ON CERTAIN REPRESENTATION OF TWISTED GROUP ALGEBRA OF SYMMETRIC GROUPS ON MULTIPARAMETRIC QUON ALGEBRAS

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Here we consider two algebras:

- a free unital associative complex algebra $\mathcal{B} = \mathcal{B}^{\mathbf{q}} = \mathbb{C}\langle e_{i_1}, \ldots, e_{i_N} \rangle$, $N \geq 0$ equiped with a multiparametric **q**-differential structure

$$\partial_i(e_j x) = \delta_{ij} x + q_{ij} e_j \partial_i(x), \quad \text{for each } x \in \mathcal{B};$$

  with $\partial_i(1) = 0$, $\partial_i(e_j) = \delta_{ij}$, ($\delta_{ij}$ is a standard Kronecker delta) ($\mathcal{B}$ is sometimes called a multiparametric quon algebra).
- a twisted group algebra

$$\mathcal{A}(S_n) = R_n \rtimes \mathbb{C}[S_n]$$

  of the symmetric group S_n with coefficients in a polynomial algebra R_n in commuting variables $X_{a\,b}$, $1 \leq a, b \leq n$

with the motivation to represent the algebra $\mathcal{A}(S_n)$ on the (generic) weight subspaces of the algebra $\mathcal{B}$ (with the aim to simplify certain computation in $\mathcal{B}$).

One of the fundamental problems in $\mathcal{B} = \mathcal{B}^{\boldsymbol{q}}$:

- describe the space of all constants

  (the elements which are annihilated by all multiparametric partial derivatives $\partial_i = \partial_i^{\boldsymbol{q}}$).

To solve this problem:

- one needs some special matrices and their factorizations in terms of simpler matrices.

A simpler approach:

- first, to study certain canonical elements in the twisted group algebra $\mathcal{A}(S_n)$;
- then to use certain natural representation of $\mathcal{A}(S_n)$ on the weight subspaces $\mathcal{B}_Q$.

In this representation some factorizations of certain canonical elements from $\mathcal{A}(S_n)$ will immediately give the corresponding matrix factorizations and also determinant factorizations.

Let $\mathcal{N} = \{i_1, \dots, i_N\} \subseteq \{0, 1, \dots\}$.

Fix a parametar map $\boldsymbol{q} \colon \mathcal{N} \times \mathcal{N} \to \mathbb{C}$, $(i, j) \mapsto q_{ij} \quad i, j \in \mathcal{N}$.

We consider **a free unital associative complex algebra**

$$\mathcal{B} = \mathbb{C}\left\langle e_{i_1}, \dots, e_{i_N} \right\rangle \qquad (\deg\, e_i = 1 \quad \text{for all} \;\; i \in \mathcal{N}).$$

together with N linear operators $\partial_i = \partial_i^{\boldsymbol{q}} \colon \mathcal{B} \to \mathcal{B}$, $i \in \mathcal{N}$ (of degree -1) defined recursively:

$$\partial_i(1) = 0, \quad \partial_i(e_j) = \delta_{ij},$$

$$\partial_i(e_j x) = \delta_{ij} x + q_{ij} e_j \partial_i(x), \quad \text{for each} \;\; x \in \mathcal{B} \quad \text{(twisted Leibniz rule)}.$$

- Since every sequence $l_1, \dots, l_n \in \mathcal{N}$, $l_1 \leq \dots \leq l_n$ can be thought of as a multiset $Q = \{l_1 \leq \dots \leq l_n\}$ over $\mathcal{N}$ of size $n = \mathrm{Card}\; Q$, each corresponding weight subspace $\mathcal{B}_Q = \mathcal{B}_{l_1 \dots l_n}$ is given by

$$\mathcal{B}_Q = span_{\mathbb{C}} \left\{ e_{j_1 \dots j_n} := e_{j_1} \cdots e_{j_n} \mid j_1 \dots j_n \in \widehat{Q} \right\}.$$

$\widehat{Q}$ = the set of all distinct permutations of the multiset Q, $\dim \mathcal{B}_Q = \mathrm{Card}\; \widehat{Q}$.

- A finer decomposition of $\mathcal{B}$ into multigraded components (= weight subspaces):

$$\mathcal{B} = \bigoplus_{n\geq 0,\, l_1\leq \cdots \leq l_n,\, l_j\in\mathcal{N}} \mathcal{B}_{l_1\ldots l_n}.$$

- Let $\mathfrak{B}_Q = \left\{ e_{\underline{j}} \mid \underline{j} \in \widehat{Q} \right\}$ denote the monomial basis of $\mathcal{B}_Q$, where $\underline{j} := j_1 \ldots j_n$.
- The action of $\partial_i = \partial_i^q$ on a monomial $e_{\underline{j}} \in \mathfrak{B}_Q$ is given explicitly by the formula:

$$\partial_i(e_{\underline{j}}) = \sum_{1\leq k\leq n,\, j_k=i} q_{ij_1} \cdots q_{ij_{k-1}}\, e_{j_1\ldots\widehat{j_k}\ldots j_n}, \tag{1}$$

where $\widehat{j_k}$ denotes the omission of the corresponding index j_k.

