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Time-like graphical models

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Abstract

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We study continuous processes indexed by a special family of graphs. Processes indexed by vertices of graphs are known as *probabilistic graphical models*. In 2011, Burdzy and Pal proposed a continuous version of graphical models indexed by graphs with an embedded time structure – so called time-like graphs. We extend the notion of time-like graphs and find properties of processes indexed by them. In particular, we solve the conjecture of uniqueness of the distribution for the process indexed by graphs with infinite number of vertices. We provide a new result showing the stochastic heat equation as a limit of the sequence of natural Brownian motions on time-like graphs. In addition, our treatment of time-like graphical models reveals connections to Markov random fields, martingales indexed by directed sets and branching Markov processes.

TABLE OF CONTENTS

Page

Part I: Construction and properties 6

Chapter 1: Geometry of time-like graphs 7
- 1.1 Basic definitions 7
- 1.2 TLG* family 10
- 1.3 Consistent representation of a TLG*-tower, spines and (re)construction . . . 16
- 1.4 Interval TLG*'s 21
- 1.5 Topology on TLG's 23
- 1.6 TLG* as a topological lattice 26
- 1.7 Cell collapse transformation and the stingy algorithm 30
- 1.8 TLG's with infinitely many vertices 39

Chapter 2: Processes indexed by time-like graphs 41
- 2.1 Spine-Markovian property 42
- 2.2 Consistent distributions on paths 45
- 2.3 Construction from a consistent family 47
- 2.4 Processes on TLG's with infinite number of vertices 59

Chapter 3: Markov properties of processes indexed by TLG's 66
- 3.1 Cell-Markov properties 66
- 3.2 Graph-Markovian and time-Markovian property 69
- 3.3 Processes on TLG's for Markov family $\mathcal{M}$ 70
- 3.4 Homogeneous Markov family $\mathcal{M}_{\mathcal{P}}$ 78
- 3.5 Three simple examples 84

Chapter 4: Filtrations, martingales and stopping times 87
- 4.1 Expanding the filtrations 88

4.2 Markov martingales 93
4.3 Optional sampling theorem for martingales indexed by directed sets 99
4.4 TLG - valued stopping times 102
4.5 A simple coupling and branching process 105

Part II: Natural Brownian motion and the stochastic heat equation 108

Chapter 5: Maximums of Gaussian processes 111
5.1 Sequence of Brownian bridges 111
5.2 Sequence of normal variables 114
5.3 Some concentration and convergence results 116

Chapter 6: Random walk and stochastic heat equation reviewed 120
6.1 Modification of the Local Limit Theorem 120
6.2 Approximations of the classical heat equation solution 124
6.3 Euler method for the stochastic heat equation 133
6.4 Convergence of interpolation of the Euler method 144
6.5 Euler method with initial value condition and no external noise 149

Chapter 7: Limit of the natural Brownian motion on a rhombus grid 152
7.1 Natural Brownian motion on a rhombus grid 152
7.2 Network of Brownian bridges 157
7.3 The main result 158

Part III: Processes on general and random time-like graphs 162

Chapter 8: Non-simple TLG's 164
8.1 New definitions 164
8.2 Embedding TLG's into simple TLG's 166
8.3 TLG** family 169

Chapter 9: Processes on non-simple TLG's 176
9.1 Processes on TLG** 176
9.2 Properties of constructed processes 183
9.3 Properties for Markov family $\mathcal{M}$ 187

9.4 Processes on time-like trees 188

Chapter 10: Galton-Watson time-like trees and the Branching Markov processes . . 194
10.1 TLG's with an infinite number of vertices 194
10.2 Galton – Watson time-like tree 195
10.3 Processes on TLG**'s with infinite number of vertices 197
10.4 Natural $\mathcal{P}$-Markov process 199
10.5 Branching $\mathcal{P}$-Markov process 200

Chapter 11: Open questions 206
11.1 Construction of process on all TLG's 206
11.2 Reconstruction of TLG's based on the process 208
11.3 Strong Markov property, parametrization, evolution over time,... 210

Appendix A: Independence and processes 212
A.1 Conditional independence and expectations 212
A.2 Construction of a conditional sequence 213
A.3 Markov and Brownian bridges 214
A.4 Markov random fields 219
A.5 White noise 220
A.6 The stochastic heat equation 221
A.7 Crump - Mode - Jagers trees 224
A.8 Branching Markov processes and branching Brownian motion 226

Bibliography 228

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INTRODUCTION

In the general theory of Markov processes (such as given in the book [6, Blumenthal-Getoor]) we have a process X indexed by some **parameter set** T:

$$(X(t) : t \in T). \tag{0.1}$$

The set T can be any set with some **order** $\preceq$. The book [34] by Khoshnevisan studies different cases of multiparameter processes ($T \subset \mathbb{R}^n$). T could, for instance, be vertices of a directed graph with the order induced by the direction of edges.

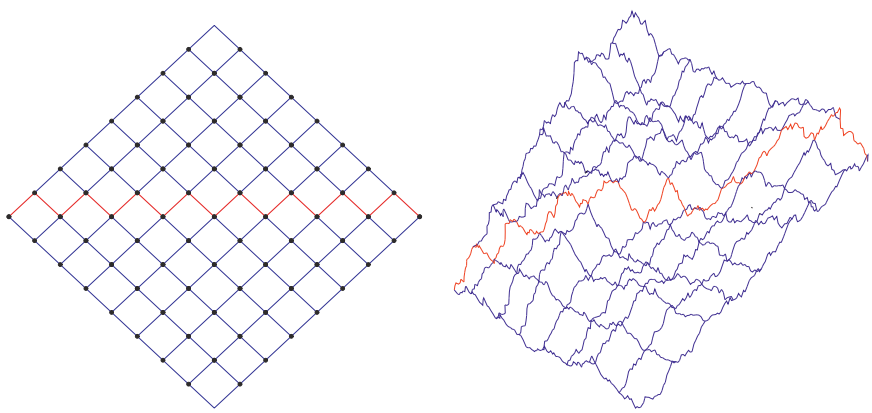


Figure 0.1: Parameter set T and the realization of the process indexed by T.

Processes indexed by vertices of graphs are well studied and are often used in machine learning ([35, Koller - Friedman], [28, Hastie et al.]) and statistics ([37, Lauritzen], [45, Studený]), where they are called *probabilistic graphical models*. In each of these models the **conditional independencies** can be read from the **structure of the graph**. (A short introduction to undirected graphical models is given in §A.4.) In probability, Markov

processes indexed by trees have been studied (see [4, Benjamini - Peres]), as well as Gibbs processes.

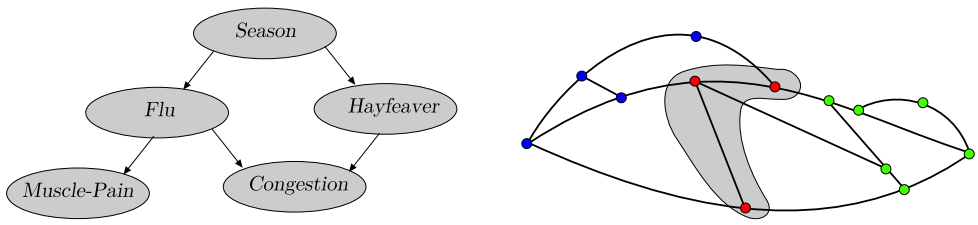


Figure 0.2: In graphical models the structure of the graph induces conditional independencies.

Similar continuous models such as the branching Brownian motion ([16]), Le Gall's Brownian snake ([16]), Brownian web ([19]) and Brownian net ([46]) have been studied. In all these models the underlying graph is a **random graph**.

In 2011, Burdzy and Pal ([7]) introduced time-like graphs (TLG's) and defined (Markov) processes on graphs with no co-terminal cells (NCC-graphs). Compared to graphical models, these were **continuous** processes (they have a random variable defined at each point of the representation), and unlike the continuous models studied in probability, the underlying graph was **deterministic**. A number of properties (induced by the structure of the underlying graph) of these processes were proved. However, the model had **strong restrictions** both on the degrees of vertices of the graph and the distribution of the process.

In this paper we expand the definition of processes onto a wider family of graphs, answer open questions asked by Burdzy and Pal, and investigate new properties and connections with some known processes.

This paper has three main parts, and it ends with a list of open problems and an appendix that contains definitions of some terms that might not be familiar to the reader.

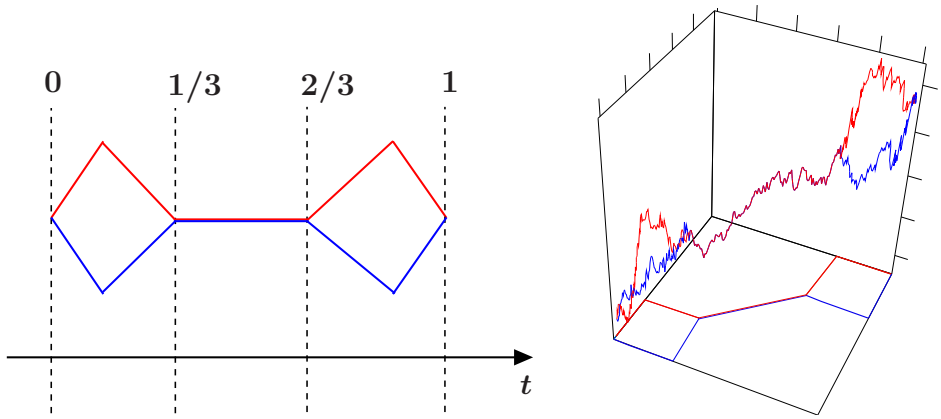


Figure 0.3: Time-like graph and a process indexed by it.

Construction and properties

In §1 we study the geometry of time-like graphs (TLG's). We are focused on the TLG* family, since the processes on this sub-family of time-like graphs can be well-defined. Many of the properties depend on the structure of the underlying time-like graph, so we investigate the **properties** and lastly give an **algorithm** for determining whether a graph belongs to the TLG* family.

In §2 we give a very general criteria for constructing a process indexed by a TLG* $\mathcal{G}$ (see §2.3). Further, we show that the constructed process has the hereditary spine-Markovian property (see §2.3.3) and we get that the distribution of the process does not depend on its construction (see Theorem 2.20). Burdzy and Pal (in [7]) **conjectured** that this holds for NCC graphs with infinitely many vertices. This is proven here in a much more general setting (Theorem 2.24).

In §3 we look into several properties of the constructed process induced by time and graph structure. Theorem 3.8 proves that a **generalized Markov property** holds, while Theorem 3.12 shows the connection between the constructed process and **Markov random fields**.

Kurtz [36] studied **martingales** that are indexed by directed sets. Theorem 1.34 shows that every TLG* $\mathcal{G}$ is a directed set, and under some conditions the process indexed by $\mathcal{G}$ will be a martingale. In §4 we develop stopping times and look at the properties of filtrations to prove the **Optional Stopping Theorem** (Theorem 4.21) for martingales indexed by TLG*'s.

Natural Brownian motion and the stochastic heat equation

In Part 2 we investigate another question from the original paper [7]. What happens when we have a process on a *dense net* that covers (a subset of) the plane? In §7 we look at a rhombus grid that covers the whole plane and the two sided Brownian motion defined on this graph. We analyze what happens when the mesh size goes to zero, and study the connection with the **stochastic heat equation** (Theorem 7.8).

In chapters §5 and §6 we develop tools to prove the result about the stochastic heat equation. §5 reviews some results about maximums of Gaussian vectors and continuous Gaussian process. §6 studies the approximation of the (stochastic) heat equation with one boundary and an initial value condition with the **Euler method** under very general conditions. The main tool for the analysis is the **simple random walk**.

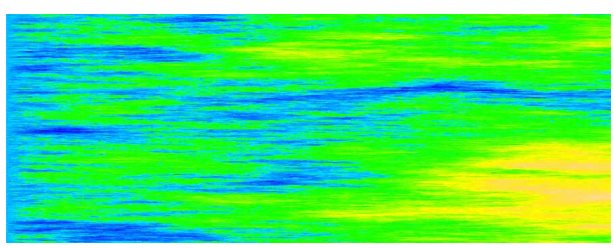


Figure 0.4: Topographical image of the simulation of a process indexed by a dense rhombus grid

Processes on general and random time-like graphs

The graphs used in Part 1 and 2 have one beginning and one end, so we can not define a process on a time-like tree. In §8 we modify our approach to define a process with nice properties on a more general family of graphs - TLG**'s. This family includes **trees**, and it turns out (see §9.4) that some properties which do not hold in general are true for time-like trees. We proceed to define **Galton-Watson time-like trees** (§10.2), and investigate (§10.5) what happens when we index the process by this type of random trees.

Open questions and appendix

This paper ends with several open questions: Under what conditions can we define a process on any TLG? If we know the process on some parts of the graph, what can we tell about the parts that are hidden from us? Do we (under some conditions) have the strong Markov property? How would we model the evolution of the process on a graph over time?

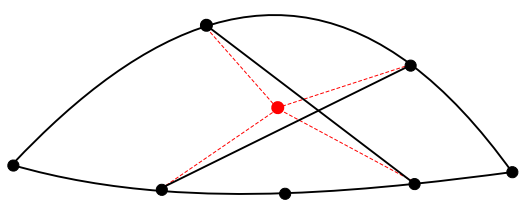


Figure 0.5: Open question: We know about the black parts of the graph and the process on it, what can we say about the part of the structure that is hidden?

The Appendix contains some definitions and known results that we will often use.

Part I

CONSTRUCTION AND PROPERTIES

Chapter 1

GEOMETRY OF TIME-LIKE GRAPHS

Most of the definitions presented in this chapter are modified from the original model presented in [7]. The crucial difference is the Definition 1.1 of time-like graphs. In the original model, Burdzy and Pal considered time-like graphs with the beginning and end vertex of degree 1, and all other vertices of degree 3. See Figure 1.1.

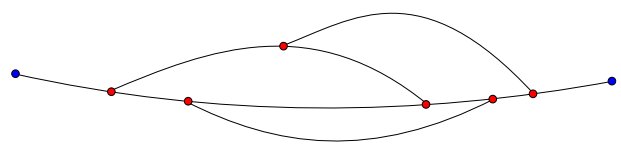


Figure 1.1:

The rest is a deeper study of geometric properties of the special family TLG*. These properties will later be vital for the construction of the processes and many of their properties.

1.1 Basic definitions

Definition 1.1. *A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ will be called a* ***time-like graph (TLG)*** *if its sets of vertices $\mathcal{V}$ and edges $\mathcal{E}$ satisfy the following properties.*

(i) The set $\mathcal{V}$ contains at least two elements, $\mathcal{V} = \{t_0, t_1, \ldots, t_N\}$, where $t_0 = A$, $t_N = B$ and for $k = 1, 2, \ldots, N-2$, $A < t_k \leq t_{k+1} < B$.

(ii) An edge between t_j and t_k will be denoted E_{jk}. We assume that there is no edge between t_j and t_k if $t_j = t_k$. E_{jk} indicates that $t_j < t_k$.

(iii) We assume that all vertices have a finite degree.

(iv) We assume that for every vertex t_k $k = 1, \ldots, N-1$ there exist edges E_{jk} and E_{kn} with $j < k < n$.

We call TLG to be a ***unit*** *TLG if $A = 0$ and $B = 1$.*

Remarks. (0) In our study of TLG's, we will assume that TLG is a unit TLG, unless specified differently. (1) We do not exclude the case $\mathcal{V} = \{t_0 = A, t_N = t_1 = B\}$.

(2) The definition implies that TLG has no loops.

(3) In (i) formally we should say that the elements have the form (k, t_k), so that (k, t_k) and $(k+1, t_{k+1})$ are distinct even if $t_k = t_{k+1}$. This notation was simplified to make writing easier.

(4) An edge between t_j and t_k $(j < k)$ will be denoted E_{jk} (if it exists), and if we are using more of them we will use the notation E^1_{jk}, E^2_{jk} ... (or something similar).

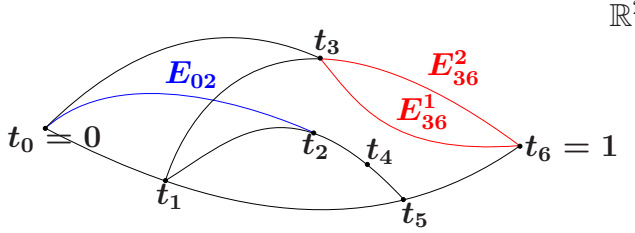


Figure 1.2: TLG $\mathcal{G}$

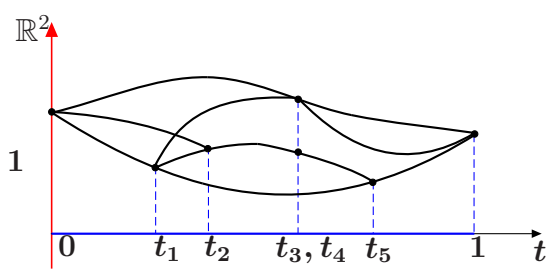


Figure 1.3: Representation of a TLG $\mathcal{G}$

The representation of a TLG in $\mathbb{R}^3$ is given by the following definition.

Definition 1.2. *By abuse of notation let $E_{jk} : [t_j, t_k] \to \mathbb{R}^2$ denote a continuous function for all $E_{jk} \in \mathcal{E}$. Assume:*

(i) That the images of the open sets (t_j, t_k) under the maps $t \mapsto (t, E_{jk}(t))$, where $E_{jk} \in \mathcal{E}$ are disjoint.

(ii) That $E_{jk}(t_k) = E_{kn}(t_k)$ if $E_{jk}, E_{kn} \in \mathcal{E}$; $E_{jk}(t_k) = E_{mk}(t_k)$ if $E_{jk}, E_{mk} \in \mathcal{E}$; and $E_{0k}(t_0) = E_{0j}(t_0)$ for $E_{0k}, E_{0j} \in \mathcal{E}$.

We will call the set

$$R(\mathcal{G}) = \{(t, E_{jk}(t)) \in [0,1] \times \mathbb{R}^2 : E_{jk} \in \mathcal{E}, t \in [t_j, t_k]\}$$

a ***representation*** *of* $\mathcal{G}$*. We will say that* $\mathcal{G}_1$ *is a subgraph of* $\mathcal{G}_2$*, and write* $\mathcal{G}_1 \subset \mathcal{G}_2$ *if there exist representations of the two such that* $R(\mathcal{G}_1) \subset R(\mathcal{G}_2)$*. We will call* $\mathcal{G}$ ***planar*** *if it has a representation* $R(\mathcal{G}) \subset \mathbb{R}^2$*.*

Let $\bar{t}_j = (t_j, E_{jk}(t_j))$ *for* $j < N$ *and* $\bar{t}_N = (t_N, E_{N-1,N}(t_N))$*.*

Remark. There are many representations for a TLG, but there is a unique TLG corresponding to a representation.

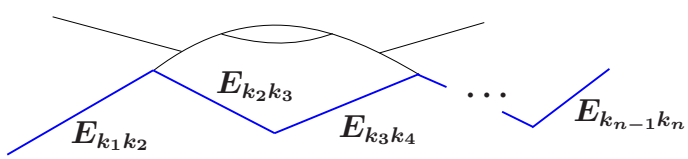


Figure 1.4: A time-path

Definition 1.3. *We will call a sequence of edges*

$$(E_{k_1k_2}, E_{k_2k_3}, \ldots, E_{k_{n-1}k_n}) \tag{1.1}$$

a ***time-path*** *if* $E_{k_jk_{j+1}} \in \mathcal{E}$ *for every* j*. We will denote the set of all paths of the form* (1.1) *by* $\sigma(k_1, k_2, \ldots, k_n)$*. This time path is* ***full time-path*** *if* $k_0 = 0$ *and* $k_n = N$*. We will denote the set of all full time-paths by* $P_{0\to 1}(\mathcal{G})$*.*

Remark. Note that the notation $\sigma(k_1, k_2, \ldots, k_n)$ does not uniquely identify the path, since there can be more than one edge between the two vertices.

Definition 1.4. *(i) Time paths* $\sigma_j \in \sigma(j_1, j_2, \ldots, j_n)$ *and* $\sigma_k \in \sigma(k_1, k_2, \ldots, k_m)$ *are* ***co-terminal*** *if* $j_1 = k_1$ *and* $j_n = k_m$*.*

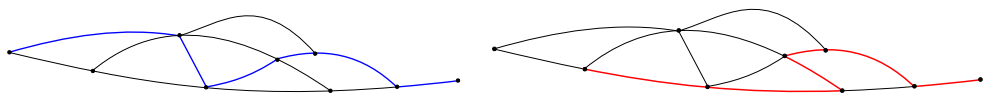


Figure 1.5: Example of a full time-path and an example of path that is not a time-path

(ii) *Co-terminal paths* $\sigma_j \in \sigma(j_1, j_2, \ldots, j_n)$ *and* $\sigma_k \in \sigma(k_1, k_2, \ldots, k_m)$ *will be form a* ***cell*** (σ_j, σ_k) *if*

$$\{j_2, j_3, \ldots, j_{n-1}\} \cap \{k_2, \ldots, k_{m-1}\} = \emptyset.$$

(iii) *We will call a cell* (σ_j, σ_k) *for* $\sigma_j \in \sigma(j_1, j_2, \ldots, j_n)$ *and* $\sigma_k \in \sigma(k_1, k_2, \ldots, k_m)$ ***simple*** *if if there does not exist a time path* $\pi \in \sigma(i_1, i_2, \ldots, i_r)$ *such that* $i_1 \in \{j_2, j_3, \ldots, j_{n-1}\}$ *and* $i_r \in \{k_2, \ldots, k_{m-1}\}$*, or* $i_1 \in \{k_2, \ldots, k_{m-1}\}$ *and* $i_r \in \{j_2, j_3, \ldots, j_{n-1}\}$.

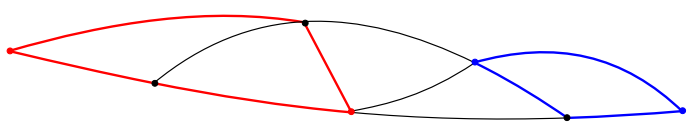


Figure 1.6: Non-simple cell and a simple cell

1.2 TLG family*

We will now describe the family of TLG graphs that is generated from minimal graph by adding vertices and adding edges between vertices connected by a time-path.

Definition 1.5. *The* ***TLG*-family*** *is given in the following inductive way.*

(i) *The minimal graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$*, with* $\mathcal{V} = \{t_0 = 1, t_N = 1\}$ *and* $\mathcal{E} = \{E_{0N}\}$ *is a TLG*.*

(ii) *Let* $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ *be a TLG*, where* $\mathcal{V}_1 = \{t_0, t_2, \ldots, t_N\}$.

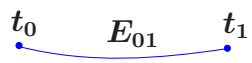


Figure 1.7: The minimal graph

(1) (ADDING A VERTEX) *If* $\tau_k \in [0,1]$, *and for some* $E_{k_1k_2} \in \mathcal{E}$ *and* $t_{k_1} < \tau_k < t_{k_2}$ *then set*

$$\mathcal{V}_2 := \mathcal{V}_1 \cup \{\tau_k\} \quad \text{and} \quad \mathcal{E}_2 := \mathcal{E}_1 \cup \{E_{k_1k}, E_{kk_2}\} \setminus \{E_{k_1k_2}\}.$$

$\mathcal{G}_2 := (\mathcal{V}_2, \mathcal{E}_2)$ *is also a TLG**.

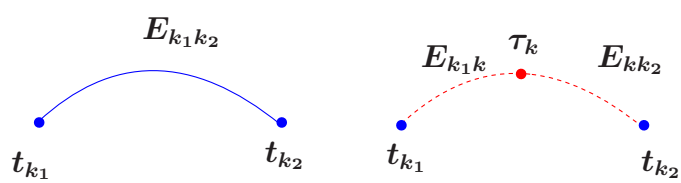


Figure 1.8: Adding a vertex

(2) (ADDING AN EDGE) *Let* $t_j, t_k \in \mathcal{V}_1$ *such that* $t_j < t_k$, *and assume that there exists a time-path* $\sigma_{jk} \in \sigma(j, \ldots, k)$ *between these vertices. Then set*

$$\mathcal{V}_2 := \mathcal{V}_1 \quad \text{and} \quad \mathcal{E}_2 := \mathcal{E}_1 \cup \{E^*_{jk}\}.$$

$\mathcal{G}_2 := (\mathcal{V}_2, \mathcal{E}_2)$ *is also a TLG**. *(*E^*_{jk} *is an new edge (not in* $\mathcal{E}_1$*).)*

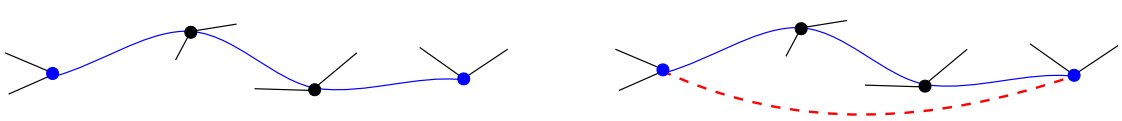


Figure 1.9: Adding the edge E^*_{jk}

(iii) We will say that $(\mathcal{G}_j)_{1 \le j \le k}$ *is a* ***tower of TLG***'s*** *or* ***TLG***-tower*** *if for* $j > 1$, $\mathcal{G}_j$ *is constructed from* $\mathcal{G}_{j-1}$ *as in (ii).*

Remarks. (1) Clearly, all TLG*'s are TLG's. (2) It is also clear that if $(\mathcal{G}_j)_{1\le j\le k}$ is a tower of TLG*'s and G_k is planar that all the graphs in this tower of TLG*'s are planar.

