Binary Orthologic with Modus Ponens Is either Orthomodular or Distributive

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Abstract. We show that binary orthologic becomes either quantum or classical logic when nothing but modus ponens rule is added to it, depending on the kind of the operation of implication used. We also show that in the usual approach the rule characterizes neither quantum nor classical logic. The difference turns out to stem from the chosen valuation on a model of a logic. Thus algebraic mappings of axioms of standard quantum logics would fail to yield an orthomodular lattice if a unary—as opposed to binary—valuation were used. Instead, non-orthomodular nontrivial varieties of orthologic are obtained. We also discuss the computational efficiency of the binary quantum logic and stress its importance for quantum computation and related algorithms.

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1 Introduction

The new field of quantum computing has developed spectacularly since its origin a few years ago. Quantum logic gates have been formulated and elementary quantum logic considered for several physical candidates for a future quantum computer. [1, 2, 3, 4] Quantum error correction theory to compensate decoherence has also been extensively formulated in the past two years. [5] Paradoxically, so far only one quantum algorithm (Shor's) is known which can be potentially used on a future quantum computer (to factor huge numbers in polynomially increasing time—as opposed to exponentially increasing time which is believed to be required by a classical computer). [6] The search for possible algorithms faces difficulties in handling quantum logic due to its particular properties.

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Both classical and quantum logics can have a probabilistic semantics, i.e., can be shown equivalent to a Kolmogorovian [7] and a quantum probability [8] theory, respectively. A simple mapping from the propositions of the logics to the interval [0,1] suffices. The problem with quantum logic—as opposed to classical—is that so obtained quantum probability cannot be used even for the simplest experiments. Quantum logic is a small algebraic core of the full probabilistic descriptions of a quantum measurement, i.e., the Hilbert space probability theory. In order to handle a superposition of states, which are the starting point of any quantum computer, the computer cannot be given the present quantum (orthomodular) logic. It can be shown that only a variety of quantum logic (quantum logic with new additional independent axioms) can be used for a description of even the simplest superposition of states if we want that a mapping from an obtained final proposition to the interval [0,1] be direct and represent a calculated result of the computer. On this mapping rather complicated conditions (*read off* from the Hilbert space structure) must be imposed although just recently a significant advance has been made when M. P. Solèr proved that an orthomodular form that has an infinite orthonormal sequence *is* a Hilbert space. [9] The reason for that is that within standard formulations of quantum logics additional axioms become rather complicated even for automated theorem proving [10] and that the problem of inferring new theorems has not been properly solved. In particular it was not clear whether modus ponens was adequately defined in various systems and whether the implications it used were appropriate.

In this paper, in Sec. 3 we show that binary orthologic with the modus ponens rule added to it is distributive when the implication used is classical and orthomodular when the implication is quantum. We then analyze the standard approach and show that the properties of logics are strongly dependent on the kind of valuation one uses in order to model a logic by a lattice. In particular, we show that the modus ponens rule in the standard approach does not structurally determine a logic. Therefore in Sec. 2 we investigate the properties of structures obtained from the standard quantum logics (using two of Kalmbach's systems as examples) by means of the unary valuation. In particular, we consider the algebra $\mathcal{A}$ of all true formulas of the logic. The algebra is a subalgebra of the algebra $\mathcal{F} = \langle \mathcal{F}^\circ, \neg, \vee \rangle$ of formulas of the logic. We investigate mapping of $\mathcal{A}$ to ortholattice OL—we call this mapping an *algebraic* mapping—by means of valuation v which turns the operations $\neg, \vee$ into $^\perp, \cup$. We obtain a collection of expressions t in OL all of which have the form $t = 1$. Because of our former result according to which the orthomodularity in an ortholattice boils down to $a \equiv b = 1 \ \Rightarrow \ a = b$ [11], it is interesting to see into which equations axioms of the logic map. It turns out that such an algebraic mapping of the axioms of any standard quantum logic is a proper non-orthomodular variety of an ortholattice. In this variety, algebraic mappings of all but one axiom turn out to be satisfied in all ortholattices. The axiom in question is however *not* the orthomodularity axiom one uses in the proof of the completeness. The algebraic mapping of the orthomodularity axiom, on the other hand, turns out to be satisfied in any ortholattice. In Sec. 3 we then show that the variety is equivalent to another non-orthomodular variety based on the *Kotas biconditional* property.

2 Standard Quantum Logic: All Modus Ponenses Are Interchangeable

In this section we approach the problem of finding an efficient modus ponens rule as the rule of inference for quantum logic by analyzing two systems of [12] which employ two different modus ponenses formulated by means of two different implications, a quantum and a classical, respectively. We first eliminate non-independent axioms and in particular an axiom (KA11 below) which has the form of the orthomodularity property on an orthomodular lattice except that the identity stands for the equality. Here we stress that axioms reflect a kind of valuation on a model of the logic. So, identity $\vdash A \equiv B$ between propositions from the logic does not mean equality of their valuations $v(A) = v(B)$ on the model (orthomodular lattice). Next we look at the structure we obtain by an algebraic mapping of the axioms of Kalmbach's two systems into an ortholattice. The structure turns out to be non-orthomodular as shown below. Also all the axioms of one of the two systems are derived within other. Hence the kind of implication one used would be irrelevant, if one did not use them to establish a lattice model of a considered logic. This is complemented by our result in Sec. 3 which shows that both the modus ponens rule (corresponding itself to the chosen binary valuation) and the operations of implication (which also correspond to the chosen valuation) structurally determine the logics we construct using them.