The number of terms in this sum is equal to the number of appearances (multiplicity) of the generator e_i in the monomial $e_{\underline{j}}$.

- In the generic case, when Q is a set, the formula (1) is reduced to:

$$\partial_i(e_{\underline{j}}) = q_{ij_1} \cdots q_{ij_{k-1}}\, e_{j_1\ldots\widehat{j_k}\ldots j_n}. \tag{2}$$

- With the motivation of treating better the matrices of $\partial_i|_{\mathcal{B}_Q}$, we introduce a multidegree operator $\partial\colon \mathcal{B} \to \mathcal{B}$ with $\partial = \sum_{i\in\mathcal{N}} e_i\,\partial_i$,

  where $e_i\colon \mathcal{B} \to \mathcal{B}$ are considered as (multiplication by e_i) operators on $\mathcal{B}$.
- The operator ∂ preserves the direct sum decomposition of the algebra $\mathcal{B}$.
- We denote by $\partial^Q\colon \mathcal{B}_Q \to \mathcal{B}_Q$ the restriction of $\partial\colon \mathcal{B} \to \mathcal{B}$ to the subspace $\mathcal{B}_Q$.

  Then for each $j_1 \dots j_n \in \widehat{Q}$ we get

$$\partial^Q\left(e_{j_1\dots j_n}\right) = \sum_{i\in\mathcal{N}} e_i\partial_i\left(e_{j_1\dots j_n}\right) = \sum_{i\in\mathcal{N}} e_i \sum_{1\le k\le n,\, j_k=i} q_{ij_1}\cdots q_{ij_{k-1}}\, e_{j_1\dots\widehat{j_k}\dots j_n}$$

$$= \sum_{1\le k\le n}\ \sum_{i\in\mathcal{N},\, i=j_k} q_{ij_1}\cdots q_{ij_{k-1}}\, e_{ij_1\dots\widehat{j_k}\dots j_n} = \sum_{1\le k\le n} q_{j_kj_1}\cdots q_{j_kj_{k-1}}\, e_{j_kj_1\dots\widehat{j_k}\dots j_n}.$$

If B_Q denotes the matrix of ∂^Q w.r.t basis $\mathfrak{B}_Q$ (totally ordered by the Johnson-Trotter ordering on permutations) of $\mathcal{B}_Q$, then we can write

$$\mathrm{B}_Q\, e_{j_1\dots j_n} = \sum_{1\le k\le n} q_{j_kj_1}\cdots q_{j_kj_{k-1}}\, e_{j_kj_1\dots\widehat{j_k}\dots j_n}. \tag{3}$$

Now we consider **a twisted group algebra** $\mathcal{A}(S_n) = R_n \rtimes \mathbb{C}[S_n]$ of the symmetric group S_n with coefficients in the polynomial ring $R_n = \mathbb{C}[X_{a\,b}, 1 \le a, b \le n]$ (here $\rtimes$ denotes the semidirect product.)

The elements of $\mathcal{A}(S_n)$ are the linear combinations $\sum_{g_i \in S_n} p_i\, g_i$ with $p_i \in R_n$.

- The multiplication in $\mathcal{A}(S_n)$ is given by

$$(p_1 g_1) \cdot (p_2 g_2) := (p_1 \cdot (g_1.p_2))\, g_1 g_2$$

where $\quad g.p = g.p(\ldots, X_{a\,b}, \ldots) = p(\ldots, X_{g(a)\,g(b)}, \ldots)\, g.$

The algebra $\mathcal{A}(S_n)$ is associative but not commutative.

- In the algebra $\mathcal{A}(S_n)$ we have introduced decorated more specific elements:

$$g^* = \left(\prod_{(a,b) \in I(g^{-1})} X_{a\,b} \right) g = \left(\prod_{a<b, g^{-1}(a) > g^{-1}(b)} X_{a\,b} \right) g$$

for every $g \in S_n$, where $I(g) = \{(a,b) \mid 1 \le a < b \le n,\ g(a) > g(b)\}$ denotes the set of inversions of the permutation g.

