Analog geometry in an expanding fluid from AdS/CFT perspective

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(Dated: November 24, 2014)

The dynamics of an expanding hadron fluid at temperatures below the chiral transition is studied in the framework of AdS/CFT correspondence. We establish a correspondence between the asymptotic AdS geometry in the the 4+1-dimensional bulk with the analog spacetime geometry on its 3+1 dimensional boundary with the background fluid undergoing a spherical Bjorken type expansion. The analog metric tensor on the boundary depends locally on the soft pion dispersion relation and the four-velocity of the fluid. The AdS/CFT correspondence provides a relation between the pion velocity and the critical temperature of the chiral phase transition.

PACS numbers: 04.70.-s ,11.25.Tq, 12.38.Mh, 47.75.+f
Keywords: analogue gravity, AdS/CFT correspondence, chiral phase transition, linear sigma model, Bjorken expansion

I. INTRODUCTION

The original formulation of gauge-gravity duality in the form of the so called AdS/CFT correspondence establishes an equivalence of a four dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills theory and string theory in a ten dimensional $\mathrm{AdS}_5 \times \mathrm{S}_5$ bulk [1–3]. However, the AdS/CFT correspondence goes beyond pure string theory as it links many other important theoretical and phenomenological issues such as fluid dynamics [4], thermal field theories, black hole physics, quark-gluon plasma [5], gravity and cosmology. In particular, the AdS/CFT correspondence proved to be useful in studying some properties of strongly interacting matter [6] described at the fundamental level by a theory called quantum chromodynamics (QCD), although $\mathcal{N} = 4$ supersymmetric Yang-Mills theory differs substantially from QCD.

Our purpose is to study in terms of the AdS/CFT correspondence a class of field theories with spontaneously broken symmetry restored at finite temperature. Spontaneous symmetry breaking is related to many phenomena in physics, such as, superfluidity, superconductivity, ferro-magnetism, Bose-Einstein condensation etc. One well known example which will be studied here in some detail is the chiral symmetry breaking in strong interactions. At low energies, the QCD vacuum is characterized by a non-vanishing expectation value [7]: $\langle\bar{\psi}\psi\rangle \approx (235\ \mathrm{MeV})^3$, the so called chiral condensate. This quantity describes the density of quark-antiquark pairs found in the QCD vacuum and its non-vanishing value is a manifestation of chiral symmetry breaking [8]. The chiral symmetry is restored at finite temperature through a chiral phase transition which is believed to be first or second order depending on the underlying global symmetry [9].

In the temperature range below the chiral transition point the thermodynamics of quarks and gluons may be investigated using the linear sigma model [10] which serves as an effective model for the low-temperature phase of QCD [11, 12]. The original sigma model is formulated as spontaneously broken φ^4 theory with four real scalar fields which constitute the $(\frac{1}{2},\frac{1}{2})$ representation of the chiral SU(2) × SU(2). Hence, the model falls in the O(4) universality class owing to the isomorphism between the groups O(4) and SU(2) × SU(2). We shall consider here a linear sigma model with spontaneously broken $\mathrm{O}(N)$ symmetry, where $N \geq 2$. According to the Goldstone theorem, the spontaneous symmetry breaking yields massless particles called *Goldstone bosons* the number of which depends on the rank of the remaining unbroken symmetry. In the case of the $\mathrm{O}(N)$ group in the symmetry broken phase, i.e., at temperatures below the point of the phase transition, there will be $N-1$ Goldstone bosons which we will call the *pions*. In the symmetry broken phase the pions, in spite of being massless, propagate slower than light owing to finite temperature effects [13–16]. Moreover, the pion velocity approaches zero at the critical temperature. In the following we will use the term "chiral fluid" to denote a hadronic fluid in the symmetry broken phase consisting predominantly of massless pions. In our previous papers [17, 18] we have demonstrated that perturbations in the chiral fluid undergoing a radial Bjorken expansion propagate in curved geometry described by an effective analog metric of the Friedmann Robertson Walker (FRW) type with hyperbolic spatial geometry

As an application of the AdS/CFT duality in terms of D7-brane embeddings [19] the chiral phase transition can be regarded as a transition from the Minkowski to black hole embeddings of the D7-brane in a D3-brane background. This has been exploited by Mateos, Myers, and Thomson [20] who find a strong first order phase transition. In this paper we consider a model of a brane world universe in which the chiral fluid lives on the 3+1

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dimensional boundary of the AdS_5 bulk. We will combine the linear sigma model with a boost invariant spherically symmetric Bjorken type expansion [21] and use AdS/CFT techniques to establish a relation between the effective analog geometry on the boundary and the bulk geometry which satisfies the field equations with negative cosmological constant. The formalism presented here could also be applied to the calculation of two point functions, Willson loops, and entanglement entropy for a spherically expanding Yang-Mills plasma as it was recently done by Pedraza [22] for a linearly expanding $\mathcal{N}=4$ supersymmetric Yang-Mills plasma.

The remainder of the paper is organized as follows. In Sec. II we describe the dynamics of the chiral fluid and the corresponding 3+1 dimensional analog geometry. In Sec. III we solve the Einstein equations in the bulk using the metric ansatz that respects the spherical boost invariance of the fluid energy-momentum tensor at the boundary. We demonstrate a relationship of our solution with the D3-brane solution of 10 dimensional supergravity. In Sec. IV we establish a connection of the bulk geometry with the analog geometry on the boundary and derive the temperature dependence of the pion velocity. In the concluding section, Sec. V, we summarize our results and discuss physical consequences.