We will turn our attention to the question which TLG's are TLG*. The following is a generalization and a new proof of the result known to Burdzy and Pal (see Theorem 2.9 (ii) in [7]).

Theorem 1.6. *All planar TLG's are TLG*'s.*

Proof. Let $\mathcal{G}$ be a planar TLG and $R(\mathcal{G})$ its representation in $\mathbb{R}^2$. We will prove the claim in several steps.

(i) Denote time-paths from $t_0 = 0$ to $t_N = 1$ in $\mathcal{G}$ with $P_{0\to 1}(\mathcal{G})$. For each $\sigma \in P_{0\to 1}(\mathcal{G})$ there exists a continuous function $g_\sigma : [0,1] \to \mathbb{R}$ such that its graph $\Gamma_{g_\sigma} = \{(x, g_\sigma(x)) : x \in [0,1]\}$ is the representation of σ in $R(\mathcal{G})$. For two paths $\sigma' \neq \sigma''$ we have $g_{\sigma'} \neq g_{\sigma''}$, and there are three possibilities

- If $g_{\sigma'} \le g_{\sigma''}$ or $g_{\sigma'} \ge g_{\sigma''}$. In the first case we say $\sigma' \le \sigma''$ and in the second case we say $\sigma' \ge \sigma''$.

- If not, $\min\{g_{\sigma'}, g_{\sigma''}\}$, $\max\{g_{\sigma'}, g_{\sigma''}\}$ are also representations of paths from 0 to 1. (These paths use the same set of edges as paths σ' and σ''.)

We define $\sigma' \wedge \sigma''$ and $\sigma' \vee \sigma''$ to be the path represented by $\min\{g_{\sigma'}, g_{\sigma''}\}$ and $\max\{g_{\sigma'}, g_{\sigma''}\}$ in $R(\mathcal{G})$. This operation is closed, commutative and associative, and further $\sigma' \wedge \sigma'' \le \sigma' \le \sigma' \vee \sigma''$ and $\sigma' \wedge \sigma'' \le \sigma'' \le \sigma' \vee \sigma''$.

(ii) We pick σ_1 to be $\wedge_{\sigma \in P_{0\to 1}(\mathcal{G})}\sigma$, and we set $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ such that all vertices and all edges of σ_1 are in $\mathcal{V}_1$ and $\mathcal{E}_1$. Clearly this is a planar TLG. Note that we choose σ_1 such that there is no σ' in $P_{0\to 1}(G)$ with $\sigma' \le \sigma_1$.

Now we continue inductively. Let $\mathcal{G}_{k-1} = (\mathcal{V}_{k-1}, \mathcal{E}_{k-1})$ be a TLG obtained in the previous step. If $\mathcal{E} \setminus \mathcal{E}_{k-1} = \emptyset$ clearly $\mathcal{G}_{k-1} = \mathcal{G}$. Otherwise, choose σ_k in $P_{0\to 1}(\mathcal{G}) \setminus P_{0\to 1}(\mathcal{G}_{k-1})$ such that there is no σ' in the same set with $\sigma' \le \sigma_k$. (The set $P_{0\to 1}(\mathcal{G}) \setminus P_{0\to 1}(\mathcal{G}_{k-1})$ is nonempty

since every edge $E \in \mathcal{E} \setminus \mathcal{E}_{k-1}$ is part of a path from 0 to 1 in $\mathcal{G}$. There is such minimal edge with respect to the given order, since this is a finite set.) We now set $\mathcal{G}_k = (\mathcal{V}_k, \mathcal{E}_k)$, where $\mathcal{V}_k$ is the set of all vertices in $\mathcal{V}_{k-1}$ and on σ_k and $\mathcal{E}_k$ is set of all edges in $\mathcal{E}_{k-1}$ and that σ_k is made of. Again, $\mathcal{G}_k$ is a planar TLG.

Since there is only a finite number of edges in $\mathcal{E}$, at some step K we will stop, and we will have $\mathcal{G}_K = \mathcal{G}$.

(iii) Note that for each k there is no edge $E_{jn} \in \mathcal{E} \setminus \mathcal{E}_k$ such that there exists $\sigma \in P_{0\to 1}(\mathcal{G}_k)$ with $E_{jn} \leq g_\sigma|_{[t_j,t_n]}$. Otherwise, there would exist a σ_l for some $l \leq k$ such that $E_{jn} \leq g_{\sigma_l}|_{[t_j,t_n]}$, and a path $\sigma' \in P_{0\to 1}(\mathcal{G})$ that contains E_{jn}, but then $\sigma' \wedge \sigma_l \leq \sigma_l$, and this contradicts the definition of σ_l.

(iv) From the definition in (i) it is clear that

$$\sigma^k_{max} = \vee_{\sigma \in P_{0\to 1}(\mathcal{G}_k)} \sigma \tag{1.2}$$

is also a path in $\mathcal{G}_k$.

(v) Now we will show that all $\mathcal{G}_k$ are TLG*'s. It is clear that $\mathcal{G}_1$ can be obtained from the minimal graph $\mathcal{G}_0$ by repeating step (ii1) in Definition 1.5.

We assume that $\mathcal{G}_{k-1}$ is a TLG*. For an edge E_{jn} in σ_k that is not in $\mathcal{E}_{k-1}$, we have by (iii)

$$g_{\sigma^{k-1}_{max}}|_{[t_j,t_k]} \leq E_{jn}. \tag{1.3}$$

Further, σ^{k-1}_{max} (see (1.2)) will have common vertices with σ_k (at least in 0 and 1). The set $T = \{t \in [0,1] : \sigma^{k-1}_{max}(t) \neq \sigma_k(t)\}$ has at exactly one connected component. Otherwise, there would exist $t_{l_1} < t_{l_2} \leq t_{l_3} < t_{l_4}$ in ∂T and we would have two sub-paths $\sigma(l_1 \dots l_2)$ and $\sigma(l_3 \dots l_4)$ that start and end at vertices that are on σ^{k-1}_{max}, but since (1.3) we have

$$g_{\sigma^{k-1}_{max}}|_{[t_{l_1},t_{l_2}]\cup[t_{l_3},t_{l_4}]} \leq g_{\sigma_k}|_{[t_{l_1},t_{l_2}]\cup[t_{l_3},t_{l_4}]}$$

(their representations lie above $R(\mathcal{G}_{k-1})$). But, now σ' is represented by

$$g_{\sigma'}(t) := \begin{cases} g_{\sigma_k}(t) & t \in [t_{l_1}, t_{l_2}] \\ g_{\sigma^{k-1}_{max}}(t) & t \in [t_{l_1}, t_{l_2}]^c \end{cases}$$

is also a path in $P_{0\to1}(\mathcal{G}) \setminus P_{0\to1}(\mathcal{G}_{k-1})$, such that $\sigma' \leq \sigma_k$. This is a contradiction, with the definition of σ_k. Therefore, T has only one connected component and σ_{max}^{k-1} and σ_k have two common vertices - t_{l_1} and t_{l_2}. Since t_{l_1} and t_{l_2} are on the path σ_{max}^{k-1} by Definition 1.5 we can add an edge between them, and after that add vertices that are on the path that connects them. All the other edges of the path σ_k (that are below the path σ_{max}^{k-1} in the representation) are already included in $\mathcal{G}_{k-1}$ (by (iii)), so we get $\mathcal{G}_k$.

■

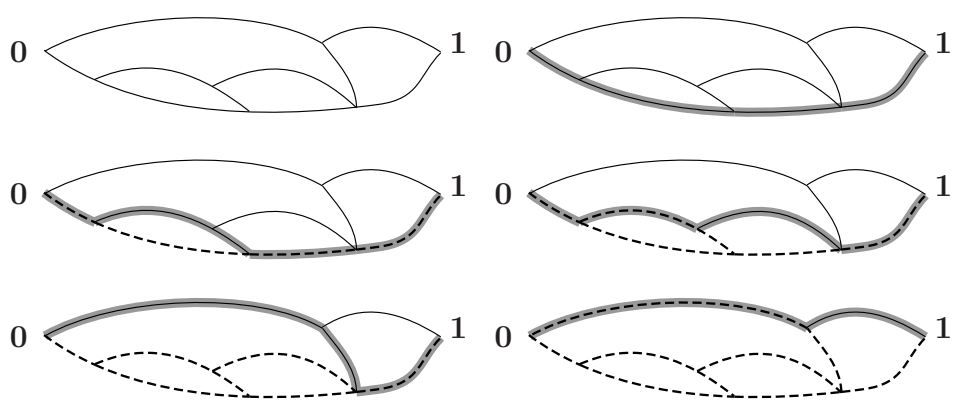


Figure 1.10: Illustration of the proof of Theorem 1.6. The the path colored in gray represents σ_k, while dashed lines represent $\mathcal{G}_{k-1}$.

Remark. The proof gives us the following algorithm for constructing a planar TLG $\mathcal{G}$ as a TLG*.

1 σ a minimal path with respect to $\leq$ in $P_{0\to 1}(\mathcal{G})$;

2 $\mathcal{G}^{\#} = (\mathcal{V}^{\#}, \mathcal{E}^{\#})$ that consists of all vertices and all edges of σ (in $\mathcal{G}$);

3 **while** $\mathcal{E} \setminus \mathcal{E}^{\#} \neq \emptyset$ **do**

4 $\quad$ σ a minimal path with respect to $\leq$ in $P_{0\to 1}(\mathcal{G}) \setminus P_{0\to 1}(\mathcal{G}^{\#})$;

5 $\quad$ add all edges and vertices that make σ (in $\mathcal{G}$) to $\mathcal{G}^{\#}$;

6 **end**

Algorithm 1: Constructing a planar TLG as a TLG*.

We have shown that the step in line 5 can be done by adding edges and vertices as described in Definition 1.5. Since $\mathcal{G}^{\#}$ is a TLG* in line 2, $\mathcal{G}^{\#}$ remains a TLG* through the whole algorithm. The illustration of this algorithm is given in Figure 1.10.

Corollary 1.7. *For a planar TLG $\mathcal{G}$ there exists a tower of planar TLG's (TLG*'s) $(\mathcal{G}_j)_{1\leq j\leq n}$. such that $\mathcal{G}_1 = (\{t_0 = 0, t_N = 1\}, \{E_{0N}\})$ and $\mathcal{G}_n = \mathcal{G}$. Further, there exists a sequence of representations $(R(G_j))_{1\leq j\leq n}$ such that $R(G_{j-1}) \subset R(G_j)$ for $j > 1$.*

Theorem 1.8. *(i) There exists a TLG that is not a TLG*.*

(ii) There exists a non-planar TLG*.

Proof. We will show the claim using examples similar to those Burdzy and Pal gave in [7].

(i) Assume the TLG $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{t_j = j/5 : j = 0, 1, \ldots, 5\}$ and

$$\mathcal{E} = \{E_{01}, E_{02}, E_{14}, E_{13}, E_{23}, E_{24}, E_{45}, E_{35}\}$$

(on the Figure 1.11.) is a TLG*. Then there exists a tower of TLG* $(\mathcal{G}_j)_{1\leq j\leq n}$ such that $\mathcal{G}_n = \mathcal{G}$. Let E^* be the edge form the set $\mathcal{E}^* = \{E_{14}, E_{13}, E_{24}, E_{23}\}$ with largest j such that $E^* \in \mathcal{E}_j \setminus \mathcal{E}_{j-1}$. ($E^*$ is the last edge from $\mathcal{E}^*$ to be added to the graph.)

In Definition 1.5. we add edges in each step, so that their vertices lie on the same path from 0 to 1 and these vertices will continue to be on the same path in future steps. Since, no three vertices from the set $\{t_1, t_2, t_3, t_4\}$ are on the same path in $\mathcal{G}$, in each step we can add only one edge from the set $\mathcal{E}^*$.

The graph $\mathcal{G}_{j-1}$ contains the vertices t_1, t_2, t_3 and t_4, since it contains three out of four edges from $\mathcal{E}^*$ connecting them.

In order to obtain $\mathcal{G}_j$ the endpoints of E^* have to be connected by a time path. It is clear that each element of the tower $(\mathcal{G}_j)_{1\leq j\leq n}$ the number of time paths between the two vertices increases. This means that the number time paths between the endpoints of $\mathcal{G}$ will be at least two, but this is not true in $\mathcal{G}$. Hence, $\mathcal{G}$ can not be a TLG*.

(ii) Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{t_j = j/7 : j = 0, 1, \ldots, 7\}$ and

$$\mathcal{E} = \{E_{01}, E_{12}, E_{23}, E_{34}, E_{45}, E_{56}, E_{67}, E_{14}, E_{25}, E_{3,6}\}.$$

It is clear that this is a TLG* and it is not planar. See Figure 1.12 ■

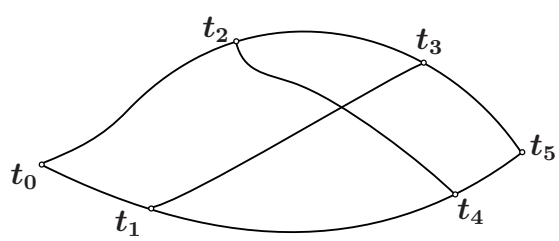


Figure 1.11: A TLG that is not a TLG*

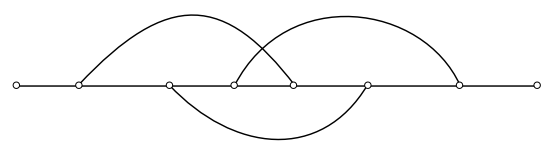


Figure 1.12: Non-planar TLG*.

1.3 Consistent representation of a TLG*-tower, spines and (re)construction

If $\mathcal{G}$ is a TLG*, then let $(\mathcal{G}_j)_{j=0}^n$ be a TLG* tower. In the corresponding sequence of representations $(R(\mathcal{G}_j))_{j=0}^n$ we could have some inconsistencies which we would like to avoid. For instance, let co-terminal edges $E^1 = E^1_{m_1m_2}$ and $E^2 = E^2_{m_1m_2}$ be present in the whole tower and the graph in the Figure 1.13 can represent part of each representation. The arcs

a and b in representation $R(\mathcal{G}_{j_1})$ might represent E^1 and E^2, while in some other representation $R(\mathcal{G}_{j_2})$ it might be the other way around. To avoid this we will only use *consistent representations* of the TLG* tower $(\mathcal{G}_j)_{j=0}^n$.

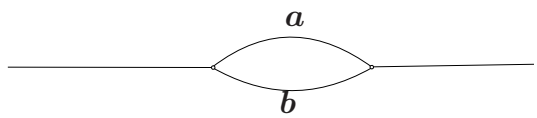


Figure 1.13: The arc a and b might not always represent the same edges.

Definition 1.9. *We will call a sequence of representations $(R(\mathcal{G}_j))_{j=0}^n$ a* ***consistent representation*** *of the TLG*-tower $(\mathcal{G}_j)_{j=0}^n$ if:*

(a) If we add a new vertex τ_k to the TLG $\mathcal{G}_{j-1}$ to obtain $\mathcal{G}_j$ by removing an edge $E_{k_1k_2}$, and replacing it with E_{k_1k} and E_{kk_2} (as in step (ii1) of Definition 1.5.), then the representation of edges E_{k_1k} and E_{kk_2} is the same as that of $E_{k_1k_2}$, i.e.*

$$E_{k_1k_2}([t_{k_1}, t_{k_2}]) = E_{k_1k}([t_{k_1}, t_k]) \cup E_{kk_2}([t_k, t_{k_2}]).$$

(b) All the edges that are in both $\mathcal{G}_{j-1}$ and $\mathcal{G}_j$, will have the same representation in $R(\mathcal{G}_{j-1})$ and $R(\mathcal{G}_j)$, i.e. for $E_{k_1k_2} \in \mathcal{E}_{j-1} \cap \mathcal{E}_j$ if $E'_{k_1k_2}$ is the representation in $R(\mathcal{G}_{j-1})$ and $E''_{k_1k_2}$ is the representation in $R(\mathcal{G}_j)$ then

$$E'_{k_1k_2}([t_{k_1}, t_{k_2}]) = E''_{k_1k_2}([t_{k_1}, t_{k_2}]).$$

The two following facts are true about consistent representations.

Proposition 1.10. *(i) If $(R(\mathcal{G}_j))_{j=0}^n$ is a consistent representation of the TLG*-tower $(\mathcal{G}_j)_{j=0}^n$ then $R(\mathcal{G}_{j-1}) \subset R(\mathcal{G}_j)$ for $j \geq 1$.*

(ii) If $(\mathcal{G}_j)_{j=0}^n$ is a TLG-tower, for a fixed representation $R(\mathcal{G}_n)$, there exists a unique consistent representation $(R(\mathcal{G}_j))_{j=0}^n$ of this TLG* tower.*

Proof. The claim (i) is clear from Definition 1.9. (ii) follows by induction on the number of edges. ■

Definition 1.11. *Let $\mathcal{G}$ be a TLG* and fix its representation $R(G)$. By Definition 1.5. of TLG*'s there exists a TLG* tower $(\mathcal{G}_j)_{j=0}^n$, where $\mathcal{G}_0$ is the minimal graph and $\mathcal{G}_n = \mathcal{G}$. By Proposition 1.10 there exists a consistent representation $(R(\mathcal{G}_j))_{j=0}^n$ where $R(\mathcal{G}_n) = R(\mathcal{G})$.*

It is easy to see that that $R(\mathcal{G}_0)$ is the representation of a full time-path σ in $\mathcal{G}$. We will call such a full time-path a ***spine****.*

The question is each full time-path a spine? In other words, can we take any full time path, and by adding vertices and edges as in the Definition 1.5 of TLG* get the TLG* $\mathcal{G}$.

Theorem 1.12. *Each full-time path in TLG* is a spine.*

Proof. We will prove this claim by induction on the number of edges $m = |\mathcal{E}|$ in $\mathcal{G}$.

For $m = 1$ the claim holds, since the spine is the whole $\mathcal{G}$.

Assume that the claim holds for $m \geq 1$. Let $\mathcal{G}$ be a TLG* with $m+1$ edges. There exists a TLG* $\mathcal{G}'$ such that by adding a vertex or edge (as in step (ii) Definition 1.5.) we get $\mathcal{G}$. (Note that in both cases $\mathcal{G}'$ has m edges.)

If we added a new vertex to $\mathcal{G}'$ there exists a representation of $R(\mathcal{G})$ that is the same as the one of $R(\mathcal{G}')$. Now it is clear, that if we pick any full time-path in $\mathcal{G}$, there is a σ' full time-path in $\mathcal{G}'$ with the same representation in $R(\mathcal{G}')$. We first construct $\mathcal{G}'$, from σ' and then we add $\mathcal{G}$ to the tower describing that construction.

If we added a new edge $E^*_{h_1h_2}$. If we pick a full time-path σ' that is in $\mathcal{G}'$, then we first construct $\mathcal{G}'$ from it and then add $\mathcal{G}$ as the last member of the tower describing that construction. If we pick a full time path σ^* containing $E^*_{h_1h_2}$, let σ' be a full time path connecting t_{h_1} and t_{h_2}, such that σ' and σ^* are the same except between t_{h_1} and t_{h_2}. We can construct $\mathcal{G}'$ from σ'. To construct $\mathcal{G}$ from σ^* we start with one edge representing σ^*, and then add vertices t_{h1} and t_{h2} and an edge between them. Now, we have a full time-path that has the same representation as σ', and we keep adding edges and vertices in the same

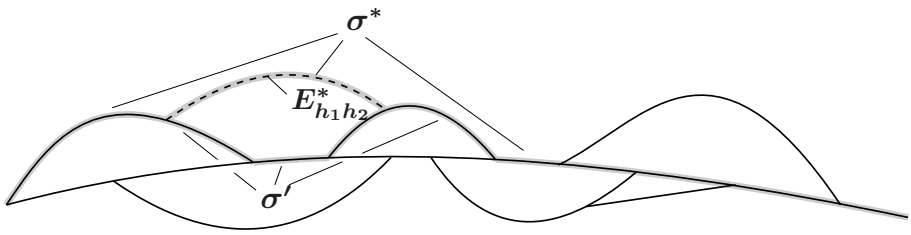


Figure 1.14: Two spines σ^* and σ'.

order as in the construction of $\mathcal{G}'$ starting with σ' (we skip the steps in which t_1 and t_2 are added). At the end we have $\mathcal{G}$. ■

We have an interesting consequence of the previous Theorem.

Corollary 1.13. *If $E_{h_1h_2}$ is an edge between the two vertices connected by a time path (not containing that edge) in $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, then $\mathcal{G}' = (\mathcal{V}, \mathcal{E} \setminus \{E_{h_1h_2}\})$ is also a TLG*.*

Proof. We pick a full time-path containing that path. Now in the construction of $\mathcal{G}$ from that time path we skip the step in which need to add the edge $E_{h_1h_2}$ and we get $\mathcal{G}'$. ■

Definition 1.14. *A **point** on $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is an element of the set*

$$\{(E_{jk}, \tau) \ : \ E_{jk} \in \mathcal{E}, \tau \in [t_j, t_k]\},$$

and the representation of the point $t = (E_{jk}, \tau)$ is the point on $R(E_{jk})$ whose time coordinate is τ. $t_1 = (E^1, \tau_1)$ and $t_2 = (E^2, \tau_2)$ are connected by a (time-)path if E^1 and E^2 are a part of some (time-)path. We will write $t_1 \preceq t_2$ if $\tau_1 \leq \tau_2$ and t_1 and t_2 are connected by a time-path.

Remark. For a point t on $\mathcal{G}$ we will write $t \in G$. Note that vertices can be represented as several points, if they are endpoints to several edges, identify them as one point. The order '$\preceq$' introduced is the **order induced by the structure of the graph**. We will write for the time of t, to simplify the notation, just t.

We will give a criteria for connectedness of two points by a time-path. This says that the two points are connected by a time-path in $\mathcal{G}$, if and only if their representations are connected from the moment that these points exist in the TLG*-tower (that leads to the construction of $\mathcal{G}$). A very similar result will hold for any path in $\mathcal{G}$ with a given time frame.

Theorem 1.15. *Let t_* and t^* be two points on $\mathcal{G}$ and let $(\mathcal{G}_k)_{k=1}^n$ be a TLG*-tower such that $\mathcal{G}_n = \mathcal{G}$ and $(R(\mathcal{G}_k))_{k=1}^n$ its consistent representation. Assume k_0 is the smallest k such that $\bar{t}_*$ and $\bar{t}^*$ are on $R(\mathcal{G}_k)$. Then t_* and t^* are connected by a time-path in $\mathcal{G}$ if and only if they are connected by a time-path in $\mathcal{G}_{k_0}$.*

Proof. If t_* and t^* are connected by a time-path in $\mathcal{G}_{k_0}$, they will remain connected by a time path in all $\mathcal{G}_k$ for $k \geq k_0$.

Let $k_* \geq k_0$ be the smallest k such that t_* and t^* are connected in $\mathcal{G}_k$. k_* exists and is less or equal n. If $k_* > k_0$, then t_* and t^* are points in $\mathcal{G}_{k_*-1}$ but are not connected. This means that an edge between two vertices t_j and t_h was added and t_* and t^* are on some time-path. But since the points t_j and t_h need to be connected in the previous step, this would not affect the connection between t_* and t^*. So t_* and t^* are connected in $\mathcal{G}_{k_*-1}$. This contradicts the definition of k_*. Therefore, $k_0 = k_*$. ■

From the last result we know that a simple cell will remain a simple cell in the TLG*-tower.

Corollary 1.16. *Let $(\mathcal{G}_k)_{k=1}^n$ be a TLG*-tower and $1 \leq k < l \leq n$. If (σ_1, σ_2) is a simple cell in $\mathcal{G}_k$ then (σ_1', σ_2') is a simple cell in $\mathcal{G}_l$, where (σ_1, σ_2) and (σ_1', σ_2') have the same representation in the consistent representation of $(\mathcal{G}_k)_{k=1}^n$.*

Definition 1.17. *For any path ρ in $\mathcal{G}$ we say that the interval $I = [a, b]$ is its* ***time-frame*** *if $R(\rho) \subset I \times \mathbb{R}^2$.*

Theorem 1.18. *Let t_* and t^* be two points on $\mathcal{G}$ and let $(\mathcal{G}_k)_{k=1}^n$ be a TLG*-tower such that $\mathcal{G}_n = \mathcal{G}$. Assume k_0 is the smallest k such that t_* and t^* are points on $\mathcal{G}_k$. Then t_* and t^**

are connected by a path ρ within the time-frame $[a, b]$ in $\mathcal{G}$ if and only if they are connected by a path within the time-frame $[a, b]$ in $\mathcal{G}_{k_0}$.

