RATIONAL TRANSFER FUNCTIONS WITH MINIMUM TIME-BANDWIDTH PRODUCTS

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ABSTRACT

Transfer functions with finite zeros giving the minimum product of the impulse response duration and the frequency bandwidth are considered. The impulse response spread is characterized by the higher order moments. For the frequency spread measure, the second order moment is used. Minimizing products of the moments, causal systems with the largest energy concentration in time for a given bandwidth are obtained. The resulting impulse response is quasi Gaussian with small and short ringing. The transfer functions' poles and zeros suitable for the filter design, up to the tenth order, are given.

Keywords: filter design, time-bandwidth product, impulse response moments, higher order moments

1 INTRODUCTION

In many applications, the systems with small time spread of the impulse response for a given bandwidth are required. The real systems of the finite order typically have ringing. The general requirement is to make the ringing small and short. In the optimization procedure, all these aspects should be present in the goal function. To have them all in an integral criterion, the use of higher order moments for characterization of the response spread is proposed. The moments have simple relations to the system function parameters. Therefore, the optimization can be carried out in the complex domain by varying pole-zero locations.

The frequency occupation of the band is determined by the signal shape and its time duration. To obtain the best shape, an optimization of the product

$$P_{nm} = \alpha_n \beta_m \tag{1}$$

of the time α_n and the frequency β_m spread should be carried out.

For the spread α_n and β_m various measures might be used [1]. The n-th central moment around delay t_d of the squared impulse response h(t) normalized to the impulse response energy

$$\alpha_n^n = \frac{\int_{-\infty}^{\infty} (t - t_d)^n h^2(t)\, dt}{\int_{-\infty}^{\infty} h^2(t)\, dt}, \quad n=2, 4, 6, ... \tag{2}$$

has been used here for the time spread. The m-th moment of the squared amplitude response normalized to the energy

$$\beta_m^m = \frac{\frac{1}{2\pi}\int_{-\infty}^{\infty} \omega^m |H(\omega)|^2\, d\omega}{\frac{1}{2\pi}\int_{-\infty}^{\infty} |H(\omega)|^2\, d\omega}, \quad m=2, 4, 6, ... \ . \tag{3}$$

has been used for the frequency spread measure.

As it is well known, the second order moments n=2 and m=2, have been used in the uncertainty principle for noncausal [1] and causal signals [2]. Here we use higher order moments n=2, 4, 6 and 8 for the impulse response characterization α_n and the second order moment β_2 for frequency response characterization. Motivation for such a decision is based on the fact that parabola $(t-t_d)^n$ is a weighting function in (2). In this way the contribution of the impulse response ringing in the spread measure α_n will be increased by n. Minimizing the product (1) with such a measure, one can expect small and short ringing.

All pole transfer functions with minimum time-bandwidth product have already been analyzed [3]. In this paper we are extending the proposed system class by one pair of finite zeros.

The frequency response spread can be expressed by the second and higher order moments. The speed in which the frequency response of a real system approaches to zero is determined by number of poles and zeros. Therefore, higher order moments will not be able to modify significantly the frequency response form in the stopband. Thus, for the measure of the frequency response spread, the second moment will be sufficient. Also, the use of the second order moment has another important consequence. Namely, the integral (3) will converge for all system orders $N \geq 2+M$, where M is the number of zeros. The integral (2) will converge regardless of used moment order for $N>M$.

2 MOMENTS AND TRANSFER FUNCTIONS

Time spread and bandwidth definitions (2) and (3), are suitable for causal functions as well. In that case the lower limit in (2) equals to 0, and $H(\omega)$ is Fourier transform of a causal function $h(t)$. Thus, we define a measure of the impulse response spread by the n-th order central moment, and bandwidth as the second order moment, both normalized to the impulse response energy.