II. EXPANDING HADRONIC FLUID

Consider a linear sigma model in a background medium at finite temperature in a general curved spacetime. The dynamics of mesons in such a medium is described by an effective action with $O(N)$ symmetry [16]

$$S_{\rm eff}=\int d^4x\sqrt{-G}\left[\frac{1}{2}G^{\mu\nu}\partial_\mu\mathbf{\Phi}\partial_\nu\mathbf{\Phi}\right.$$
$$\left.-\frac{c_\pi}{f^2}\left(\frac{m_0^2}{2}\mathbf{\Phi}^2+\frac{\lambda}{4}(\mathbf{\Phi}^2)^2\right)\right], \tag{1}$$

where $\mathbf{\Phi}$ denotes a multicomponent scalar field $\mathbf{\Phi}\equiv(\Phi_1,...,\Phi_N)$. The effective metric tensor, its inverse, and its determinant are

$$G_{\mu\nu}=\frac{f}{c_\pi}[g_{\mu\nu}-(1-c_\pi^2)u_\mu u_\nu], \tag{2}$$

$$G^{\mu\nu}=\frac{c_\pi}{f}\left[g^{\mu\nu}-(1-\frac{1}{c_\pi^2})u^\mu u^\nu,\right], \tag{3}$$

$$G\equiv\det G_{\mu\nu}=\frac{f^4}{c_\pi^2}\det g_{\mu\nu}, \tag{4}$$

respectively, where u^μ is the velocity of the fluid and $g^{\mu\nu}$ is the background metric. The coefficient f and the pion velocity c_π depend on the local temperature T and on the parameters λ and m_0^2 of the model, and may be calculated in perturbation theory. At zero temperature the medium is absent in which case $f=c_\pi=1$ and $G_{\mu\nu}$ is identical to $g_{\mu\nu}$.

If $m_0^2<0$ the symmetry will be spontaneously broken. At zero temperature the Φ_i fields develop non-vanishing vacuum expectation values such that

$$\sum_i\langle\Phi_i\rangle^2=-\frac{m_0^2}{\lambda}\equiv f_\pi^2. \tag{5}$$

We redefine the fields

$$\Phi_i(x)\to\langle\Phi_i\rangle+\varphi_i(x), \tag{6}$$

so that φ_i represent fluctuations around the vacuum expectation values $\langle\Phi_i\rangle$. It is convenient to choose here

$$\langle\Phi_i\rangle=0\ \text{ for }\ i=1,2,..,N-1,\qquad\langle\Phi_N\rangle=f_\pi. \tag{7}$$

At nonzero temperature the quantity $\langle\Phi_N\rangle$, usually referred to as the condensate, is temperature dependent and vanishes at the point of phase transition. In view of the usual physical meaning of the φ-fields in the chiral $SU(2)\times SU(2)$ sigma model it is customary to denote the $N-1$ dimensional vector $(\varphi_1,...,\varphi_{N-1})$ by $\boldsymbol{\pi}$, the field φ_N by σ, and the condensate $\langle\Phi_N\rangle$ by $\langle\sigma\rangle$. We obtain the effective Lagrangian in which the $O(N)$ symmetry is explicitly broken down to $O(N-1)$:

$$\mathcal{L}_{\rm eff}=\frac{1}{2}G^{\mu\nu}\partial_\mu\boldsymbol{\varphi}\partial_\nu\boldsymbol{\varphi}-\frac{c_\pi}{f^2}\left(\frac{m_\sigma^2}{2}\sigma^2+\frac{m_\pi^2}{2}\boldsymbol{\pi}^2\right.$$
$$\left.+\lambda\langle\sigma\rangle\sigma\boldsymbol{\varphi}^2+\frac{\lambda}{4}(\boldsymbol{\varphi}^2)^2+V_0\right), \tag{8}$$

where V_0 is a constant that depends on $\langle\sigma\rangle$,

$$V_0=\frac{\lambda}{4}\langle\sigma\rangle^4-\frac{\lambda}{2}f_\pi^2\langle\sigma\rangle^2. \tag{9}$$

At and above a critical temperature $T_{\rm c}$ the symmetry will be restored and all the mesons will have the same mass. Below the critical temperature the meson masses are given by

$$m_\pi^2=0,\quad m_\sigma^2=2\lambda\langle\sigma\rangle^2. \tag{10}$$

The temperature dependence of $\langle\sigma\rangle$ is obtained by minimizing the thermodynamic potential

$$\Omega=-\frac{T}{V}\ln\int[d\varphi]\exp\left(-\int d^4x\mathcal{L}_{\rm E}[\varphi]\right) \tag{11}$$

with respect to $\langle\sigma\rangle$ [12]. The integral in the exponent goes over the large space volume V and the euclidean time interval $(0,1/T)$. The Lagrangian $\mathcal{L}_{\rm E}$ is the Euclidean version of (8) in flat spacetime,

$$\mathcal{L}_{\rm E}=\frac{1}{2}(\partial_\mu\boldsymbol{\varphi})^2+\frac{m_\sigma^2}{2}\sigma^2+\frac{m_\pi^2}{2}\boldsymbol{\pi}^2$$
$$+\lambda\langle\sigma\rangle\sigma\boldsymbol{\varphi}^2+\frac{\lambda}{4}(\boldsymbol{\varphi}^2)^2+V_0. \tag{12}$$

Given N, f_π, and m_σ, the extremum condition can be solved numerically at one loop order [12]. In this way, the value $T_c = 183$ MeV of the critical temperature was found [18] for $N = 4$, $f_\pi = 92.4$ MeV, and $m_\sigma = 1$ GeV as a phenomenological input.

Next we assume that the background medium, consisting of predominantly pions and a small number of heavier hadrons, is going through a Bjorken type expansion. A realistic hydrodynamic model of heavy ion collisions involves a transverse expansion superimposed on a longitudinal boost invariant expansion. Here we will consider a radial boost invariant expansion [23] in Minkowski background spacetime. A similar model has been previously studied in the context of disoriented chiral condensate [21]. Our approach is in spirit similar to that of Janik and Peschanski [24, 25] who consider a hydrodynamic model based on a longitudinal Bjorken expansion and neglect the transverse expansion. A spherically symmetric Bjorken expansion considered here is certainly not the best model for description of high energy heavy ion collisions but is phenomenologically relevant in the context of hadron production in e^+e^-.