Proof. The proof is the same as in Theorem 1.15. We look a the first member of the tower when t_* and t^* are connected by a path within the time frame $[a, b]$, if this is not k_0, then the connection was established by adding an edge between some vertices t_j and t_k, but these had to already be connected by a time-path. So the connection existed in the previous member of the tower. Which proves the claim. ■

1.4 Interval TLG*'s

In this section we will show the interval property of TLG*'s.

Definition 1.19. *Let $\mathcal{G}$ be a TLG, and $\tau_1 \leq \tau_2$ vertices on a TLG. We define $\mathcal{G}[\tau_1, \tau_2]$ the* ***interval*** *$[\tau_1, \tau_2]$ of $\mathcal{G}$ to be the graph $(\mathcal{V}[\tau_1, \tau_2], \mathcal{E}[\tau_1, \tau_2])$ such that $\mathcal{V}[\tau_1, \tau_2]$ are all the vertices t_k such that there exist a time-paths $\sigma_{\tau_1 t_k}$ and $\sigma_{t_k \tau_2}$, and $\mathcal{E}[\tau_1, \tau_2]$ are edges from $\mathcal{E}$ that connect vertices from $\mathcal{V}[\tau_1, \tau_2]$.*

Remark. Note that if τ_1 and τ_2 are not connected by a time-path then $\mathcal{V}[\tau_1, \tau_2] = \emptyset$.

The following result will show that interval TLG*'s are TLG*.

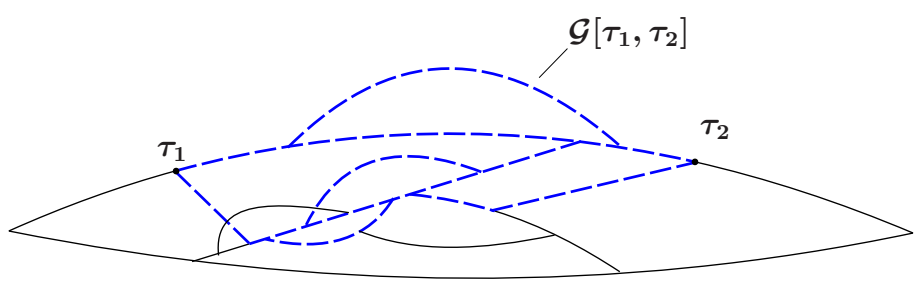


Figure 1.15: The interval graph $\mathcal{G}[\tau_1, \tau_2]$.

Theorem 1.20. *Let $\mathcal{G}$ be a TLG* and $\tau_1 \leq \tau_2$ be two vertices connected by a time path. Then $\mathcal{G}[\tau_1, \tau_2]$ is a TLG*.*

Proof. Pick a spine/full time-path σ that contains τ_1 and τ_2. (It will exist since τ_1 and τ_2 are connected by a time-path.) Now, pick a TLG* tower $(\mathcal{G}_j)_{j=1}^n$ that starts with σ and ends with $\mathcal{G}$.

We will show by induction that $\mathcal{G}_j[\tau_1, \tau_2]$ is a TLG* for all j. Without loss of generality we can assume that $\mathcal{G}_1$ contains all vertices on σ in $\mathcal{G}$.

It is clear that the claim holds for $j = 1$. Assume it holds for $j \geq 1$, and let's prove it for $j + 1$. There are 4 cases to consider:

(1) If we added an vertex to $\mathcal{G}_j$ to obtain $\mathcal{G}_{j+1}$ the claim clearly holds.

(2) We added an edge that is not connecting vertices in $\mathcal{V}_j[\tau_1, \tau_2]$. Then $\mathcal{G}_{j+1}[\tau_1, \tau_2]$ is the same as $\mathcal{G}_j[\tau_1, \tau_2]$.

(3) We added an edge that is connecting vertices in $\mathcal{V}_j[\tau_1, \tau_2]$, then these two vertices are connected by a time-path in $\mathcal{G}_j$, and hence they are connected by a time-path in $\mathcal{G}_j[\tau_1, \tau_2]$. This is the same as if we added a new edge on $\mathcal{G}_j[\tau_1, \tau_2]$ to obtain $\mathcal{G}_{j+1}[\tau_1, \tau_2]$.

(4) We added an edge that is connecting a vertex in $\mathcal{V}_j[\tau_1, \tau_2]$ and a vertex not in $\mathcal{V}_j[\tau_1, \tau_2]$. In this case $\mathcal{G}_{j+1}[\tau_1, \tau_2]$ is the same as $\mathcal{G}_j[\tau_1, \tau_2]$, because the vertex not in $\mathcal{V}_j[\tau_1, \tau_2]$, by Theorem 1.15, can't be in $\mathcal{V}_{j+1}[\tau_1, \tau_2]$.

Since in all cases $\mathcal{G}_{j+1}$ is either the same as $\mathcal{G}_j[\tau_1, \tau_2]$, or obtained from $\mathcal{G}_j[\tau_1, \tau_2]$ by adding and edge or a vertex, it is a TLG*.

This proves that $\mathcal{G}[\tau_1, \tau_2]$ is a TLG*. ■

From this proof we can get the following conclusion.

Corollary 1.21. *When we erase the repeating elements the sequence* $(\mathcal{G}_j[\tau_1, \tau_2])_{j=1}^n$ *is a TLG*-tower for* $\mathcal{G}[\tau_1, \tau_2]$.

Corollary 1.22. *For a TLG* $\mathcal{G}$ and vertices τ_1 and τ_2 on a spine σ we have that there exists a TLG*-tower $(\mathcal{G}_j)_{j=1}^n$ with consistent representation $(R(\mathcal{G}_j))_{j=1}^n$ such that for some $n_0 \leq n$*

$$R(\mathcal{G}_0) = R(\sigma), \quad R(\mathcal{G}_{n_0}) = R(\mathcal{G}[\tau_1, \tau_2]) \cup R(\sigma).$$

That is after the spine σ, we can construct $\mathcal{G}[\tau_1, \tau_2]$, and then the rest of $\mathcal{G}$.

Proof. We first construct the spine σ, and then construct TLG* $\mathcal{G}[\tau_1, \tau_2]$. Now, we apply steps from the proof of Theorem 1.20. that are using edges and vertices that haven't yet been constructed. In each of these steps when we add an edge time-path connectedness is already guaranteed since the TLG* that we have is a sup-graph of the TLG* when the step was done in the proof of Theorem 1.20. ■

1.5 Topology on TLG's

For some things that follow we will need a notion of a limit of points on a TLG. In order to define a limit we need to define a topology.

Definition 1.23. *For a point t on a TLG $\mathcal{G}$, and $0 < \delta < \min\{|t_k - t| : t_k \in \mathcal{V} \setminus \{t\}\}$, we say that the* **ball** *$B_\delta(t)$ centered at t with radius δ is the set of all points s on a TLG, such that:*

- *t and s are on a time-path;*
- *the absolute value of the time difference $|t - s|$ is less than δ.*

The following is a classical definition of open sets.

Definition 1.24. *For a set U of points on a TLG $\mathcal{G}$ we say it is an* **open set***, if for each $t \in U$ there exists a $\delta > 0$ such that $B_\delta(t) \subset U$.*

We define $\mathcal{T}_\mathcal{G}$ to be the set of all open sets in TLG $\mathcal{G}$.

Lemma 1.25. *Let $\mathcal{G}$ be a TLG, and fix its representation $R(\mathcal{G})$. U is an open set in if and only if $R(U)$ is an open set in $R(\mathcal{G})$.*

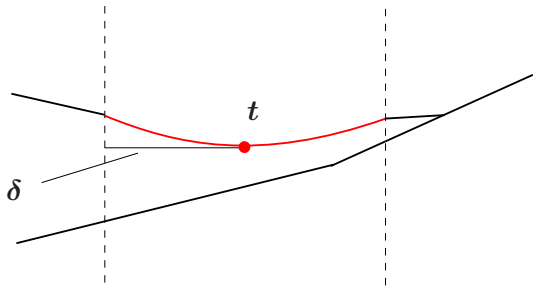


Figure 1.16: Ball in a TLG

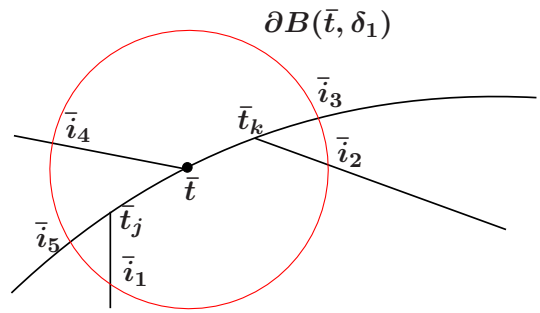


Figure 1.17: The representation intersected by a sphere. In this case we have $I = \{i_1, \ldots, i_5\}$.

Proof. If $U \in \mathcal{T}_{\mathcal{G}}$, then pick arbitrary $\bar{t} \in R(U)$. There are only finitely many paths that don't pass through t, and the union of their representations is a compact set K in $\mathbb{R}^3$. Now, we pick $\delta_1 = d(\bar{t}, K)/2$ (where d is the usual metric in $\mathbb{R}^3$). Also, we pick $\delta_2 > 0$ such that $B_{\delta_2}(t) \subset U$. For $\delta = \min\{\delta_1, \delta_2\}$, $\{s \in R(\mathcal{G}) : d(s, \bar{t}) < \delta\} \subset R(U)$. Hence $R(U)$ is opened.

If $R(U)$ is opened, then we pick $t \in U$. Pick $\delta_1 > 0$ such that $B(\bar{t}, \delta_1) = \{s \in R(G) : d(s, \bar{t}) < \delta_1\} \subset R(U)$. There exists finitely many full time-paths $\pi_1, \ldots, \pi_k$ that contain t. Let I be the points on $\mathcal{G}$ whose representations are at the intersection of $R(\pi_1), \ \ldots \ , R(\pi_k)$ with $\partial B(\bar{t}, \delta_1)$. (See Figure 1.17.) I is finite, and now pick $\delta = \min\{|t - z| : z \in I \cup (\mathcal{V} \setminus \{t\})\}/2$. Since $t \notin I$, $\delta > 0$. Hence, $B_\delta(t) \subset U$. ■

Proposition 1.26. *$\mathcal{T}_{\mathcal{G}}$ is a topology on $\mathcal{G}$.*

Proof. Note that $t \mapsto \bar{t}$ is a bijection. Hence, if $(U_\alpha : \alpha \in A)$ is in $\mathcal{T}_{\mathcal{G}}$, then since

$$R(\bigcup_{\alpha \in A} U_\alpha) = \bigcup_{\alpha \in A} R(U_\alpha)$$

is an open set so is $\bigcup_{\alpha \in A} U_\alpha$. We can use the same approach for the finite intersection. ■

Corollary 1.27. *$t \mapsto \bar{t}$ is a homeomorphism (i.e. a continuous bijective function with a continuous inverse) from $\mathcal{G}$ to $R(\mathcal{G})$.*

Corollary 1.28. *The topological space $(\mathcal{G}, \mathcal{T}_{\mathcal{G}})$ is metrizable.*

Proof. Fix the representation $R(\mathcal{G})$, and set $d_{\mathcal{G}}(s,t) := d_{\mathbb{R}^3}(\bar{s}, \bar{t})$. $d_{\mathcal{G}}$ is a metric and the topology induced by $d_{\mathcal{G}}$ is $\mathcal{T}_{\mathcal{G}}$. ■

Corollary 1.29. *$(\mathcal{G}, \mathcal{T}_{\mathcal{G}})$ is a Hausdorff space.*

Proof. Follows from the fact that this space is metrizable. ■

We define limit on TLG's in the following natural way.

Definition 1.30. *We say that the sequence of points (t_n) converges to the point t in TLG if:*

- *there exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$ the points t_n and t are connected by a time-path;*
- *the absolute value of the time difference $|t_n - t|$ converges to 0.*

Remark. The time-path that connects t_n and t can depend on n. and can be a different time-path for different n's. (It will always contain t.)

We will show that this is also the limit in the topology that we defined.

Theorem 1.31. *Let $\mathcal{G}$ be a TLG, and $R(\mathcal{G})$ be its representation. A sequence of points (t_n) converges to t in $\mathcal{G}$ if and only if their representations $(\bar{t}_n)$ converge to $\bar{t}$.*

Proof. If $t_n \to t$ in $\mathcal{G}$. There are finitely many paths $\sigma_1, \ldots, \sigma_k$ going through t. In the representation each path σ_j is represented by a graph of some continuous function f_{σ_j}. But now since

$$(t_n, f_{\sigma_j}(t_n)) \to (t, f_{\sigma_j}(t)) = \bar{t},$$

and for each $\bar{t}_n$ there is k_n such that $\bar{t}_n = (t_n, f_{\sigma_{k_n}}(t_n))$, the claim follows.

Let $\bar{t}_n \to \bar{t}$ in $R(\mathcal{G})$. Now, there are only finitely many paths that don't pass through t, and the union of their representations is a compact set K. Now we pick $\delta = d(\bar{t}, K)/2$. Now, there exists n_0 such that for all $n \geq n_0$ $\bar{t}_n \in B_\delta(\bar{t})$, but this implies that all t_n are connected by a time-path to t. It is clear that the absolute value of the time difference $|t - t_n|$ converges to 0. ■

Corollary 1.32. *$t_n \to t$ in $\mathcal{G}$ if and only if $t_n \to t$ in $(\mathcal{G}, \mathcal{T}_\mathcal{G})$.*

Proof. Fix a representation $R(\mathcal{G})$, and define a metric $d_\mathcal{G}$ as in Corollary 1.28. It is clear from Theorem 1.31 that we have convergence if and only if $d_\mathcal{G}(t_n, t) = d_{\mathbb{R}^3}(\bar{t}_n, \bar{t}) \to 0$. ■

1.6 TLG as a topological lattice*

In this section we will show that TLG*'s are topological lattices.

Definition 1.33. *A Hausdorff space X with some order '$\leq$' is called a* ***topological lattice*** *if for $x_1, x_2 \in X$:*

- *there exists a unique element $x_1 \wedge x_2$ such that*

$$\{x \in X : x \leq x_1\} \cap \{x \in X : x \leq x_2\} = \{x \in X : x \leq x_1 \wedge x_2\};$$

- *there exists a unique element $x_1 \vee x_2$ such that*

$$\{x \in X : x \geq x_1\} \cap \{x \in X : x \geq x_2\} = \{x \in X : x \geq x_1 \vee x_2\}.$$

and $x_1 \wedge x_2$ and $x_1 \vee x_2$ are continuous mappings of $X \times X$ (with product topology) onto X.

Theorem 1.34. *A TLG* $\mathcal{G}$ is a topological lattice with respect to the order $\preceq$ induced by the structure of $\mathcal{G}$.*

Proof. Let $(\mathcal{G}_k)_{k=0}^n$ be a TLG*-tower starting with the minimal graph $\mathcal{G}_0$ and ending with $\mathcal{G}_n = \mathcal{G}$.

We will prove the claim by induction. Clearly, $\mathcal{G}_0$ is a topological lattice. Let's assume $\mathcal{G}_k$ is a topological lattice.

If we added a new vertex to $\mathcal{G}_k$ in order to get $\mathcal{G}_{k+1}$, then clearly $\mathcal{G}_{k+1}$ is also a topological lattice.

If we added a new edge to $\mathcal{G}_k$ in order to get $\mathcal{G}_{k+1}$, then take two points $t, s \in \mathcal{G}_{k+1}$. If $t, s \in \mathcal{G}_k$, then by assumption there exist $t \wedge s$ and $t \vee s$, the same is clear if t, s are points of the new edge E^*_{jk}. The only case that remains to be checked is when $t \in E^*_{jk}$ and $s \in \mathcal{G}_k$. If $t \preceq s$, then $t \wedge s = t$ and $t \vee s = s$. Similarly when $s \preceq t$. Otherwise, we have $\{\tau \in \mathcal{G}_{k+1} : \tau \preceq s\}$ is in $\mathcal{G}_k$, so

$$\begin{aligned}
&\{\tau \in \mathcal{G}_{k+1} : \tau \preceq s\} \cap \{\tau \in \mathcal{G}_{k+1} : \tau \preceq t\} \\
=&\{\tau \in \mathcal{G}_k : \tau \preceq s\} \cap \{\tau \in \mathcal{G}_k : \tau \preceq t\} \\
=&\{\tau \in \mathcal{G}_k : \tau \preceq s\} \cap \{\tau \in \mathcal{G}_k : \tau \preceq t_j\} \\
=&\{\tau \in \mathcal{G}_k : \tau \preceq s \wedge t_j\},
\end{aligned}$$

therefore, we have $s \wedge t = s \wedge t_j$. In the same way we can show that $s \vee t = s \vee t_k$. The uniqueness follows from the fact that if $u \preceq v$ and $v \preceq u$ we have $u = v$.

Let (t^1_n) and (t^2_n) be a sequence of points converging respectively to t_1 and t_2 on $\mathcal{G}$. If $t_1 = t_2$ both sequences converge to the same point, and so will $(t^1_n \vee t^2_n)$ and $(t^1_n \wedge t^2_n)$. If t_1 and t_2 are on the same time-path, assume $t_1 \prec t_2$. Now, by the definition of convergence, there will exist a n_0 such that for $n \geq n_0$ we have $t^1_n \prec t^2_n$, hence

$$t^1_n \vee t^2_n = t^1_n \to t_1, \quad t^1_n \wedge t^2_n = t^2_n \to t_2$$

If t_1 and t_2 are not connected by a time-path, let $\delta < \min\{|t_1 - t|/2 : t \in \mathcal{V} \setminus \{t_1\}\} \wedge \min\{|t_2 - t|/2 : t \in \mathcal{V} \setminus \{t_2\}\}$, it is not hard to see that for $t' \in B_\delta(t_1)$ and $t'' \in B_\delta(t_2)$, we have $t' \vee t'' = t_1 \vee t_2$ and $t' \wedge t'' = t_1 \wedge t_2$. So for large n, the sequences will have the values $t_1 \vee t_2$ and $t_1 \wedge t_2$. ■

It is not hard to see, that the TLG that is not a TLG* from Figure 1.11 is not a topological lattice – there is no unique $t_1 \vee t_2$ and $t_3 \wedge t_4$.

Lemma 1.35. *There exists a topological lattice TLG, that is not a TLG*.*

Proof. The TLG in the Figure 1.18 is an example of a topological lattice TLG, that is not a TLG*.

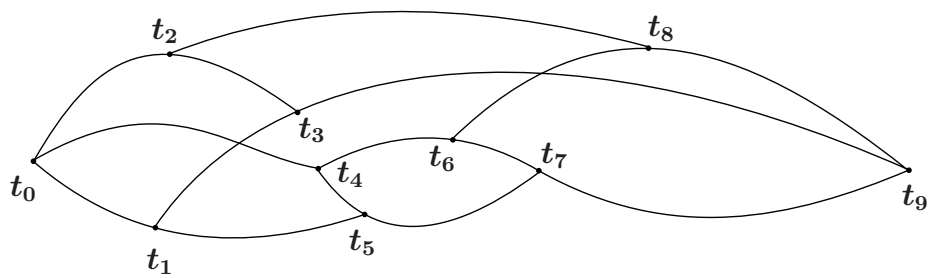


Figure 1.18: Topological lattice TLG that is not a TLG*.

It is easy to see that $t_0 \wedge t_j = t_0$ and $t_0 \vee t_j = t_j$, and similarly $t_9 \wedge t_j = t_j$ and $t_9 \vee t_j = t_9$. The following table will show what $t_k \wedge t_j$ and $t_k \vee t_j$ are. In the table above the main diagonal (for $k < j$) $t_k \wedge t_j$ is calculated, and below (for $k > j$) $t_k \vee t_j$. The diagonal is omitted, since $t_j \vee t_j = t_j \wedge t_j = t_j$.

	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8
t_1	$\circ$	t_0	t_1	t_0	t_1	t_0	t_1	t_0
t_2	t_3	$\circ$	t_2	t_0	t_0	t_0	t_0	t_2
t_3	t_3	t_3	$\circ$	t_0	t_1	t_0	t_1	t_2
t_4	t_5	t_8	t_9	$\circ$	t_4	t_4	t_4	t_4
t_5	t_5	t_9	t_9	t_5	$\circ$	t_4	t_5	t_4
t_6	t_7	t_8	t_9	t_6	t_7	$\circ$	t_6	t_6
t_7	t_7	t_9	t_9	t_7	t_7	t_7	$\circ$	t_6
t_8	t_9	t_8	t_9	t_8	t_9	t_8	t_9	$\circ$

This is not a TLG*, since by applying the cell collapse transformation, see Definition 1.40 on the cell $(t_4 - t_5 - t_7, t_4 - t_6 - t_7)$ we will no longer have a topological lattice, since $t_3 \wedge t_8$ will

no longer be unique. Therefore the transformed graph is no longer a TLG* which contradicts Lemma 1.43. if this is a TLG*. ■

A natural question that will be useful later is if we have a sequence of points (t_k) does there exist their infinitum and supreme, that is

$$\wedge_{k=1}^{\infty} t_k \quad \text{and} \quad \vee_{k=1}^{\infty} t_k.$$

Lemma 1.36. *The order in which we take apply $\wedge$ and $\vee$ doesn't matter, that is*

$$(t_1 \wedge t_2) \wedge t_3 = t_1 \wedge (t_2 \wedge t_3) \quad \textit{and} \quad (t_1 \vee t_2) \vee t_3 = t_1 \vee (t_2 \vee t_3).$$

Proof. Let $t_* = (t_1 \wedge t_2) \wedge t_3$ and $t^* = t_1 \wedge (t_2 \wedge t_3)$. It is clear that $t_* \preceq t_3$, and $t_* \preceq t_1 \wedge t_2$ implies $t_* \preceq t_2$ and $t_* \preceq t_1$. By definition it is clear that $t_* \preceq (t_2 \wedge t_3)$, again using the same we have $t_* \preceq t_1 \wedge (t_2 \wedge t_3) = t^*$. In the same way, we can get $t^* \preceq t_*$, and this implies $t_* = t^*$. Hence, the first equality follows. The second equality follows by similar arguments. These equalities imply the other statements. ■

Lemma 1.37. *Let $(t_k)_{k=1}^{\infty}$ be a sequence of points in a TLG*. We define the sequences (t_k^-) and (t_k^+) by $t_1^- = t_1$, and $t_k^- = t_k \wedge t_{k-1}^-$, and $t_1^+ = t_1$, and $t_k^+ = t_k \wedge t_{k-1}^+$. Sequence (t_k^-) and (t_k^+) will converge to limits t_* and t^*. Further for any bijection $f : \mathbb{N} \to \mathbb{N}$ the sequences (t_k^{f-}) and (t_k^{f+}) obtained from $(t_{f(k)})$ in the same way will converge respectively to t_* and t^*.*

Proof. By definition, for each n the points $(t_k^-)_{k=1}^n$ there exists a full time-path σ, such that these points are all on σ. Further, the sequence of times (t_k^-) converges to a time t_*. On the TLG* $\mathcal{G}$ there are only finitely many points with that time, name them $t_{1*}, \ldots, t_{m*}$. Let $\varepsilon = \min\{|t_* - t_{j*} \vee t_{k*}| : k \neq j\}$ where the minimum is taken over the time distances. Now, if we pick k_0 such that $|t_* - t_k^-| < \varepsilon$ (time distance) for $k \geq k_0$, then there will be only one t_{j*} in the future of t_k^-'s for $k \geq k_0$. We set it to be t_*, and it is not hard to see that all the points are on the unique path between t_* and $t_{k_0}^-$. Now it is clear, since the topology on that path is the same as the one on the open segment, that $t_k^- \to t_*$.