- Of particular interest are the elements $t^*_{b,a} \in \mathcal{A}(S_n)$, $1 \le a \le b \le n$, where for $a < b$, $t_{b,a} \in S_n$ denotes the inverse of the cyclic permutation $t_{a,b} \in S_n$ i.e

$$t_{b,a} = t_{a,b}^{-1} = \begin{pmatrix} 1 & \cdots & a-1 & a & a+1 & \cdots & b-1 & b & b+1 & \cdots & n \\ 1 & \cdots & a-1 & a+1 & a+2 & \cdots & b & a & b+1 & \cdots & n \end{pmatrix}$$

- Every permutation $g \in S_n$ can be decomposed into cycles (from the left) as follows ($k_j \geq j$):

$$g = t_{k_n,n} \cdot t_{k_{n-1},n-1} \cdots t_{k_j,j} \cdots t_{k_2,2} \cdot t_{k_1,1} \left(= \overleftarrow{\prod_{1 \leq j \leq n}} t_{k_j,j} \right)$$

- Now we introduce canonical element α_n^* in the twisted group algebra by formula

$$\alpha_n^* := \sum_{g \in S_n} g^*, \quad n \geq 1.$$

Then we get the following factorizations:

1. if we define simpler elements $\beta_k^* \in \mathcal{A}(S_n)$ $(1 \leq k \leq n)$ as follows

$$\beta_{n-k+1}^* = t_{n,k}^* + t_{n-1,k}^* + \cdots + t_{k+1,k}^* + t_{k,k}^*$$

   then: $\alpha_n^* = \beta_1^* \cdot \beta_2^* \cdots \beta_n^*$;

2. if we define yet simpler elements in $\mathcal{A}(S_n)$ for all $1 \leq k \leq n-1$

$$\gamma_{n-k+1}^* = \left(id - t_{n,k}^*\right) \cdot \left(id - t_{n-1,k}^*\right) \cdots \left(id - t_{k+1,k}^*\right),$$

$$\delta_{n-k+1}^* = \left(id - (t_k^*)^2\, t_{n,k+1}^*\right) \cdot \left(id - (t_k^*)^2\, t_{n-1,k+1}^*\right) \cdots \left(id - (t_k^*)^2\, t_{k+1,k+1}^*\right),$$

   then we get further factorization: $\beta_{n-k+1}^* = \delta_{n-k+1}^* \cdot \left(\gamma_{n-k+1}^*\right)^{-1}$.

We have used: $t_{k+1,k+1}^* = id$ and $(t_k^*)^2 = X_{\{k,\,k+1\}}\, id$ because

$$(t_k^*)^2 = (X_{k\,k+1}\, t_k) \cdot (X_{k\,k+1}\, t_k) = X_{k\,k+1} \cdot X_{k+1\,k}\, (t_k)^2 = X_{\{k,\,k+1\}}\, id$$

- Recall that for any vector space V over a field F, $\mathrm{End}(V)$ denotes the algebra of all F-endomorphisms of V;
- for any associative algebra A a **representation of A on V is any algebra homomorphism** $\varphi\colon A \to \mathrm{End}(V)$.

Our next task is to define a representation $\varrho\colon \mathcal{A}(S_n) \to \mathrm{End}(\mathcal{B}_Q)$, where

$$\mathcal{B}_Q = span_{\mathbb{C}} \left\{ e_{j_1 \dots j_n} \mid j_1 \dots j_n \in \widehat{Q} \right\}.$$

Since $\mathcal{A}(S_n) = R_n \rtimes \mathbb{C}[S_n]$ we will consider first a representation ϱ_1 of R_n and second a representation ϱ_2 of $\mathbb{C}[S_n]$:

$$\varrho_1\colon R_n \to \mathrm{End}(\mathcal{B}_Q),$$
$$\varrho_2\colon \mathbb{C}[S_n] \to \mathrm{End}(\mathcal{B}_Q)$$

as follows.

We first denote by: $Q_{a\,b}$, $1 \le a, b \le n$ **a diagonal operator on** $\mathcal{B}_Q$ defined by

$$Q_{a\,b}\, e_{j_1 \dots j_n} := q_{j_a j_b}\, e_{j_1 \dots j_n}.$$

Note that these operators commute $(Q_{a\,b} \cdot Q_{c\,d} = Q_{c\,d} \cdot Q_{a\,b})$.

Definition

We define a representation $\varrho_1 : R_n \to \mathsf{End}(\mathcal{B}_Q)$ on the generators $X_{a\,b}$ of R_n by the formula

$$\varrho_1(X_{a\,b}) := Q_{a\,b} \qquad 1 \leq a, b \leq n.$$

Note that: $\varrho_1(X_{a\,b})\, e_{j_1\ldots j_n} = Q_{a\,b}\, e_{j_1\ldots j_n} = q_{j_a j_b}\, e_{j_1\ldots j_n}.$

Definition

We define a linear operator $\varrho_2 : \mathbb{C}[S_n] \to \mathsf{End}(\mathcal{B}_Q)$ by

$$\varrho_2(g)\, e_{j_1\ldots j_n} := e_{j_{g^{-1}(1)}\cdots j_{g^{-1}(n)}} \qquad \text{for every} \;\; g \in S_n.$$

In fact ϱ_2 is a (right) regular representation on $\mathcal{B}_Q$, $Q =$ generic.