The Bjorken expansion is defined by a specific choice of the fluid 4-velocity which may be regarded as a coordinate transformation in Minkowski spacetime. In radial coordinates $x^\mu = (t, r, \vartheta, \varphi)$ the fluid four-velocity of the radial Bjorken expansion is given by

$$u^\mu = (\gamma, \gamma v_r, 0, 0) = (t/\tau, r/\tau, 0, 0), \tag{13}$$

where $v_r = r/t$ is the radial three-velocity and $\tau = \sqrt{t^2 - r^2}$ is the *proper time*. It is convenient to introduce the so called *radial rapidity* variable y and parameterize the four-velocity as

$$u^\mu = (\cosh y, \sinh y, 0, 0), \tag{14}$$

so that the radial three-velocity is

$$v_r = \tanh y. \tag{15}$$

Now, it is natural to use the spherical rapidity coordinates (τ, y) defined by the following transformation

$$\begin{aligned} t &= \tau \cosh y, \\ r &= \tau \sinh y. \end{aligned} \tag{16}$$

As in these coordinates the velocity components are $u^\mu = (1, 0, 0, 0)$, the new coordinate frame is comoving. The transformation from $(t, r, \vartheta, \varphi)$ to $(\tau, y, \vartheta, \varphi)$ takes the background Minkowski metric into

$$g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -\tau^2 & & \\ & & -\tau^2 \sinh^2 y & \\ & & & -\tau^2 \sinh^2 y \sin^2 \vartheta \end{pmatrix}, \tag{17}$$

which describes the geometry of the Milne cosmological model [26] – a homogeneous, isotropic, expanding universe with the cosmological scale $a = \tau$ and negative spatial curvature.

The functional dependence of the fluid temperature T on τ can be derived from energy-momentum conservation. First, we assume that our fluid is conformal, i.e., that its energy momentum is traceless $T^\mu_\mu = 0$. Assuming quite generally that the pressure is not isotropic

$$T^\mu_\nu = \mathrm{diag}(\rho, -p_y, -p_\perp, -p_\perp), \tag{18}$$

the tracelessness implies

$$\rho - p_y - 2p_\perp = 0. \tag{19}$$

On the other hand, the energy momentum conservation

$$T^{\mu\nu}{}_{;\nu} = 0 \tag{20}$$

yields

$$\partial_\tau \rho + \frac{1}{\tau}(p_y + 2p_\perp + 3\rho) = 0, \quad p_\perp = p_y \equiv p. \tag{21}$$

This implies that the fluid is perfect with energy-momentum tensor

$$T_{\mu\nu} = (p + \rho) u_\mu u_\nu - p g_{\mu\nu}, \tag{22}$$

where p and ρ denote respectively the pressure and the energy density of the fluid. Equation (21) is equivalent to

$$u^\mu \rho_{;\mu} + (p + \rho) u^\mu{}_{;\mu} = 0, \tag{23}$$

where the subscript $;\mu$ denotes the covariant derivative associated with the background metric (17). Either from (21) or (23) one finds

$$\frac{\partial \rho}{\partial \tau} + \frac{4\rho}{\tau} = 0, \tag{24}$$

with solution

$$\rho = \left(\frac{c_0}{\tau}\right)^4. \tag{25}$$

The dimensionless constant c_0 may be fixed from the phenomenology of high energy collisions. For example, with a typical value of $\rho = 1$ GeVfm$^{-3} \approx 5$ fm^{-4} at $\tau \approx 5$ fm [27] we find $c_0 = 7.5$.

Now, the energy momentum tensor in comoving coordinates reads

$$T^{\mathrm{conf}}_{\mu\nu} = \frac{c_0^4}{3\tau^4} \begin{pmatrix} 3 & & & \\ & \tau^2 & & \\ & & \tau^2 \sinh^2 y & \\ & & & \tau^2 \sinh^2 y \sin^2 \vartheta \end{pmatrix}. \tag{26}$$

Equation (25) implies that the temperature of the expanding chiral fluid is, to a good approximation, proportional to τ^{-1}. This follows from the fact that the expanding hadronic matter is dominated by massless pions, and hence, the pressure of the fluid may be approximated by [28]

$$p = \frac{1}{3}\rho = \frac{\pi^2}{90}(N - 1) T^4. \tag{27}$$

This approximation is justified as long as we are not very close to $T = 0$ in which case we may neglect the contribution of the condensate with vacuum energy equation of state $p = -\rho$. Moreover, this approximation is consistent with the conformal fluid assumption which also fails at and near $T = 0$, because the energy momentum tensor of the vacuum is proportional to the metric tensor and hence $T^\mu_\mu \neq 0$ in the vicinity of $T = 0$. Combining (27) with (25) one finds

$$T = \left(\frac{30}{\pi^2(N-1)}\right)^{1/4} \frac{c_0}{\tau}. \tag{28}$$

Hence, the temperature and proper time are uniquely related. For example, there is a unique proper time τ_c which corresponds to the critical temperature T_c of the chiral phase transition, so that

$$\frac{T}{T_c} = \frac{\tau_c}{\tau}. \tag{29}$$

If we adopt the value $c_0 = 7.5$ estimated above and $T_c = 0.183$ GeV $= 0.927$ fm^{-1} [18] as the critical temperature for $N = 4$, the corresponding proper time will be $\tau_c = 8.2$ fm $= 41$ GeV^{-1}.

If we relax the conformal fluid condition $T^\mu_\mu = 0$ and add the contribution of the vacuum to the conformal part, the energy momentum tensor will read

$$T_{\mu\nu} = T^{\rm conf}_{\mu\nu} + \rho_{\rm vac} g_{\mu\nu}, \tag{30}$$

where $\rho_{\rm vac}$ is a constant vacuum energy density and $T^{\rm conf}_{\mu\nu}$ is the conformal part given by (26). Then, instead of (24), we obtain

$$\frac{\partial\rho}{\partial\tau} + \frac{4(\rho - \rho_{\rm vac})}{\tau} = 0, \tag{31}$$

with solution

$$\rho = \left(\frac{c_0}{\tau}\right)^4 + \rho_{\rm vac}. \tag{32}$$

In this case, instead of (27), the equation of state reads

$$p + \rho_{\rm vac} = \frac{1}{3}(\rho - \rho_{\rm vac}) = \frac{\pi^2}{90}(N-1)T^4, \tag{33}$$

and we obtain precisely the same relation (29) between the temperature and the proper time.

