(x,y,z)+wf_{n-1}(x,y,z)+\dots+w^{n-k}f_k(x,y,z)=0, \tag{1}$$

and the tangent cone at O is given by

$$f_k(x,y,z) = 0. \tag{2}$$

In the paper [2] the author studied the quartics Φ_4 in E^3 which have a triple point and touch the plane at infinity through the absolute conic. The surfaces were classified according to the type of the tangent cone $\mathscr{T}$ at the triple point. The following cases were observed: $\mathscr{T}$ is a proper 3-order cone, $\mathscr{T}$ splits into a proper 2-order cone and a real plane, $\mathscr{T}$ splits into three planes. In this paper we give the generalization of the third class of surfaces Φ_4.

2. SURFACES $\mathscr{S}_{2n}$ WITH A $(2n-1)$-FOLD POINT TOUCHING THE PLANE AT INFINITY THROUGH THE ABSOLUTE CONIC

In the real projective space $P^3(\mathbb{R})$ the Euclidean metric defines the Euclidean space E^3 with the absolute conic given by the equations: $x^2 + y^2 + z^2 = 0, w = 0$.

Theorem 1 *A surface $\mathscr{S}_{2n}$ given by the equation*

$$A_2(x,y,z)^n + wH_{2n-1}(x,y,z) = 0, \qquad (3)$$

where $A_2(x,y,z) = x^2+y^2+z^2$ and $H_{2n-1}(x,y,z)$ is a product of $2n-1$ linear homogeneous polynomials, is a surface of order $2n$ which has a $(2n-1)$-fold point in the origin and intersects the plane at infinity only in the absolute conic.

Proof. By comparing (3) with (1) it is evidently that the origin $O(0:0:0:1)$ is the $(2n-1)$-fold point of $\mathscr{S}_{2n}$ at which the tangent cone splits into $2n-1$ planes. The intersection of the plane at infinity ($w=0$) and the surface $\mathscr{S}_{2n}$ is given by $A_2(x,y,z)^n = 0$. It is the absolute conic with the intersection multiplicity n. □

Theorem 2 *There are only $2(2n-1)$ straight lines through the origin lying entirely on the surface $\mathscr{S}_{2n}$. They are the intersections of cones given by $A_2(x,y,z)^n = 0$ and $H_{2n-1}(x,y,z) = 0$. They are imaginary in pairs.*

Proof. Let a line p through $O(0:0:0:1)$ be spanned by O and a further point $P(a:b:c:1) \neq O$. The line p is parametrized by

$$p \quad \ldots \quad (x:y:z:1) = (at:bt:ct:1), t \in \mathbb{R}.$$

It lies on $\mathscr{S}_{2n}$ if and only if

$$A_2(at,bt,ct)^n + H_{2n-1}(at,bt,ct) = 0,$$

for every $t \in \mathbb{R}$. This is precisely when

$$t^{2n-1}[tA_2(a,b,c)^n + H_{2n-1}(a,b,c)] = 0,$$

for every $t \in \mathbb{R}$. It follows that $A_2(a,b,c)^n = 0$, $H_{2n-1}(a,b,c) = 0$. Therefore, $A_2(at,bt,ct)^n = 0$, $H_{2n-1}(at,bt,ct) = 0$, for every $t \in \mathbb{R}$. Evidently the line p lies on the cones given by equations $A_2(x,y,z)^n = 0$ and $H_{2n-1}(x,y,z) = 0$. We conclude: the only lines through the origin that lie on $\mathscr{S}_{2n}$ are the isotropic lines in the tangent planes at the origin. □

Theorem 3 *The surface $\mathscr{S}_{2n}$ has only one real singular point.*

Proof. Let us suppose that there is a k-fold point $T \neq O$ of $\mathscr{S}_{2n}$, $k \geq 2$. The real line OT intersects $\mathscr{S}_{2n}$ in the point O with the intersection multiplicity $2n-1$ and the point T with the intersection multiplicity k, or entirely lies on $\mathscr{S}_{2n}$. The first option is not possible because the order of the $\mathscr{S}_{2n}$ is $2n < 2n-1+k$, while the second option is in contradiction with Theorem 2. □