Propositions we use are based on elementary propositions $p_0, p_1, p_2, \ldots$ and the following primitive connectives: $\neg$ (negation) and $\vee$ (disjunction). The set of propositions is defined formally as follows:

p_j is a proposition for $j = 0, 1, 2, \ldots$

$\neg A$ is a proposition iff A is a proposition.

$A \vee B$ is a proposition iff A and B are propositions.

The conjunction is introduced by the following definition:

Definition 2.1. $A \wedge B \stackrel{\text{def}}{=} \neg(\neg A \vee \neg B)$.

Our metalanguage consists of axioms and rules from the object language as elementary metapropositions and of compound metapropositions built up by means of the following metaconnectives: $\sim$ (*not*), & (*and*), $\vee$ (*or*), $\Rightarrow$ (*if..., then*), and $\Leftrightarrow$ (*iff*), with the usual *classical* meaning.

The operation of implication is one of the following:

Definition 2.2.

$$A \to_0 B \stackrel{\text{def}}{=} \neg A \vee B \qquad \text{(classical)}$$

$$A \to_1 B \stackrel{\text{def}}{=} \neg A \vee (A \wedge B) \qquad \text{(Mittelstaedt)}$$

$$A \to_2 B \stackrel{\text{def}}{=} B \vee (\neg A \wedge \neg B) \qquad \text{(Dishkant)}$$

$$A \to_3 B \stackrel{\text{def}}{=} ((\neg A \wedge B) \vee (\neg A \wedge \neg B)) \vee \big(A \wedge (\neg A \vee B)\big) \qquad \text{(Kalmbach)}$$

$$A \to_4 B \stackrel{\text{def}}{=} ((A \wedge B) \vee (\neg A \wedge B)) \vee \big((\neg A \vee B) \wedge \neg B\big) \qquad \text{(non-tollens)}$$

$$A \to_5 B \stackrel{\text{def}}{=} ((A \wedge B) \vee (\neg A \wedge B)) \vee (\neg A \wedge \neg B) \qquad \text{(relevance)}$$

Identity is defined as follows:

Definition 2.3. $A \equiv B \stackrel{\text{def}}{=} (A \wedge B) \vee (\neg A \wedge \neg B)$

We also define $\equiv$ and $\to_i$ for the lattice algebra using the analogs of their logic definitions.

The following two lemmas are well-known. [13]

Lemma 2.4. *The following holds in any quantum (orthomodular) logic:*

$$A \equiv B \;=\; A \leftrightarrow_i B \stackrel{\text{def}}{=} (A \to_i B) \wedge (B \to_i A) \qquad i = 1, \ldots, 5$$

Lemma 2.5. *The following holds in any classical (distributive) logic:*

$$A \equiv B \;=\; A \leftrightarrow_i B \stackrel{\text{def}}{=} (A \to_i B) \wedge (B \to_i A) \qquad i = 0, \ldots, 5$$

Connectives bind from weakest to strongest in the order $\to$, $\leftrightarrow$, $\equiv$, $\vee$, $\wedge$, and $\neg$, with similar bindings for the lattice algebra analogs.

Let us first consider the following system K0 with a "classical" (using $A \to_0 B$) modus ponens rule. [12]

Axioms.

KA1. $\vdash A \equiv A$

KA2. $\vdash A \equiv B \to_0 (B \equiv C \to_0 A \equiv C)$

KA3. $\vdash A \equiv B \to_0 \neg A \equiv \neg B$

KA4. $\vdash A \equiv B \to_0 A \wedge C \equiv B \wedge C$

KA5. $\vdash A \wedge B \equiv B \wedge A$

KA6. $\vdash A \wedge (B \wedge C) \equiv (A \wedge B) \wedge C$

KA7. $\vdash A \wedge (A \vee B) \equiv A$

KA8. $\vdash \neg A \wedge A \equiv (\neg A \wedge A) \wedge B$

KA9. $\vdash A \equiv \neg\neg A$

KA10. $\vdash \neg(A \vee B) \equiv \neg A \wedge \neg B$

KA11. $\vdash A \vee (\neg A \wedge (A \vee B)) \equiv A \vee B$

KA12. $\vdash (A \equiv B) \equiv (B \equiv A)$

KA13. $\vdash A \equiv B \to_0 (A \to_0 B)$

Rule of Inference.

KMP0. $\vdash A \quad \& \quad \vdash A \to_0 B \quad \Rightarrow \quad \vdash B$

K0 is characterized by the class of orthomodular lattices in the sense that $\vdash A \ \Leftrightarrow\ v(A) = 1$ (the unary valuation). In this section we shall look at the role of the orthomodular law in the unary valuation, using system K0 as a convenient formalization for our purpose.