In the comoving coordinate frame defined by the coordinate transformation (16) the analog metric (2) is diagonal with line element

$$ds^2 = G_{\mu\nu}dx^\mu dx^\nu = a d\tau^2 - b\tau^2(dy^2 + \sinh^2 y d\Omega^2), \tag{34}$$

where

$$a = f c_\pi, \quad b = \frac{f}{c_\pi}. \tag{35}$$

Here, the parameters f and c_π are functions of τ through the temperature dependence on τ (28). Hence, this metric represents an FRW spacetime with negative spatial curvature.

At nonzero temperature, f and c_π can be derived from the finite temperature self energy $\Sigma(q, T)$ in the limit when the external momentum q approaches zero and can be expressed in terms of second derivatives of $\Sigma(q, T)$ with respect to q_0 and q_i. The quantities f and c_π as functions of temperature have been calculated at one loop level by Pisarski and Tytgat [13] in the low temperature approximation

$$f \sim 1 - \frac{T^2}{12 f_\pi^2} - \frac{\pi^2}{9}\frac{T^4}{f_\pi^2 m_\sigma^2}, \quad c_\pi \sim 1 - \frac{4\pi^2}{45}\frac{T^4}{f_\pi^2 m_\sigma^2}, \tag{36}$$

and by Son and Stephanov for temperatures close to the chiral transition point [14, 15] (see also [17]). Whereas the low temperature result (36) does not depend on N, the result near the critical temperature does. In the limit $T \to T_c$ in $d = 3$ dimensions one finds [14, 15]

$$f \propto (1 - v)^{\nu - 2\beta}, \quad c_\pi \propto (1 - v)^{\nu/2} \tag{37}$$

where $v = \tau_c/\tau = T/T_c$ and the critical exponents ν and β depend on N. For example, $\nu = 0.749$ and $\beta = 0.388$ for the O(4) universality class [29, 30]. Combining these limiting cases with the numerical results at one loop order [18], a reasonable fit in the entire range $0 \leq T \leq T_c$ is provided by

$$f = (1 - v^4)^{\nu - 2\beta}, \qquad c_\pi = (1 - v^4)^{\nu/2}. \tag{38}$$

With this we have $f = c_\pi = 1$ at $T = 0$, $c_\pi^2 \simeq 1 - \nu(T/T_c)^4$ near $T = 0$ as predicted by the one loop low temperature approximation [13], and we recover the correct behavior (37) near $T = T_c$.

Note that the spacetime described by (34) with (35) and (37) has a curvature singularity at $\tau = \tau_c$. The Ricci scalar corresponding to (34) is given by

$$R = \frac{6}{\tau^2 a} - \frac{6}{\tau^2 b} - \frac{3\dot{a}}{\tau a^2} + \frac{12\dot{b}}{\tau a b} - \frac{3}{2}\frac{\dot{a}\dot{b}}{a^2 b} + \frac{3\ddot{b}}{ab}, \tag{39}$$

where the overdot denotes a derivative with respect to τ. Using (37) in the limit $\tau \to \tau_c$ one finds that R diverges as

$$R \sim (\tau - \tau_c)^{2\beta - 3\nu/2 - 2}. \tag{40}$$

III. HOLOGRAPHIC DESCRIPTION OF THE HADRONIC FLUID

We now turn to AdS/CFT correspondence and look for a five-dimensional bulk geometry dual to the four-dimensional spherically expanding chiral fluid described by the energy momentum tensor (22). A general asymptotically AdS metric in Fefferman-Graham coordinates [31] is of the form

$$ds^2 = g_{AB}dx^A dx^B = \frac{\ell^2}{z^2}\left(h_{\mu\nu}dx^\mu dx^\nu - dz^2\right), \tag{41}$$

where we use the uppercase Latin alphabet for bulk indices and the Greek alphabet for 3+1 spacetime indices. Our curvature conventions are as follows: $R^a{}_{bcd} = \partial_c\Gamma^a_{db} - \partial_d\Gamma^a_{cb} + \Gamma^e_{db}\Gamma^a_{ce} - \Gamma^e_{cb}\Gamma^a_{de}$ and $R_{ab} = R^s{}_{asb}$, so that Einstein's equations are $R_{ab} - \frac{1}{2}Rg_{ab} = +\kappa T_{ab}$.

The length scale ℓ is the AdS curvature radius related to the cosmological constant by

$$\Lambda = -6/\ell^2. \tag{42}$$

The four dimensional tensor $h_{\mu\nu}$ may be expanded near the boundary at $z = 0$ as [32]

$$h_{\mu\nu} = g^{(0)}_{\mu\nu} + z^2 g^{(2)}_{\mu\nu} + z^4 g^{(4)}_{\mu\nu} + z^6 g^{(6)}_{\mu\nu} + \dots\ , \tag{43}$$

where $g^{(0)}_{\mu\nu}$ is the background metric on the boundary.

A. Ricci flat boundary

Let us assume now that the boundary geometry is described by the Ricci flat spacetime metric (17). According to the holographic renormalization rules [32] in this case $g^{(2)}_{\mu\nu} = 0$ and $g^{(4)}_{\mu\nu}$ is proportional to the vacuum expectation value of the energy-momentum tensor

$$g^{(4)}_{\mu\nu} = -\frac{4\pi G_5}{\ell^3}\langle T^{\rm conf}_{\mu\nu}\rangle, \tag{44}$$

where G_5 is the five dimensional Newton constant and the expectation value on the righthand side is assumed to be equal to the energy momentum tensor (26). This equation is an explicit realization of the AdS/CFT prescription that the field dual to the energy momentum tensor $T_{\mu\nu}$ should be the four-dimensional metric $g_{\mu\nu}$.

