THE ANALOGUE OF THEOREMS RELATED TO WALLACE-SIMSON'S LINE IN QUASI-HYPERBOLIC PLANE

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ABSTRACT: The quasi-hyperbolic plane is one of nine projective-metric planes where the absolute figure is the ordered triple $j_1,\ j_2,\ F$. It is dual to the pseudo-Euclidean plane. It is known for the fact that a pencil of parabolas, in the Euclidean and pseudo-Euclidean plane, can be set according to lines $a,\ b,\ c$. The focus points of all parabolas in the pencil lie on the circle circumscribed to the triangle given by lines $a,\ b,\ c$. The connection between the pencil of parabolas, Wallace-Simson lines and Steiner deltoid curve are studied and proved in [2]. Analogues theorems are valid in the pseudo-Euclidean plane. In this paper the dual theorems will be proved in quasi-hyperbolic plane.

Keywords: Quasi-hyperbolic plane, pencil of parabolas, Wallace-Simson line, point A

1. INTRODUCTION

It is known that there exist nine geometries in plane with projective metric on a line and on a pencil of lines which are denoted as Cayley-Klein projective metrics. Hence, these plane geometries differ according to the type of the measure of distance between points and measure of angles which can be parabolic, hyperbolic, or elliptic. The plane geometry with hyperbolic measure of distance and parabolic measure of angle is denote as the quasi- hyperbolic plane (further in text qh-plane). Furthermore, each of the Cayley-Klein projective metrics can be embedded in the real projective plane $\mathscr{P}_2(\mathbb{R})$ where then the metric is induced by an absolute figure which is given as non-degenerated or degenerated conic [5], [6], [7]. The absolute figure in the qh-plane is a real point F and a pair of real lines j_1 and j_2 incidental with F. In order to simplify the constructions the extended model of qh-plane where all points and lines of the qh-plane embedded in the real projective plane $\mathscr{P}_2(\mathbb{R})$ are observed. In [1] some basic geometric notions of the qh-plane are introduced, also the classification of qh-conics with respect to their position to the absolute figure is given and some basic constructions are presented. In [2] we have studied and proved the connection between the pencil of parabolas, Wallace-Simson lines and Steiner deltoid curve. It is not difficult to conclude that in the pseudo-Euclidean plane the analogous theorems are valid. In this paper we will not proved them. The main aim of the paper is to proof the dual of above mentioned theorems in the qh-plane, by using the notions defined in [1]. Following notions need to be highlighted :

- The lines incidental with the absolute point F are called **isotropic lines**.
- Two points A and B in qh-plane are called **perpendicular points** if they lie on a pair of isotropic lines a and b that are in harmonic relation with the absolute lines j_1 and j_2.
- A **qh-circle** is a qh-conic for which the tangents from the absolute point F are the absolute lines j_1 and j_2.
- A **qh-parabola** is a qh-conic passing through the absolute point F i.e. both isotropic tangent lines coincide. A **special**

parabola is a qh-parabola whose isotropic tangent is an absolute line.

- The **directrices** of a qh-conic are (non-absolute) lines incident with the isotropic points of the qh-conic, i.e. lines incidental with the intersection points of the qh-conic with the absolute lines j_1 and j_2. A qh-conic can have none, one, two or four directrices f_i, $i \in \{1, 2, 3, 4\}$. The directrices are the dual of the pseudo-Euclidean focuses.

2. THE ANALOGUE OF THEOREMS RELATED TO WALLACE-SIMSON'S LINE

Theorem 1. *Let M, N and P be three non-isotropic points, incidental with three different isotropic lines. If the pencil of qh-parabolas is set according to points M, N and P then the envelope of the directrices of the qh-parabolas is the qh-circle inscribed in the trilateral mnp.*