System K0 has redundant axioms. Axiom KA1 is obviously redundant, being derivable from KA2, KA9, KA12, KA13, and KMP0.

Less obvious is a main result of this section, which is that KA11 is redundant. This is somewhat surprising, since the resemblance between KA11 and a "proper" orthomodular axiom at first suggests it might be essential.

Theorem 2.6. *Axiom* KA11 *is derivable from* KA1–KA10*,* KA12–KA13*, and* KMP0*.*

Proof. $\vdash_{\mathrm{K0}} A$ or just $\vdash A$ will mean "A is a theorem of K0." In our proofs we will sometimes not mention possible use of KA2, KA12, KA13, or KMP0. These, together with KA3, KA4, KA9, and KA10, allow us to prove, by induction on formula length, an analog to equality in lattice algebra, so that from $\vdash A \equiv B$ we may infer $\vdash \_\_A\_\_ \equiv \_\_B\_\_$. Using this equality metatheorem, the equality analogs provided by KA1, KA2 and KA12, and the obvious ortholattice analogs KA5–KA9, it is easy to see how to prove any other ortholattice analog in a manner paralleling an ortholattice proof. We will say that any theorem constructed in this manner is proved "by ortholattice analogy." Of course ortholattice analogs can include only theorems of K0 of the form $\vdash A \equiv B$.

From KA8 we have $\vdash \neg A \wedge A \equiv (\neg A \wedge A) \wedge (\neg B \wedge B)$ and $\vdash \neg B \wedge B \equiv (\neg B \wedge B) \wedge (\neg A \wedge A)$. Using KA5 to connect the right-hand sides we conclude $\vdash \neg A \wedge A \equiv \neg B \wedge B$. This theorem together with the equality metatheorem above allow us to define constants $0 \stackrel{\text{def}}{=} \neg A \wedge A$ and $1 \stackrel{\text{def}}{=} \neg 0$.

We define $\vdash A \leq B \stackrel{\text{def}}{=} \ \vdash A \vee B \equiv B$.

(i) We have the law of excluded middle $\vdash A \vee \neg A$ which follows from KA1, KA13, and KA5. This can be restated as $\vdash 1$.

(ii) By ortholattice analogy we have that $\vdash A \leq B$, i.e. $\vdash A \vee B \equiv B$, implies $\vdash A \wedge B \equiv A$ (conjoin A to both sides and apply KA7).

(iii) $\vdash A \vee (\neg A \wedge (A \vee B)) \leq A \vee B$ follows by ortholattice analogy: in $\vdash \neg A \wedge (A \vee B) \leq A \vee B$, place A in a disjunction on both sides.

(iv) $\vdash \neg(A \vee B) \vee (A \vee (\neg A \wedge (A \vee B))) \equiv 1$ is proved as follows. By DeMorgan's laws we have

$$\vdash \neg A \wedge (A \vee B) \equiv \neg(A \vee \neg(A \vee B)).$$

By commutativity of disjunction we have

$$\vdash \neg(A \vee B) \vee A \equiv A \vee \neg(A \vee B).$$

The disjunction of these yields

$$\vdash (\neg(A \vee B) \vee A) \vee (\neg A \wedge (A \vee B)) \equiv 1$$

and associativity of disjunction gives us the result.

(v) From $\vdash A \leq B$ and $\vdash \neg B \vee A \equiv 1$ we can infer $\vdash A \equiv B$, proved as follows. From the first hypothesis: by definition of $\leq$ and DeMorgan's laws, we have $\vdash \neg A \wedge \neg B \equiv \neg B$; by (ii) we have $\vdash A \wedge B \equiv A$. The disjunction of both sides yields $\vdash (A \equiv B) \equiv A \vee \neg B$. From this and the second hypothesis we obtain $\vdash (A \equiv B) \equiv 1$. Using KA12, KA13, and KMP0, we detach excluded middle (i) to obtain the result.

(vi) $\vdash A \vee (\neg A \wedge (A \vee B)) \equiv A \vee B$ follows immediately from (iii), (iv), and (v). This is axiom KA11, the desired result. □

Looking closer at KA11, we find another surprise: its mapping has nothing to do with the orthomodular law.

Theorem 2.7. *The mapping for* KA11,

$$a \cup (a^\perp \cap (a \cup b)) \equiv a \cup b = 1$$

is true in all ortholattices.

Proof. In a manner exactly analogous to sections (iii) and (iv) of the previous proof, we obtain for ortholattices

$$a \cup (a^\perp \cap (a \cup b)) \leq a \cup b$$

and

$$(a \cup b)^\perp \cup (a \cup (a^\perp \cap (a \cup b))) = 1$$

By analogy to section (v) of the previous proof, but omitting the final detachment of excluded middle, we obtain that

$$a \leq b \text{ and } b^\perp \cup a = 1 \text{ implies } a \equiv b = 1$$

in all ortholattices. The desired result follows immediately from these. □

Given this result, it is natural to ask whether the orthomodular law is needed at all in the algebra underlying K0. The answer is in the following theorem.