For optimization procedure in the complex domain, the criterion (1) should be expressed by the transfer function poles p_j, and zeros, z_i. The system function of the N-th order, with M zeros is given by

$$H(s) = H_0 \frac{\prod_{i=1}^{M}(s - z_i)}{\prod_{j=1}^{N}(s - p_j)} \quad . \tag{4}$$

If the poles are simple, and M<N, the impulse response is

$$h(t) = \sum_{r=1}^{N} K_r e^{p_r t} \quad , \quad K_r = H_0 \frac{\prod_{i=1}^{M}(p_r - z_i)}{\prod_{\substack{j=1 \\ j \neq r}}^{N}(p_r - p_j)} \quad , \tag{5}$$

where K_r, r=1,2,...,N are the pole residues. Now, the n-th order moment of the squared impulse response,

$$m_n = \int_0^{\infty} (t - t_m)^n \, h^2(t) \, dt \quad , \tag{6}$$

can be expressed as a function of poles and residues:

$$m_n(h) = (-1)^{n+1} \sum_{i=1}^{N} \sum_{j=1}^{N} K_i K_j \sum_{k=0}^{n} \frac{n!}{k!} \frac{t_d^k}{(p_i + p_j)^{n-k+1}} \, . \tag{7}$$

Impulse response energy can be obtained from (7) with n=0.

The second order moment of the frequency response can be expressed by the impulse response derivative, using Parseval's relation

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \omega^2 |H(j\omega)|^2 d\omega = \int_{-\infty}^{\infty} \left[\frac{dh}{dt}\right]^2 dt \ . \tag{8}$$

Expression (8) can also be computed from (7) using $K_r p_r$ that are residua of the impulse response derivative, i. e.

$$L\left\{\frac{dh}{dt}\right\} = sH(s) \, , \quad \frac{dh}{dt} = \sum_{r=1}^{N} K_r p_r e^{p_r t} \quad . \tag{9}$$

Thus, the second order moment of the amplitude response (8) is equal to zeroth moment of the impulse response derivative (9)

$$m_0\left(\frac{dh}{dt}\right) = -\sum_{i=1}^{N} \sum_{j=1}^{N} \frac{K_i p_i K_j p_j}{p_i + p_j} \quad . \tag{10}$$

To ensure convergence of the moment integral (3), the number of zeros and poles should satisfy inequality

$$2M + 2 \leq 2N - 2 \ \text{ or } \ N \geq M + 2 \ . \tag{11}$$

3 OPTIMIZATION PROCEDURE

Pole and zero positions of causal filters with minimum time-bandwidth product can be found by solving the problem

$$\min_{z_i, p_j} P_{n2}[z_i, p_j] \ , \ i=1,..,M \text{ and } j=1,...,N. \tag{12}$$

It is more practical to use goal function with real variables. Therefore, the complex poles and zeros in (4) were separated into their real and imaginary parts; $z_i = z_{\sigma i} + i \cdot z_{\varepsilon i}$ and $p_j = p_{\sigma j} + i \cdot p_{\varepsilon j}$. Using this notation, the optimization problem was formed as

$$\min_{z_{\sigma i}, z_{\varepsilon i}, p_{\sigma j}, p_{\varepsilon j}} P_{n2}[z_{\sigma i}, z_{\varepsilon i}, p_{\sigma j}, p_{\varepsilon j}] \ , \quad i=1,..,M/2 \text{ and } j=1,...,N/2 \ , \tag{13}$$

$$\min_{z_{\sigma i}, z_{\varepsilon i}, p_{\sigma j}, p_{\varepsilon j}, p_{\sigma N}} P_{n2}[z_{\sigma i}, z_{\varepsilon i}, p_{\sigma j}, p_{\varepsilon j}, p_{\sigma N}] \ , \quad i=1,..,M/2 \text{ and } j=1,...,(N-1)/2. \tag{14}$$

for even and odd systems order, respectively. The complex poles and zeros of real systems come in conjugate pairs. Therefore, the indexes i and j in (13) and (14) are running to M/2 and N/2 only.