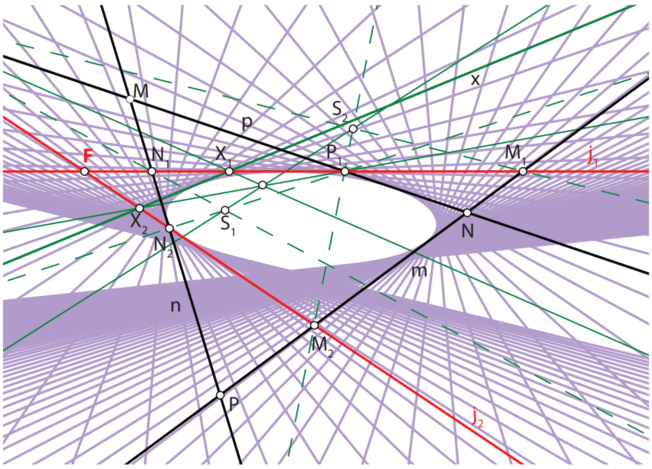

Figure 1:

Proof: Let the point X_1 be an arbitrary chosen point on a absolute line e.g. $X_1 \in j_1$. The qh-parabola from the given pencil is set according to points F, M, N, P and X_1. Let the intersection point of the qh- parabola and the absolute line j_2 be denoted as X_2. The line $x := X_1X_2$ is its directix (see Fig. 1). The ranges of isotropic points (j_1) and (j_2) are determineted by all the parabolas from the given pencil $(X_1 \in (j_1), X_2 \in (j_2))$. The ranges (j_1) and (j_2) are projectively linked, and the result of its correspondence will be the 2^{nd} class curve [8]. In the given pencil of qh-parabolas there are two special qh-parabolas in the cases when X_1 or X_2 coincides with F, respectively. Its directrix coincides with the absolute line that is different from its isotropic tangent line. Therefore, the absolute lines are the tangent lines of mentioned 2^{nd} class curve i.e. the envelope of the directrices of the qh-parabolas from the pencil is a qh-circle. The sides MN, NP, MP of the given trilateral are the tangent lines to a qh-circle and they coincide with the directrices of three degenerated qh-parabolas from the pencil. □

By using the results of the previous theorem we can prove the following dual of Wallace - Simsons theorem [2].

Theorem 2. *Let mnp be a trilateral with non-isotropic sides m, n, p and non isotropic vertices M, N, P and k its inscribed qh-circle. Let x be an arbitrary tangent line to k, and M_1, N_1 and P_1 its respective perpendicular points to the vertices M, N and P. Then the lines M_1M, N_1N and P_1P intersect at the point A.*

Proof: Without loss of generality, to simplify the construction, qh-circle is represented with the Euclidean circle. Notice that the pencil of qh-parabolas can be determined by vertices M, N and P. By previous theorem it follows that the qh-circle, inscribed in trilateral mnp, is the envelope of the directrices of the qh-parabolas from the pencil. In the given pencil there are three degenerated qh-parabolas, a pair of lines (MN, PF), (NP, MF) and (MP, NF). Its directrices coincide with the lines MN, NP and MP, respectively. The tangent line x is the directrix of the qh-parabola, denoted as p_1, that passes through the vertices M, N, P. The qh-parabola p_1 is incidental with the intersection points of the directrix x with the absolute lines j_1 and j_2, denoted as X_1 and X_2. The point F_1 is the focus of the qh-parabola (see Fig. 2). By using the definition of the pedal transformation in the qh-plane in [3] it is proved that the pedal qh-curve of a qh-parabola, with respect to an arbitrary polar line of the pedal transformation, is a 3^{rd} class curve. If the polar line of the pedal transformation coincide with the directrix x, the 3^{rd} class pedal curve degenerates into three points i.e. three pencils of line (X_1), (X_2) and the pencil at the vertex of a qh-parabola. In our case point A is the vertex of the qh-parabola determined by points M, N, P and directrix x. □