Theorem 2.8. *(i) The algebraic mappings for* KA1, KA3, KA5–KA13, *and* KMP0 *are true or sound in all ortholattices. (ii) The algebraic mappings for* KA2 *and* KA4 *do not hold in all ortholattices.*

Proof. KMP0: This rule is sound in all ortholattices. [12]

KA1, KA5–KA10, KA12: Each theorem has an obvious equational analog in ortholattices. Since $a = b$ implies $a \equiv b = 1$ in all ortholattices (if $a = b$, then $a \equiv b = b \equiv b = 1$), we obtain the desired result. For example, for KA5 we have the analog $a \cup b = b \cup a$, from which we deduce its mapping $a \cup b \equiv b \cup a = 1$, which is true in all ortholattices.

KA3, KA13: We use the fact that $a \le b$ implies $a^\perp \cup b = 1$ in all ortholattices (if $a \cup b = b$, then $a^\perp \cup a \cup b = a^\perp \cup b$). For KA13, $a \cap b \le b$ and $a^\perp \cap b^\perp \le a^\perp$, so $a \equiv b \le a^\perp \cup b$.

KA11: Theorem 2.7.

KA2, KA4: Each of their mappings

$$(a \equiv b)^\perp \cup (b \equiv c)^\perp \cup (a \equiv c) = 1$$

$$(a \equiv b)^\perp \cup (a \cap c \equiv b \cap c) = 1$$

violates the non-orthomodular ortholattice of Fig. 1 with the assignment $a = x$, $b = z$, $c = y$ showing that they do not hold in all ortholattices.

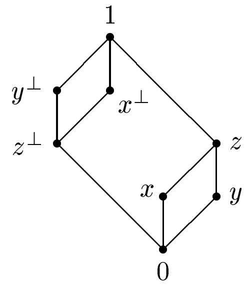

Figure 1: Ortholattice violated by mappings of KA2, KA4

□

Finally, we ask whether the full strength of the orthomodular law is embodied in the algebraic semantics for system K0. The answer is no.

Definition 2.9. *We define a* weakly orthomodular (WOM) lattice *as an ortholattice extended with the following two laws:*

W1. $(a \equiv b)^\perp \cup (a \cap c \equiv b \cap c) = 1$

W2. $(a \equiv b)^\perp \cup (b \equiv c)^\perp \cup (a \equiv c) = 1$

Because these are the mappings of KA4 and KA2, and since all other mappings of K0 are true or sound in ortholattices, WOM lattice algebra is sufficient to model K0. (Later we shall show W2 is redundant; see Theorem 2.14 below.)

Theorem 2.10. *(i) Every orthomodular lattice is a WOM lattice. (ii) Every WOM lattice is an ortholattice.*

Proof. (i) W1 and W2 are true in every orthomodular lattice. See the proofs of the KA4 and KA2 mappings in Ref. [12]. (ii) By definition. □

However, no theorem of WOM lattice algebra is as strong as the orthomodular law, as shown by the following theorem. This shows that K0 can be modeled by an algebra strictly weaker than orthomodular lattice algebra.

Theorem 2.11. *(i) There exist WOM lattices that are not orthomodular. (ii) There exist ortholattices that are not WOM lattices.*

Proof. (i) The non-orthomodular ortholattice O6 (Fig. 2) is not violated by W1 and W2. (ii) Theorem 2.8(ii).

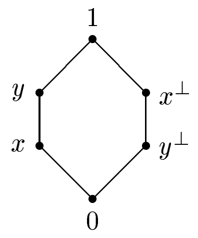

Figure 2: Ortholattice O6

□

Theorem 2.12. *No collection of theorems of orthomodular lattices of the form*

$$t = 1$$

(where t is a term, such as $a \cup a^\perp$), when added to an ortholattice, determines an orthomodular lattice.

Proof. By completeness of K0 [12], any theorem of orthomodular lattices of the form $t = 1$ is provable from the mappings of the axioms and inference rule of K0. Thus WOM lattice algebra, which is strictly weaker than orthomodular lattice algebra by Theorem 2.11 (i), is sufficient to prove any such theorem. □

Indeed, WOM lattice algebra is equivalent to ortholattice algebra extended with all orthomodular lattice theorems of the form $t = 1$ [where t is a term such as $a \cup (a^\perp \cap (a \cup b)) \equiv a \cup b$].

Kalmbach defines another system, which we shall call K1, with a modus ponens rule using the implication $A \to_3 B$. K1 is defined as K0 minus KMP0, plus the following axioms and inference rule: [12]

Axioms.

KA14. $\vdash (A \to_0 B) \to_3 (A \to_3 (A \to_3 B))$

KA15. $\vdash (A \to_3 B) \to_0 (A \to_0 B)$

Rule of Inference.

KMP3. $\vdash A \quad \& \quad \vdash A \to_3 B \quad \Rightarrow \quad \vdash B$

Kalmbach shows that KA14, KA15, and KMP3 can be derived in K0, and also that KMP0 can be derived in K1. Thus K0 and K1 are logically equivalent, and the above results apply to K1 as well as K0. We mention that KA1, KA11, and KA15 are redundant in K1.