Let $\varrho\colon \mathcal{A}(S_n) \to \mathrm{End}(\mathcal{B}_Q)$ be a map defined on decomposable elements by

$$\varrho(pg) := \varrho_1(p) \cdot \varrho_2(g)$$

for every $p \in R_n$ and $g \in S_n$ and extended by additivity. In the trivial cases we have

(i) $\varrho(1 \cdot g)\, e_{j_1 \dots j_n} = \varrho_1(1) \cdot \varrho_2(g)\, e_{j_1 \dots j_n} = 1 \cdot e_{j_{g^{-1}(1)} \cdots j_{g^{-1}(n)}} = e_{j_{g^{-1}(1)} \cdots j_{g^{-1}(n)}}$,

(ii) $\varrho(X_{a\,b}\, e)\, e_{j_1 \dots j_n} = \varrho_1(X_{a\,b}) \cdot \varrho_2(e)\, e_{j_1 \dots j_n} = Q_{a\,b}\, e_{j_1 \dots j_n} = q_{j_a j_b}\, e_{j_1 \dots j_n}$.

Note that the basic instance of the multiplication $(p_1 g_1) \cdot (p_2 g_2) = (p_1 \cdot (g_1 . p_2))\, g_1 g_2$ in $\mathcal{A}(S_n)$ reads as follows:

$$(X_{a\,b}\, g_1) \cdot (X_{c\,d}\, g_2) = \left(X_{a\,b} \cdot X_{g_1(c)\, g_1(d)}\right)\, g_1 g_2$$

which are the consequences of the following two types of basic relations:

$$X_{a\,b} \cdot X_{c\,d} = X_{c\,d} \cdot X_{a\,b},$$

$$g . X_{a\,b} = X_{g(a)\, g(b)}\, g.$$

Theorem

A map $\varrho\colon \mathcal{A}(S_n) \to \mathrm{End}(\mathcal{B}_Q)$ *is a representation.*

Proof.

It is enough to check that ϱ preserves the previously listed basic relations, where we will apply the formula $\varrho(pg) = \varrho_1(p) \cdot \varrho_2(g)$ and properties of representations ϱ_1 and ϱ_2.

(i) Note that: $\varrho(X_{a\,b} \cdot X_{c\,d}) = Q_{a\,b} \cdot Q_{c\,d} = Q_{c\,d} \cdot Q_{a\,b} = \varrho(X_{c\,d} \cdot X_{a\,b})$.

(ii) Now we will show that $\varrho(g.X_{a\,b})\, e_{j_1\ldots j_n} = \varrho(X_{g(a)\,g(b)}\, g)\, e_{j_1\ldots j_n}$.

$$\begin{aligned} L \equiv \varrho(g.X_{a\,b})\, e_{j_1\ldots j_n} &= \varrho_2(g) \cdot \varrho_1(X_{a\,b})\, e_{j_1\ldots j_n} = \varrho_2(g)\, q_{j_a j_b}\, e_{j_1\ldots j_n} \\ &= q_{j_a j_b}\, \varrho_2(g)\, e_{j_1\ldots j_n} \\ &= q_{j_a j_b}\, e_{j_{g^{-1}(1)}\cdots j_{g^{-1}(n)}}; \end{aligned}$$

$$\begin{aligned} D \equiv \varrho(X_{g(a)\,g(b)}\, g)\, e_{j_1\ldots j_n} &= \varrho_1(X_{g(a)\,g(b)}) \cdot \varrho_2(g)\, e_{j_1\ldots j_n} \\ &= Q_{g(a)\,g(b)}\, e_{j_{g^{-1}(1)}\cdots j_{g^{-1}(n)}} \\ &= q_{j_{g^{-1}(g(a))} j_{g^{-1}(g(b))}}\, e_{j_{g^{-1}(1)}\cdots j_{g^{-1}(n)}} \\ &= q_{j_a j_b}\, e_{j_{g^{-1}(1)}\cdots j_{g^{-1}(n)}}. \end{aligned}$$

$\square$

Lemma

The representation ϱ *applied to element* $g^* = \left(\prod_{(a,b)\in I(g^{-1})} X_{a\,b} \right) g$ *is given by*

$$\varrho(g^*)\, e_{j_1 \ldots j_n} = \prod_{(a,b)\in I(g)} q_{j_b j_a}\, e_{j_{g^{-1}(1)} \cdots j_{g^{-1}(n)}}.$$

Proof.