Table I. Poles and zeros of the systems based on the second order moment, t_d=1.

N	p_j	z_i	ω_{3dB}
4	-1.6419±3.6500i -1.9188±1.1834i	1.6862±8.6033i	2.0960
5	-1.9299±5.1496i -2.2893±2.4785i -2.3798	2.1762±8.1422i	2.3578
6	-2.1233±6.6307i -2.5549±3.8224i -2.7151±1.2540i	2.7984±8.8747i	2.6245
7	-2.2477±8.1113i -2.7506±5.2033i -2.9723±2.5545i -3.0385	3.4840±9.7917i	2.8873
8	-2.3188±9.5898i -2.8929±6.6100i -3.1714±3.8912i -3.2934	4.2104±10.7574i	3.1449
9	-2.3471±11.0643i -2.9919±8.0340i -3.3246±5.2551i -3.4971±2.6010i -3.5512	4.9654±11.7327i	3.3856
10	-2.3428±12.5323i -3.0574±9.4696i -3.4420±6.6406i -3.6605±3.9417i -3.7609±1.3074i	5.7505±12.7015i	3.6223

For searching minimum Quasi-Newton method with BFGS formula for Hessian matrix update [4] was used. Analytic expressions for gradients were used in order to avoid numerical errors that finite difference approximation might have caused. Gradient was calculated using forward mode of automatic differentiation, as it can be found for example in [5].

To get causal filters with minimum time-bandwidth product, the optimization is carried out for systems up to tenth order and moment orders n=2, 4, 6 and 8. Optimization will force impulse response to concentrate around t_d and practically extend to $2t_d$. The parameter t_d is chosen to be 1. This will not change the generality of the solution.

4 OPTIMIZATION RESULTS

The numerical values of poles and zeros are given in Table I - IV for systems of the fourth up to the tenth order, with one pair of complex zeros.

For rational transfer functions with t_d=1, the examples of zero-pole positions are shown in Figure 1. It is interesting to note that the poles are very nearly placed on ellipses with ellipses center located at the complex plane origin. The imaginary parts of poles are nearly equidistant. The deviation of equality is 10%, 6.8%, 3.1% and 1.7% for the moment order n=2, 4, 6 and 8, respectively. Such properties are typical for linear phase systems and systems with symmetric impulse response [6].

Table II. Poles and zeros of the systems based on the fourth order moment, t_d=1.

N	p_j	z_i	ω_{3dB}
4	-1.8981±3.3475i -2.2203±1.1184i	1.0485±9.4955i	2.0758
5	-2.0179±4.5249i -2.4637±2.2691i -2.5903	0.4623±8.3770i	2.2548
6	-2.0098±5.6907i -2.5768±3.4291i -2.8070±1.1425i	-0.0158±8.0113i	2.4089
7	-1.9425±7.0070i -2.6265±4.6604i -2.9464±2.3244i -3.0424	-0.0436±7.9044i	2.5604
8	-1.8999±8.4590i -2.6386±5.9519i -3.0407±3.5518i -3.2166±1.1813i	0.0164±8.2558i	2.7352
9	-1.9051±9.9125i -2.6226±7.2728i -3.1036±4.8086i -3.3520±2.3962i -3.4293	0.0223±8.7728i	2.9178
10	-1.9360±11.3153i -2.5954±8.6308i -3.1411±6.0922i -3.4561±3.6389i -3.6004±1.2108i	0.0189±9.3072i	3.1012