The mapping for KA14 is not a theorem of ortholattice algebra. In particular the mapping for KA14

$$a^\perp \cup b \to_3 (a \to_3 (a \to_3 b)) = 1$$

violates the non-orthomodular ortholattice of Fig. 3 with the assignment $a = y$, $b = w$.[3] (The mapping for KA14 is of course true in all WOM lattices.)

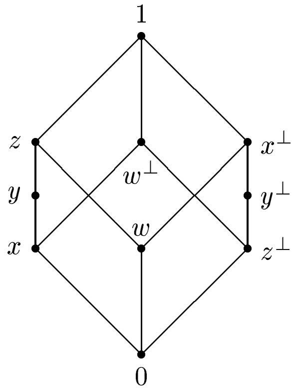

Figure 3: Ortholattice violated by mapping of KA14

The mapping for KA15 is true in all ortholattices: from $a^\perp \cap b \le b$, $a^\perp \cap b^\perp \le a^\perp$ it follows that $(a^\perp \cap b) \cup (a^\perp \cap b^\perp) \le a^\perp \cup b$; this and $a \cap (a^\perp \cup b) \le a^\perp \cup b$ imply $a \to_3 b \le a^\perp \cup b$. Thus $(a \to_3 b)^\perp \cup (a^\perp \cup b) = 1$. Since KMP3 follows from KMP0 applied twice to KA15, and KMP0 is sound in all ortholattices, KMP3 is also sound in all ortholattices.

Given our result that the orthomodular law is needed only for KA2 and KA4 in the algebra for system K0, we next show that it can be eliminated from one of them. We define a new system, K0′, as consisting of KA3–KA10, KA12, KA13, and KMP0, along with two axioms

Axioms.

KA2a. $\vdash (A \vee C \equiv B \vee C) \equiv (C \vee A \equiv C \vee B)$

KA2b. $\vdash (A \vee C \equiv B \vee C) \equiv (\neg(\neg A \wedge \neg C) \equiv \neg(\neg B \wedge \neg C))$

The mappings of all axioms of K0′, with the exception of KA4, are true in all ortholattices. Thus KA4 alone contains the essential non-ortholattice character of the algebraic semantics for K0′, and by the next theorem, also for K0.

Theorem 2.13. *Systems* K0 *and* K0′ *are logically equivalent.*

Proof. The proofs of KA2a and KA2b in system K0 are straightforward. The derivation of K0 from K0′ is more involved.

(i) From KA4 and KMP0 we have that $\vdash A \equiv B$ implies $\vdash A \wedge C \equiv B \wedge C$.

(ii) From (i), KA2a, KA2b, KA3, KA13, and KMP0, we have that $\vdash A \equiv B$ implies both $\vdash A \vee C \equiv B \vee C$ and $\vdash C \vee A \equiv C \vee B$.

[3]The authors are grateful to William McCune of Argonne National Laboratory, Argonne IL, USA for finding this lattice, using the matrix-finding program MACE (`http://www.mcs.anl.gov/home/mccune/ar/ortholattice/`).

(iii) The law

$$\vdash A \equiv B \quad \& \quad \vdash B \equiv C \quad \Rightarrow \quad \vdash A \equiv C$$

is derived as follows. From $\vdash A \equiv B$, (i), and (ii), we obtain $\vdash (A \wedge C) \vee (\neg B \wedge \neg C) \equiv (B \wedge C) \vee (\neg B \wedge \neg C)$ i.e. $\vdash (A \wedge C) \vee (\neg B \wedge \neg C) \equiv (B \equiv C)$. From this, $\vdash B \equiv C$, KA12, KA13, and KMP0, we obtain $\vdash (A \wedge C) \vee (\neg B \wedge \neg C)$. From $\vdash A \equiv B$, KA3, KMP0, (i), and (ii) we obtain $\vdash (A \equiv C) \equiv (A \wedge C) \vee (\neg B \wedge \neg C)$. From these two, KA12, KA13, and KMP0, we obtain $\vdash A \equiv C$.

(iv) KA1 follows from (iii), KA9, KA12, KA13, and KMP0.

(v) We can use (iii) in place of KA2 in the work of Theorem 2.6 to construct proofs by "ortholattice analogy." In particular, we can prove KA11.

(vi) Having KA11 available as an analog to the orthomodular law, we can further construct proofs by "orthomodular lattice analogy." In particular we can prove the analog of the algebra mapping for KA2,

$$\vdash (A \equiv B \to_0 (B \equiv C \to_0 A \equiv C)) \equiv 1$$

by analogy to the proof in Ref. [12].

(vii) We prove $\vdash 1$ as in Theorem 2.6. Detaching it from (vi), we derive KA2. □

Theorem 2.14. *Axiom* W2 *for WOM lattices is redundant.*

Proof. W2 is the mapping for KA2. By analogy to the proof of Theorem 2.13 [through step (vi)] in ortholattices, we can prove W2 from W1 (the mapping for KA4). □

WOM lattices are characterized by a somewhat simpler law, as the following theorem shows.