By using $\varrho(pg) = \varrho_1(p) \cdot \varrho_2(g)$ on g^* we obtain

$$\begin{aligned}
\varrho\,(g^*)\; e_{j_1 \ldots j_n} &= \prod_{(a',b')\in I(g^{-1})} \varrho_1\,(X_{a'\,b'}) \cdot \varrho_2(g)\, e_{j_1 \ldots j_n} \\
&= \prod_{(a',b')\in I(g^{-1})} q_{j_{g^{-1}(a')} j_{g^{-1}(b')}}\; e_{j_{g^{-1}(1)} \cdots j_{g^{-1}(n)}} \\
&= \prod_{(b,a)\in I(g)} q_{j_a j_b}\; e_{j_{g^{-1}(1)} \cdots j_{g^{-1}(n)}} = \prod_{(a,b)\in I(g)} q_{j_b j_a}\; e_{j_{g^{-1}(1)} \cdots j_{g^{-1}(n)}}
\end{aligned}$$

with $a = g^{-1}(a'),\, b = g^{-1}(b')$. Now it is easy to check that

$(a',b') \in I(g^{-1}) \;\Rightarrow\; a' < b',\, g^{-1}(a') > g^{-1}(b')$ i.e $g(a) < g(b),\, a > b$, so: $(b,a) \in I(g)$. □

A direct consequence of the Lemma:

- the element $\varrho(t^*_{b,a}) \in \mathrm{End}(\mathcal{B}_Q)$ is given by

$$\varrho(t^*_{b,a})\, e_{j_1 \dots j_a j_{a+1} \dots j_b \dots j_n} = \prod_{a \le i \le b-1} q_{j_b j_i}\, e_{j_1 \dots j_b j_a \dots j_{b-1} \dots j_n}$$

- and in special case: $\varrho(t^*_a)\, e_{j_1 \dots j_a j_{a+1} \dots j_n} = q_{j_{a+1} j_a}\, e_{j_1 \dots j_{a+1} j_a \dots j_n}$

  (where: $t^*_a = t^*_{a+1,a}$). If we denote by $\sigma_{j_a j_{a+1}} := q_{j_a j_{a+1}} q_{j_{a+1} j_a}$, then

$$\varrho((t^*_a)^2)\, e_{j_1 \dots j_n} = \sigma_{j_a j_{a+1}}\, e_{j_1 \dots j_n}.$$

Theorem

Let $\varrho \colon \mathcal{A}(S_n) \to \mathrm{End}(\mathcal{B}_Q)$ be the twisted regular representation on the generic weight space $\mathcal{B}_Q$. Then the $(\underline{k}, \underline{j})$-entry of the matrix $\mathbf{A}_Q$ of the $\varrho(\alpha^*_n)$ is given by

$$(\mathbf{A}_Q)_{\underline{k}, \underline{j}} = \prod_{(a,b) \in I(g)} q_{j_b j_a}$$

where g is such that $\underline{k} = g.\underline{j}$ $(\underline{j} = j_1 \dots j_n \in \widehat{Q}$, $\underline{k} = k_1 \dots k_n \in \widehat{Q})$.

Recall, that $\alpha^*_n = \sum_{g \in S_n} g^*, \quad n \ge 1.$

Factorization of the matrix A_Q

Let us denote

$$\mathbf{T}_{b,a} := \varrho(t^*_{b,a}), \qquad \mathbf{T}_a := \varrho(t^*_a).$$

If $b = a$ then $\mathbf{T}_{b,a} = \mathbf{I}$.

- The $(\underline{k}, \underline{j})$-entry of the matrices $\mathbf{T}_{b,a}$, $1 \leq a < b \leq n$ and $\mathbf{T}_a$, $1 \leq a \leq n-1$:

$$(\mathbf{T}_{b,a})_{\underline{k},\underline{j}} = \begin{cases} \prod\limits_{a \leq i \leq b-1} q_{j_b j_i} & \text{if } \underline{k} = t_{b,a}.\underline{j} \\ 0 & \text{otherwise} \end{cases}$$

  with $t_{b,a}.\underline{j} = j_1 \ldots j_b j_a \ldots j_{b-1} \ldots j_n$,

$$(\mathbf{T}_a)_{\underline{k},\underline{j}} = \begin{cases} q_{j_{a+1} j_a} & \text{if } \underline{k} = t_a.\underline{j} \\ 0 & \text{otherwise} \end{cases}$$

  with $t_a.\underline{j} = j_1 \ldots j_{a+1} j_a \ldots j_n$.

- Now it is easy to see that

$$(\mathbf{T}_a)^2 \, e_{\underline{j}} = \sigma_{j_a j_{a+1}} \, e_{\underline{j}}$$

  is the diagonal matrix with $\sigma_{j_a j_{a+1}}$ as $\underline{j}$-th diagonal entry.