Theorem 4 *The surface $\mathscr{S}_{2n}$ touches the plane at infinity through the absolute conic.*

Proof. In Theorem 1 it stated that the absolute conic is the intersection of $\mathscr{S}_{2n}$ and the plane at infinity with the intersection multiplicity n. It is left to show that the absolute conic is not the singular line of $\mathscr{S}_{2n}$. If the absolute conic was the singular line of $\mathscr{S}_{2n}$, its every point would be the singular point of $\mathscr{S}_{2n}$ and every isotropic line through the origin O would lie on the surface $\mathscr{S}_{2n}$. This is not possible since there are only $2(2n-1)$ straight lines through the origin lying entirely on $\mathscr{S}_{2n}$. □

According to Theorem 3 there is no real double point on the surface $\mathscr{S}_{2n}$ and therefore there is no selfintersections of $\mathscr{S}_{2n}$. Hence, the surface consists of the separated parts sharing only $(2n-1)$-fold point O. These parts we will call *petals*.

Theorem 5 *The maximum number of petals of the surface $\mathscr{S}_{2n}$ in the $(2n-1)$-fold point equals $2n^2-3n+2$.*

Proof. Each petal of $\mathscr{S}_{2n}$ lies on one side of one tangent plane at the $(2n-1)$-fold point $O(0:0:0:1)$. Therefore, the number of petals is twice less then the number of parts into which space is divided by 2n-1 planes passing through one point. Let us first show that the maximum number of parts of space divided by k copunctal planes equals k^2-k+2. The number of the parts will be the largest when no three planes contain a common line. We will assume that this condition is fulfilled. The plane is divided by k concurrent lines into $2k$ parts. Let the number of parts of the space divided by k plane be denoted by $\overline{k}$. The additional $(k+1)^{st}$ plane intersects the first k planes into k lines which divide the plane into $2k$ regions. Therefore, $\overline{k+1} = \overline{k} + 2k$. We have

$$\begin{aligned}
\overline{1} &= 2 \\
\overline{2} &= \overline{1} + 2 \\
&\vdots \\
\overline{k} &= \overline{k-1} + 2(k-1). \qquad (4)
\end{aligned}$$

By summation of these equations we obtain the following: $\overline{k} = 2+2+4+...+2(k-1) = 2+2\cdot\frac{(k-1)k}{2} = k^2-k+2$. By substituting k with $2n-1$, we get $\overline{2n-1} = 4n^2-6n+4$ and obtain the claimed result. □

Remark. If no three tangent planes at the origin share a common line, the number of petals equals $2n^2-3n+2$. If three planes intersect at a line, the number of petals decreases. Let us prove that if l planes pass through the same line, the number of petals is decreased by $\frac{(l-1)(l-2)}{2}$. Let us first take into consideration $k-l$ planes such that no three planes share a common line. Than we add two of l planes with a common line, and at the end we add remaining $k-l-2$ planes. The list of equations (4) now becomes

$$\begin{aligned}
\overline{1} &= 2 \\
\overline{2} &= \overline{1} + 2 \\
&\vdots \\
\overline{k-l} &= \overline{k-l-1} + 2(k-l-1) \\
\overline{k-l+1} &= \overline{k-l} + 2(k-l) \\
\overline{k-l+2} &= \overline{k-l+1} + 2(k-l+1) \\
\overline{\overline{k-l+3}} &= \overline{k-l+2} + 2(k-l+1) \\
&= \overline{k-l+2} + 2(k-l+2) - 2 \\
\overline{\overline{k-l+4}} &= \overline{\overline{k-l+3}} + 2(k-l+1) \\
&= \overline{\overline{k-l+3}} + 2(k-l+3) - 4 \\
&\vdots \\
\overline{\overline{k-1}} &= \overline{\overline{k-2}} + 2(k-l+1) \\
&= \overline{\overline{k-2}} + 2(k-2) - 2(l-3) \\
\overline{\overline{k}} &= \overline{\overline{k-1}} + 2(k-l+1) \\
&= \overline{\overline{k-1}} + 2(k-1) - 2(l-2).
\end{aligned}$$

Therefore, $\overline{\overline{k}} = 2+2\cdot\frac{(k-1)k}{2} - 2\cdot\frac{(l-2)(l-1)}{2} = k^2-k+2-(l-1)(l-2)$. Since the number of the petals of $\mathscr{S}_{2n}$ is two times less then the number of the parts into which space is divided by the tangent planes at the origin, we can conclude the following: if l of $2n-1$ tangent planes intersect in one line, the number of the petals is decreased by $\frac{(l-1)(l-2)}{2}$.