Instead of a linear boost invariance of [24, 25, 33], we impose a spherically symmetric boost invariance in the bulk. The most general metric respecting the spherically symmetric boost invariance in Fefferman-Graham coordinates is of the form

$$ds^2 = \frac{\ell^2}{z^2}\left[A(z,\tau)d\tau^2 - \tau^2 B(z,\tau)(dy^2 + \sinh^2 y d\Omega^2) - dz^2\right]. \tag{45}$$

Equations (43) and (44) together with (26) imply the conditions near the $z = 0$ boundary

$$A = 1 - 3kz^4/\tau^4 + \dots\ , \quad B = 1 + kz^4/\tau^4 + \dots\ , \tag{46}$$

where

$$k = \frac{4\pi G_5 c_0^4}{3\ell^3} \tag{47}$$

is a dimensionless constant.

Using the metric ansatz (45) we solve Einstein's equations with negative cosmological constant

$$R_{AB} - \left(\frac{1}{2}R - \frac{6}{\ell^2}\right)g_{AB} = 0. \tag{48}$$

By inspecting the components of (48) subject to (45), it may be verified that Einstein's equations are invariant under simultaneous rescaling

$$\tau \to \lambda\tau, \quad z \to \lambda z, \tag{49}$$

for any real positive λ. In other words, if $A = A(z,\tau)$ and $B = B(z,\tau)$ are solutions to (48), then so are $A = A(\lambda z, \lambda\tau)$ and $B = B(\lambda z, \lambda\tau)$. This implies that A and B are functions of a single scaling variable

$$v = \frac{z}{\tau}. \tag{50}$$

From the zz and $z\tau$ components of Einstein's equations we find two independent differential equations for A and B

$$\frac{2A'}{vA} + \frac{A'^2}{A^2} + \frac{6B'}{vB} + \frac{3B'^2}{B^2} - \frac{2A''}{A} - \frac{6B''}{B} = 0, \tag{51}$$

$$\frac{6A'}{A} - \frac{vA'B'}{AB} - \frac{3vB'^2}{B^2} + \frac{6vB''}{B} = 0, \tag{52}$$

where the prime denotes a derivative with respect to v. It may be easily verified that the functions

$$A(v) = \frac{(1 - kv^4)^2}{1 + kv^4}, \tag{53}$$

$$B(v) = 1 + kv^4. \tag{54}$$

satisfy (51), (52), and the remaining set of Einstein's equations, k being an arbitrary constant. Equations (53) and (54) satisfy the boundary conditions (46) if we identify k with the constant defined in (47). The line element (45) becomes

$$ds^2 = \frac{\ell^2}{z^2}\left[\frac{(1 - kz^4/\tau^4)^2}{1 + kz^4/\tau^4}d\tau^2 - (1 + kz^4/\tau^4)\tau^2(dy^2 + \sinh^2 y d\Omega^2) - dz^2\right]. \tag{55}$$

This type of metric is a special case of a more general solution derived by Apostolopoulos, Siopsis, and Tetradis [34, 35] with an arbitrary FRW cosmology at the boundary.

It is useful to compare the solution (55) with the static Schwarzschild-AdS$_5$ metric [36]

$$ds^2 = \frac{\ell^2}{z^2}\left[\frac{(1 - z^4/z_0^4)^2}{1 - \kappa z^2/(2\ell^2) + z^4/z_0^4}d\tau^2 - (1 - \kappa z^2/(2\ell^2) + z^4/z_0^4)\ell^2 d\Omega_3^2(\kappa) - dz^2\right], \tag{56}$$

where $\kappa = 0, 1, -1$ for a flat, spherical and hyperbolic boundary geometry with

$$d\Omega_3^2(\kappa) = \begin{cases} dy^2 + \sinh^2 y d\Omega^2, & \kappa = -1 \\ dy^2 + y^2 d\Omega^2, & \kappa = 0 \\ dy^2 + \sin^2 y d\Omega^2, & \kappa = 1. \end{cases} \tag{57}$$

The location of the horizon z_0 is related to the BH mass as

$$z_0^4 = \frac{16\ell^4}{4\mu + \kappa^2}, \tag{58}$$

where μ is the BH mass in units of ℓ^{-1}. It has been noted [37] that (55) is obtained from (56) by keeping the conformal factor ℓ^2/z^2 and elsewhere making the replacements $\kappa \to \kappa + 1$ and $\ell \to \tau$. Hence, the constant k in (55) is related to the BH mass of the static solution as $k = \mu/4$.

It is worth emphasizing the difference between the spherically symmetric solution (55) and the solution of the similar form found in the case of linear boost invariance [24]. First, our solution (55) is *exact* and valid at all times. In contrast, the solution found in [24] of the form (53) and (54) with $v \sim z/\tau^{1/3}$ is valid only in the asymptotic regime $\tau \to \infty$. A similar late time asymptotic solution was found by Sin, Nakamura, and Kim [38] for the case of a linear anisotropic expansion described by the Kasner metric. In a related recent work Fischetti, Kastor, and Traschen [39] have constructed solutions that expand spherically and approach the Milne universe at late times. Their solution, obtained by making use of a special type of ideal fluid in addition to the negative cosmological constant in the bulk, gives rise to open FRW cosmologies at the boundary and on the Poincaré slices (which correspond to the z-slices in Fefferman Graham coordinates) of a late time asymptotic AdS_5.

Another remarkable property of the solution (55) is that the induced metric on each z-slice is equivalent to the Milne metric. This may be seen as follows. The 3+1 dimensional metric induced on a z-slice is, up to a multiplicative constant, given by

$$ds^2 = \frac{(1 - kz^4/\tau^4)^2}{1 + kz^4/\tau^4} d\tau^2 - (1 + kz^4/\tau^4)\tau^2(dy^2 + \sinh^2 y d\Omega^2). \tag{59}$$

This line element is of the form (34) and the corresponding 3+1 spacetime at a given z-slice may be regarded as an FRW spacetime. Then the coordinate transformation

$$\tilde{\tau}(\tau) = \tau(1 + kz^4/\tau^4)^{1/2} \tag{60}$$

brings the metric (59) to the Milne form (17).