Theorem 2.15. *An ortholattice in which the following law*

WR1. $a \equiv b = 1$ *implies* $a \cup c \equiv b \cup c = 1$

holds is a WOM lattice and vice-versa.

Proof. This law follows directly from W1 (we obtain the $\cup$ version of W1 using DeMorgan's laws and the mapping for KA3, and use the mapping of KMP0 to finally obtain the law). Conversely, using this law in the role of KA4, we construct an ortholattice proof analogous to that for Theorem 2.13 through step (vi), except that in step (vi) we follow the proof for the mapping of KA4 rather than KA2 in Ref. [12], to obtain W1. □

WOM lattices are also characterized by a "weaker-looking" but equivalent law.

Theorem 2.16. *An ortholattice in which the following law*

WR1′. $a \equiv b = 1$ *implies* $a \cup c \equiv b \cup c \ge a$

holds is a WOM lattice and vice-versa.

Proof. This law follows immediately from WR1. Conversely, using this law twice we obtain that $a \equiv b = 1$ implies $a \cup b \le a \cup c \equiv b \cup c$. WR1 follows from this and the following equation, true in all ortholattices, whose proof we leave to the reader:

$$(a \cup b) \cup (a \cup c \equiv b \cup c) = 1$$

□

By a similar proof we can also show that

W1′. $(a \equiv b)^{\perp} \cup (a \cup c \equiv b \cup c) \ge a$

can replace W1.

3 Binary Orthologic: Modus Ponens Rule Determines Logics

In this section we use binary orthologic, as an efficient formulation of implicational logic, in order to formulate the appropriate modus ponens rule. We then show that this modus ponens rule turns orthologic into quantum logic. Thereupon we show that binding quantum bi-implication to the quantum identity gives an orthologic which is weaker than quantum logic and which is contained in the logic we obtained in the previous section.

Binary logics from the references [14], [15], [11] and [16] are the logics of all pairs $A \vdash B$ that in an associated standard logic S would satisfy $\vdash_S A \to_i B$ for $i = 0, \dots, 5$. Quantum (classical) logic is characterized by the class of orthomodular (distributive) lattices in the sense that $A \vdash B \;\Leftrightarrow\; v(A) \le v(B)$ for all valuations v on all orthomodular (distributive) lattices. By means of this characterization quantum implications determine quantum logic and the classical implication classical logic, in the following way:

Theorem 3.1 (Pavičić, 1987). *An orthologic in which*

$$\vdash A \to_i B \qquad \Leftrightarrow \qquad A \vdash B$$

holds is a classical logic for $i = 0$ and a quantum logic for $i = 1, \dots, 5$. Here, $\vdash A$ means $C \vee \neg C \vdash A$.

We also have a corresponding result for lattices:

Theorem 3.2 (Pavičić, 1993). *An ortholattice in which*

$$a \to_i b \;=\; 1 \qquad \Leftrightarrow \qquad a \le b$$

holds is a distributive lattice for $i = 0$ and an orthomodular lattice for $i = 1, \dots, 5$.

Theorem 3.1 enables us to axiomatize classical and quantum logic in the following way:

Axioms.

A1. $A \vdash \neg\neg A$

A2. $\neg\neg A \vdash A$

A3. $A \vdash A \vee B$

A4. $B \vdash A \vee B$

A5. $B \vdash A \vee \neg A$

Rules of Inference.

R1. $A \vdash B \quad \Rightarrow \quad \neg B \vdash \neg A$

R2. $A \vdash B \quad \& \quad B \vdash C \quad \Rightarrow \quad A \vdash C$

R3. $A \vdash C \quad \& \quad B \vdash C \quad \Rightarrow \quad A \vee B \vdash C$

R4(i). $\vdash A \to_i B \quad \Rightarrow \quad A \vdash B$

Here the system A1–A5 + R1–R3 is orthologic (OL), also called minimal quantum logic. OL + R4(i) is quantum logic (QL) for $i = 1, \dots, 5$ and classical logic (CL) for $i = 0$.

In the previous section we have shown that the traditional forms of modus ponens rule for implications $\to_i$ for $i = 0, 3$ (KMP0, KMP3) can be derived within an orthologic. The following theorem shows that this is also valid for the other implications.

Theorem 3.3. *Modus ponens rules of the form*

$$\mathbf{MP}(i). \qquad \vdash A \quad \& \quad \vdash A \to_i B \quad \Rightarrow \quad \vdash B$$

hold in any OL *for any* $i = 0, \dots, 5$.

Proof. The proof is trivial. So, we only sketch it for $i = 0, 1, 2$.