2.1 Parametric equations of surfaces $\mathscr{S}_{2n}$

By substituting $\omega = 1$ into the equation (3) we get the following equation of the surface $\mathscr{S}_{2n}$:

$$A_2(x,y,z)^n + H_{2n-1}(x,y,z) = 0. \qquad (5)$$

If we use the spherical coordinates (ρ, ϕ, θ):

$$x = \rho\cos\phi\sin\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\theta,$$

the equation (5) takes the following form:

$$\rho^{2n-1}\left(\rho + H_{2n-1}(\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)\right) = 0.$$

For every point of of the surface $\mathscr{S}_{2n}$, except for the $(2n-1)$-fold point $O(0,0,0)$, it holds

$$\rho = -H_{2n-1}\left(\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta\right),$$

and therefore the surface $\mathscr{S}_{2n}$ is given by the parametric equations:

$$\begin{aligned}
x(\phi,\theta) &= -H_{2n-1}(\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)\cos\phi\sin\theta, \\
y(\phi,\theta) &= -H_{2n-1}(\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)\sin\phi\sin\theta, \\
z(\phi,\theta) &= -H_{2n-1}(\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)\cos\theta, \\
&\phi, \theta \in [0,\pi]\times[0,\pi]. \qquad (6)
\end{aligned}$$

$H_{2n-1}(x,y,z)$ is the product of $2n-1$ linear polynomials in x, y, z and therefore determined by $3(2n-1)$ coefficients.

2.2 Visualization of surfaces $\mathscr{S}_{2n}$

Based on the equations (5) or (6), we can visualize any surface $\mathscr{S}_{2n}$ with the program *Mathematica*.

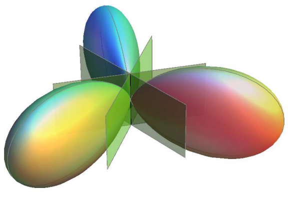

Figure 1: Three surfaces $\mathscr{S}_{2n}$, where $n = 2,3,4$, are shown in this figure. The tangent cone at the origin splits into $2n-1$ planes that intersect through the axis z. Each tangent cone has $4n-1$ planes of symmetry, and corresponding surface $\mathscr{S}_{2n}$ has $2n-1$ petals and $2n$ planes of symmetry.

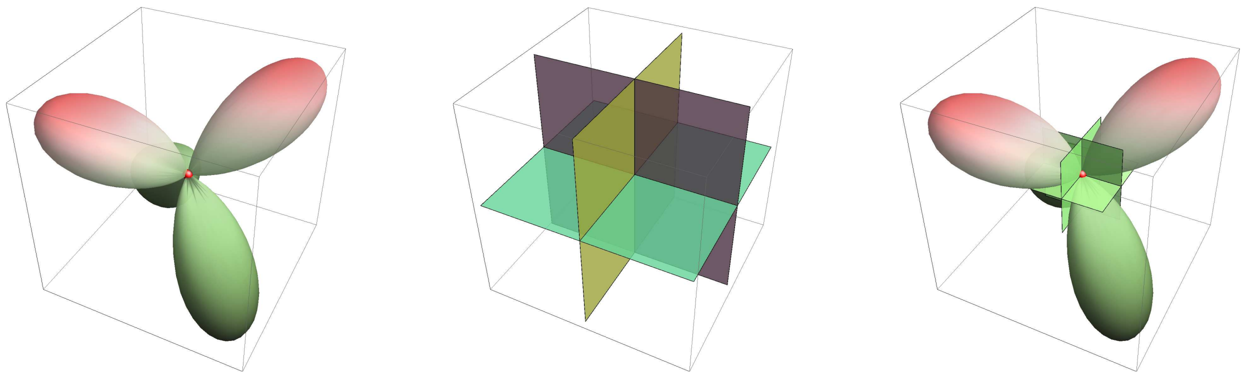

Figure 2: This figure shows one surface $\mathscr{S}_4$ and its tangent cone at the origin that splits into three coordinate planes. The surface is given by the following implicit equation: $A_2(x,y,z)^2 + xyz = 0$. Since three tangent planes at the origin have only one common point, the surface has 4 petals.