- Now we consider the elements $\beta^*_k \in \mathcal{A}(S_n)$, $1 \leq k \leq n$.

  (recall: $\beta^*_{n-k+1} = t^*_{n,k} + t^*_{n-1,k} + \cdots + t^*_{k+1,k} + \underbrace{t^*_{k,k}}_{=id}$).

- Then the corresponding elements $\varrho(\beta^*_{n-k+1}) \in \mathrm{End}(\mathcal{B}_Q)$ are given by

$$\varrho(\beta^*_{n-k+1})\, e_{\underline{j}} = \sum_{k+1 \leq m \leq n} \varrho(t^*_{m,k})\, e_{\underline{j}} + e_{\underline{j}} \tag{4}$$

- Let

$$\mathbf{B}_{Q,l} := \varrho(\beta^*_l), \quad 1 \leq l \leq n,$$

  with $\mathbf{B}_{Q,1} = \varrho(\beta^*_1) = \varrho(id) = \mathbf{I}$. Then in the matrix notation (4) can be written as

$$\mathbf{B}_{Q,n-k+1} = \sum_{k+1 \leq m \leq n} \mathbf{T}_{m,k} + \mathbf{I}$$

- The $(\underline{k}, \underline{j})$-entry of the matrix $\mathbf{B}_{Q,n-k+1}$ of $\varrho(\beta^*_{n-k+1})$, $1 \leq k \leq n-1$ is given by

$$\left(\mathbf{B}_{Q,n-k+1}\right)_{\underline{k},\underline{j}} = \begin{cases} \prod_{k \leq i < m} q_{j_m j_i} & \text{if } \underline{k} = t_{m,k}.\underline{j} \quad k \leq m \leq n \\ 0 & \text{otherwise} \end{cases}$$

  for each $1 \leq k \leq n-1$ (recall: $t_{m,k}.\underline{j} = j_1 \ldots j_m j_k \ldots j_{m-1} \ldots j_n$).

- In the special case for $k = 1$ we have

$$\mathbf{B}_{Q,n} = \sum_{1 \le m \le n} \mathbf{T}_{m,1} = \mathbf{T}_{n,1} + \mathbf{T}_{n-1,1} + \cdots + \mathbf{T}_{3,1} + \mathbf{T}_{2,1} + \mathbf{I}$$

(where $\mathbf{T}_{1,1} = \mathbf{I}$) so the $(\underline{k}, \underline{j})$-entry of $\mathbf{B}_{Q,n}$ is given by

$$(\mathbf{B}_{Q,n})_{\underline{k},\underline{j}} = \begin{cases} q_{j_m j_1} \cdots q_{j_m j_{m-1}} & \text{if } \underline{k} = t_{m,1}.\underline{j} \quad 1 \le m \le n \\ 0 & \text{otherwise} \end{cases}$$

(recall: $t_{m,1}.\underline{j} = j_m j_1 \ldots j_{m-1} j_{m+1} \ldots j_n$).

1. We get: $\mathbf{B}_{Q,n}\, e_{\underline{j}} = \sum_{1 \le m \le n} q_{j_m j_1} \cdots q_{j_m j_{m-1}}\, e_{j_m j_1 \ldots j_{m-1} j_{m+1} \cdots j_n}$.
2. Recall that the matrix of $\partial^Q : \mathcal{B}_Q \to \mathcal{B}_Q$ is given by

$$\mathrm{B}_Q\, e_{j_1 \ldots j_n} = \sum_{1 \le k \le n} q_{j_k j_1} \cdots q_{j_k j_{k-1}}\, e_{j_k j_1 \ldots \widehat{j_k} \ldots j_n}.$$

Then: $\mathbf{B}_{Q,n} = \mathrm{B}_Q$. It turns out that:

- the factorization of the matrix $\mathbf{B}_{Q,n}$ is equivalent to factorization of B_Q (i.e the matrix of ∂^Q w.r.t monomial basis of $\mathcal{B}_Q \subset \mathcal{B}$);
- the problem of computing $\det \mathrm{B}_Q$ can be reduced to the problem of computing $\det \mathbf{B}_{Q,n}$.

With this motivation we are going to find a formula for the factorization of $\mathbf{B}_{Q,n}$ and also its determinant.