$i = 0 \qquad \vdash A \;\Rightarrow\; \neg B \vdash A \wedge \neg B \;\Rightarrow\; \neg A \vee B \vdash B \;\Rightarrow\; \vdash B$

$i = 1 \qquad \vdash A \;\Rightarrow\; \neg(A \wedge B) \vdash A \wedge \neg(A \wedge B) \;\Rightarrow\; \neg A \vee (A \wedge B) \vdash B \;\Rightarrow\; \vdash B$

$i = 2 \qquad \vdash A \;\Rightarrow\; \vdash A \vee B \;\Rightarrow\; \neg B \vdash (A \vee B) \wedge \neg B \;\Rightarrow\; B \vee (\neg A \wedge \neg B) \vdash B$ ☐

Thus traditional modus ponens rules do not characterize logics structurally. Loosely speaking this is so because modus ponens rule cannot take care of a proposition to the left of '$\vdash$'. As the following lemma shows, we also cannot use the object language modus ponens usually called lattice theoretic modus ponens [17] instead of the above traditional rules.

Lemma 3.4. *Any* OL *with lattice theoretic modus ponens*

$$\mathbf{MP}(i). \qquad A \wedge (A \to_i B) \vdash B$$

is QL *for* $i = 1, 2, 4, 5$ *and* CL *for* $i = 0, 3$. *On the other hand,* MP(i), $i = 0, 3$, *is satisfied in* CL *and* MP(i), $i = 1, 2, 4, 5$ *in* QL.

Proof. To prove that R4(i) holds in OL + MP(i), $i = 0, 1, 2, 4, 5$ we make use of

X1. $\quad A \vdash B \quad \Rightarrow A \wedge C \vdash B \wedge C$ [Def. 2.1, A1–A5,R1–R3]

We infer the conclusion $A \vdash B$ of R4(i) from the premise $\vdash A \to_i B$ of R4(i):

$\neg A \vee A \vdash A \to_i B$ [premise of R4(i)]

$A \vdash A \wedge A \to_i B$ [X1,Def. 2.1,A1–A5,R1–R3]

$A \vdash B$ [MP(i),R2]

Theorem 3.2 yields the $i = 3$ case straightforwardly. The proof that MP(i) holds in OL + R4(i) (or CL for i=3) is straightforward and we omit it. E.g., MP(1) is a well-known orthomodularity axiom [15]. ☐

However, in the binary logic we can straightforwardly construct the appropriate modus ponens rule in the following way.

In an associated standard logic S of our binary logic all elementary propositions have the form $\vdash_S A \to_i B$ for $i = 0, \dots, 5$. So, MP(i) from Theorem 3.3 would read $\vdash_S A \to_i B \;\&\; \vdash_S (A \to_i B) \to_i (C \to_i D) \quad \Rightarrow \quad \vdash_S C \to_i D$. This modus ponens rule turns out to be the proper one because it structures logics so as to make an orthologic a quantum logic for $i = 1, \dots, 5$ and a classical logic for $i = 0$, as the following theorem shows.

Theorem 3.5. *An orthologic in which the following modus ponens rule*

$$\mathbf{MPR}(i). \qquad A \vdash B \quad \& \quad A \to_i B \vdash C \to_i D \quad \Rightarrow \quad C \vdash D$$

holds is a quantum logic for $i = 1, \dots, 5$ *and a classical logic for* $i = 0$ *and vice versa.*

Proof. By choosing $B = A$ we obtain $\vdash C \to_i D \quad \Rightarrow \quad C \vdash D$ and the claim follows by Theorem 3.1. As for the vice versa part, from the first premise, $\vdash A \to_i B$ follows by Theorem 3.1. From this, $\vdash C \to_i D$ follows from the second premise. By applying Theorem 3.1 again we obtain the conclusion $C \vdash D$. □

We again see that the interplay between '$\to_i$', $i = 1, \dots, 5$ and '$\vdash$' plays a particular role in any orthomodular logic. When there is no such connection in a system then the system is not orthomodular as, e.g., the systems elaborated in the previous section. However, we can still unify all five implications by adding the following rules of inference to OL thus forming what we are going to call weak quantum logic (WQL).

Rules of Inference.

WR4a(i). $\vdash A \to_i B \quad \Rightarrow \quad \vdash A \vee C \to_i B \vee C \qquad i = 1, \dots, 5$

WR4b(i). $\vdash A \leftrightarrow_i B \quad \Rightarrow \quad \vdash A \equiv B \qquad i = 1, \dots, 5$

Theorem 3.6. WQL *(i.e., the system* A1–A5+R1–R3+WR4a+WR4b*) is a non-trivial variety of orthologic which is not orthomodular.*

Proof. We show the proof only for $i = 1$. The cases $i = 2, \dots, 5$ one proves analogously. The non-triviality of the varieties also follows from the next theorem and the results of Sec. 2.

WR4a violates the non-orthomodular ortholattice of Fig. 1 with the assignment $a = x^\perp$, $b = y^\perp$, and $c = z^\perp$. Then the premise $\neg A \vee (A \wedge B)$ is equal to 1, while the consequence is equal to $z^\perp$.

WR4b violates the non-orthomodular ortholattice of Fig. 1 with the assignment $a = x^\perp$, $b = y^\perp$. Then the premises $\neg A \vee (A \wedge B)$ and $\neg B \vee (A \wedge B)$ are both equal to 1, while the consequence is equal to $z^\perp$.