By what we have just proven (t_k^{f-}) converges to some point t_*^f. But then, we can show by definition, that $t_*^f \preceq t_*$ and $t_* \preceq t_*^f$, which implies $t_*^f = t_*$. ■

Definition 1.38. *For a finite sequence $(t_k)_{k=1}^n$ we define*

$$\wedge_{k=1}^n t_k := t_{p(1)} \wedge (t_{p(2)} \wedge (\ldots (t_{p(n-1)} \wedge t_{p(n)}))) \quad \text{and} \quad \vee_{k=1}^n t_k = t_{p(1)} \vee (t_{p(2)} \vee (\ldots (t_{p(n-1)} \vee t_{p(n)}))).$$

where p is any permutation of the set $\{1, 2, \ldots, n\}$. For a sequence $(t_k)_{k=1}^\infty$ we define and any bijection $f : \mathbb{N} \to \mathbb{N}$ we define

$$\wedge_{k=1}^\infty t_k := \lim_{n\to\infty} \wedge_{k=1}^n t_{f(k)} \quad \text{and} \quad \vee_{k=1}^\infty t_k := \lim_{n\to\infty} \vee_{k=1}^n t_{f(k)}.$$

Corollary 1.39. *The terms $\wedge_{k=1}^n t_k$, $\vee_{k=1}^n t_k$, $\wedge_{k=1}^\infty t_k$ and $\vee_{k=1}^\infty t_k$ are well defined for any sequence (t_k).*

Proof. Follows from Lemma 1.36. and Lemma 1.37. ■

1.7 Cell collapse transformation and the stingy algorithm

Another property of TLG* will be introduced in this section. This will be a transformation on TLG's that will map TLG*'s into TLG*'s.

Definition 1.40. *We will call the map $\mathcal{G} \mapsto \mathcal{G}^\circ$ from TLG's into TLG's a* ***cell collapse transformation*** *if:*

Pick a cell $(\sigma_{uv}, \sigma_{uv}^1)$ (starting at t_u and ending at t_v). The transformation that we will describe, basically, glues σ_{uv}^1 with its vertices to σ_{uv}, while keeping most of the connections between vertices in the graph.

We construct the graph $\mathcal{G}^\circ = (\mathcal{V}^\circ, \mathcal{E}^\circ)$ in the following way:

- *In the first step we are maping the cell into a time-path.*

Let $t_u = t_{w_1} \leq \ldots \leq t_{w_h} = t_v$ *be the set of vertices on the time-paths* σ_{uv} *and* σ^1_{uv} *ordered with respect to time. We will map* t_{w_j} *into* $(t_{w_j})^\circ$ *in* $\mathcal{V}^\circ$ *so that the vertices with the same time are mapped into same vertices, that is if* $t_{w_{j_1}} = t_{w_{j_2}}$ *then* $(t_{w_{j_1}})^\circ \equiv (t_{w_{j_2}})^\circ$.

We will use the notation $(t_{w_{j_1}})^\circ = t^\circ_{w^\circ_{j_1}}$. *Note that if* $t_{w_{j_1}} = t_{w_{j_2}}$, *then* $w^\circ_{j_1} = w^\circ_{j_2}$.

We add an edge in $\mathcal{E}^\circ$ *between* $t^\circ_{w^\circ_j}$ *and* $t^\circ_{w^\circ_{j+1}}$ *if their times are different. (Note that in this way all the vertices in* $\{(t_{w_j})^\circ : j = 1, 2, \ldots n\}$ *are on the same time-path.)*

- *Every other vertex* t_j *from* $\mathcal{V}$ *not contained on the paths* σ_{uv} *and* σ^1_{uv} *is mapped into* $(t_j)^\circ$ *in* $\mathcal{V}^\circ$ *so that the time is preserved, and these vertices are mapped into different vertices and disjoint from where the vertices on* σ_{uv} *and* σ^1_{uv} *were maped.*

- *For each edge* E *in* $\mathcal{E}$ *not a part of* σ_{uv} *or* σ^1_{uv} *we add a* E° *in* $\mathcal{E}^\circ$ *between the corresponding vertices. We color* E° *in red if* E *is adjacent to a vertex from* σ_{uv}, *or in blue if it is adjacent to the vertex from* σ^1_{uv}.

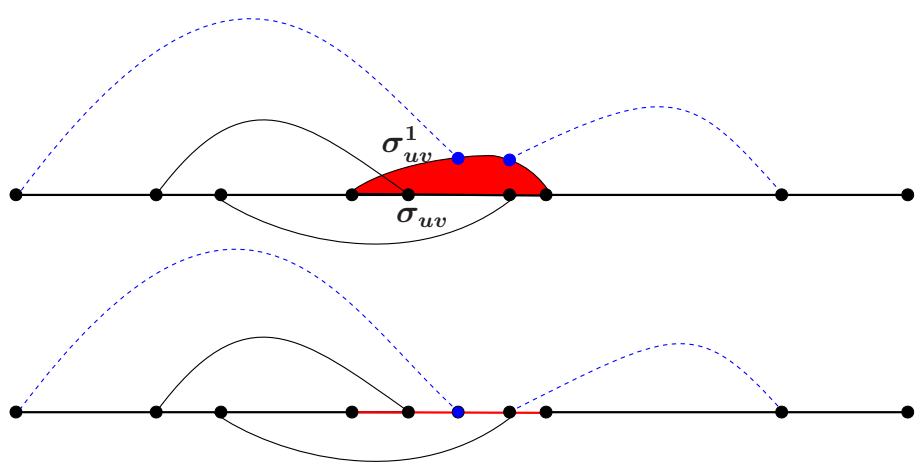


Figure 1.19: Transformation from $\mathcal{G}$ to $\mathcal{G}^\circ$.

Definition 1.41. *A cell (σ_1, σ_2) in TLG $\mathcal{G}$ starting at t_{k_1} and ending at t_{k_2} will be called* ***truly simple*** *, if there is no path in $\mathcal{G}[t_{k_1}, t_{k_2}]$ connecting the interior of σ_1 and σ_2.*

Remark. The path in question does not have to be a time-path. If there exists a a time path between σ_1 and σ_2 then it will be in $\mathcal{G}[t_{k_1}, t_{k_2}]$, so a truly simple cell is a simple cell.

Before we prove the main result of this section we will prove the following lemma.

Lemma 1.42. *(a) Let (σ_1, σ_2) starting at t_{k_1} and ending at t_{k_2} in in TLG* $\mathcal{G}$ be a truly simple cell. Then*

$$R(\mathcal{G}[t_{k_1}, t_{k_2}]) \setminus \{\bar{t}_{k_1}, \bar{t}_{k_2}\}$$

has at least two connected components.

(b) Let $\mathcal{G}$ be a TLG*, and let $R(\mathcal{G}) \setminus \{\bar{t}_0, \bar{t}_N\}$ have two connected components. Closure of each of these components, is a representation of a TLG*.

Proof. (a) Since (σ_1, σ_2) is a truly simple cell, there is no path between the interior vertices of σ_1 and σ_2. So $R(\sigma_1)$ and $R(\sigma_2)$ are connected only through $\bar{t}_{k_1}$ and $\bar{t}_{k_2}$. Therefore, $R(\sigma_1) \setminus \{\bar{t}_{k_1}, \bar{t}_{k_2}\}$ and $R(\sigma_2) \setminus \{\bar{t}_{k_1}, \bar{t}_{k_2}\}$ are in two different connected components of $R(\mathcal{G}[t_{k_1}, t_{k_2}]) \setminus \{\bar{t}_{k_1}, \bar{t}_{k_2}\}$.

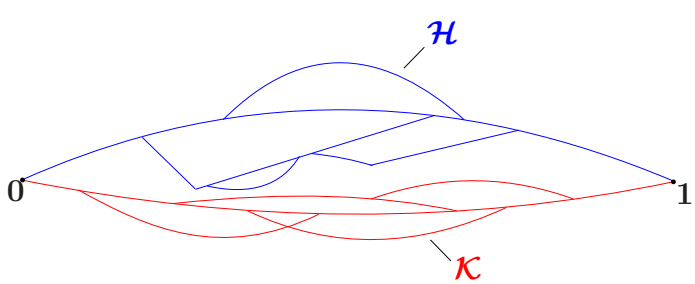


Figure 1.20: $\mathcal{H}$ and $\mathcal{K}$ are TLG*'s.

(b) Pick a component, and let $\mathcal{H}$ be the sub-graph of $\mathcal{G}$ that represents this component and the union of $\{\bar{t}_1, \bar{t}_N\}$. Pick a TLG*-tower $(\mathcal{G}_j)_{j=0}^n$ that starts with a minimal edge and

ends with $\mathcal{G}$. Let $(\mathcal{G}_{j_k})_{k=1}^{n_1}$ be the subsequence of all members of $(\mathcal{G}_j)_{j=0}^{n}$ such that an edge or a vertex whose representation intersects $R(\mathcal{H}) \setminus \{\bar{t}_1, \bar{t}_N\}$ has been added to $\mathcal{G}_{j_k-1}$ to obtain $\mathcal{G}_{j_k}$.

By the definition of the sequence $(\mathcal{G}_{j_k})$, an edge has been added to $\mathcal{G}_{j_1-1}$ in order to obtain $\mathcal{G}$. Since the representation of that edge intersects $R(\mathcal{H}) \setminus \{\bar{t}_1, \bar{t}_N\}$ which is a disconnected component of $R(\mathcal{G}) \setminus \{\bar{t}_1, \bar{t}_N\}$. So therefore that edge needs to be between t_1 and t_N. Set

$$\mathcal{H}_1 := (\mathcal{V}_{j_1} \cap \mathcal{V}_{\mathcal{H}}, \mathcal{E}_{j_1}|_{\mathcal{V}_{j_1} \cap \mathcal{V}_{\mathcal{H}}}),$$

where $\tilde{\mathcal{E}}|_{\tilde{\mathcal{V}}}$ represents the subset of edges in $\tilde{\mathcal{E}}$ that are connecting vertices in $\tilde{\mathcal{V}}$. It is clear that $\mathcal{H}_1$ is a minimal graph.

Further, define $\mathcal{H}_k = (\mathcal{V}_{j_k} \cap \mathcal{V}_{\mathcal{H}}, \mathcal{E}_{j_k}|_{\mathcal{V}_{j_k} \cap \mathcal{V}_{\mathcal{H}}})$ for $k = 2, \ldots, n_1$. We will show that $(\mathcal{H}_k)_{k=1}^{n_1}$ is a TLG*-tower. $\mathcal{H}_{n_1}$ by construction equals $\mathcal{H}$. $\mathcal{H}_1$ is a TLG*. Let's assume $\mathcal{H}_k$ is a TLG* (for $k \geq 1$) and show that $\mathcal{H}_{k+1}$ is a TLG*. If a new vertex has been added to $\mathcal{G}_{j_{k+1}-1}$ to obtain $\mathcal{G}_{j_{k+1}}$, this is, by construction, the same as if we added a new vertex to $\mathcal{H}_k$ in order to obtain $\mathcal{H}_{k+1}$. If we added a new edge, the representation of that edge intersects $R(\mathcal{H}) \setminus \{\bar{t}_1, \bar{t}_N\}$, and therefore is in that component. Since the new edge is connecting two vertices connected by a time-path in $R(\mathcal{H}) \cap R(\mathcal{G}_{j_{k+1}-1})$ these vertices are in $\mathcal{H}$, and they are connected in $\mathcal{H}_k$. Hence, we added an edge to $\mathcal{H}_k$ between two vertices connected by a time-path. In both cases $\mathcal{H}_{k+1}$ is a TLG* obtained from $\mathcal{H}_k$. Hence, $\mathcal{H}$ is a TLG*. ■

Theorem 1.43. *If $\mathcal{G}$ is a TLG* and $\circ$ is collapsing a truly simple cell, then $\mathcal{G}^\circ$ is also a TLG*. Further, if $\circ$ is collapsing a simple cell, then $\mathcal{G}^\circ$ doesn't have to be a TLG*.*

Proof. Pick a spine σ that contains σ_{uv} side of the chosen cell $(\sigma_{uv}, \sigma_{uv}^1)$. We know from Theorem 1.20 that $\mathcal{G}[t_u, t_v]$ is a TLG*. By Lemma 1.42 $\mathcal{G}[t_u, t_v]$ is a union of two or more TLG*'s that only have vertices t_u and t_v in common, and are otherwise disconnected. σ_{uv} and σ_{uv}^1 are in two different TLG* components and they are also spines in these components. Therefore, we can construct $\mathcal{G}[t_u, t_v]$, from the TLG* whose representation is $R(\sigma_{uv}) \cup R(\sigma_{uv}^1)$. We start with the two edges that have the representation $R(\sigma_{uv})$ and $R(\sigma_{uv}^1)$, and then we

first construct the component that contains σ_{uv}, then the one that contains σ^1_{uv}, and possible other components. At the end we get $\mathcal{G}[t_u, t_v]$. But then, for any full-time path σ that contains t_u and t_v we can construct the TLG* whose representation is $R(\sigma) \cup R(\mathcal{G}[t_u, t_v])$ starting with the TLG*

$$\mathcal{G}_1 = (\{t_0, t_u, t_v, t_N\}, \{E_{0u}, E^1_{uv}, E^2_{uv}, E_{vN}\}),$$

and later, by Corollary 1.22, we can construct $\mathcal{G}$. Hence, there exists a TLG*-tower $(\mathcal{G}_j)_{j=1}^n$ such that ends with $\mathcal{G}$, and its consistent representation has the representation of the cell $(\sigma_{uv}, \sigma^1_{uv})$ at each level. Now, we define $\circ$-transformation to collapse the cell whose representation is $R(\sigma_{uv}, \sigma^1_{uv})$. We will show that $(\mathcal{G}^\circ_j)_{j=1}^n$ is a TLG*-tower.

It is clear that $\mathcal{G}^\circ_1$ is a TLG* and that images of all points connected by a time path in $\mathcal{G}_1$ are connected in $\mathcal{G}^\circ_1$. Let's assume $\mathcal{G}^\circ_k$ is a TLG* and that images of all points connected by a time path in $\mathcal{G}_k$ are connected in $\mathcal{G}^\circ_k$.

If we added a vertex to $\mathcal{G}_k$ in order to obtain $\mathcal{G}_{k+1}$, then $\mathcal{G}^\circ_{k+1}$ is either the same as $\mathcal{G}_k$ or it has an added vertex. It is clear in this case that images of all the points that are connected in $\mathcal{G}_{k+1}$ by a time-path are connected by a time-path in $\mathcal{G}^\circ_{k+1}$.

If we added an edge to $\mathcal{G}_k$ in order to obtain $\mathcal{G}_{k+1}$, then $\mathcal{G}^\circ_{k+1}$ is the same as if added an edge to $\mathcal{G}^\circ_k$. Since this edge is connecting image of two points in $\mathcal{G}_k$ that are connected by a time-path, they are also connected by a time-path in $\mathcal{G}^\circ_k$. Hence, $\mathcal{G}^\circ_{k+1}$ is also a TLG*. Images of all the time-path connected points in $\mathcal{G}_{k+1}$ that are not on the edge added, are connected by a time path in $\mathcal{G}^\circ_{k+1}$. (This is inherited from $\mathcal{G}_k$.) The points on the edge are connected through the endpoints, and since the image of the edge is connected through the image of the endpoints, the connectedness follows.

Hence $(\mathcal{G}^\circ_k)$ is a TLG*-tower ending with $\mathcal{G}^\circ$.

The example when we collapse a simple cell in a TLG* an don't obtain a TLG* is given on Figure 1.21. The second figure is not a topological lattice, so it is not a TLG*.

■

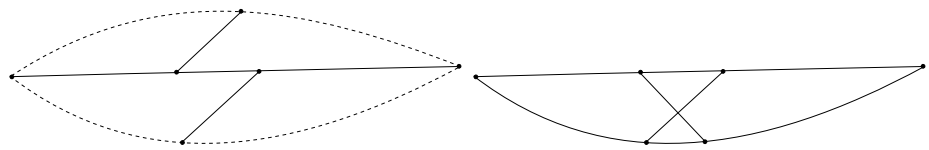


Figure 1.21: Planar TLG$^{(*)}$, when we collapse the dashed (simple) cell we no longer have a TLG*.

We obtain following from the previous proof.

Corollary 1.44. *Let $(\mathcal{G}_j)_{j=1}^n$ be a TLG*-tower. If there exists a truly simple cell, in $\mathcal{G}_n$ such that the representation of this cell is truly simple in each member of the tower, then for $\circ$ the cell collapsing transformation of this cell $(\mathcal{G}_j^\circ)_{j=1}^n$ is also a TLG*-tower.*

Proof. We first construct a pre-tower $(\mathcal{G}_j)_{j=-m}^1$, where $\mathcal{G}_{-m}$ is the minimal TLG* containing the cell. Now, $(\mathcal{G}_j)_{j=-m}^n$ is a TLG*-tower, and in the same way as in the previous proof we can show that $(\mathcal{G}_j^\circ)_{j=-m}^n$ is a tower, and the claim follows. ■

In what follows we will define an algorithm which will give us the criteria for understanding is something a TLG* or not.

Definition 1.45. *For a TLG $\mathcal{G}$ and a full-time path σ in $\mathcal{G}$, the following algorithm will be called the* ***stingy algorithm for the TLG $\mathcal{G}$ with respect to the full-time path*** *σ:*

```
σ a full time-path (i.e. in P_{0→1}(G));
G# = (V#, E#) a TLG that consists of all vertices and all edges of σ (in G);
while E \ E# ≠ ∅ do
    σ_kl a time-path in G and not in G# between t_k and t_l in V# such that |t_l − t_k| is
    minimal;
    add all edges and vertices that make σ_kl (in G) to G#;
end
```

Algorithm 2: Stingy algorithm for constructing $\mathcal{G}$ with respect to σ

Lemma 1.46. *The stingy algorithm for any $\mathcal{G}$ and any full-time path σ in $\mathcal{G}$ terminates in finitely many steps.*

Proof. It is clear that as long the condition $\mathcal{E} \setminus \mathcal{E}^{\#} \neq \emptyset$ is satisfied, we can find σ_{kl} as in the line 4 (it may not be unique, but it will exist). Since in each **while** loop execution we add at least one edge, eventually we will have $\mathcal{E} = \mathcal{E}^{\#}$. Clearly, at that point we have $\mathcal{G} = \mathcal{G}^{\#}$. ■

The key to answering is $\mathcal{G}$ a TLG* is in line 4. We claim that if $\mathcal{G}$ is a TLG*, then for the chosen σ_{kl} the vertices t_k and t_l are connected by a time path in $\mathcal{G}^{\#}$ (constructed before we picked σ_{kl}).

Theorem 1.47. *If $\mathcal{G}$ is a TLG* and σ a spine in $\mathcal{G}$, then in the stingy algorithm for $\mathcal{G}$ with respect to σ, each time line 4 is executed we pick a time-path between two points connected by a time-path in $\mathcal{G}^{\#}$.*

Proof. Let n be the sum of degrees of vertices in $\mathcal{G}$ whose degree is at least 3, that is

$$n(\mathcal{G}) = \sum_{v \in \mathcal{V}, d(v) \geq 3} d(v).$$

We will prove the following claim by induction on n:

For a TLG* $\mathcal{G}$ where $n(\mathcal{G}) = n$, when we run the algorithm on $\mathcal{G}$ for any spine σ in line 4 the chosen σ_{kl} is such that t_k and t_l are connected by a time path in $\mathcal{G}^{\#}$ from the previous iteration.

For $n = 0$ this claim is clearly true (then we have a TLG* with one spine). Assume that this claim holds for all $n \leq m$ where $m \geq 0$.

Let's show that this claim holds for $n = m + 1$. If there is no such TLG* $\mathcal{G}$, then we say that the claim holds trivially. Otherwise, let $\mathcal{G}$ be such a TLG*, and σ its arbitrary spine from $P_{0\to 1}(\mathcal{G})$.

We pick t_u and t_v on σ that are connected by a time-path σ^1_{uv} in $\mathcal{G}$ outside of σ such that $|t_u - t_v|$ is minimal. Let σ_{uv} be the time-path between t_u and t_v on σ. Note that, by

the construction, the cell $(\sigma_{uv}, \sigma^1_{uv})$ is truly simple. (Otherwise, if the sides σ_{uv} and σ^1_{uv} are connected by a path in $\mathcal{G}[t_u, t_v]$ that would contradict the minimality of $t_v - t_u$.)

The graph constructed by the simple cell collapsing transformation with respect to $(\sigma_{uv}, \sigma^1_{uv})$ - $\mathcal{G}^\circ$ is by Lemma 1.43 a TLG* and we have $n(\mathcal{G}^\circ) < n(\mathcal{G})$ (it is clear that $d((t_u)^\circ) < d(t_u)$ and $d((t_v)^\circ) < d(t_v)$). So by induction assumption we can apply the algorithm to $\mathcal{G}^\circ$ and in this way show that it is a TLG*.

We will parallely run the algorithm on $\mathcal{G}^\circ$ and $\mathcal{G}$ with the given spine $(\sigma)^\circ$ and the corresponding spine σ.

We will assume that in the first iteration of the **while** loop in line 3 (of the algorithm on $\mathcal{G}$) time-path σ^1_{uv} was chosen.

Let p denote the number of iterations of the **while** loop in line 3, and $\mathcal{G}^\#_p$ the graph constructed until that point when we run the algorithm on $\mathcal{G}$.

Now, we will show that if σ°_{kl} was chosen in the p-th iteration of the while loop on $\mathcal{G}^\circ$, then we can choose σ_{kl} in $p+1$-st iteration of the **while** loop on $\mathcal{G}$.

For $p = 1$ this holds, σ°_{kl} is connecting $(t_k)^\circ$ and $(t_l)^\circ$, and by the construction of $\mathcal{G}^\circ$, t_k and t_l are connected by a time path in $\mathcal{G}$. (Otherwise, we t_k and t_l would be points on different sides of the cell, connected by a the time path σ_{kl}, and the cell $(\sigma_{uv}, \sigma^1_{uv})$ wouldn't be minimal.)

Assume this holds for $p = r \geq 1$.

For $p = r+1$ let σ°_{kl}, be chosen. By assumption we know that $(t_k)^\circ$ and $(t_l)^\circ$ are connected by a time path in $(\mathcal{G}')^\#_{p-1}$, we know, that σ_{kl} is a path connecting t_k and t_l in $\mathcal{G}$, there can't be a path whose time difference is smaller, because such would exist in $\mathcal{G}^\circ$ also. The only thing that we need to show is that t_k and t_l are connected by a time-path in $\mathcal{G}^\#_p$.

Assume the opposite. This would mean, by construction that there exists $t_{k'} \in \sigma_{uv}$ and $t_{l'} \in \sigma^1_{uv}$ on different sides of the cell $(\sigma^1_{uv}, \sigma_{uv})$, such that $t_k \leq t_{k'} \leq t_{l'} \leq t_l$ (time order), and there exist paths $\sigma_{kk'}$ and $\sigma_{l'l}$. Now this would mean, since $\mathcal{G}$ is a TLG*, and therefore

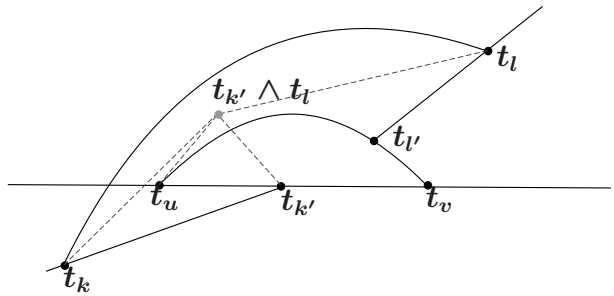


Figure 1.22:

by Theorem 1.34 a topological lattice , that

$$\{t \in \mathcal{G} : t \preceq t_k\} \cap \{t \in \mathcal{G} : t \preceq t_u\} \subset \{t \in \mathcal{G} : t \preceq t_{k'}\} \cap \{t \in \mathcal{G} : t \preceq t_l\} = \{t \in \mathcal{G} : t \leq t_{k'} \wedge t_l\},$$

Hence, in $\mathcal{G}$ there exists a vertex $t_{k'} \wedge t_l$ connected by time paths to t_k, t_u, $t_{k'}$ and $t_{l'}$. Now, $t_{k'} \wedge t_l$ has to be on σ, or otherwise t_u, $t_{k'} \wedge t_l$ and $t_{k'}$ form a cell, that will be a smaller cell whose on side is on the spine σ in $\mathcal{G}$. But this contradicts the choice of t_u and t_v. Now, if $t_{k'} \wedge t_l$ is on σ, this contradicts the choice of t_k and t_l, since $t_k < t_{k'} \wedge t_l < t_l$, because $t_{k'} \wedge t_l$ and t_l are in $\mathcal{G}_p^{\#}$, they are connected in $\mathcal{G}$ and their time difference is less than $t_l - t_k$.

Hence t_k and t_l have to be connected in $\mathcal{G}_p^{\#}$.

This shows that the algorithm will be making a connection between two connected vertices in each step.

Finally, this proves the claim. ■

Corollary 1.48. *The following algorithm determines is a TLG $\mathcal{G}$ a TLG* or not:*

```
σ a full time-path (i.e. in P_{0→1}(G));
G# = (V#, E#) a TLG that consists of all vertices and all edges of σ (in G);
while E \ E# ≠ ∅ do
    σ_kl a time-path in G and not in G# between t_k and t_l in V# such that |t_l − t_k| is
    minimal;
    if t_k and t_l are connected by a time-path in G# then
        add all edges and vertices that make σ_kl (in G) to G#;
    else
        return This is not a TLG*;
    end
end
return This is a TLG*;
```

Algorithm 3: Determine is TLG $\mathcal{G}$ a TLG* or not.