The solution (55) is closely related to the D3-brane solution of 10 dimensional supergravity corresponding to a stack of $N_{\rm D}$ coincident D3-branes. A near-horizon nonextremal D3-brane metric is given by [40]

$$ds^2 = \frac{U^2}{L^2}\left[\left(1 - \frac{U_0^4}{U^4}\right)dt^2 - \frac{L^4}{U^4}\left(1 - \frac{U_0^4}{U^4}\right)^{-1} dU^2 - \sum_{i=1}^{3} dy_i^2\right] - L^2 d\Omega_5^2, \tag{61}$$

where

$$L^2 = \ell_s^2\sqrt{4\pi g_s N_{\rm D}}, \tag{62}$$

g_s is the string coupling constant, and $\ell_s = \sqrt{\alpha'}$ is the fundamental string length. Ignoring the five sphere which decouples throughout the spacetime (61), the remaining five dimensional spacetime is equivalent to the standard AdS_5 Schwarzschild spacetime in the limit of large BH mass [41] and is asymptotically AdS_5. By identifying L with the AdS curvature radius ℓ, rescaling the coordinates

$$t = \frac{\ell^2}{U_0 W_0}\tau, \quad y_i = \frac{\ell^2}{U_0 W_0}x_i, \tag{63}$$

and making a coordinate transformation $U \to W$

$$U = \frac{U_0 W_0}{W}, \tag{64}$$

the metric of the asymptotically AdS_5 bulk of (61) turns into

$$ds^2 = \frac{\ell^2}{W^2}\left[\left(1 - \frac{W^4}{W_0^4}\right)d\tau^2 - \left(1 - \frac{W^4}{W_0^4}\right)^{-1} dW^2 - \sum_{i=1}^{3} dx_i^2\right], \tag{65}$$

with the horizon at W_0 and the inverse horizon temperature

$$\beta = \pi W_0. \tag{66}$$

Furthermore, by another coordinate transformation

$$W = \frac{z}{\sqrt{1 + z^4/z_0^4}}, \tag{67}$$

where $z_0 = \sqrt{2}W_0$, we find

$$ds^2 = \frac{\ell^2}{z^2}\left[\frac{(1 - z^4/z_0^4)^2}{1 + z^4/z_0^4}d\tau^2 - (1 + z^4/z_0^4)\sum_{i=1}^{3} dx_i^2 - dz^2\right]. \tag{68}$$

This coincides with equation (56) for $\kappa = 0$ if we set $z_0^4 = 4\ell^4/\mu$. In this coordinate representation, the BH horizon is at $z = z_0$ and the inverse horizon temperature is

$$\beta = \frac{\pi}{\sqrt{2}} z_0. \tag{69}$$

In the case $\kappa = +1$ or -1 the metric (68) may be regarded as a large BH mass limit of a Schwarzschild-AdS_5 metric given by (56) with (57). In the limit $\mu \to \infty$ we have $z_0/\ell \to 0$, so $z^2/l^2 \ll 1$ for z close to the horizon and the quadratic term in the metric coefficients in (56) vanishes in that limit. Hence, taking $\mu \to \infty$ in (56) for $\kappa = +1$ or -1 one finds

$$ds^2 = \frac{\ell^2}{z^2}\left[\frac{(1 - z^4/z_0^4)^2}{1 + z^4/z_0^4}d\tau^2 - (1 + z^4/z_0^4)\ell^2 d\Omega_3^2(\kappa) - dz^2\right]. \tag{70}$$

Comparing this with (55), the geometry (55) appears as a dynamical black hole with the location of the horizon $z_0 = \tau/k^{1/4}$ moving in the bulk with velocity $k^{-1/4}$. Then the horizon temperature depends on time as

$$T = \frac{\sqrt{2}k^{1/4}}{\pi\tau}, \tag{71}$$

in agreement with Tetradis [35].

B. De Sitter boundary

It is important to analyze more closely the late time asymptotics which corresponds to low temperatures. Up until now we have studied a conformal fluid in a flat boundary spacetime and, as a consequence, the quadratic term in the expansion (43) has been absent. However, as we have mentioned in section II, in the limit $T \to 0$ the vacuum energy contribution becomes dominant and the trace of the energy momentum tensor does not vanish. Hence, the nonvanishing of the trace will certainly change the late time asymptotics. To see this in more detail, assume that at late times the energy momentum tensor is dominated by the vacuum energy term, i.e.,

$$T_{\mu\nu} = \rho_{\rm vac} g^{(0)}_{\mu\nu}. \tag{72}$$

Assume in addition that the boundary metric $g^{(0)}_{\mu\nu}$ is a solution to Einstein's equations with $T_{\mu\nu}$ as the source. Then $g^{(0)}_{\mu\nu}$ describes the de Sitter universe with line element in the $\kappa = -1$ representation

$$ds^2 = g^{(0)}_{\mu\nu} dx^\mu dx^\nu = d\tau^2 - \sinh^2\tau(dy^2 + \sinh^2 y d\Omega^2). \tag{73}$$

The coordinates τ and y in this expression are measured in units of the de Sitter curvature radius $\ell_{\rm dS}$ which is related to $\rho_{\rm vac}$ by $3/\ell^2_{\rm dS} \equiv \Lambda_{\rm dS} = 8\pi G_{\rm N}\rho_{\rm vac}$. The solution in the bulk corresponding to the above boundary geometry is given by [34]

$$ds^2 = \frac{\ell^2}{z^2}\left[\frac{F_-(\tau,z)^2}{F_+(\tau,z)}d\tau^2 - F_+(\tau,z)\sinh^2\tau d\Omega_3^2 - dz^2\right], \tag{74}$$

where

$$F_\pm(\tau,z) = 1 - \frac{z^2}{2\ell^2_{\rm dS}} + \frac{1}{16}\left(1 \pm \frac{4\mu}{\sinh^4\tau}\right)\frac{z^4}{\ell^4_{\rm dS}}. \tag{75}$$