- Now we will represent the elements γ_l^*, δ_l^*, $2 \leq l \leq n$ belonging to $\mathcal{A}(S_n)$.
- The corresponding elements $\varrho(\gamma_{n-k+1}^*), \varrho(\delta_{n-k+1}^*) \in \mathrm{End}(\mathcal{B}_Q)$, $\quad 1 \leq k \leq n-1$ are given by

$$\varrho(\gamma_{n-k+1}^*)\, e_{\underline{j}} = \big(id - \varrho(t_{n,k}^*)\big) \cdot \big(id - \varrho(t_{n-1,k}^*)\big) \cdots \big(id - \varrho(t_{k+1,k}^*)\big)\, e_{\underline{j}}$$

$$\varrho(\delta_{n-k+1}^*)\, e_{\underline{j}} = \big(id - \varrho((t_k^*)^2)\, \varrho(t_{n,k+1}^*)\big) \cdot \big(id - \varrho((t_k^*)^2)\, \varrho(t_{n-1,k+1}^*)\big) \cdots \big(id - \varrho((t_k^*)^2)\, \varrho(t_{k+2,k+1}^*)\big) \cdot \big(id - \varrho((t_k^*)^2)\big)\, e_{\underline{j}}$$

which in matrix notation leads to the following matrix factorizations

$$\mathbf{C}_{Q,n-k+1} = (\mathbf{I} - \mathbf{T}_{n,k}) \cdot (\mathbf{I} - \mathbf{T}_{n-1,k}) \cdots (\mathbf{I} - \mathbf{T}_{k+1,k})$$

$$\mathbf{D}_{Q,n-k+1} = \big(\mathbf{I} - (\mathbf{T}_k)^2\, \mathbf{T}_{n,k+1}\big) \cdot \big(\mathbf{I} - (\mathbf{T}_k)^2\, \mathbf{T}_{n-1,k+1}\big) \cdots \big(\mathbf{I} - (\mathbf{T}_k)^2\big)$$

where

$$\mathbf{C}_{Q,l} := \varrho(\gamma_l^*), \qquad \mathbf{D}_{Q,l} := \varrho(\delta_l^*), \qquad 2 \leq l \leq n.$$

Clearly, $(\mathbf{T}_k)^2 = (\mathbf{T}_{k+1,k})^2$ is the diagonal matrix.

Thus we obtain

1. $$\mathbf{B}_{Q,n-k+1} = \mathbf{D}_{Q,n-k+1} \cdot \left(\mathbf{C}_{Q,n-k+1}\right)^{-1} \qquad \text{for all} \quad 1 \le k \le n-1.$$

   what we can write in the form

   $$\mathbf{B}_{Q,n-k+1} = \prod_{k+1 \le m \le n}^{\leftarrow} \left(\mathbf{I} - (\mathbf{T}_k)^2 \, \mathbf{T}_{m,k+1}\right) \cdot \prod_{k+1 \le m \le n}^{\rightarrow} \left(\mathbf{I} - \mathbf{T}_{m,k}\right)^{-1}.$$

2. $$\mathbf{A}_Q = \prod_{1 \le k \le n-1}^{\leftarrow} \left(\mathbf{B}_{Q,n-k+1}\right) \left(= \prod_{2 \le k \le n} \mathbf{B}_{Q,k} \right)$$

   i.e

   $$\mathbf{A}_Q = \prod_{1 \le k \le n-1}^{\leftarrow} \left(\prod_{k+1 \le m \le n}^{\leftarrow} \left(\mathbf{I} - (\mathbf{T}_k)^2 \, \mathbf{T}_{m,k+1}\right) \cdot \prod_{k+1 \le m \le n}^{\rightarrow} \left(\mathbf{I} - \mathbf{T}_{m,k}\right)^{-1} \right).$$

Now it is easy to see that

- for computing $\det \mathbf{B}_{Q,n-k+1}$ and $\det \mathbf{A}_Q$ it is enough to compute

  $$\det(\mathbf{I} - \mathbf{T}_{b,a}) \quad \text{and} \quad \det(\mathbf{I} - (\mathbf{T}_{a-1})^2 \, \mathbf{T}_{b,a}), \quad \text{for all } 2 \le a < b \le n.$$

Lemma

We have the following formulas

$$(i) \qquad \det(\mathbf{I} - \mathbf{T}_{b,a}) = \prod_{T \in \binom{Q}{b-a+1}} (1 - \sigma_T)^{(b-a)! \cdot (n-b+a-1)!} \qquad (1 \leq a < b \leq n)$$

$$(ii) \qquad \det(\mathbf{I} - (\mathbf{T}_{a-1})^2\, \mathbf{T}_{b,a}) = \prod_{T \in \binom{Q}{b-a+2}} (1 - \sigma_T)^{(b-a)! \cdot (b-a+2) \cdot (n-b+a-2)!}$$

$$(1 < a \leq b \leq n),$$

where for any subset T, $\sigma_T = \prod_{\{i \neq j\} \subset T} \sigma_{ij} = \prod_{i \neq j \in T} q_{ij}.$

- This Lemma is the twisted group algebra analogue of the Lemma 1.9.1 in the paper of Svrtan and Meljanac [1].
  Therefore the proof will be similar to the proof of Lemma 1.9.1 (only it will be here use the factorizations in different direction).