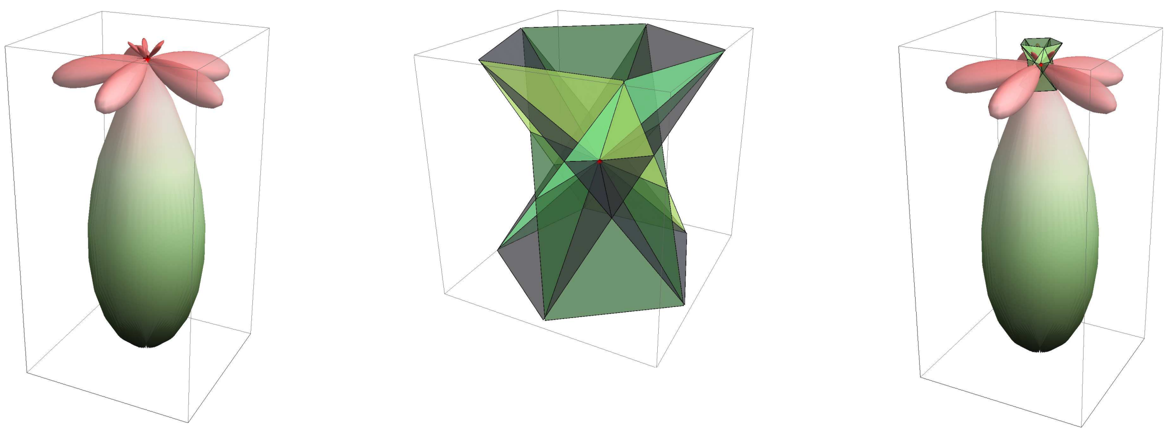

Figure 3: This figure shows one surface $\mathscr{S}_6$ and its tangent cone at the origin that splits into 5 planes which have only one common points. The surface has the largest number of petals, i.e. 11 petals.

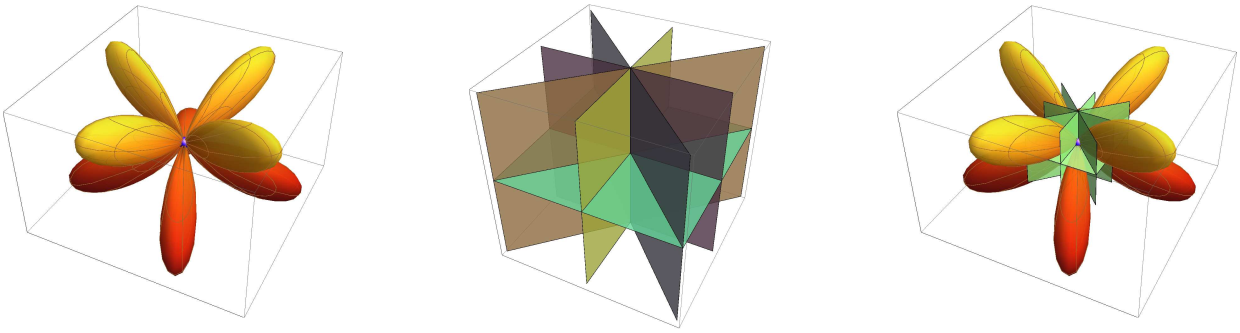

Figure 4: This figure shows one surface $\mathscr{S}_8$ and its tangent cone at the origin that splits into 5 planes, where 4 planes share the common axis z. Therefore, the largest number of petals (11) decreases to 8.