By making use of the holographic renormalization prescription [32] we find the coefficients of the quadratic and quartic terms in the expansion (43)

$$g^{(2)}_{\mu\nu} = -\frac{1}{2\ell^2_{\rm dS}} g^{(0)}_{\mu\nu}, \tag{76}$$

$$g^{(4)}_{\mu\nu} = \frac{1}{16\ell^4_{\rm dS}} g^{(0)}_{\mu\nu} - \frac{\mu}{4\sinh^4\tau\,\ell^4_{\rm dS}} t_{\mu\nu}, \tag{77}$$

where $t_{\mu\nu}$ is a traceless tensor

$$t_{\mu\nu} = \begin{pmatrix} 3 & & & \\ & \sinh^2\tau & & \\ & & \sinh^2\tau\sinh^2 y & \\ & & & \sinh^2\tau\sinh^2 y\sin^2\vartheta \end{pmatrix}. \tag{78}$$

Then, the vacuum expectation value of the energy momentum tensor at the boundary is expressed as

$$\langle T_{\mu\nu}\rangle = \frac{\ell^3\mu}{16\pi G_5 \ell^4_{\rm dS}\sinh^4\tau} t_{\mu\nu} + \frac{3\ell^3}{64\pi G_5\ell^4_{\rm dS}} g^{(0)}_{\mu\nu}. \tag{79}$$

The first term on the right-hand side of this expression is just the conformal part of the energy momentum tensor corresponding to (26) and is obviously suppressed at large τ with respect to the second term. The second term is related to the conformal anomaly.

For consistency with (72) we demand

$$\rho_{\rm vac} = \frac{3G_5}{\pi\ell^3}\frac{1}{G^2_{\rm N}}. \tag{80}$$

This equation has an interesting implication in the cosmological context as it can be related to the holographic bound of Cohen, Kaplan, and Nelson [42]

$$\rho_{\rm vac} \lesssim \Lambda^4_{\rm UV} \lesssim \frac{m^2_{\rm Pl}}{L^2}, \tag{81}$$

where $\Lambda_{\rm UV}$ and L denote the ultraviolet and long distance cutoffs, respectively. If we compare (80) with (81) and identify L with $l_{\rm dS}$ we obtain

$$\frac{G_5}{\ell^3} \lesssim \frac{1}{l^2_{\rm dS} m^2_{\rm Pl}}, \tag{82}$$

which may be regarded as a holographic bound on the dimensionless quantity G_5/ℓ^3.

IV. THE PION VELOCITY

We next exploit the finite temperature AdS/CFT correspondence by making use of the following assumptions:

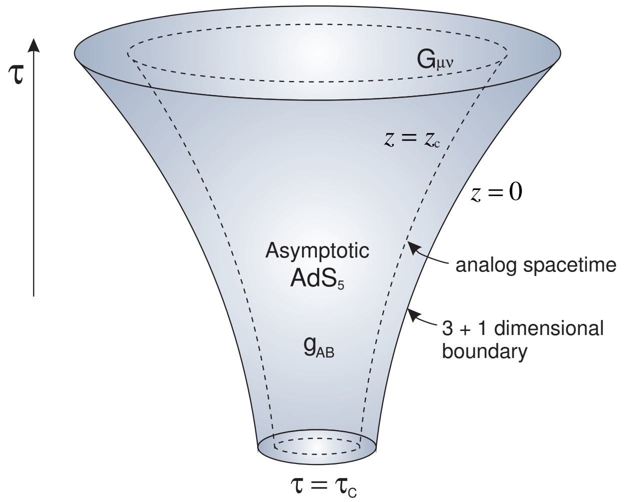

FIG. 1. An illustration of the correspondence between the asymptotic AdS geometry in the bulk with the analog spacetime geometry on its $3+1$ boundary. The induced metric on the z_c-slice corresponds to the effective metric (34) in the symmetry broken phase ($\tau > \tau_c$).

a) The horizon temperature (71) is proportional to the physical temperature of the expanding conformal fluid.

b) A correspondence exists between the fifth coordinate z and the energy scale Q such that z is proportional to $1/Q$.

c) There exist a maximal z equal to $z_{\rm c} = k^{-1/4}\tau_{\rm c}$ where the critical proper time $\tau_{\rm c}$ corresponds to the critical temperature $T_{\rm c}$.

d) The induced metric (59) on the z_c-slice corresponds to the effective metric (34) in the symmetry broken phase ($\tau > \tau_c$) in which the perturbations (massless pions) propagate.

The geometry is illustrated in Fig. 1. The first assumption stems from the relation (28) and is obviously in agreement with the Bjorken dynamics. The assumptions b) and c) are similar to those of Erlich et al. [43] who assumed the infrared cutoff at some $z = z_m$ ("infrared brane"). Our key assumption d) is motivated by the apparent resemblance of the effective analog metric (34) with (38) to the induced metric (59) where we identify

$$f = 1 - \tau_{\rm c}^4/\tau^4, \qquad c_\pi = \frac{1-\tau_{\rm c}^4/\tau^4}{1+\tau_{\rm c}^4/\tau^4}. \tag{83}$$

Owing to (29), these quantities may be written as functions of temperature

$$f = 1 - T^4/T_{\rm c}^4, \qquad c_\pi = \frac{1-T^4/T_{\rm c}^4}{1+T^4/T_{\rm c}^4}. \tag{84}$$

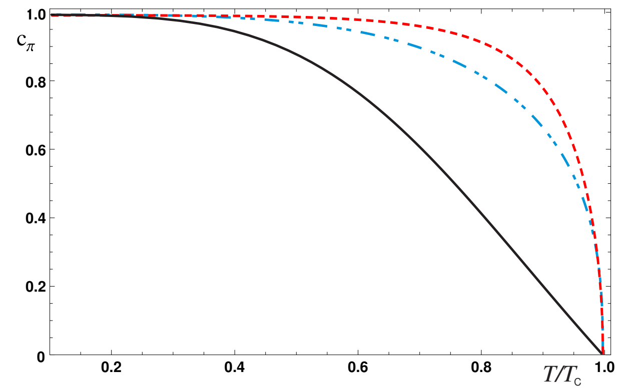

FIG. 2. Pion velocity versus $v \equiv T/T_{\rm c}$ described by three different functions: approximate model $(1-v^4)^{\nu/2}$ based on the O(4) critical exponent $\nu = 0.749$ (dot-dashed); the model based on numerical results at one loop level [18] (dashed); the AdS/CFT model (84) (full line).