1.8 TLG's with infinitely many vertices

We will allow t_0 and t_N to take values in $\mathbb{R} \cup \{-\infty, \infty\}$.

Definition 1.49. *(i) Suppose that the vertex set of a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is infinite. We will call $\mathcal{G}$ a time-like graph (TLG) if it satisfies the following conditions.*

(a) There is a sequence of TLG's $\mathcal{G}_n = (\mathcal{V}_n, \mathcal{E}_n)$ with finite vertex set $\mathcal{V}_n$, $n \geq 1$, and for some representations of $\mathcal{G}_n$'s and G we have

$$\bigcup_{n=1}^{\infty} R(\mathcal{G}_n) = R(\mathcal{G}).$$

(b) The graph $\mathcal{G}$ is locally finite, i.e. it has a representation $R(\mathcal{G})$ such that for any compact $K \subset \mathbb{R}^3$ a finite number of edges intersects K.

(ii) A TLG $\mathcal{G}$ with infinite vertex set will be called an TLG* if it satisfies the following conditions.

(a) *We can choose a sequence of TLG***'s* $\mathcal{G}_n$ *in (i). (In the sense of the Definition 1.5.(iii), i.e.* $(\mathcal{G}_j)_{1\leq j\leq n}$ *is a tower of TLG***'s for all* n*.)*

(b) *Let* $\mathcal{V}_n = \{t_{0,n}, t_{1,n}, \ldots, t_{N_n,n}\}$*. The initial vertices* $t_{0,n} \in \mathcal{V}_n$ *and* $t_{N_n,n} \in \mathcal{V}_n$ *are the same for all* $\mathcal{G}_n$*, i.e. for all* $m, n \geq 1$

$$t_{0,n} = t_{0,m} \quad \text{and} \quad t_{N_n,n} = t_{N_m,m}.$$

(c) *The initial and terminal edges form a decreasing sequence in the representations of* $\mathcal{G}_n$*'s, i.e. if* $n > m$

$$E_{t_{0,n},t_{1,n}}((t_{0,n}, t_{1,n})) \subset E_{t_{0,m},t_{1,m}}((t_{0,m}, t_{1,m}))$$

and

$$E_{t_{N_n-1,n},t_{N_n,n}}((t_{N_n-1,n}, t_{N_n,n})) \subset E_{t_{N_m-1,m},t_{N_m,m}}((t_{N_m-1,m}, t_{N_m,m})).$$

The following lemma will be useful for the construction of processes.

Lemma 1.50. *Let* $(\mathcal{G}_n)$ *and* $(\mathcal{G}'_n)$ *be two TLG***-towers that lead to the construction of* $\mathcal{G}$*. Let* $\mathcal{H}$ *be a sub-graph (not necessarily a TLG***) of some* $\mathcal{G}_{n_0}$ *whose vertices have a finite time. Then there exists* $\mathcal{G}'_{n_1}$ *such that* $R(\mathcal{H}) \subset R(\mathcal{G}'_{n_1})$ *and all the vertices of* $\mathcal{H}$ *are contained in* $\mathcal{G}'_{n_1}$*.*

Proof. Since $\mathcal{G}$ is locally finite, there are finitely many vertices with representation on $R(\mathcal{H})$, also these vertices are of finite degree. For each such vertex v, by same argument, there has to be $\mathcal{G}'_{n_v}$ such v in $\mathcal{G}'_{n_v}$ has that degree. Now if n_1 is the maximum of n_v over each such vertex v the claim follows. ■

Chapter 2

PROCESSES INDEXED BY TIME-LIKE GRAPHS

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a TLG*. In this chapter we construct a stochastic process on $\mathcal{G}$ in such a way that we have a random variable defined at every point of the representation. (See Figure 2.1. for illustration.)

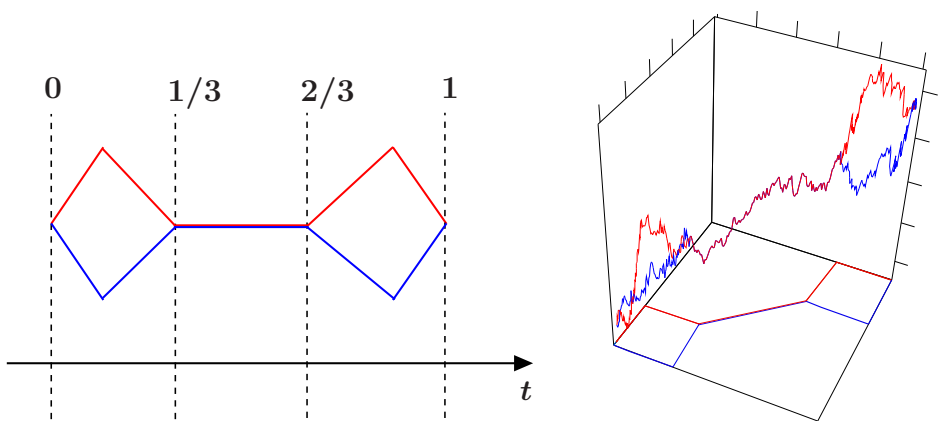


Figure 2.1: Time-like graph $\mathcal{G}$ and a process indexed by it.

Definition 2.1. *We define $X = (X(t) : t \in \mathcal{G})$ as a collection of random variables with*

$$X = (X_E(t) : E = E_{jk} \in \mathcal{E}, t \in [t_j, t_k]).$$

We will assume the following things.

- *If $E_{jk}, E_{kn} \in \mathcal{E}$ then $X_{E_{jk}}(t_k) = X_{E_{kn}}(t_k)$.*
- *If $E_{jk}, E_{nk} \in \mathcal{E}$ then $X_{E_{jk}}(t_k) = X_{E_{nk}}(t_k)$.*

- *Finally, if* $E_{0j}, E_{0k} \in \mathcal{E}$ *then* $X_{0j}(t_0) = X_{0k}(t_0)$.

For a path $\sigma_1 \in \sigma(k_1, k_2, \ldots, k_n)$ *we use the notation*

$$X_{\sigma_1}(t) = X_{E_{k_{j-1}k_j}}(t),$$

for all $j = 2, 3, \ldots, n$ *and* $t \in [t_{k_{j-1}}, t_{k_j}]$.

Remark. (1) If there are two edges E^q_{jk} and E^p_{jk} with the same endpoints we will denote processes on them by X^q_{jk} and X^p_{jk}.

(2) We will write $X(t)$ instead of $X_{jk}(t)$ or X_σ when this will not cause any confusion.

(3) In an infinite graph case we will do the same thing, but we will not define the process at t_0 and t_N, if they are not in $\mathbb{R}$.

If $\mathcal{P}$ is the distribution of a Markov process $(Y(t) : t \in [t_0, t_N])$, note that for every TLG there exists a $\mathcal{P}$-process on $\mathcal{G}$. Trivial example of a $\mathcal{P}$-process on a TLG can be constructed by taking a Markov process $(Y(t) : t \in [t_0, t_N])$ with distribution $\mathcal{P}$ and then letting $X_\sigma(t) = Y(t)$ for all full time-paths $\sigma \in P_{0\to 1}(\mathcal{G})$.

We will require some properties to hold for the process to be non-trivial.

2.1 Spine-Markovian property

Definition 2.2. *Let* σ *be any full-time path (from 0 to 1) in the TLG* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. *Let* $\mathcal{G}_-$ *be a subgraph (not necessarily a TLG) of* $\mathcal{G}$ *whose representation is a closure of a connected component of* $R(\mathcal{G}) \setminus R(\sigma)$. *Let* W *be the set of vertices -* ***roots*** *connecting* $\mathcal{G}_-$ *to* σ *and let* $\mathcal{G}_+$ *denote the graph represented by* $R(\mathcal{G}) \setminus R(\mathcal{G}_-)$.

We say that the process X *on a TLG* $\mathcal{G}$ *is* ***spine-Markovian*** *if for each such* σ *and* $\mathcal{G}_-$ *the processes* $(X(t) : t \in \mathcal{G}_-)$ *and* $(X(t) : t \in \mathcal{G}_+)$ *given* $(X(t) : t \in W)$ *are independent.*

Proposition 2.3. *Let* σ, $\mathcal{G}_-$, $\mathcal{G}_+$ *and* W *in a TLG* $\mathcal{G}$ *be as in the Definition 2.2. Then for any* σ*-algebra* $\mathcal{F}$ *such that* $\sigma(X(t) : t \in W) \subset \mathcal{F} \subset \sigma(X_\sigma)$, *If the process* X *on* $\mathcal{G}$ *is spine-Markovian then the processes* $(X(t) : t \in \mathcal{G}_-)$ *and* $(X(t) : t \in \mathcal{G}_+)$ *given* $\mathcal{F}$ *are independent.*

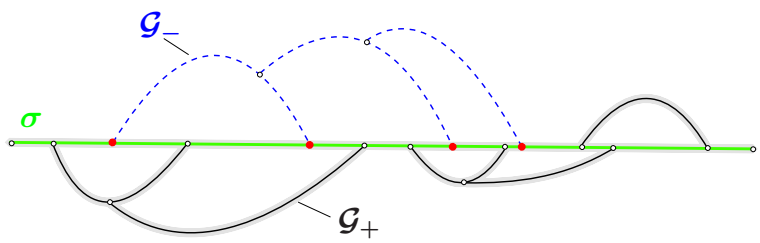


Figure 2.2: Spine-Markovian property: The set of roots W is illustrated by bullet points (•).

Proof. Let Y_- and Y_+ denote bounded random variables respectively measurable in $\sigma(X(t) : t \in \mathcal{G}_-)$ and $\sigma(X(t) : t \in \mathcal{G}_+)$. For $A \in \mathcal{F}$, $Y_+\mathbf{1}_A$ is a bounded $\sigma(X(t) : t \in \mathcal{G}_+)$-measurable random variable, and we have

$$\begin{aligned}
\mathbb{E}(Y_-Y_+\mathbf{1}_A) &= \mathbb{E}(\mathbb{E}(Y_-Y_+\mathbf{1}_A|(X(t) : t \in W))) \\
&= \mathbb{E}(\mathbb{E}(Y_-|(X(t) : t \in W))\mathbb{E}(Y_+\mathbf{1}_A|(X(t) : t \in W))) \\
&= \mathbb{E}(\mathbb{E}(Y_-|(X(t) : t \in W))\mathbb{E}(\mathbb{E}(Y_+\mathbf{1}_A|\mathcal{F})|(X(t) : t \in W))) \\
&= \mathbb{E}(\mathbb{E}(Y_-|(X(t) : t \in W))\mathbb{E}(\mathbb{E}(Y_+|\mathcal{F})\mathbf{1}_A|(X(t) : t \in W))) \\
&= \mathbb{E}(\mathbb{E}(Y_-\mathbb{E}(Y_+|\mathcal{F})\mathbf{1}_A|(X(t) : t \in W))) = \mathbb{E}(Y_-\mathbb{E}(Y_+|\mathcal{F})\mathbf{1}_A) \\
&= \mathbb{E}(\mathbb{E}(Y_-\mathbb{E}(Y_+|\mathcal{F})\mathbf{1}_A|\mathcal{F})) = \mathbb{E}(\mathbb{E}(Y_-|\mathcal{F})\mathbb{E}(Y_+|\mathcal{F})\mathbf{1}_A).
\end{aligned}$$

■

Remark. Note that $\mathcal{G}_+$ is a TLG while $\mathcal{G}_-$ does not have to be (it is still a connected graph). Also, $\mathcal{G}_+$ contains σ, so we can find $\mathcal{G}_-^2$ a connected component of $R(\mathcal{G}_+) \setminus R(\sigma)$, and so on... So, the TLG $\mathcal{G}$ can be decomposed into $\mathcal{G}_-^1, \ldots, \mathcal{G}_-^n$ that are connected components of $R(\mathcal{G}) \setminus R(\sigma)$ and the spine σ.

Definition 2.4. *We will call $(\sigma; \mathcal{G}_-^1, \ldots, \mathcal{G}_-^n)$ the* ***decomposition*** *of the TLG $\mathcal{G}$ with respect to σ. The elements of this decomposition (including σ) we will call* ***components****.*

Remark. Notice that the decomposition, given σ, is unique up to an order of $\mathcal{G}_-^1, \ldots, \mathcal{G}_-^n$.

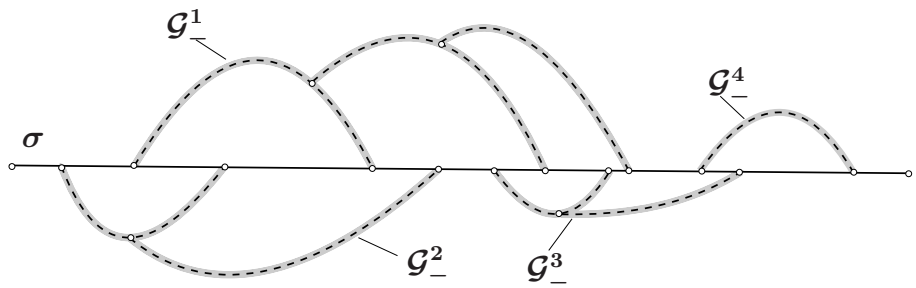


Figure 2.3: The decomposition of $\mathcal{G}$ with respect to σ.

Proposition 2.5. *Let $\mathcal{G}'$ be a graph that is the union of some graphs $\mathcal{G}_-^{j_1}, \ldots, \mathcal{G}_-^{j_k}$ in the decomposition, and $\mathcal{G}''$ the union of all the other components in the decomposition. Then the process $(X(t) : t \in \mathcal{G}')$ and $(X(t) : t \in \mathcal{G}'')$ are independent given $(X(t) : t \in W_{j_1} \cup \ldots \cup W_{j_k})$, where W_j is the set of roots of $\mathcal{G}_j$.*

Proof. For $l = 1, \ldots, k$ let Y_l be a bounded $\sigma(X(t) : t \in \mathcal{G}_{j_l})$-measurable random variable, Z a bounded $\sigma(X(t) : t \in \mathcal{G}'')$-measurable random variable and $A \in \sigma(X(t) : t \in W_{j_1} \cup \ldots \cup W_{j_k})$. Using the spine-Markovian property for each $\mathcal{G}_{j_l}$ $l = 1, 2, \ldots, k$ at a time with respect to σ we get

$$\begin{aligned}
\mathbb{E}(Y_1 Y_2 \ldots Y_k Z \mathbf{1}_A) &= \mathbb{E}(\mathbb{E}(Y_1 | X_{W_{j_1}}) Y_2 \ldots Y_k Z \mathbf{1}_A) \\
&= \mathbb{E}(\mathbb{E}(Y_1 | X_{W_{j_1}}) \mathbb{E}(Y_2 | X_{W_{j_2}}) \ldots Y_k Z \mathbf{1}_A) \\
&\ldots \\
&= \mathbb{E}(\mathbb{E}(Y_1 | X_{W_{j_1}}) \mathbb{E}(Y_2 | X_{W_{j_2}}) \ldots \mathbb{E}(Y_k | X_{W_{j_k}}) Z \mathbf{1}_A).
\end{aligned}$$

Now, taking the conditional expectation with respect to $\sigma(X_{W_{j_1}} \ldots X_{W_{j_k}})$

$$\begin{aligned}
&= \mathbb{E}(\mathbb{E}(\mathbb{E}(Y_1 | X_{W_{j_1}}) \ldots \mathbb{E}(Y_k | X_{W_{j_k}}) Z \mathbf{1}_A | X_{W_{j_1}} \ldots X_{W_{j_k}})) \\
&= \mathbb{E}(\mathbb{E}(Y_1 | X_{W_{j_1}}) \ldots \mathbb{E}(Y_k | X_{W_{j_k}}) \mathbb{E}(Z | X_{W_{j_1}} \ldots X_{W_{j_k}}) \mathbf{1}_A)
\end{aligned}$$

Now, again using the spine-Markovian property on each graph in the union we get

$$= \mathbb{E}(Y_1 \dots \mathbb{E}(Y_k | X_{W_{j_k}}) \mathbb{E}(Z | X_{W_{j_1}} \dots X_{W_{j_k}}) \mathbf{1}_A)$$

$$\dots$$

$$= \mathbb{E}(Y_1 \dots Y_k \mathbb{E}(Z | X_{W_{j_1}} \dots X_{W_{j_k}}) \mathbf{1}_A).$$

Which, finally, gives us

$$= \mathbb{E}(\mathbb{E}(Y_1 \dots Y_k \mathbb{E}(Z | X_{W_{j_1}} \dots X_{W_{j_k}}) \mathbf{1}_A | X_{W_{j_1}} \dots X_{W_{j_k}}))$$

$$= \mathbb{E}(\mathbb{E}(Y_1 \dots Y_k | X_{W_{j_1}} \dots X_{W_{j_k}}) \mathbb{E}(Z | X_{W_{j_1}} \dots X_{W_{j_k}}) \mathbf{1}_A).$$

Now from the Monotone Class Theorem the claim follows. ■

We will need a stronger property for some proofs.

Definition 2.6. *For a TLG* $\mathcal{G}$ we define $S^*(\mathcal{G})$ to be the set of all TLG*'s $\mathcal{H}$ such that there exists a TLG*-tower $(\mathcal{K}_k)_{k=0}^n$ that starts with $\mathcal{K}_0 = \mathcal{H}$ and ends with $\mathcal{K}_n = \mathcal{G}$.*

Definition 2.7. *The process $(X(t) : t \in \mathcal{G})$ has a* ***hereditary spine-Markovian property*** *if $(X(t) : t \in \mathcal{H})$ is a spine-Markovian process for each $\mathcal{H} \in S^*(\mathcal{G})$.*

2.2 Consistent distributions on paths

Definition 2.8. *Let $\mathcal{G}$ be a TLG, for a family of distributions of stochastic processes on $[0, 1]$*

$$\{\mu_\sigma : \sigma \in H\},$$

where $H \subset P_{0\to 1}(\mathcal{G})$ (a subset of the set of full time-paths), we say that it is ***consistent*** *if for $\sigma_1, \sigma_2 \in H$*

$$\mu_{\sigma_1} \circ \pi_T^{-1} = \mu_{\sigma_2} \circ \pi_T^{-1},$$

where $T = \{t : t \in E, E \in \sigma_1 \ \& \ E \in \sigma_2\}$.

Proposition 2.9. *If μ is the distribution of the process X on a TLG $\mathcal{G}$, then*

$$\{\mu_\sigma = \mathbb{P} \circ X_\sigma^{-1} : \sigma \in P_{0\to 1}(\mathcal{G})\} \tag{2.1}$$

is a consistent family.

Remark. It is not hard to see that the family of distributions given by (2.1) does not uniquely determine μ - the distribution on $\mathcal{G}$. For example if we take a Markov process $\mathcal{P}$ on $[0,1]$, and we take the TLG graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V} = \{0,1\}$ and $\mathcal{E} = \{E^1_{01}, E^2_{01}\}$. Let Y^1 be a Markov process on $[0,1]$ with distribution $\mathcal{P}$, and Y^2 a $\mathcal{P}$-Markov bridge starting at $Y^1(0)$ and ending at $Y^1(1)$ conditionally independent given $Y^1(0)$ and $Y^1(1)$. (This can be done as in Theorem A.15.) Now, the process X^1 such that $X^1_{E^1_{01}} = Y^1$ and $X^1_{E^2_{01}} = Y^1$, has the same distributions along the full-time paths as X^2 given by $X^2_{E^1_{01}} = Y^1$ and $X^2_{E^2_{01}} = Y^2$. But, these two processes are clearly different in distribution. (See Figure 2.4.)

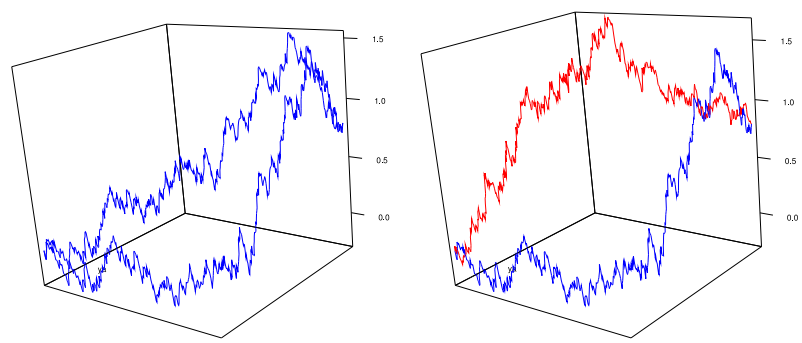


Figure 2.4: Processes X^1 and X^2

Corollary 2.10. *Let $\mathcal{P}$ be a distribution of some process on $[0,1]$. If $\mu_\sigma = \mathcal{P}$ for each full-time path σ in TLG $\mathcal{G}$, then*

$$\{\mu_\sigma : \sigma \in P_{0\to 1}(\mathcal{G})\}$$

is a consistent family.

2.3 Construction from a consistent family

The interesting question is if we have a consistent family

$$\mathcal{M} := \{\mu_\sigma : \sigma \in P_{0\to 1}(\mathcal{G})\},$$

under what conditions can we construct a process X on $\mathcal{G}$ such that X_σ has the distribution μ_σ. We will call X to be an **$\mathcal{M}$-process**

We will show that such a process exists under the following assumptions:

T1 $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a TLG*.

T2 $\mathcal{M}$ is a consistent family of measures that induce continuous or RCLL processes.

T3 For each simple cell (σ_1, σ_2) in $\mathcal{G}$ starting at t_* and ending at t^*, if σ is a full-time path containing σ_1 (or σ_2) then the μ_σ-distributed process

$$(Y(t) : t \in [0, 1]) \tag{2.2}$$

has the property that $(Y(t) : t \in [0, t_*] \cup [t^*, 1])$ and $(Y(t) : t \in [t_*, t^*])$ given $Y(t_*)$ and $Y(t^*)$ are independent.

Conditions (T1)-(T3) we will call **(3T) conditions**.

Remark. Condition (T2) is needed so that we could define a conditional distribution when needed. So other $\mathcal{M}$ can be a family of other types of processes for which this would be possible (for example all the arguments would work for discrete processes).

The condition (T3) can be rewritten in a different way.

Lemma 2.11. *The process given by* (2.2) *has the property that the distribution* $(Y(t) : t \in [t_*, t^*])$ *given* $(Y(t) : t \in [0, t_*] \cup [t^*, 1])$ *depends only on* $(Y(t_*), Y(t^*))$*, in other words if* Z *is a bounded* $\sigma(Y(t) : t \in [t_*, t^*])$*-measurable random variable then*

$$\mathbb{E}(Z|Y(t) : t \in [0, t_*] \cup [t^*, 1]) = \mathbb{E}(Z|Y(t_*), Y(t^*)).$$

Proof. Let $A \in \sigma(Y(t) : t \in [0, t_*] \cup [t^*, 1])$ and define $U := \mathbb{E}(Z|Y(t) : t \in [0, t_*] \cup [t^*, 1])$. Using the definition of the conditional expectation, and the property of Y

$$\begin{aligned}
&\mathbb{E}(U\mathbf{1}_A) \\
&= \mathbb{E}(Z\mathbf{1}_A) = \mathbb{E}(\mathbb{E}[Z\mathbf{1}_A|Y(t_*), Y(t^*)]) \\
&= \mathbb{E}(\mathbb{E}[Z|Y(t_*), Y(t^*)]\mathbb{E}[\mathbf{1}_A|Y(t_*), Y(t^*)]) \\
&= \mathbb{E}(\mathbb{E}[\mathbb{E}[Z|Y(t_*), Y(t^*)]\mathbf{1}_A|Y(t_*), Y(t^*)]) \\
&= \mathbb{E}(\mathbb{E}[Z|Y(t_*), Y(t^*)]\mathbf{1}_A).
\end{aligned}$$

The claim follows from the a.s. uniqueness of the conditional expectation. ■

2.3.1 Construction

We will define a $\mathcal{M}$-process on a TLG* $\mathcal{G}$ with finite sets $\mathcal{V}$ and $\mathcal{E}$, where $t_0 = 0$ and $t_N = 1$.

Definition 2.12. *Let $(\mathcal{G}_l)_{0 \le l \le n}$ be a tower of TLG* where $\mathcal{G}_0$ is a minimal graph $\mathcal{V}_0 = \{t_0 = 0, t_N = 1\}$, $\mathcal{E}_0 = \{E_{0N}\}$ and $\mathcal{G}_n = \mathcal{G}$. Further let $\mathcal{M}$ be a family of distributions satisfying (3T) conditions.*

- *On $\mathcal{G}_0$ we define a process X^0 with $\mu_{E_{0n}}$ distribution.*
- *If we have already defined X^l on $\mathcal{G}_l$ (for some $l < n$), then we define X^{l+1} on $\mathcal{G}_{l+1}$ in the following way depending how we constructed $\mathcal{G}_{l+1}$ from $\mathcal{G}_l$ (recall part (ii) of Definition 1.5.).*

  (1) In the construction a new vertex $\tau_l \in [0, 1] \setminus \mathcal{V}_l$ was added to graph $\mathcal{G}_l$, by subdividing some E_{jk} such that $t_j < \tau_l < t_k$, into E_{jl} and E_{lk} to get G_{l+1}. In this case, the two graphs $\mathcal{G}_l$ and $\mathcal{G}_{l+1}$ have a common representation, $R(\mathcal{G}_l) = R(\mathcal{G}_{l+1})$, and we can define X^{l+1} on $\mathcal{G}_{l+1}$ to have the same values on this representation as X^l.