On the other hand, non-orthomodular ortholattice O6 (Fig. 2) is not violated by mappings of the either rule. □

In WQL we can derive the logical equivalent of WR1 (see Theorem 2.15) and therefore all the axioms of K0 as the following theorem shows (see Definition 2.9 and Theorems 2.14 and 2.15).

Theorem 3.7. *Rule*

WRL1. $\vdash A \equiv B \quad \Rightarrow \quad \vdash A \vee C \equiv B \vee C$

holds in WQL *and vice versa:* OL *in which* WRL1 *holds is* WQL.

Proof.

$\vdash A \to_i B \quad \& \quad \vdash B \to_i A$ [premise,A1–A5,R1–R3]

$\vdash A \vee C \to_i B \vee C \quad \& \quad \vdash B \vee C \to_i A \vee C$ [WR4a]

$\vdash A \vee C \equiv B \vee C$ [WR4b]

Conversely, by completeness of K0, the unary subset of QL can be proved in K0. Since WR4a and WR4b are unary laws of QL, they can be proved in K0. We previously showed WOM lattice algebra is sufficient to model K0; hence it is sufficient to model WR4a and WR4b, i.e. WR4a and WR4b are sound in all WOM lattices. □

An immediate consequence is:

Corrolary 3.8. WOM *lattices are the models of* WQL.

4 Conclusion

Our study shows that in order to construct a sound and efficient quantum logical system we have to chose both the valuation (v) of propositions on a model and the form of the axioms and rules of inference so as to correspond to the ordering relation on the model. So we strengthen our previous result—according to which the Birkhoff-von Neumann's requirement, i.e., the correspondence $\vdash A \to_i B \ \Leftrightarrow\ v(A) \le v(B)$ alone turns an orthologic into a quantum logic for $i = 1, \ldots, 5$ and into a classical logic for $i = 0$—by showing that axioms and rules of inference should also reflect this correspondence. In the standard approach this amounts to saying that all propositions in a quantum logic with such a correspondence are of the form $A \to_i B$ and that rules of inference also deal only with such propositions. Thus, the modus ponens rule takes the form (see Theorem 3.5)

MPR(i). $\quad A \vdash B \quad \& \quad A \to_i B \vdash C \to_i D \quad \Rightarrow \quad C \vdash D$

which turns orthologic into quantum logic for $i = 1, \ldots, 5$ and into classical logic for $i = 0$.

In contradistinction to these results for the binary logic, our study—in Sec. 2—of the standard quantum logic under the unary valuation shows that the latter logic does not have similar structural features. The theorems of the logic which have an "orthomodular form," with the object language identity $\equiv$ standing for the lattice equality, form a set F of formulas of propositions. This set serves for proving the completeness by means of the Lindenbaum-Tarski algebra $F/\equiv$. However, as we proved by Theorem 2.12, the formulas of the logic cannot be mapped *directly* to an ortholattice so as to form an orthomodular lattice and, as we proved by Theorem 2.7, the algebraic mapping of the axiom KA11—which yields the orthomodularity property of the model lattice within the completeness proof—is true in all ortholattices. Hence unary logic lacks a direct correspondence between the object language implication and the ordering in the lattice which models the logic; so, there is no point in talking about Birkhoff-von Neumann's requirement within such a logic except in an indirect sense as explained below. There is no point also because for the completeness proof one does not need the operation of implication at all. The identity, independently defined by Definition 2.3, suffices for the completeness proof and the logic itself can be formulated with the help of any implication, even the classical one.

In Sec. 2 we identify a nontrivial variety of orthologic in which one can obtain axioms of Kalmbach's system K0 (or K1—the two systems are shown derivable within each other and 5 axioms from the systems are shown redundant). In Sec. 3 we show that the variety is equivalent to another nonorthomodular system (see Theorem 3.6) which we obtain by adding the correspondence $\vdash A \leftrightarrow_i B \ \Rightarrow\ \vdash A \equiv B$, $i = 1, \ldots, 5$ (together with an implicational property) to orthologic (note that the direction $\Leftarrow$ is valid in any orthologic). Hence, the so-called *Kotas biconditional* property [13] (given by Lemma 2.4) is not *necessarily* a property of an orthomodular system. Thus, already in our WQL a completeness proof based on $\equiv$ [12] follows from the one based on $\leftrightarrow$ [18]. In addition, the authors possess an explicit proof that Dishkant's [18] and Kalmbach's [12] systems are syntactically equivalent and derivable within each other.

Comparing the above correspondence with our definitions of the orthomodularity $\vdash A \leftrightarrow B \ \Leftrightarrow\ A \dashv\vdash B \ \Leftrightarrow\ \vdash A \equiv B$ [11] we see that one can embed quantum logic QL in our weak quantum logic WQL by invoking Definition 2.9, Theorem 2.15, and Theorem 3.6. In other words, a theorem of the form $\vdash A \equiv B$ holds in WQL, iff $A \dashv\vdash B$ holds in QL (extended from WQL by adding the orthomodularity to it). This explains why one is able to prove the completeness in the unary approach although the Birkhoff-von Neumann's requirement is not directly fulfilled.

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