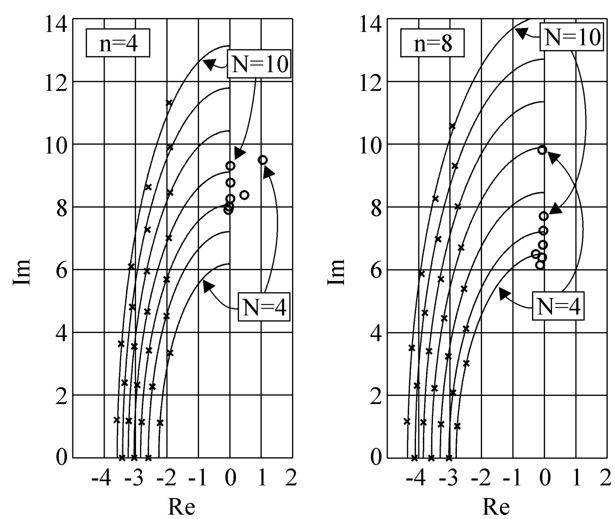

Figure 1. Pole-zero positions of the optimum systems based on the fourth and the eighth order moment, normalized to t_d=1.

4.1 Systems based on the fourth order moment

The impulse responses of the optimum systems based on the fourth order moment order are shown in Figure 2. It is a quasi Gaussian response, with small time

Table III. Poles and zeros of the systems based on the sixth order moment, t_d=1.

N	p_j	z_i	ω_{3dB}
4	-2.1849±3.1558i -2.4936±1.0625i	0.3470±9.7438i	2.0202
5	-2.1999±4.2515i -2.6451±2.1500i -2.7776	-0.2480±6.9285i	2.0889
6	-2.2075±5.5284i -2.7628±3.3232i -3.0012±1.1086i	-0.1451±6.4718i	2.2151
7	-2.2549±6.8973i -2.8559±4.5572i -3.1825±2.2734i -3.2832	-0.0672±6.7370i	2.3851
8	-2.3278±8.2595i -2.9231±5.8280i -3.3246±3.4788i -3.5091±1.1578i	-0.0414±7.1890i	2.5696
9	-2.4064±9.5901i -2.9775±7.1306i -3.4341±4.7164i -3.6910±2.3520i -3.7725	-0.0289±7.6828i	2.7539
10	-2.4814±10.8921i -3.0376±8.4518i -3.5223±5.9785i -3.8426±3.5730i -3.9941±1.1895i	-0.0194±8.1846i	2.9343

Table IV. Poles and zeros of the systems based on the eighth order moment, t_d=1.

N	p_j	z_i	ω_{3dB}
4	-2.4851±3.0239i -2.7715±1.0193i	-0.0681±9.8197i	1.9934
5	-2.4927±4.1240i -2.9100±2.0852i -3.0383	-0.2735±6.4988i	2.0203
6	-2.5570±5.3931i -3.0603±3.2423i -3.2905±1.0826i	-0.1364±6.1485i	2.1431
7	-2.6523±6.7094i -3.1883±4.4535i -3.5008±2.2250i -3.6001	-0.0741±6.3931i	2.3068
8	-2.7543±8.0186i -3.2917±5.7013i -3.6692±3.4104i -3.8503±1.1360i	-0.0481±6.7958i	2.4833
9	-2.8498±9.3099i -3.3812±6.9757i -3.8031±4.6289i -4.0535±2.3107i -4.1346	-0.0332±7.2464i	2.6611
10	-2.9343±10.5854i -3.4662±8.2637i -3.9111±5.8719i -4.2210±3.5130i -4.3723±1.1700i	-0.0232±7.7110i	2.8363

spread, and small and short ringing. Undershoots are smaller than 0.7% for N≥5. Higher system orders apparently give the response with smaller time spread and higher symmetry. The step response, Figure 3, is practically monotonic with overshoots smaller than 0.1% for N≥5. Generally, the impulse response undershoots and step response overshoots are few times smaller than in the all pole case given in [3].

Amplitude and group delay responses are shown in Figure 4 and Figure 5, respectively, in a form suitable for comparison with classic filter approximations, given, for example, in [7]. The amplitude response is quasi Gaussian. The group delay curves illustrate an approximation of a constant.