The above expression for the pion velocity gives a roughly correct overall behavior in the temperature interval $(0, T_c)$ (Fig. 2). It is worth analyzing our predictions in the limiting cases of temperatures near the endpoints of this interval.

In the limit $T \to 0$ the pion velocity in (84) will agree with the low temperature approximation (36) if we identify

$$T_{\rm c} = \left(\frac{45}{2\pi^2} f_\pi^2 m_\sigma^2\right)^{1/4}. \tag{85}$$

Our result confirms the expectation [13, 44] that the deviation of the velocity squared from unity is proportional to the free energy density, or pressure which for massless pions is given by (27). Given f_π and m_σ, equation (85) can be regarded as a prediction for the critical temperature. The Particle Data Group [45] gives a rather wide range 400-1500 MeV of the sigma meson masses. With the lowest value $m_\sigma \simeq 400$ MeV and $f_\pi = 92.4$ MeV one finds the lower bound $T_{\rm c} \simeq 230$ MeV which is somewhat larger than lattice results which range between 150 and 190 MeV.

It is important to note here that we do not recover the quadratic term in the low temperature approximation (36) of the function f. The reason may be that by assuming exact conformal invariance, i.e., the condition $T^\mu_\mu = 0$, we discarded the contribution of the vacuum energy (including the condensate) which actually dominates at low temperatures or equivalently at late times as is evident in (79).

As to the limit $T \to T_{\rm c}$, the behavior of our solution differs in two aspects from what one finds in other treatments based on conventional calculations. First, the induced metric (59) being equivalent to the Milne metric is Ricci flat so the singularity at $\tau = z$ is just a coordinate singularity. In contrast, as we have mentioned at the end of Sec. II, the analog metric (34) obtained from the linear sigma model exhibits a curvature singularity at the critical point $\tau = \tau_c$. Second, we do not recover the critical exponents predicted by conventional calculations. From Fig. 2 it is clear that the critical behavior differs significantly from the prediction based on the O(4) critical exponents or the one loop sigma model prediction. In the vicinity of the critical point the function (84) approaches zero as

$$c_\pi \sim T_\mathrm{c} - T. \tag{86}$$

In contrast, the sigma model at one loop order [18] gives $c_\pi \sim (T_\mathrm{c} - T)^{1/4}$, whereas the Monte Carlo calculations of the critical exponents for the O(4) universality class [29] yields $c_\pi \sim (T_\mathrm{c} - T)^{0.37}$.

As a side remark, our bulk spacetime is free of curvature singularities. Clearly, the Ricci scalar $R = 20$ is regular everywhere. However, as noted in [24], there is a potential singularity of the Riemann tensor squared $\Re^2 \equiv R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}$ at the hypersurface $z = \tau$. A straightforward calculation yields

$$\Re^2 = 8\left(5 + \frac{144k^2v^8}{(1+kv^4)^4}\right), \tag{87}$$

which is regular everywhere. Remarkably, if one substitutes $w^4 = 3z^4/\tau^{4/3}$ for our $kv^4 = kz^4/\tau^4$ in (87) the resulting expression for $\Re^2$ will be precisely of the form obtained in the asymptotic regime $\tau \to \infty$ [24] for the case of a perfect fluid undergoing a longitudinal Bjorken expansion.

V. CONCLUSIONS

We have investigated a spherically expanding hadronic fluid in the framework of AdS_5/CFT correspondence. According to the holographic renormalization, the energy momentum tensor of the spherically expanding conformal fluid is related to the bulk geometry described by the metric (55) which satisfies the field equations with negative cosmological constant. It is remarkable that the exact correspondence exists at all times $0 \le \tau < \infty$. Based on this solution and analogy with the AdS-Schwarzschild black hole, we have established a relation between the effective analog geometry on the boundary and the bulk geometry. Assuming that the chiral fluid dynamics at finite temperature is described by the linear sigma model as the underlying field theory, we obtain a prediction for the pion velocity in the range of temperatures below the phase transition point. Compared with the existing conventional calculations, a reasonable agreement is achieved generally for those quantities, such as the pion velocity and the critical temperature, that do not substantially depend on the number of scalars N. In particular, our prediction at low temperature confirms the expectation [13, 44] that the deviation of the pion velocity from the velocity of light is proportional to the free energy density. The agreement is of course not so good for the critical exponents since their values crucially depend on N. The estimate of the critical temperature is close to but somewhat higher then the lattice QCD prediction.

Obviously, our results are based on a crude simplification that the hadronic fluid is a perfect conformal fluid undergoing a spherically symmetric radial expansion. A realistic hadronic fluid is neither perfect nor conformal. First, a hadronic fluid in general has a non vanishing shear viscosity which is neglected here. Second, our model is based on a scalar field theory with broken symmetry which is only approximately conformal in the vicinity of the critical point where the condensate vanishes and all particles (mesons and quarks) become massless. Hence, in this way we could not have obtained more then a rough estimate of the critical temperature and the pion velocity at finite temperature.

Acknowledgments

The work of S.D. and D.T. was supported by the Ministry of Science, Education and Sport of the Republic of Croatia and the work of N.B. and D.T. was partially supported by the ICTP-SEENET-MTP grant PRJ-09 “Strings and Cosmology“ in the frame of the SEENET-MTP Network.

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