  (2) In the construction a new edge between two vertices $t_j < t_k$ in $\mathcal{V}_l$ that are connected by a time path in $\mathcal{G}_l$, was added to get G_{l+1}. So, $\mathcal{G}_{l+1}$ has a new edge $E^* = E^*_{jk}$. Let $Z_j = X^l(t_j)$ and $Z_k = X^l(t_k)$.

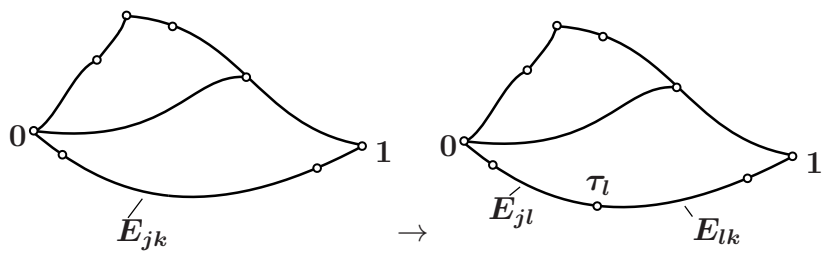


Figure 2.5: Case (1) in the construction.

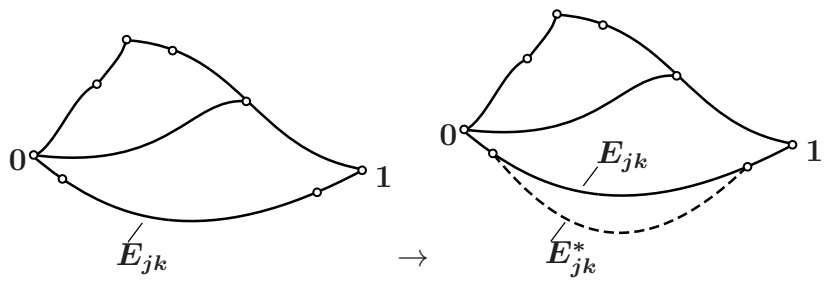


Figure 2.6: Case (2) in the construction.

Now we pick a full-time path σ that contains E^*. Now we define $\mu_*(\cdot|x,y)$ to be the conditional probability of the process with the distribution $\mu_\sigma \circ \pi^{-1}_{[t_j,t_k]}$ conditioned to have values x at t_j and y at t_k. So we construct the process X^{l+1} in such a way that X^{l+1} on $R(\mathcal{G}_l)$ is equal to X^l and $X^{l+1}_{E^*}$ is the process with distribution $\mu_*(\cdot|Z_j, Z_k)$ and otherwise independent of X^l given Z_j and Z_k.

Since n is finite this procedure will end and we will have a process $X = X^n$ defined on $\mathcal{G}$.

Remark. Note that this construction, i.e. the definition of X on $\mathcal{G}$ *depends* on the choice of the TLG* tower $(\mathcal{G}_l)_{0\le l\le n}$.

2.3.2 Constructed process is an $\mathcal{M}$-process

Definition 2.13. *If $(\mathcal{G}_k)_{k=0}^n$ is a TLG*-tower where $\mathcal{G}_n = \mathcal{G}$. If $\mathcal{M}$ is a family of distributions on full time-paths of $\mathcal{G}$. This naturally induces a family $\mathcal{M}(\mathcal{G}_k)$ of distributions on full time-*

paths of $\mathcal{G}_k$.

Remark. This is well-defined since a representation of every full time-path in $\mathcal{G}_k$, is a representation of a full time-path in $\mathcal{G}$ (in the consistent representation of the TLG*-tower $(\mathcal{G}_k)_{k=0}^n$).

The only question remains will the family induced by $\mathcal{M}$ have the same properties as $\mathcal{M}$. This is shown to be true.

Lemma 2.14. *If $\mathcal{M}$ is a family of distributions on full time-paths of a TLG* $\mathcal{G}$ *satisfying properties (T1)-(T3), then for any $\mathcal{H} \in S^*(\mathcal{G})$ the family $\mathcal{M}(\mathcal{H})$ also satisfies properties (T1)-(T3).*

Proof. (T1) is clearly satisfied since $\mathcal{H}$ is a TLG*. (T2) is satisfied since in the consistent representation all the full time paths in $\mathcal{H}$ are full time paths in $\mathcal{G}$. By Corollary 1.16, in a consistent representation a representation of a simple cell in $\mathcal{H}$ is a representation of a simple cell in $\mathcal{G}$. Therefore (T3) holds. ■

Lemma 2.15. *The process X on $\mathcal{G}$ defined in 2.3.1 is an $\mathcal{M}$-process.*

Proof. It is clear that X^0 is a $\mathcal{M}(\mathcal{G}_0)$-process on the minimal graph $\mathcal{G}_0$.

For, $l < n$ we assume X^l is a $\mathcal{M}(\mathcal{G}_l)$-process on $\mathcal{G}_l$. If we got X^{l+1} using step (1) in the construction, then we inherited this property from X^l, since $\mathcal{M}(\mathcal{G}_l) = \mathcal{M}(\mathcal{G}_{l+1})$. If we got X^{l+1} using step (2), recall that $\mathcal{G}_l$ contains a time-path σ_{jk} connecting t_j and t_k, so there is a full path σ' in $\mathcal{G}_{l+1}$ that starts with a time-path σ_{0j} from t_0 to t_j, contains σ_{jk}, and ends with a time-path σ_{kN}.

Now for every full time-path σ that contains the new edge $E^* = E^*_{jk}$ that was added in the construction, $\mu_{E^*} = \mu_\sigma \circ \pi^{-1}_{[t_j,t_k]}$ is well defined since $\mathcal{M}$ is a consistent family, and μ_{E^*} doesn't depend on the choice of σ.

Since, (E^*, σ_{jk}) is a simple cell, by property (T3) of $\mathcal{M}$ we have that $\mu_*(\cdot|x, y)$ is the conditional distribution of a μ_{E^*}-distributed process on $[t_j, t_k]$ conditioned to have value x at t_j and y at t_k.

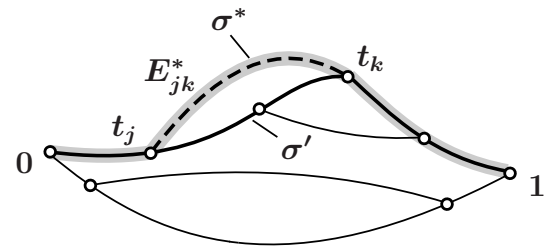


Figure 2.7:

The process $X^{l+1}_{E^*}$, by construction, is independent given (t_j, Z_j) and (t_k, Z_k). By property (T3) of $\mathcal{M}$ the distribution of $X^{l+1}_{\sigma'}$ where σ' is the union of σ_{0j}, E^*, σ_{k1} is given by $\mu_{\sigma'}$: Let A_0 be an event in the path σ-algebra on $[0, t_j]$, A_1 an event in the path σ-algebra on $[t_k, 1]$, and B an event in the path σ-algebra on $[t_j, t_k]$ we have:

$$\begin{aligned}
\mathbb{P} \circ (X^{l+1}_{\sigma'})^{-1}(A_0 \cap B \cap A_1) &= \mathbb{E}(\mathbf{1}_{A_0}(X^{l+1}_{\sigma_{0j}})\mathbf{1}_B(X^{l+1}_{E^*})\mathbf{1}_{A_1}(X^{l+1}_{\sigma_{k1}})) \\
&= \mathbb{E}(\mathbb{E}(\mathbf{1}_{A_0}(X^{l+1}_{\sigma_{0j}})\mathbf{1}_B(X^{l+1}_{E^*})\mathbf{1}_{A_1}(X^{l+1}_{\sigma_{k1}})|Z_j, Z_k)) \\
&= \mathbb{E}(\mathbf{1}_{A_0}(X^{l+1}_{\sigma_{0j}})\mathbb{E}(\mathbf{1}_B(X^{l+1}_{E^*})|Z_j, Z_k)\mathbf{1}_{A_1}(X^{l+1}_{\sigma_{k1}})) \\
&\overset{(T3)}{=} \int_{A_0 \times A_1} \mu_*(B|\pi_{t_j}(x), \pi_{t_k}(x))\mu_\sigma \circ \pi^{-1}_{[0,t_j]\cup[t_k,1]}(dx) \\
&\overset{(T2)}{=} \int_{A_0 \times A_1} \mu_*(B|\pi_{t_j}(x), \pi_{t_k}(x))\mu_{\sigma'} \circ \pi^{-1}_{[0,t_j]\cup[t_k,1]}(dx) \\
&\overset{(T3)}{=} \mu_{\sigma'}(A_0 \cap B \cap A_1).
\end{aligned}$$

By Monotone Class Theorem $\mathbb{P} \circ (X^{l+1}_{\sigma'})^{-1} = \mu_{\sigma'}$. ■

Remark. Note that just for the existence of an $\mathcal{M}$-process on the TLG* we could weaken condition (T3). If we fix a construction to a TLG*-tower, then only some full time-paths need to have the described property, but then we would lose some properties of the constructed process.

2.3.3 The constructed process is a spine-Markovian process

Lemma 2.16. *The process X on $\mathcal{G}$ defined in 2.3.1 is a spine-Markovian process.*

Proof. X^0 is trivially an spine-Markovian process. Let's assume that X^l is spine-Markovian. We have two cases to study to show that X^{l+1} is spine-Markovian.

($\bullet$1) If we added a new vertex to the graph $\mathcal{G}_l$ to obtain $\mathcal{G}_{l+1}$. Then the spine-Markovian property is directly inherited from the process X^l, since W can't contain the new vertex.

($\bullet$2) We added a new edge E^* to the graph $\mathcal{G}_l$ between two existing time-path connected vertices to obtain $\mathcal{G}_{l+1}$. Pick a full time path σ, and the subgraphs $\mathcal{G}_-$ and $\mathcal{G}_+$ in the graph $\mathcal{G}_{l+1}$ (in the sense of the Definition 2.2). First, note that from the construction the process $X^{l+1}_{E^*}$ is independent from $X^{l+1}_{\mathcal{G}_+}$ given the values of the process at the endpoints of E^*. We will call this property *edge-Markovian for the edge E^* (in $\mathcal{G}_{l+1}$)*. (This property does not need to hold for other edges.) This will be used often during the this proof. We have the following cases.

($\circ$1) If the new edge is the only edge in $\mathcal{G}_-$, i.e. E^* is connecting two vertices on σ. The claim follows from the edge-Markovian property for E^*.

($\circ$2) The new $E^* = E^*_{t_1^* t_2^*}$ edge is in $\mathcal{G}_- = (\mathcal{V}_-, \mathcal{E}_-)$, but one of the vertices that E^* is connecting is on σ. (See Figure 2.8.) Let $\mathcal{G}^*_-$ be the graph in $\mathcal{G}_l$ that has the edges $\mathcal{E}_- \setminus \{E^*\}$. From Proposition 2.3. and the spine-Markovian property of X^l we know $(X^{l+1}(t) : t \in \mathcal{G}^*_-)$ and $(X^{l+1}(t) : t \in \mathcal{G}_+)$ given $(X(t) : t \in W)$ are independent. (Note that one vertex in W may not be in $\mathcal{G}^*_-$.) Now, let Y^*_- be a bounded $\sigma(X^{l+1}(t) : t \in \mathcal{G}^*_-)$-measurable, Y_* a bounded $\sigma(X^{l+1}(t) : t \in E^*)$-measurable, and Y_+ a bounded $\sigma(X^{l+1}(t) : t \in \mathcal{G}_+)$ measurable random variable. For $A \in \sigma(X^{l+1}(t) : t \in W)$ we have using edge-Markov property for E^*:

$$\mathbb{E}(Y^*_- Y_* Y_+ \mathbf{1}_A) = \mathbb{E}(Y^*_- \mathbb{E}(Y_* | X_{t_1^*}, X_{t_2^*}) Y_+ \mathbf{1}_A).$$

Now, since $Y^*_- \mathbb{E}(Y_* | X_{t_1^*}, X_{t_2^*}) \in \sigma(X^{l+1}(t) : t \in \mathcal{G}^*_-) \vee \sigma(X(t) : t \in W)$, and this is independent

of $(X^{l+1}(t) : t \in \mathcal{G}_+)$ given $(X(t) : t \in W)$. So,

$$\begin{aligned}
\mathbb{E}(Y_-^*\mathbb{E}(Y_*|X_{t_1^*}, X_{t_2^*})Y_+\mathbf{1}_A) &= \mathbb{E}(\mathbb{E}(Y_-^*\mathbb{E}(Y_*|X_{t_1^*}, X_{t_2^*})Y_+\mathbf{1}_A|(X(t) : t \in W))) \\
&= \mathbb{E}(\mathbb{E}(Y_-^*\mathbb{E}(Y_*|X_{t_1^*}, X_{t_2^*})Y_+|(X(t) : t \in W))\mathbf{1}_A) \\
&= \mathbb{E}(\mathbb{E}(Y_-^*\mathbb{E}(Y_*|X_{t_1^*}, X_{t_2^*})|(X(t) : t \in W))\mathbb{E}(Y_+|(X(t) : t \in W))\mathbf{1}_A) \\
&= \mathbb{E}(Y_-^*\mathbb{E}(Y_*|X_{t_1^*}, X_{t_2^*})\mathbb{E}(Y_+|(X(t) : t \in W))\mathbf{1}_A)
\end{aligned}$$

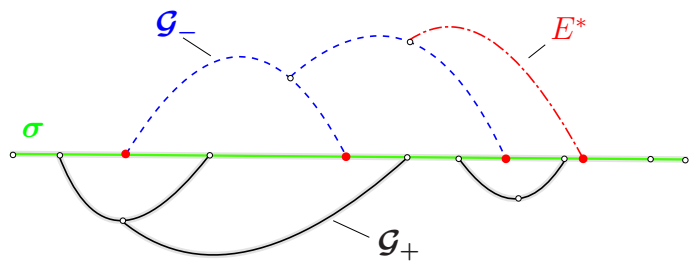


Figure 2.8:

Again using edge-Markovian property for E^* we get

$$\begin{aligned}
&\mathbb{E}(Y_-^*\mathbb{E}(Y_*|X_{t_1^*}, X_{t_2^*})\mathbb{E}(Y_+|(X(t) : t \in W))\mathbf{1}_A) \\
&= \mathbb{E}(Y_-^*Y_*\mathbb{E}(Y_+|(X(t) : t \in W))\mathbf{1}_A) \\
&= \mathbb{E}(\mathbb{E}(Y_-^*Y_*\mathbb{E}(Y_+|(X(t) : t \in W))\mathbf{1}_A|(X(t) : t \in W))) \\
&= \mathbb{E}(\mathbb{E}(Y_-^*Y_*|(X(t) : t \in W))\mathbb{E}(Y_+|(X(t) : t \in W))\mathbf{1}_A).
\end{aligned}$$

This proves the claim for ($\circ$2).

($\circ$3) The new E^* edge is in $\mathcal{G}_- = (\mathcal{V}_-, \mathcal{E}_-)$, both of the vertices that E^* is connecting are not on σ but are on $\mathcal{G}_-$. In this case we fist use the edge-Markov property for E^* and then in the similar way as in ($\circ$2) we use the spine-Markovian property or Theorem 2.5 if the graph $(\mathcal{V}_-, \mathcal{E}_- \setminus \{E^*\})$ is made of two components.

($\circ$4) The new E^* edge is in $\mathcal{G}_+ = (\mathcal{V}_+, \mathcal{E}_+)$ and not a part of σ. Using the spine-Markov property of X^l we know that $(X^{l+1}(t) : t \in \mathcal{E}_+ \setminus \{E^*\})$ and $(X^{l+1}(t) : t \in \mathcal{E}_-)$ are

independent given $(X^{l+1}(t) : t \in W)$. Using the edge-Markovian property for X^{l+1} we get that $(X^{l+1}(t) : t \in \mathcal{E}_+)$ and $(X^{l+1}(t) : t \in \mathcal{E}_-)$ are independent given $(X^{l+1}(t) : t \in W)$. (This is proven similar as in ($\circ$2).)

($\circ$5) If $E^* = E_{t_1^* t_2^*}$ is a part of the spine σ. By the construction of E^* we know that there exists a time-path going through vertices t_1^* and t_2^*, and therefore there is a full time-path σ' which contains whole of σ except E^*. Let σ'_{12} be the part of σ' connecting t_1^* and t_2^*. We will use the spine-Markov property for σ' on $\mathcal{G}_l$ to prove the one for σ on $\mathcal{G}_{l+1}$. Take $\mathcal{G}_-$ and W in $\mathcal{G}_{l+1}$ relative to σ. Clearly, none of the vetrices in W are on E^*. If none of them are on σ'_{12} (except maybe t_1^* and t_2^*), we can apply the spine-Markovian property relative to σ' in the case ($\circ$3), and we are done.

If some of the vertices in $W \setminus \{t_1^*, t_2^*\}$ are on σ'_{12}, then the whole σ'_{12} is in $\mathcal{G}_-$. Let's decompose $\mathcal{G}_l$ with the respect to σ'. Now, the graph $\mathcal{G}_-$ is a union of some components $\mathcal{G}^1_{l-}$, $\ldots$, $\mathcal{G}^k_{l-}$ and σ'_{12}. $\mathcal{G}_+$ is a union of some other components $\mathcal{G}^{k+1}_{l-}$, $\ldots$, $\mathcal{G}^h_{l-}$ and σ. Now we look a the following parts of $\mathcal{G}$ (for a illustration see Figure 2.11.)

- $A = \mathcal{G}^1_{l-} \cup \ldots \cup \mathcal{G}^k_{l-}$, $W_A = W_1 \cup \ldots \cup W_k$.
- $B = \mathcal{G}^{k+1}_{l-} \cup \ldots \cup \mathcal{G}^h_{l-}$, $W_B = W_{k+1} \cup \ldots \cup W_h$. Note that $W_B \setminus \{t_1^*, t_2^*\}$ contains no vertices on the path σ_{12}
- C is the graph containing σ_{12}.
- D is the graph containing σ' without σ_{12}. This is the same as a graph containing σ without E^*.
- E is the graph containing E^*.

Let's review which parts we have in the important graphs

$\mathcal{G}_-$	$\mathcal{G}_+$	σ	σ'	E^*
A, C	B, D, E	D, E	C, D	E

.

Note that W the roots of $\mathcal{G}_-$ are $(W_A \backslash (t_1^*, t_2^*)) \cup \{t_1^*, t_2^*\}$. Let Y_H be a bounded $\sigma(X(t) : t \in H)$-measurable random variable, for $H = A, B, C, D, E$, and let $T \in \sigma(X_W)$. Now, we go step by step, using the right Markovian properties. First we use the edge-Markovian property for edge $E^* = E_{t_1^* t_2^*}$, hence Y_E is independent of the rest of the Y-variables given $X(t_1^*)$ and $X(t_2^*)$

$$\mathbb{E}(Y_A Y_B Y_C Y_D Y_E \mathbf{1}_T) = \mathbb{E}(Y_A Y_B Y_C Y_D \mathbb{E}(Y_E | X(t_1^*), X(t_2^*)) \mathbf{1}_T).$$

Now, we apply the spine-Markovian property for on A and B relative to σ':

$$\begin{aligned} &= \mathbb{E}(\mathbb{E}(Y_A | X_{W_A}) Y_B Y_C Y_D \mathbb{E}(Y_E | X(t_1^*), X(t_2^*)) \mathbf{1}_T) \\ &= \mathbb{E}(\mathbb{E}(Y_A | X_{W_A}) \mathbb{E}(Y_B | X_{W_B}) Y_C Y_D \mathbb{E}(Y_E | X(t_1^*), X(t_2^*)) \mathbf{1}_T). \end{aligned} \tag{2.3}$$

Note, that X_{W_A}, X_{W_B}, Y_C, Y_D, and $X(t_1^*)$, $X(t_2^*)$, are all $\sigma(X_{\sigma'})$-measurable, and we can use the (T3) property of the process $X_{\sigma'}$ (X on the path σ'). Let $\mathcal{F}_{\sigma'}(t_1^*) = \sigma\{X_{\sigma'}(t) : t \leq t_1^*\}$ and $\mathcal{G}_{\sigma'}(t_2^*) = \sigma\{X_{\sigma'}(t) : t \geq t_2^*\}$. Now, we take the conditional expectation in (2.3) with respect to $\mathcal{F}_{\sigma'}(t_1^*) \vee \mathcal{G}_{\sigma'}(t_2^*)$. Note that Y_D, $\mathbf{1}_T$ and X_{W_B} are $\mathcal{F}_{\sigma'}(t_1^*) \vee \mathcal{G}_{\sigma'}(t_2^*)$-measurable. Hence,

$$\begin{aligned} &= \mathbb{E}(\mathbb{E}[\mathbb{E}(Y_A | X_{W_A}) \mathbb{E}(Y_B | X_{W_B}) Y_C Y_D \mathbb{E}(Y_E | X(t_1^*), X(t_2^*)) \mathbf{1}_T | \mathcal{F}_{\sigma'}(t_1^*) \vee \mathcal{G}_{\sigma'}(t_2^*)]), \\ &= \mathbb{E}(\mathbb{E}[\mathbb{E}(Y_A | X_{W_A}) Y_C | \mathcal{F}_{\sigma'}(t_1^*) \vee \mathcal{G}_{\sigma'}(t_2^*)] \mathbb{E}(Y_B | X_{W_B}) Y_D \mathbb{E}(Y_E | X(t_1^*), X(t_2^*)) \mathbf{1}_T). \end{aligned} \tag{2.4}$$

Using, the spine-Markovian property of B with respect to σ' and the edge-Markovian property of E^*, respectively we get

$$\begin{aligned} &= \mathbb{E}(\mathbb{E}[\mathbb{E}(Y_A | X_{W_A}) Y_C | \mathcal{F}_{\sigma'}(t_1^*) \vee \mathcal{G}_{\sigma'}(t_2^*)] Y_B Y_D \mathbb{E}(Y_E | X(t_1^*), X(t_2^*)) \mathbf{1}_T) \\ &= \mathbb{E}(\mathbb{E}(\mathbb{E}(Y_A | X_{W_A}) Y_C | \mathcal{F}_{\sigma'}(t_1^*) \vee \mathcal{G}_{\sigma'}(t_2^*)) Y_B Y_D Y_E \mathbf{1}_T) \end{aligned} \tag{2.5}$$

It remains to show that $\mathbb{E}[\mathbb{E}(Y_A | X_{W_A}) Y_C | \mathcal{F}_{\sigma'}(t_1^*) \vee \mathcal{G}_{\sigma'}(t_2^*)]$ is $\sigma(X_W)$ measurable. Let $W_A = W_A' \cup W_A^*$, where $W_A' \subset [0, t_1^*] \cup [t_2^*, 1]$, and $W_A^* = W_A \setminus W_A'$. We can assume $X_{W_A} =$

$(X_{W'_A}, X_{W^*_A})$ If $f_{Y_A}(x_{W'_A}, x_{W^*_A}) = \mathbb{E}(Y_A|X_{W_A} = (x_{W'_A}, x_{W^*_A}))$, then

$$\begin{aligned}
&\mathbb{E}[\mathbb{E}(Y_A|X_{W_A})Y_C|\mathcal{F}_{\sigma'}(t_1^*) \vee \mathcal{G}_{\sigma'}(t_2^*)] \\
=&\mathbb{E}[f(X_{W_A})Y_C|\mathcal{F}_{\sigma'}(t_1^*) \vee \mathcal{G}_{\sigma'}(t_2^*)] \\
=&\int f(X_{W'_A}, x_{W^*_A})y_C\mathbb{P}(X_{W^*_A} \in dx_{W^*_A}, Y_C \in dy_C|\mathcal{F}_{\sigma'}(t_1^*) \vee \mathcal{G}_{\sigma'}(t_2^*)) \qquad (2.6)
\end{aligned}$$

Now, since W_A^* and Y_C are $\sigma(X_\sigma(t) : t \in [t_1^*, t_2^*])$-measurable, using the (T3) (note that C and E form a simple cell) and Lemma 2.11, we have

$$\mathbb{P}(X_{W^*_A} \in dx_{W^*_A}, Y_C \in dy_C|\mathcal{F}_{\sigma'}(t_1^*) \vee \mathcal{G}_{\sigma'}(t_2^*)) = \mathbb{P}(X_{W^*_A} \in dx_{W^*_A}, Y_C \in dy_C|X(t_1^*), X(t_2^*)).$$

This and (2.6) implies that $\mathbb{E}[\mathbb{E}(Y_A|X_{W_A})Y_C|\mathcal{F}_{\sigma'}(t_1^*) \vee \mathcal{G}_{\sigma'}(t_2^*)]$ is $\sigma(X_W)$-measurable, since $W = W'_A \cup \{t_1^*, t_2^*\}$. Now taking the conditional expectation in (2.5) with respect to X_W we get

$$\begin{aligned}
&\mathbb{E}(\mathbb{E}(\mathbb{E}(\mathbb{E}(Y_A|X_{W_A})Y_C|\mathcal{F}_{\sigma'}(t_1^*) \vee \mathcal{G}_{\sigma'}(t_2^*))Y_BY_DY_E\mathbf{1}_T|X_W)) \\
=&\mathbb{E}(\mathbb{E}(\mathbb{E}(Y_A|X_{W_A})Y_C|\mathcal{F}_{\sigma'}(t_1^*) \vee \mathcal{G}_{\sigma'}(t_2^*))\mathbb{E}(Y_BY_DY_E|X_W)\mathbf{1}_T)
\end{aligned}$$

Using the fact that X_W is $\mathcal{F}_{\sigma'}(t_1^*) \vee \mathcal{G}_{\sigma'}(t_2^*)$-measurable, we have

$$\begin{aligned}
&\mathbb{E}(\mathbb{E}(\mathbb{E}(Y_A|X_{W_A})Y_C\mathbb{E}(Y_BY_DY_E|X_W)\mathbf{1}_T|\mathcal{F}_{\sigma'}(t_1^*) \vee \mathcal{G}_{\sigma'}(t_2^*))) \\
=&\mathbb{E}(\mathbb{E}(Y_A|X_{W_A})Y_C\mathbb{E}(Y_BY_DY_E|X_W)\mathbf{1}_T)
\end{aligned}$$

Applying the spine-Markovian property to A with respect to σ' we get

$$\begin{aligned}
&\mathbb{E}(\mathbb{E}(Y_A|X_{W_A})Y_C\mathbb{E}(Y_BY_DY_E|X_W)\mathbf{1}_T) \\
=&\mathbb{E}(Y_AY_C\mathbb{E}(Y_BY_DY_E|X_W)\mathbf{1}_T).
\end{aligned}$$

Finally, taking the conditional expectation with respect to X_W we get

$$\begin{aligned}
&\mathbb{E}(\mathbb{E}(Y_AY_C\mathbb{E}(Y_BY_DY_E|X_W)\mathbf{1}_T|X_W)) \\
=&\mathbb{E}(\mathbb{E}(Y_AY_C|X_W)\mathbb{E}(Y_BY_DY_E|X_W)\mathbf{1}_T).
\end{aligned}$$

From the Monotone Class Theorem the claim follows. ■

2.3.4 The constructed process is a hereditary spine-Markovian process

Recall how we defined $S^*(\mathcal{G})$ and the hereditary spine-Markovian property. (See Definition 2.6 and Definition 2.7. on page 45.)