4.2 Properties of the obtained systems

The time bandwidth products, i. e. magnitude of the goal function (1) in optimum, obtained by the use of various moment orders are given in Figure 5. The curves in the diagram are given for all-pole systems presented in [3] and new results obtained with one pair of complex zeros. They show an asymptotic approach to the values 0.5, 0.648, 0.753 and 0.834 for n=2, 4, 6 and 8 respectively. It is obvious that time-bandwidth product is practically constant for systems with order N≥7. The presence of zeros reduces the time-bandwidth product. However, the contribution of the second pair of zeros is negligible. This is the reason why we found sufficient to give here the complete data for systems with one pair of zeros only. We also found that the finite real zeros can not improve the time-bandwidth product. Namely, the optimization procedure gives their position very far from the origin, i. e. far from the rest of poles and complex zeros.

The contribution of the complex zeros is also visible on Figure 6 where impulse responses are shown for eighth order fourth moment systems with various numbers of zeros. Adding one pair of complex zeros significantly reduce undershoot compared to the all pole case, while the reduction caused by the second pair of zeros is negligible.

Impulse response undershoots and step response overshoots are much smaller than in classical linear phase approximation. The data for various system orders together with the data for Bessel filters are shown in Figure 7 and Figure 8. Impulse response undershoots of the optimum filters are smaller than 1%, and generally decrease for higher system order. The similar behavior can be notices in the step response overshoots, which are smaller than 0.12% for N≥5. The obtained product P_{n2} is a measure for the optimization performance. The conventional and more practical value for the time and frequency spread is for example the rise-time t_r (10%-

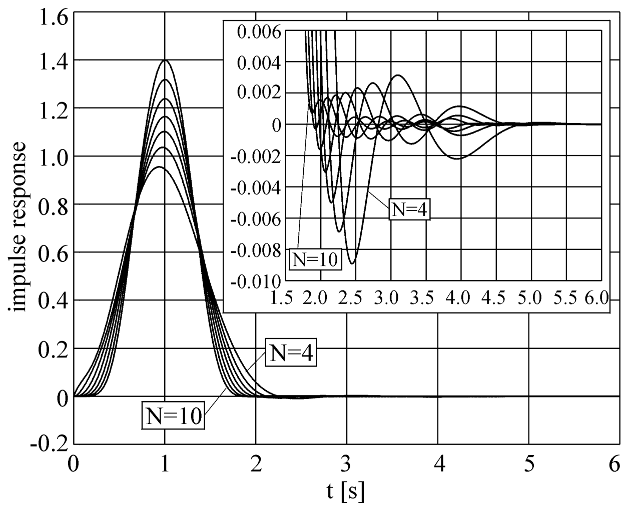

Figure 2. Impulse response of the optimum systems based on the fourth order moment, normalized to $t_d=1$.

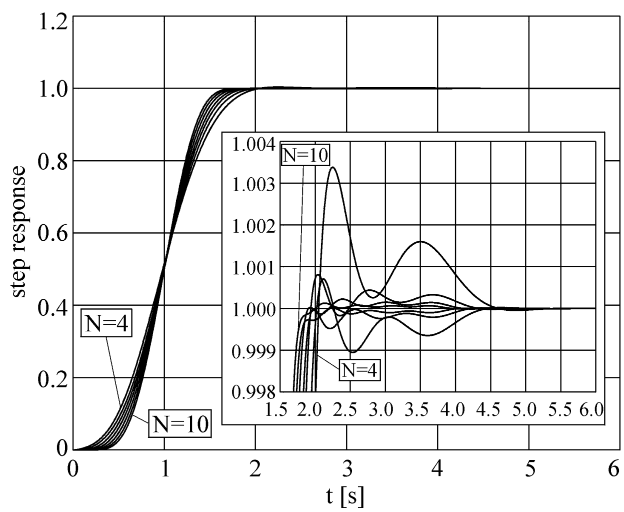

Figure 3. Step response of the optimum systems based on the fourth order moment, normalized to $t_d=1$.