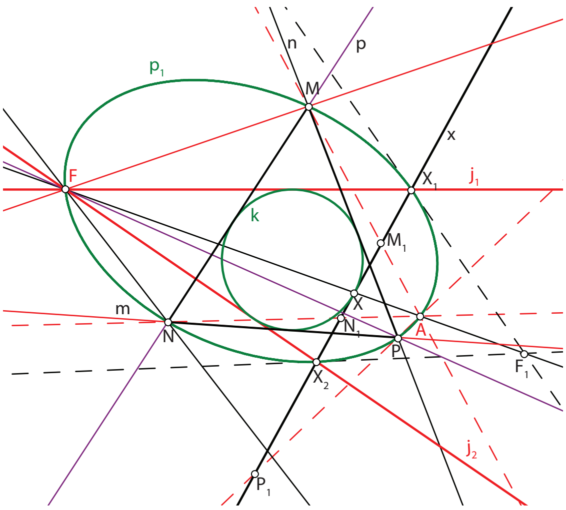

Figure 2:

Remark 1. *The curve in the qh-plane is an entirely circular if its isotropic tangent lines coincide with the absolute lines, and there are no other isotropic tangent lines.*

Theorem 3. *All the points A from theorem 2 lie on an entirely circular 4^{th} class cubic (see Fig. 3).*

Proof: Let mnp be a trilateral and k its inscribed qh-circle. According to construction of a point A, with respect to an arbitrary chosen tangent line x to k, the result of $(1-2)$ -correspondence between two pencils of lines (M) and (N) (or (M) and (P) or (N) and (P)) is a curve consisting of all points A. According to $(1-2)$ - correspondence each line from the first pencil corresponds to two lines from the second pencil, conversely [4]. Hence, let m_1 be an arbitrary line from the pencil (M), and the point M_1 a unique point on m_1 perpendicular to the point M (see Fig. 3). Let the tangent lines to k from the point M_1, be denoted as t_1 and t_2. Let the points $N_1 \in t_1$ and $N_2 \in t_2$ be perpendicular to the point N. The lines $n_1 := NN_1$ and $n_2 := NN_2$ are corresponded to the line $m_1 \in (M)$. Let the point of intersection of the lines n_1, n_2 with the line m_1 be denoted as A_1 and A_2, respectively. The points A_1 and A_2 are correspond to the tangent lines t_1 and t_2. The result of this correspondence between the pencils of lines (M) and (N) is a 4^{th} order curve. Since, the line MN is corresponding to itself, the 4^{th} order curve degenerates into a cubic and the line MN. Furthermore, if the line $m_1 \in (M)$ is incidental with the absolute point F, then the point M_1 coincides with F, the lines t_1 and t_2 coincide with the absolute lines j_1 and j_2 and the points N_1 and N_2 coincide with F. Therefore, the point F is a double point of cubic, so it is 4^{th} class cubic. Since, on the absolute lines j_1 and j_2 there are no other points A except an absolute point F, the absolute lines are the tangent lines to cubic at the point F. Hence, the cubic is entirely circular [3]. Notice that theorem 3 is the dual of the Steiners deltoid curve theorem [2] □

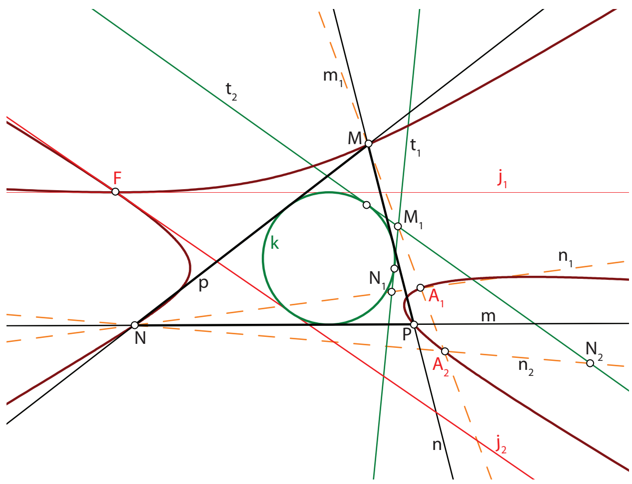

Figure 3:

Corollary 1. *The vertices of the qh-parabolas from the pencil lie on an entirely circular* 4^{th} *class cubic.*

3. CONCLUSIONS

The main aim of the paper is to proof the dual results of Wallace-Simpson line and Steiners deltoid curve theorem in the quasi-hyperbolic plane, by using the notions defined in [1].

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