Proposition 2.17. *The process X on $\mathcal{G}$ defined as in 2.3.1. is hereditary spine-Markovian.*

Proof. Fix an arbitrary TLG* $\mathcal{G}$ and an TLG* tower $(\mathcal{G}_k)_{k=0}^n$ such that $\mathcal{G}_0$ is the minimal graph and $\mathcal{G}_n = \mathcal{G}$.

Clearly, X^0 is spine-Markovian, and the claim holds since $S^*(\mathcal{G}_0) = \{\mathcal{G}_0\}$. Now, we will show that if the process X^{k-1} on $\mathcal{G}_{k-1} \neq \mathcal{G}$ is hereditary spine-Markovian, so is X^k on $\mathcal{G}_k$.

• If we got $\mathcal{G}_k$ by adding a new vertex to $\mathcal{G}_{k-1}$ then we are done, since the distribution of the process didn't change on the joint representation of these two TLG*.

• Let's view the case when we added a new edge E^* (between the existing vertices) to $\mathcal{G}_{k-1}$ to obtain $\mathcal{G}_k$. Take any $\mathcal{H} \in S^*(\mathcal{G}_k)$. If $\mathcal{H} \in S^*(\mathcal{G}_{k-1})$, then we are done. Otherwise, $\mathcal{H} = (\mathcal{V}_H, \mathcal{E}_H)$ contains the new edge E^*, i.e. $E^* \in \mathcal{E}_H$. Let $E^* = E_{t_1^* t_2^*}$.

($\circ$1) If there exists a path σ_{12} connecting t_1^* and t_2^* (not containing E^*), then $\mathcal{H}' = (\mathcal{V}_H, \mathcal{E}_H \setminus \{E^*\})$ is a TLG* (Corollary 1.13.) and in $S^*(\mathcal{G}_{k-1})$. This implies that $(X^k(t) : t \in \mathcal{H}')$ is spine-Markovian, and in the same way as in the Lemma 2.16, we can show that $(X^k(t) : t \in \mathcal{H})$ is spine-Markovian.

($\circ$2) If a path σ_{12} connecting t_1^* and t_2^* does not exist, then take any tower $(\mathcal{K}_l)_{l=0}^n$ such that $\mathcal{K}_0 = \mathcal{H}$ and $\mathcal{K}_n = \mathcal{G}$.

Let k be a minimum l such t_1^* and t_2^* are connected in K_l by some path not containing E^*. Such a k exists, because for the construction of E^* t_1^* and t_2^* need to be connected by a time-path in $\mathcal{G}_{k-1}$, so this is also true in $\mathcal{G}_k = \mathcal{K}_n$. But then, we just added a new edge $E'_{t_1^* t_2^*}$ to $\mathcal{K}_{m-1}$. Now we can first add an edge $E'_{t_1^* t_2^*}$ to $\mathcal{K}_0$, and after that add vertices and edges in the order we added them to obtain $\mathcal{K}_{m-1}$ from $\mathcal{K}_0$.

In this way, we would still get $\mathcal{K}_m$ at the end. This shows that a TLG* $(\mathcal{V}_H, \mathcal{E} \cup \{E'_{t_1^* t_2^*}\})$ (the TLG* that we get when we add a new edge connecting t_1^* and t_2^* to $\mathcal{K}_0$) is in $S^*(\mathcal{G}_k)$.

Now, we are previous case ($\circ 1$): X^k on $(\mathcal{V}_H, \mathcal{E} \cup \{E'_{t_1^* t_2^*}\})$ is spine-Markovian.

To prove that X^k on $\mathcal{H}$ is spine-Markovian we need to consider two cases: If a spine σ in $\mathcal{H}$ contains E^*, then $E'_{t_1^* t_2^*}$ is just one of the components (disjoint from others) in $(\mathcal{V}_H, \mathcal{E} \cup \{E'_{t_1^* t_2^*}\})$ with respect to σ. For any other spine σ not containing E^*, since $E'_{t_1^* t_2^*}$ will be an extra part of some component in $(\mathcal{V}_H, \mathcal{E} \cup \{E'_{t_1^* t_2^*}\})$ with respect to σ. This shows that X^k on $\mathcal{H}$ is spine-Markovian. ■

2.3.5 Uniqueness in law of hereditary spine-Markovian $\mathcal{M}$-processes

Lemmas 2.15 and 2.16 give the following proposition.

Proposition 2.18. *The process X on $\mathcal{G}$ defined in 2.3.1 is a hereditary spine-Markovian $\mathcal{M}$-process.*

We will finish this discussion by showing uniqueness in law of hereditary spine-Markovian $\mathcal{M}$-processes.

As we noticed in the Remark after the Definition 2.12, the definition of the process X on $\mathcal{G}$ depends on the choice of the TLG* tower, on which we inductively define the process. It turns out, that the distribution of the process X is unique, and therefore it doesn't depend on the choice of the TLG* tower.

First, let's prove the following lemma.

Lemma 2.19. *Let X be a hereditary spine-Markovian $\mathcal{M}$-process on a TLG* $\mathcal{G}$. If $\mathcal{G}$ can be obtained from a TLG* $\mathcal{G}'$, by adding a new edge or vertex as in Definition 1.5.(ii), then X' a restriction of X to $\mathcal{G}'$ is also a hereditary spine-Markovian $\mathcal{M}$-process.*

Proof. Any full-time path in $\mathcal{G}'$ is also a full-time path in $\mathcal{G}$. Since, $S^*(\mathcal{G}') \subseteq S^*(\mathcal{G})$, it is clear that $(X(t) : t \in \mathcal{G}')$ is hereditary spine-Markovian. ■

Theorem 2.20. *A hereditary spine-Markovian $\mathcal{M}$-process (satisfying (3T) properties) on a TLG* $\mathcal{G}$ has a unique distribution.*

Proof. We will prove this using the induction on the number of edges n of the TLG*.

For $n = 1$, we have a minimal graph and its distribution is clearly uniquely given.

For $n > 1$, suppose $\mathcal{G}$ can be obtained from $\mathcal{G}'$ by adding a new edge or vertex as in Definition 1.5.(ii). If we just added a vertex to $\mathcal{G}'$ in order to obtain $\mathcal{G}$, then we are done since these two graphs have the same representation $R(\mathcal{G}) = R(\mathcal{G}')$. Since $\mathcal{G}'$ has $n-1$ edge, the distribution on it is unique, and so is on $\mathcal{G}$.

If we added a new edge between the existing E^* two vertices t_1 and t_2 on $\mathcal{G}'$. We are done since, there has to exist a full time-path σ in $\mathcal{G}'$ containing t_1 and t_2. But now, $\mathcal{G}_- = E^*$ is a component in the decomposition of $\mathcal{G}$ with the respect to σ and $\mathcal{G}_+ = \mathcal{G}'$ is the rest of $\mathcal{G}$. Now, the processes $(X(t) : t \in \mathcal{G}')$ and $(X(t) : t \in E^*)$ are independent given $X(t_1)$ and $X(t_2)$. By Lemma 2.19 $(X(t) : t \in \mathcal{G}')$ is a hereditary spine-Markovian $\mathcal{M}$-process, so its distribution is unique. The distribution of $(X(t) : t \in E^*)$ given $X(t_1)$ and $X(t_1)$ is also uniquely given because of the consistency (i.e. (T2) property) of $\mathcal{M}$.

Hence, the distribution of X on $\mathcal{G}$ is unique. ■

Definition 2.21. *We define the process constructed in §2.3.1 to be the natural $\mathcal{M}$-process on the TLG* $\mathcal{G}$.*

2.4 Processes on TLG's with infinite number of vertices

In Section 1.8 (see Definition 1.49) we introduced TLG's and TLG* with infinitely many vertices. As in the case where we had only a finite number of vertices, here also we will construct a process on TLG* graphs.

2.4.1 Construction

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ a TLG* such that $\mathcal{V}$ is infinite. According to the definition, there exists a tower of TLG*'s $\mathcal{G}_n = (\mathcal{V}_n, \mathcal{E}_n)$, $n \geq 1$, such that $\mathcal{V}_n$ is finite, where $\mathcal{V} = \bigcup_{n\geq 1} \mathcal{V}_n$.

Let

$$\mathcal{M} = \{\mu_\sigma : \sigma \in P_{0\to 1}(\mathcal{G})\} \tag{2.7}$$

be a family of distributions of Markov processes along full-time paths in $\mathcal{G}$ satisfying conditions (T1)-(T3) given in Section 2.3. (Although 0 and 1 don't have to be the start and the end of time in $\mathcal{G}$, we will still use the notation $P_{0\to 1}(\mathcal{G})$ for full-time paths in $\mathcal{G}$.)

Since

$$\mathcal{M}(\mathcal{G}_n) = \{\mu_\sigma : \sigma \in P_{0\to 1}(\mathcal{G}_n)\}$$

is well-defined, and we can show similarly as in Lemma 2.14 that $\mathcal{M}(\mathcal{G}_n)$ satisfies (T1)-(T3), we can define a hereditary spine-Markovian process X^n on $\mathcal{G}_n$, such that for each $\sigma \in P_{0\to 1}(\mathcal{G}_n)$ the process X^n_σ has the distribution μ_σ. Further, the restriction of this process to $\mathcal{G}_k$ ($k \le n$) has the same distribution as the $\mathcal{M}(\mathcal{G}_k)$-process X^k defined on $\mathcal{G}_k$ in the similar manner.

Now, Kolomogorov's consistency theorem shows, that there exists a process X on $\mathcal{G}$ such that the restriction of X to any $\mathcal{G}_k$ has same distribution as X^k. Note, that since each $\sigma \in P_{0\to 1}(\mathcal{G})$ is in some of the $\mathcal{G}_k$'s we have X_σ has the distribution μ_σ.

2.4.2 Uniqueness of the distribution

Lemma 2.22. *Let $\mathcal{G}_0$, $\mathcal{H}$ and $\mathcal{G}_1$ be TLG*'s with the following properties:*

(1) $\mathcal{G}_0 \in S^*(\mathcal{G}_1)$;

(2) $\mathcal{V}_{\mathcal{G}_0} \subset \mathcal{V}_{\mathcal{H}} \subset \mathcal{V}_{\mathcal{G}_1}$;

(3) $R(\mathcal{G}_0) \subset R(\mathcal{H}) \subset R(\mathcal{G}_1)$.

Then $\mathcal{G}_0 \in S^*(\mathcal{H})$.

Proof. We will show the claim by induction on

$$n(\mathcal{G}_0) = \sum_{v \in \mathcal{V}_{\mathcal{G}_0}, d(v) \geq 3} d(v).$$

If $n = 0$ the claim is $\mathcal{G}_0$ represents a spine in $\mathcal{H}_0$, and the claim follows by Theorem 1.12.

Assume the claim holds for $n \leq k$. We will prove the claim for $n = k + 1$. Pick a truly simple cell (π_{uv}, π^1_{uv}) (recall Definition 1.41) in $\mathcal{G}_0$ (for example pick a spine π and then a time path π^1_{uv} not contained in π connecting t_u and t_v such that $|t_u - t_v|$ is minimal), the representation of this cell will remain a truly simple cell in $\mathcal{G}_1$ (by Theorem 1.18.) and therefore also in $\mathcal{H}$.

Hence, we define a cell collapsing transformation $\circ$ that is collapsing this cell.

For the TLG*'s $\mathcal{G}_0^\circ$, $\mathcal{H}^\circ$ and $\mathcal{G}_1^\circ$ property (1) holds by Corollary 1.44, while (2) and (3) are clear. Now, since $n(\mathcal{G}_0^\circ) < n(\mathcal{G}_0)$, by induction assumption $\mathcal{G}_0^\circ \in S^*(\mathcal{H}^\circ)$.

We follow the construction from $\mathcal{G}_0^\circ$ to $\mathcal{H}^\circ$, to obtain a TLG*-tower going from $\mathcal{G}_0$ to $\mathcal{H}$. Let $(\mathcal{K}'_j)_{j=0}^n$ be TLG*-tower starting with $\mathcal{K}'_0 = \mathcal{G}_0^\circ$ and $\mathcal{K}'_n = \mathcal{H}^\circ$. Now we construct a TLG*-tower $(\mathcal{K}_l)$ staring with $\mathcal{K}_0 = \mathcal{G}_0$. The idea of the construction is the following: if on $\mathcal{K}'_j$ to obtain $\mathcal{K}'_{j+1}$ we added

- a vertex, then add an appropriate vertex to $\mathcal{K}_j$ to obtain $\mathcal{K}_{j+1}$;
- an edge, then connect two appropriate vertices in $\mathcal{K}_j$ by an edge to obtain $\mathcal{K}_{j+1}$.

The main question is: When we add an edge, are we connecting two vertices that are connected by a time-path? That means that in K_j two vertices t_k and t_l are not connected by a time path, but $(t_k)^\circ$ and $(t_l)^\circ$ are connected by a time-path in $\mathcal{K}'_j$. So we have a situation like on the Figure 2.13. (Other situations are similar.)

By Theorem 1.34 $\mathcal{H}$ is a topological lattice, there exists a unique vertex $t_{u'} := t_{k'} \wedge t_l$. By definition since t_u and t_k are in the past of t_l and t'_k, t_u and t_k are connected by a time-path to $t_{u'}$.

Using this property, again, we know that in $\mathcal{H}$ there exists $t_{v'} = t_{u'} \vee t_{l'}$. And know by the same argumentation $t_{v'}$ is connected by a time path to t_l and t_v.

Note that $t_{u'}$ and $t_{v'}$ are in the time frame $[t_u, t_v]$. In order for the cell (π_{uv}, π^1_{uv}) to remain truly simple, $t_{u'} = t_u$ or $t_{v'} = t_v$ (otherwise the path $t_{k'} - t_{u'} - t_{v'} - t_{l'}$ will go from one side of the cell to the other within time frame $[t_u, t_v]$).

But, since t_v and t_l or t_u and t_k are not connected by a time path in $\mathcal{K}_j$ (since t_k and t_l are not), it follows that their images under the transformation are not connected in $\mathcal{K}'_j$. Hence this is a contradiction.

Therefore, in our procedure we construct a TLG*-tower. ■

Remark. The conditions (2) and (3) are not sufficient to imply the conclusion of the Lemma. The example is given on Figure 2.14. The whole line graph with vertices, and the whole graph are TLG*'s (since they are planar), but we can't construct the second from the first, since a simple cell is not a simple cell in the second.

Lemma 2.23. *Let $\mathcal{G}$ be a TLG* with infinitely many vertices and $(\mathcal{G}^1_j)$ and $(\mathcal{G}^2_j)$ two TLG*-towers that construct $\mathcal{G}$. For any points $\tau_1 \prec \tau_2$ on $\mathcal{G}_1$ with finite times and $j_1 \geq 1$, the distribution of the natural $\mathcal{M}$-processes X^1 and X^2 restricted on $R(\mathcal{G}^1_{j_1}[\tau_1, \tau_2])$ is unique.*

Proof. First, we know that $\mathcal{G}^h_j[\tau_1, \tau_2]$ $(h = 1, 2)$ is a TLG* (see Theorem 1.20), also note that all of its vertices have finite time.

By Lemma 1.50, there is a $\mathcal{G}^2_{k_1}$ such that $R(\mathcal{G}_{j_1}[\tau_1, \tau_2]) \subset R(\mathcal{G}^2_{k_1})$, and $\mathcal{V}^1_{j_1} \subset \mathcal{V}^2_{k_1}$. Further, note that $R(\mathcal{G}^1_{j_1}[\tau_1, \tau_2]) \subset R(\mathcal{G}^2_{k_1}[\tau_1, \tau_2])$. Using the same idea, we can find j_2 such that $R(\mathcal{G}^2_{k_1}[\tau_1, \tau_2]) \subset R(\mathcal{G}^1_{j_2}[\tau_1, \tau_2])$ and $\mathcal{V}^2_{k_1} \subset \mathcal{V}^1_{j_2}$. In this way $\mathcal{G}^1_{j_1}[\tau_1, \tau_2]$, $\mathcal{G}^2_{k_1}[\tau_1, \tau_2]$ and $\mathcal{G}^1_{j_2}[\tau_1, \tau_2]$ satisfy the properties of Lemma 2.22. Therefore, we can construct $\mathcal{G}^2_{k_2}[\tau_1, \tau_2]$ from $\mathcal{G}^1_{j_1}[\tau_1, \tau_2]$.

By Corollary 1.22, we can construct a spine π going through τ_1 and τ_2, then $\mathcal{G}^h_j[\tau_1, \tau_2]$ $(h = 1, 2)$ on that spine, and after that the rest of $\mathcal{G}^h_j$. Since, $\mathcal{M}(\mathcal{G}^h_j[\tau_1, \tau_2])$ – the restriction of the family $\mathcal{M}$ on $\mathcal{G}^h_j[\tau_1, \tau_2]$, is a (3T) family, X^h restricted on $G^h_j[\tau_1, \tau_2]$ is a natural $\mathcal{M}(\mathcal{G}^h_j[\tau_1, \tau_2])$-process.

Hence, X^2 on $\mathcal{G}^2_{k_1}([\tau_1,\tau_2])$ is distributed as a natural $\mathcal{M}(\mathcal{G}^2_{k_1}[\tau_1,\tau_2])$-process. Since $\mathcal{G}^2_{k_1}[\tau_1,\tau_2])$ can be constructed from $\mathcal{G}^1_{j_1}[\tau_1,\tau_2]$, X^2 restricted on $R(\mathcal{G}^1_{j_1}[\tau_1,\tau_2])$ is a natural $\mathcal{M}(\mathcal{G}^1_{j_1}[\tau_1,\tau_2])$-process. Therefore, X^2 has the same distribution as X^1 on $R(\mathcal{G}^1_{j_1}[\tau_1,\tau_2])$. ■

Burdzy and Pal were able to prove the uniqueness only in the case of planar NCC TLG's with infinite vertex set. The following proves their conjecture (see the sentence before Theorem 3.9. in [7]) that this is true in general case (including the non-planar case).

Theorem 2.24. *Let $\mathcal{G}=(\mathcal{V},\mathcal{E})$ be a TLG*'s with infinitely many vertices in $\mathcal{V}$, and let X^1 and X^2 be two $\mathcal{M}$-processes constructed using the TLG*-towers $(\mathcal{G}^1_n)$ and $(\mathcal{G}^2_n)$, then X^1 and X^2 have the same distribution.*

Proof. Pick points $\tau_1^{(n)} \prec \tau_2^{(n)}$ on $\mathcal{G}^1_1$ with finite time such that $\tau_1^{(n)} \downarrow -\infty$ and $\tau_2^{(n)} \uparrow +\infty$ (in time). Now, the distributions of X^1 and X^2 on $R(\mathcal{G}_n[\tau_1^{(n)},\tau_2^{(n)}])$ are the same, and since

$$\bigcup_{n=1}^{\infty} R(\mathcal{G}_n[\tau_1^{(n)},\tau_2^{(n)}]) = R(\mathcal{G}),$$

by Kolmogorov's consistency theorem we have that X^1 and X^2 have the same distribution.

■

Remark. To use the Kolmogorov's consistency theorem we need to look at finite dimensional vectors $(X^1(t_1),\ldots,X^1(t_m))$ and $(X^2(t_1),\ldots,X^2(t_m))$ for a finite number of points $t_1,\ldots,t_m \in \mathcal{G}$ with finite time. Since each point is in some subgraph of $\mathcal{G}$, there exists a n such that

$$\{t_1,\ldots,t_m\} \subset \mathcal{G}_n[\tau_1^{(n)},\tau_2^{(n)}]$$

and hence the random vectors have the same distribution.

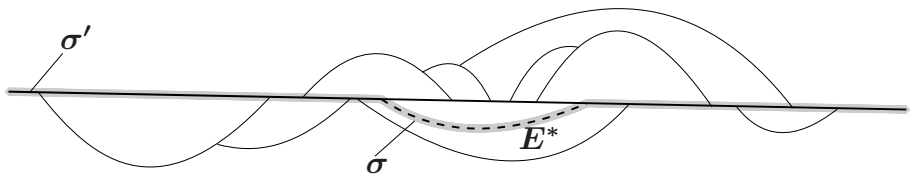


Figure 2.9: The two spines σ and σ'.

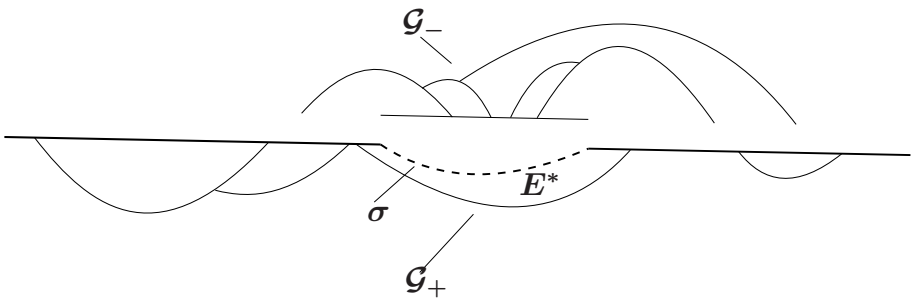


Figure 2.10: $\mathcal{G}_-$ and $\mathcal{G}_+$ with respect to σ.

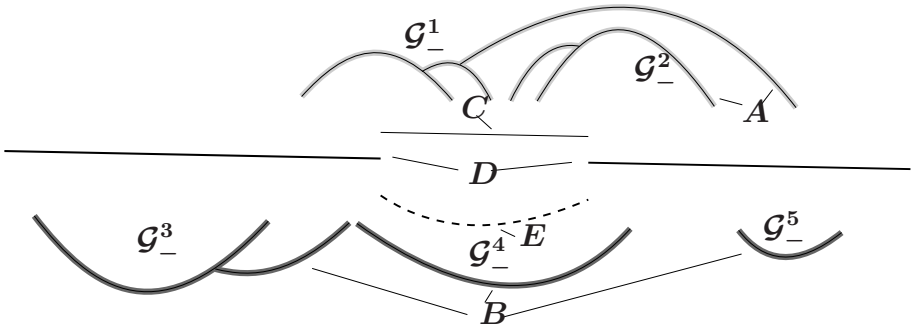


Figure 2.11: A, B, C, D and E parts of $\mathcal{G}$.