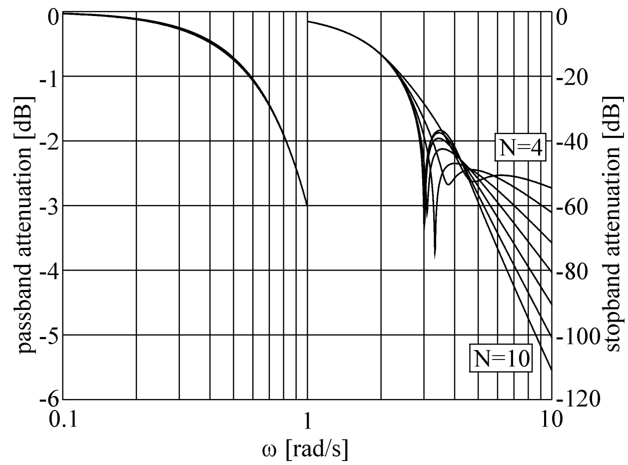

Figure 4. Amplitude response of the optimum systems based on the fourth order moment, normalized to $\omega_{3dB}=1$.

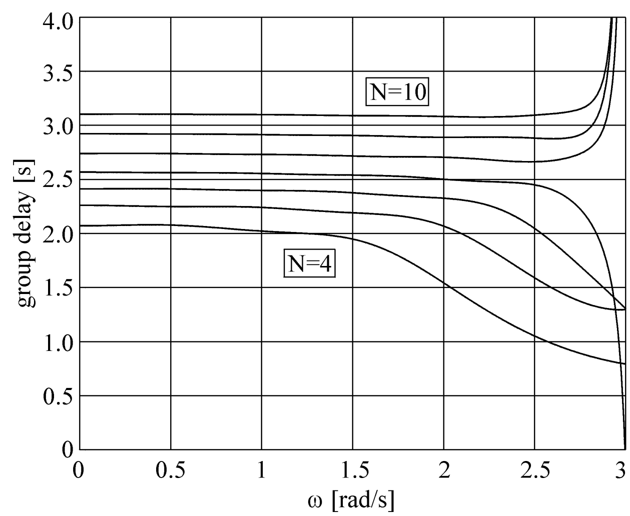

Figure 5. Group delay of the optimum systems based on the fourth order moment, normalized to $\omega_{3dB}=1$.

90%) and bandwidth $f_{3dB}=\omega_{3dB}/(2\pi)$. The product is practically constant and it equals to $t_r f_{3dB}=0.339$ to $t_r f_{3dB}=0.348$. An example can be seen from Table II and Figure 3.

Amplitude attenuation in the stop band $\omega >> \omega_{3dB}$ is smaller for higher order moments. In that band the attenuation slope is generally smaller than in the all pole case, but before zero dip the slope is higher.

5 CONCLUSION

By the minimization of the time-bandwidth product using higher order moments, a new class of finite order systems has been obtained. Here we considered the influence of zeros, which we found useful for improvement of the time bandwidth product. We have shown that only one pair of complex zeros is practically interesting. Obtained systems have the largest energy concentration in time for a given bandwidth. The impulse response and step responses have small and short ringing. Time domain properties of the obtained filter families can be favorably compared to similar filters with linear phase.

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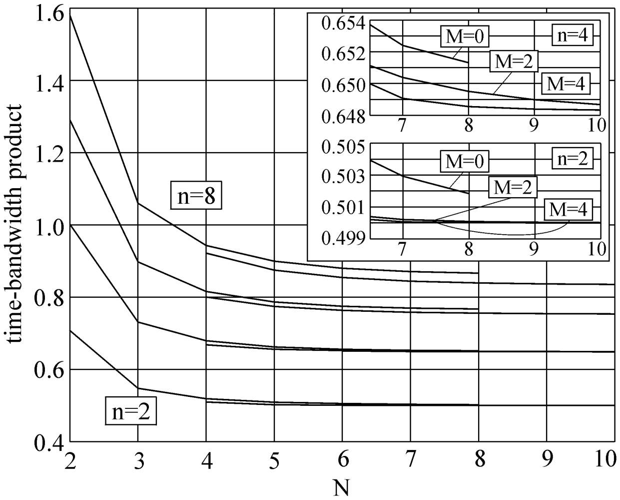

Figure 5. Time-bandwidth products of the optimum systems based on various moment orders.