[1] Meljanac, S., and Svrtan, D., **Determinants and inversion of Gram matrices in Fock representation of q_{kl}-canonical commutation relations and applications to hyperplane arrangements and quantum groups. Proof of an extension of Zagier's conjecture**, 2003, arXiv:math-ph/0304040vl

Theorem

Let $\varrho\colon \mathcal{A}(S_n) \to \mathrm{End}(\mathcal{B}_Q)$ *be a twisted regular representation* (*where* $\mathcal{B}_Q$ *is generic subspace of* $\mathcal{B}$). *Then we have*

$$(i) \qquad \det \mathbf{A}_Q = \prod_{2 \le m \le n} \prod_{T \in \binom{Q}{m}} (1 - \sigma_T)^{(m-2)! \cdot (n-m+1)!},$$

$$(ii) \qquad \det \mathbf{B}_{Q,n-k+1} = \prod_{2 \le m \le n-k+1} \prod_{T \in \binom{Q}{m}} (1 - \sigma_T)^{(m-2)! \cdot (n-m)!}, \quad (1 \le k \le n-1).$$

Here we will use the following properties

$$\mathbf{B}_{Q,n-k+1} = \mathbf{D}_{Q,n-k+1} \cdot (\mathbf{C}_{Q,n-k+1})^{-1} \qquad \text{for all} \quad 1 \le k \le n-1,$$

$$\mathbf{A}_Q = \overset{\leftarrow}{\prod_{1 \le k \le n-1}} (\mathbf{B}_{Q,n-k+1}) \left(= \prod_{2 \le k \le n} \mathbf{B}_{Q,k} \right)$$

(recall: $\mathbf{B}_{Q,1} = \mathbf{I}$).

Proof.

First part:

By using previous Lemma we get the following:

$$\begin{aligned}\det \mathbf{C}_{Q,n-k+1} &= \prod_{k+1\leq p\leq n} \det(\mathbf{I} - \mathbf{T}_{p,k}) \\ &= \prod_{k+1\leq p\leq n} \prod_{T\in\binom{Q}{p-k+1}} (1-\sigma_T)^{(p-k)!\cdot(n-p+k-1)!} \\ &= \prod_{2\leq m\leq n-k+1} \prod_{T\in\binom{Q}{m}} (1-\sigma_T)^{(m-1)!\cdot(n-m)!}\end{aligned}$$

$$\begin{aligned}\det \mathbf{D}_{Q,n-k+1} &= \prod_{k+1\leq p\leq n} \det(\mathbf{I} - (\mathbf{T}_k)^2\, \mathbf{T}_{p,k+1}) \\ &= \prod_{k+1\leq p\leq n} \prod_{T\in\binom{Q}{p-k+1}} (1-\sigma_T)^{(p-k-1)!\cdot(p-k+1)\cdot(n-p+k-1)!} \\ &= \prod_{2\leq m\leq n-k+1} \prod_{T\in\binom{Q}{m}} (1-\sigma_T)^{(m-2)!\cdot m\cdot(n-m)!}\end{aligned}$$

□

Proof.

Second part:

Therefore by applying the formula $\det \mathbf{B}_{Q,n-k+1} = \dfrac{\det \mathbf{D}_{Q,n-k+1}}{\det \mathbf{C}_{Q,n-k+1}}$ we get:

$$\det \mathbf{B}_{Q,n-k+1} = \prod_{2\le m\le n-k+1} \prod_{T\in\binom{Q}{m}} (1-\sigma_T)^{(m-2)!\cdot(n-m)!}.$$

On the other hand by applying $\det \mathbf{A}_Q = \overleftarrow{\prod_{1\le k\le n-1}} \det \mathbf{B}_{Q,n-k+1}$ we get

$$\begin{aligned}\det \mathbf{A}_Q &= \overleftarrow{\prod_{1\le k\le n-1}} \prod_{2\le m\le n-k+1} \prod_{T\in\binom{Q}{m}} (1-\sigma_T)^{(m-2)!\cdot(n-m)!} \\ &= \prod_{2\le m\le n} \prod_{T\in\binom{Q}{m}} (1-\sigma_T)^{(m-2)!\cdot(n-m)!\cdot(n-m+1)}\end{aligned}$$

i.e

$$\det \mathbf{A}_Q = \prod_{2\le m\le n} \prod_{T\in\binom{Q}{m}} (1-\sigma_T)^{(m-2)!\cdot(n-m+1)!}.$$

□

THANK YOU FOR YOUR ATTENTION