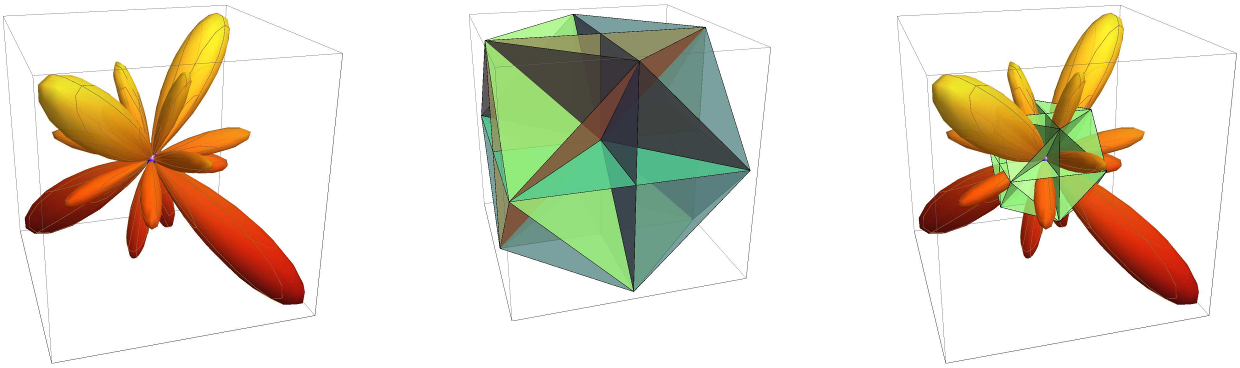

Figure 5: This figure shows one surface $\mathscr{S}_8$ and its tangent cone at the origin that splits into 7 planes and there exist 6 lines which are the intersections of 3 tangent planes. Therefore, the largest number of petals (22) decreases to 16.

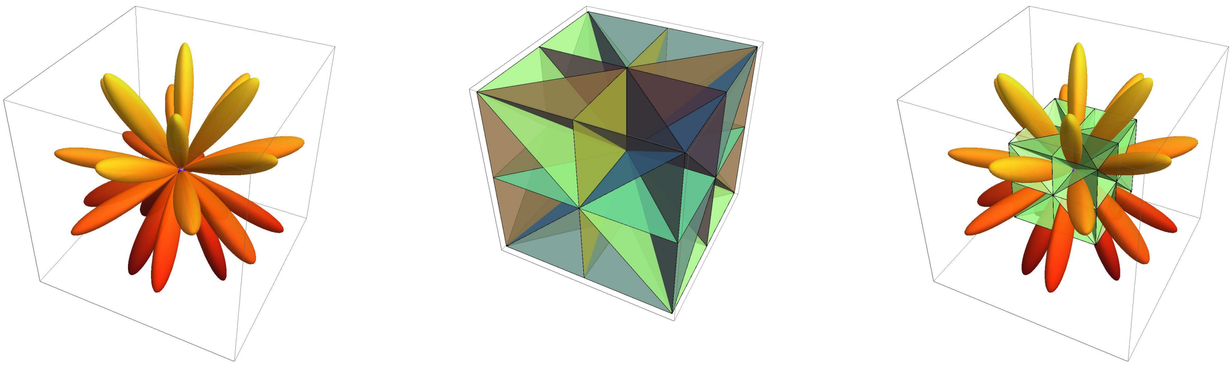

Figure 6: This figure shows one surface $\mathscr{S}_{10}$ and its tangent cone at the origin that splits into 9 planes. There exist 3 lines that are the intersections of 4 tangent planes, and 4 lines which are the intersections of 3 tangent planes. Therefore, the largest number of petals (37) decreases to 24.

3. CONCLUSIONS

In this paper we observe a special class of surfaces $\mathscr{S}_{2n}$ in the Euclidean space, given by the equation of the form

$$A_2(x,y,z)^n + wH_{2n-1}(x,y,z) = 0,$$

where $A_2(x,y,z) = x^2+y^2+z^2$ and $H_{2n-1}(x,y,z)$ is a product of $2n-1$ linear homogeneous polynomials. We show that $\mathscr{S}_{2n}$ is a surface of order $2n$ which has a $(2n-1)$-fold point in the origin and touches the plane at infinity through the absolute conic. The surfaces consists of the separated parts (petals) sharing only one real $(2n-1)$-fold point. We prove that the largest number of petals of $\mathscr{S}_{2n}$ equals $2n^2-3n+2$ and show how this number decreases if some tangent planes at the origin pass through same straight line.

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