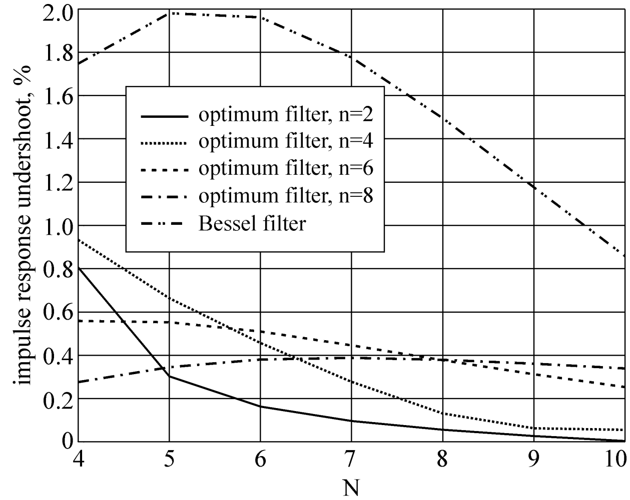

Figure 7. Impulse response undershoots of Bessel filters and the optimum filters based on various moment orders.

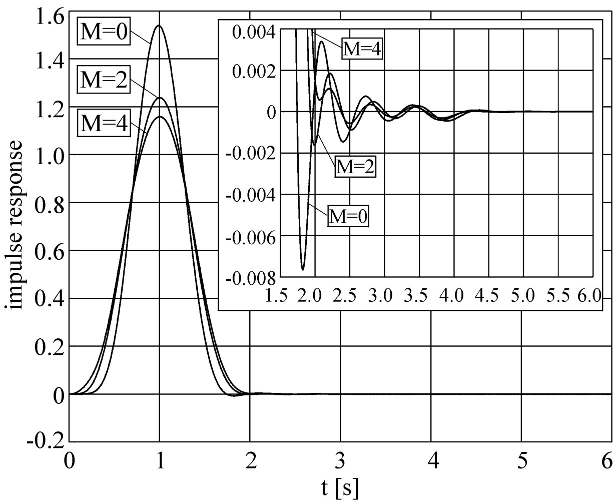

Figure 6. Impulse response of the optimum systems for various numbers of zeros, N=8, n=4, t_d=1.

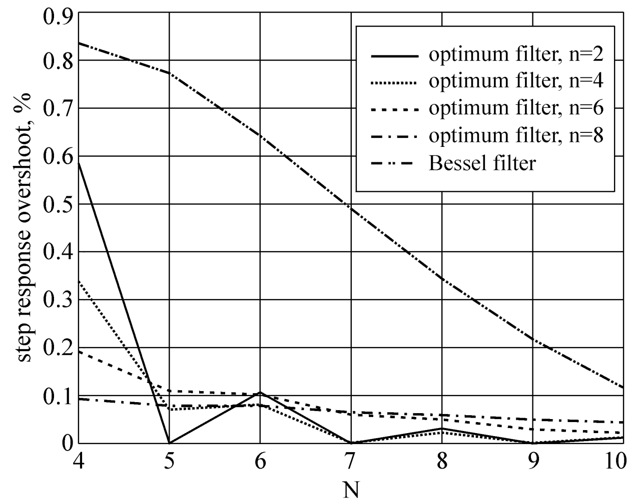

Figure 8. Step response overshoots of Bessel filters and the optimum filters based on various moment orders.

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