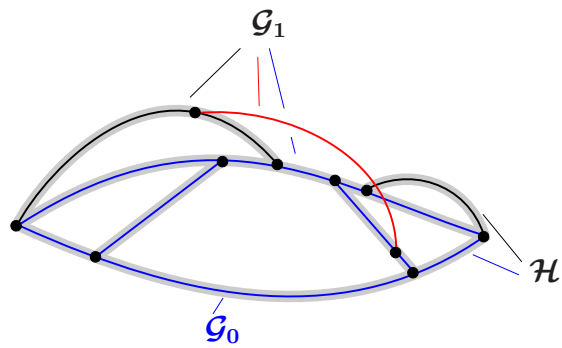


Figure 2.12:

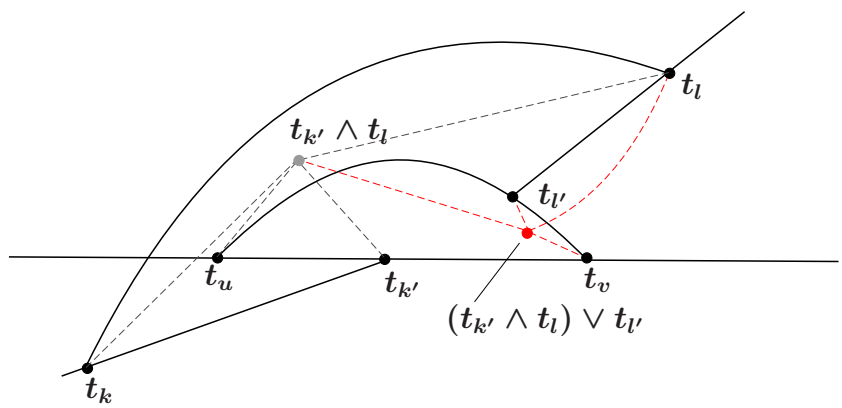


Figure 2.13:

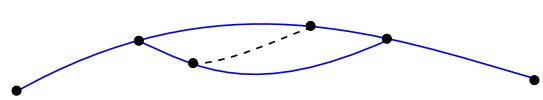


Figure 2.14:

Chapter 3

MARKOV PROPERTIES OF PROCESSES INDEXED BY TLG'S

From §2.3.3. and §2.3.4. we know that the constructed process has a (hereditary) spine-Markovian property. This property is induced by the graph structure and as we will see there is one more property this process has when $\mathcal{M}$ is a (3T)-family. If $\mathcal{M}$ has some additional properties we will have some additional properties of the process on the TLG* $\mathcal{G}$.

3.1 Cell-Markov properties

Recall, truly simple cell has been defined in Definition 1.41.

Definition 3.1. *We will say that a process X on a TLG $\mathcal{G}$ is* ***cell-Markovian*** *if for any truly simple cell (σ_1, σ_2) starting at t_* and ending at t^* the processes X_{σ_1} and X_{σ_2} are conditionally independent, given the values $X(t_*)$ and $X(t^*)$.*

Definition 3.2. *We will say that a process X on a TLG $\mathcal{G}$ is* ***strong cell-Markovian*** *if for for any truly simple cell (σ_1, σ_2) starting at t_* and ending at t^* the processes is cell-Markovian and $(X(t) : t \in \mathcal{G}[t_*, t^*])$ and $(X(t) : t \in \mathcal{G}[0, t_*] \cup \mathcal{G}[t^*, 1])$ are independent, given the values $X(t_*)$ and $X(t^*)$.*

Before we prove the that the strong cell-Markovian property holds, we will prove the following lemma.

Lemma 3.3. *Let $T = A \cup B \cup C \cup D$, and $X = (X(t) : t \in T)$ a stochastic process. If*

(1) $X_A = (X(t) : t \in A)$ and $X_C = (X(t) : t \in C)$ are independent given $X_B = (X(t) : t \in B)$

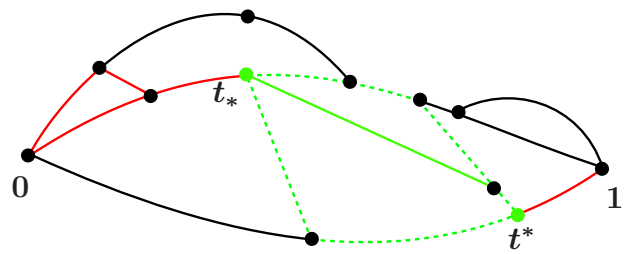


Figure 3.1: Strong cell-Markovian property: $X_{\mathcal{G}[t_*,t^*]} \perp X_{\mathcal{G}[0,t_*]\cup\mathcal{G}[t^*,1]} | (X(t_*), X(t^*))$

(2) for some subset $C' \subset C$ $X_{A\cup B\cup C}$ and X_D are independent given $X_{C'}$

then X_A and $X_{C\cup D}$ are independent given X_B.

Proof. Let Y_S be a bounded $\sigma(X_S)$-measurable function, for $S = A, B, C, D$, and U be an element in $\sigma(X_B)$. Then, using (2) we have

$$\mathbb{E}(Y_A Y_B Y_C Y_D \mathbf{1}_U) = \mathbb{E}(Y_A Y_C \mathbb{E}(Y_D | X_{C'}) \mathbf{1}_U).$$

Using (1) we get

$$\mathbb{E}(Y_A Y_C \mathbb{E}(Y_D | X_{C'}) \mathbf{1}_U) = \mathbb{E}(\mathbb{E}(Y_A | X_B) Y_C \mathbb{E}(Y_D | X_{C'}) \mathbf{1}_U),$$

and again using (2) and the fact that $\mathbb{E}(Y_A | X_B) Y_C \mathbf{1}_U$ is a bounded $\sigma(X_{A\cup B\cup C})$-measurable random variable we get

$$\mathbb{E}(\mathbb{E}(Y_A | X_B) Y_C \mathbb{E}(Y_D | X_{C'}) \mathbf{1}_U) = \mathbb{E}(\mathbb{E}(Y_A | X_B) Y_C Y_D \mathbf{1}_U).$$

Finally, conditioning everything (under the expectation) on X_B we get

$$\mathbb{E}(\mathbb{E}(Y_A | X_B) Y_C Y_D \mathbf{1}_U) = \mathbb{E}(\mathbb{E}[\mathbb{E}(Y_A | X_B) Y_C Y_D \mathbf{1}_U | X_B]) = \mathbb{E}(\mathbb{E}(Y_A | X_B) \mathbb{E}(Y_C Y_D | X_B) \mathbf{1}_U).$$

Now, using the Monotone Class Theorem the claim follows. ■

Theorem 3.4. *The process X on $\mathcal{G}$ defined in §2.3.1 is strong cell-Markovian process.*

Proof. It is known from Corollary 1.22 that there exists a TLG*-tower $(\mathcal{G}_k)_{k=0}^n$ that starts with a spine π containing t_*, t^*, there exists n_0 such that $\mathcal{G}_{n_0}$ such that $R(\mathcal{G}[t_*, t^*]) \cup R(\pi) = R(\mathcal{G}_{n_0})$, and then we can construct the rest of $\mathcal{G}$ (i.e. $\mathcal{G}_n = \mathcal{G}$). Since, $\mathcal{G}_{n_0}$ is a TLG*, we know by Theorem that X^{n_0} the natural $\mathcal{M}(\mathcal{G}_{n_0})$-process on $\mathcal{G}_{n_0}$ is the same as the restriction of the process X on $\mathcal{G}_{n_0}$.

Assume that π is the spine that contains σ_1. Since σ_2 will in a decomposition component $\mathcal{G}_-$ with roots t_* and t^*, by the spine-Markovian property, $X_{\sigma_1}^{n_0}$ is independent of $X_{\pi}^{n_0}$ given $X^{n_0}(t_*)$ and $X^{n_0}(t^*)$. This proves the cell-Markovian property.

We use induction to show that $(X(t) : t \in \mathcal{G}_k[t_*, t^*])$ is independent of $(X(t) : t \in \mathcal{G}_k[0, t_*] \cup \mathcal{G}_k[t^*, 1])$. For $n = 0$ the claim follows from (T3) property. For $k = 1, \ldots, n_0$ the process on every edge that we add will depend only on the value of the process $(X(t) : t \in \mathcal{G}_{k-1}[t_*, t^*])$ at its endpoints, so the claim will follow by Lemma 3.3. For $k > n_0$ we have the following cases:

- We added an vertex - nothing changes since the representation is the same.
- We added an edge not in $\mathcal{G}[0, t_*] \cup \mathcal{G}[t^*, 1])$ - this has no impact.
- We added an edge E that connects two vertices in $\mathcal{G}_{k-1}[0, t_*] \cup \mathcal{G}_{k-1}[t^*, 1])$. Then the process depends only on the values of X at the endpoints, the claim is true by Lemma 3.3.

Since the distribution of the process, by Theorem 2.20, doesn't depend on the construction the claim follows. ■

Corollary 3.5. *For the process X on $\mathcal{G}$ defined in §2.3.1, if (σ_1, σ_2) is a truly simple cell starting at 0 and ending at t^*, then the processes $(X(t) : t \in \mathcal{G}[0, t^*])$ and $(X(t) : t \in \mathcal{G}[t^*, 1])$ are independent given the values of $X(0)$ and $X(t^*)$.*

3.2 Graph-Markovian and time-Markovian property

First, we introduce the graph-Markovian property, a version of the global Markov property in graphical models (see Definition A.18 (c)).

Definition 3.6. *Suppose that $W \subset R(\mathcal{G})$ is a finite non-empty set such that $R(\mathcal{G}) \setminus W$ is disconnected. Some edges of $\mathcal{G}$ are cut by W into two or more components. Let us call this new collection of edges $\mathcal{E}_0$. Suppose that $\mathcal{E}_1$ and $\mathcal{E}_2$ are disjoint sets of edges with the union equal $\mathcal{E}_0$. We will call a process X on a TLG graph $\mathcal{G}$ a* ***graph-Markovian process*** *if for all W, $\mathcal{E}_1$, $\mathcal{E}_2$, the conditional distribution of $(X_t : t \in E, E \in \mathcal{E}_1)$ given $(X_t : t \in E, E \in \mathcal{E}_2)$ depends only on $(X_t : t \in W)$.*

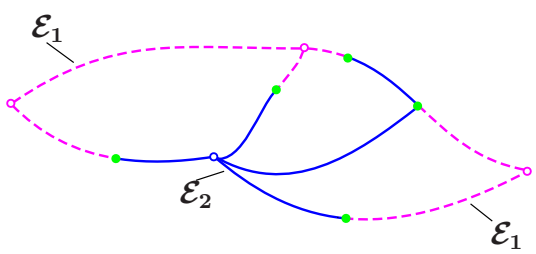


Figure 3.2: Graph-Markovian property: Black points (•) represent W, dashed lines $\mathcal{E}_1$, and full lines $\mathcal{E}_2$.

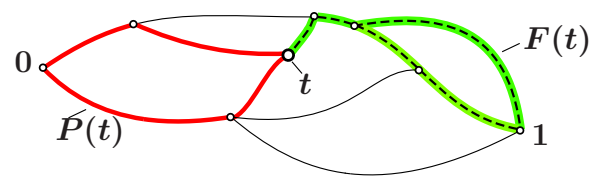


Figure 3.3: Time-Markovian property: The past $P(t)$ - of t, and the future - $F(t)$ of t.

The second property is the time-Markovian property.

Definition 3.7. *(a) Let t be a point in $\mathcal{G}$.*

(i) (THE FUTURE) $F(t) = \{s \in \mathcal{G} : s \succeq t\}$ is the set of all points with times $s \geq t$, such that there is a full path passing through t and s.

(ii) (THE PAST) $P(t) = \{s \in \mathcal{G} : s \preceq t\}$ is the set of all points with times $s \leq t$, such that there is a full path passing through t and s.

(b) We will say a process X on a TLG graph $\mathcal{G}$ is a ***time-Markovian process*** *if for every t, if the conditional distributions of $(X(s) : s \in P(t))$ and $(X(s) : s \in F(t))$ given $X(t)$ are independent.*

Remark. Note that if $(X(t) : t \in \mathcal{G})$ is time-Markovian, then for every full time-path π the process $X_\pi = (X(t) : t \in \pi)$ is a Markov process.

3.3 Processes on TLG's for Markov family $\mathcal{M}$

Some additional properties will hold if the distributions in the family $\mathcal{M}$ are all distributions of Markov processes.

Note that in this case the property (T3) is automatically satisfied, so the only thing that we need for the construction is the fact that $\mathcal{M}$ is a consistent family of distributions of Markov processes that are continuous or RCLL (or any other that we can define conditional distributions on) on a TLG* $\mathcal{G}$.

In the next few subsections we will show that in this case we have additional properties - edge-Markovian and time-Markovian properties.

3.3.1 The constructed process is a time-Markovian process

Theorem 3.8. *The process X defined on $\mathcal{G}$ defined in §2.3.1 for a Markov family $\mathcal{M}$ is a time-Markovian process.*

Proof. Let t be a point on $\mathcal{G}$. We can assume it is a vertex in $\mathcal{V}$. We will expand the vertex set $\mathcal{V}$, by adding the vertex $t_{-\varepsilon} = -\varepsilon$, i.e. $\mathcal{V}_\varepsilon = \mathcal{V} \cup \{t_{-\varepsilon}\}$. Further, we will expand the edge set by adding edges connecting $t_{-\varepsilon}$ with 0 and t, i.e. $\mathcal{E}_\varepsilon = \mathcal{E} \cup \{E_{-\varepsilon,0}, E_{-\varepsilon,t}\}$. (See Figure 3.4.) It is not hard to see that $\mathcal{G}_\varepsilon = (\mathcal{V}_\varepsilon, \mathcal{E}_\varepsilon)$ is a TLG*. We define $X(-\varepsilon) = 0$ and $X_{E_{-\varepsilon,0}}$ and $X_{E_{-\varepsilon,t}}$ to be interpolations between the values of the processes at the end points. $X_{\mathcal{G}_\varepsilon}$ is a continuous or RCLL process with Markov processes along full time-paths, and since $X_{\mathcal{G}}$ is a hereditary time-Markovian, so is $X_{\mathcal{G}_\varepsilon}$.

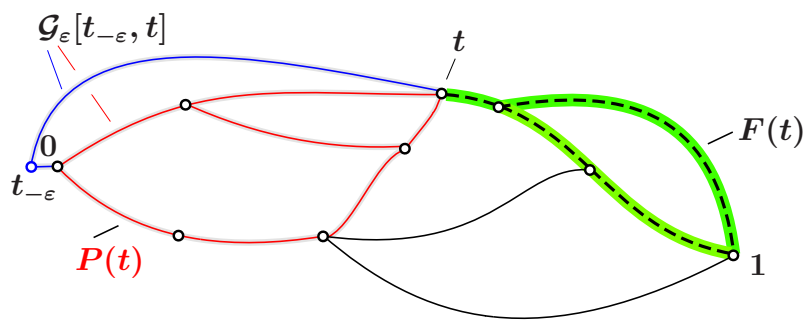


Figure 3.4: Idea of the proof of time Markovian property

For any path π between 0 and t, $(E_{-\varepsilon,0}\pi, E_{-\varepsilon,t})$ is a truly simple cell. Now, using the strong cell-Markovian property, we have that $(X(t) : t \in \mathcal{G}_\varepsilon[-\varepsilon, t]))$ and $(X(t) : t \in \mathcal{G}_\varepsilon[t, 1]))$ are independent given $X(t)$ and $X(-\varepsilon)$. Since $X(-\varepsilon)$ is deterministic, $\mathcal{G}_\varepsilon[t, 1] = \mathcal{G}[t, 1] = F(t)$ and $P(t) = \mathcal{G}[0, t] \subset \mathcal{G}_\varepsilon[-\varepsilon, t])$, the claim follows. ■

3.3.2 Moralized graph-Markovian property

In graphical models when we turn Bayes nets into Markov random fields, we *moralize* the graph (see §4.5 [35, Koller, Friedman]). It turns out that the Markov processes on TLG*'s, in general, don't satisfy the graph-Markovian property described in Section 3.2 (see discussion given in Subsection 3.4.1).

But under the modification of the graph, that we will call *moralization*, we will have a similar property.

Definition 3.9. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a TLG. The graph $\mathcal{G}^\heartsuit = (\mathcal{V}^\heartsuit, \mathcal{E}^\heartsuit)$ given by $\mathcal{V}^\heartsuit = \mathcal{V}$ and*

$$\mathcal{E}^\heartsuit = \mathcal{E} \cup \{E^\heartsuit_{ij} \ : \ i \text{ and } j \text{ are begining and end of a truly simple cell in } \mathcal{G}\}$$

will be called a ***moralized graph****.*

Remark. Note that for a TLG* $\mathcal{G}$, $\mathcal{G}^\heartsuit$ is also a TLG* - we are adding edge between points that are connected by a time-path.

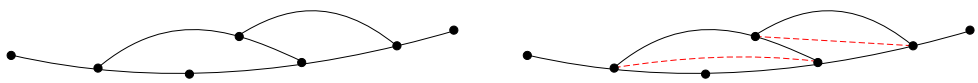


Figure 3.5: Moralization of a TLG $\mathcal{G}$ into $\mathcal{G}^{\heartsuit}$.

Definition 3.10. *Let $\mathcal{G}$ be a TLG, and $\mathcal{G}^{\heartsuit}$ its moralization. Suppose that $W \subset R(\mathcal{G}) \subset R(\mathcal{G}^{\heartsuit})$ is a finite non-empty set such that $R(\mathcal{G}^{\heartsuit}) \setminus W$ is disconnected. Some edges of $\mathcal{G}$ are cut by W into two or more components. Let us call this new collection of edges $\mathcal{E}_0$. Suppose that $\mathcal{E}_1$ and $\mathcal{E}_2$ are disjoint sets of edges with the union equal $\mathcal{E}_0$. We will call a process X on a TLG graph $\mathcal{G}$ a* ***moralized graph-Markovian process*** *if for all W, $\mathcal{E}_1$, $\mathcal{E}_2$, the conditional distribution of $(X_t : t \in E, E \in \mathcal{E}_1)$ given $(X_t : t \in E, E \in \mathcal{E}_2)$ depends only on $(X_t : t \in W)$.*

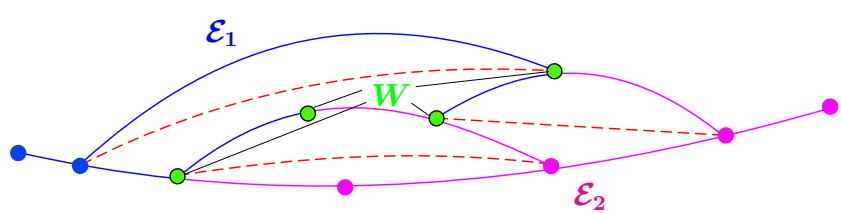


Figure 3.6: $(X(t) : t \in \mathcal{E}_1)$ is independent of $(X(t) : t \in \mathcal{E}_2)$ given X_W.

Before, we prove the moralized graph-Markovian property, we will prove the following lemma.

Lemma 3.11. *Let $T = A \cup B$, and a stochastic process $X = (X(t) : t \in T)$ such that*

(1) there exist A_1 and A_2 subsets of A such that $X_{A_1} = (X(t) : t \in A_1)$ is independent of $X_{A_1^c} = (X(t) : t \in A \setminus A_1)$ given $X_{A_2} = (X(t) : t \in A_2)$;

(2) there exists A_b subset of $A \setminus A_1$ such that X_A is independent of X_B given X_{A_b};

then $X_{B\cup A_1^c}$ *is independent of* X_{A_1} *given* X_{A_2}.

Proof. Let Y_S be a bounded $\sigma(X_S)$-measurable random variable, and $U \in \sigma(X_{A_2})$. Now, using (2) we have

$$\mathbb{E}(Y_B Y_{A_1} Y_{A_1^c} \mathbf{1}_U) = \mathbb{E}(\mathbb{E}(Y_B|X_{A_b}) Y_{A_1} Y_{A_1^c} \mathbf{1}_U),$$

and using (1) we get

$$\mathbb{E}(\mathbb{E}(Y_B|X_{A_b}) Y_{A_1} Y_{A_1^c} \mathbf{1}_U) = \mathbb{E}(\mathbb{E}(Y_B|X_{A_b}) \mathbb{E}(Y_{A_1}|X_{A_2}) Y_{A_1^c} \mathbf{1}_U).$$

Using, (2) once more we have

$$\mathbb{E}(\mathbb{E}(Y_B|X_{A_b}) \mathbb{E}(Y_{A_1}|X_{A_2}) Y_{A_1^c} \mathbf{1}_U) = \mathbb{E}(Y_B \mathbb{E}(Y_{A_1}|X_{A_2}) Y_{A_1^c} \mathbf{1}_U),$$

and now conditioning everything under the expectation on X_{A_2} we get

$$\mathbb{E}(Y_B \mathbb{E}(Y_{A_1}|X_{A_2}) Y_{A_1^c} \mathbf{1}_U) = \mathbb{E}(\mathbb{E}(Y_B Y_{A_1^c}|X_{A_2}) \mathbb{E}(Y_{A_1}|X_{A_2}) \mathbf{1}_U).$$

From the Monotone Class Theorem the claim follows. ■

Theorem 3.12. *For a Markov family* $\mathcal{M}$*, the natural* $\mathcal{M}$*-process on a TLG** $\mathcal{G}$ *is a moralized graph-Markovian process.*

Proof. We use induction on the number of edges $|E|$ for a TLG* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. For $|E| = 1$, the claim is clearly true. Assume that the claim is true for $|E| = k \geq 1$. Let's show the claim for $|E| = k + 1$. Pick $\mathcal{G}$ and W a set of points $\mathcal{G}$, such that $R(\mathcal{G}^\heartsuit) \setminus R(W)$ is disconnected. We need to consider the following cases:

If we got $\mathcal{G}$ by adding a new vertex to some TLG* $\mathcal{H}$. In that case, since the representation of $\mathcal{H}$ and $\mathcal{G}$ is the same, the claim follows.

If we got $\mathcal{G}$ by adding a new edge E_* between the vertices t_* and t^* in some TLG* $\mathcal{H}$, we first have to note that t_* and t^* are the begining and the end of a (truly) simple cell whose one side is E_*. Hence, t_* and t^* are both in one of the following $\mathcal{E}_1 \cup W$ or $\mathcal{E}_2 \cup W$.

We have the following cases to consider:

- $R(E_*) \cap R(W) = \emptyset$ then E_* will entierly be in one of $\mathcal{E}_1$ or $\mathcal{E}_2$. We will assume $E_* \in \mathcal{E}_1$, and let $\mathcal{E}_1' = \mathcal{E}_1 \setminus \{E_*\}$. In we use the spine-Markovian property with roots t_* and t^*, hence X_{E_*} is independent of $X_{\mathcal{E}_1' \cup \mathcal{E}_2}$ given $X(t_*)$ and $X(t^*)$. Now since $X_{\mathcal{E}_2}$ is independent of $X_{\mathcal{E}_1'}$ given X_W, by Lemma 3.3. it follows that $X_{\mathcal{E}_1}$ is independent of $X_{\mathcal{E}_2}$ given X_W.

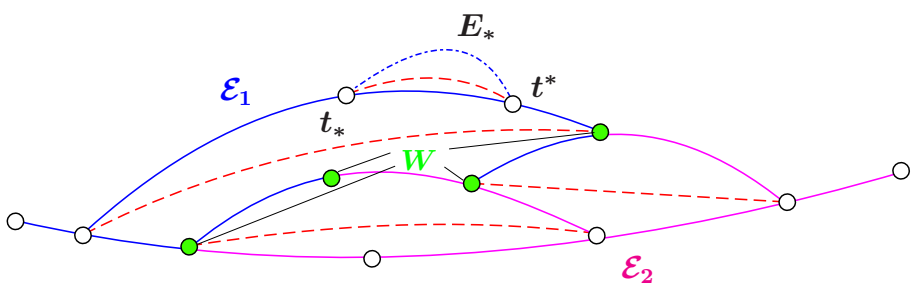


Figure 3.7: The new edge E^* doesn't contain points from W.

- $R(E_*) \cap R(W) \neq \emptyset$, then we assume $t_*, t^* \in \mathcal{E}_1 \cup W$. Denote, $W_{\mathcal{H}}$ the points represented by $R(W) \cap R(\mathcal{H})$. Note that these points separate $\mathcal{H}$.

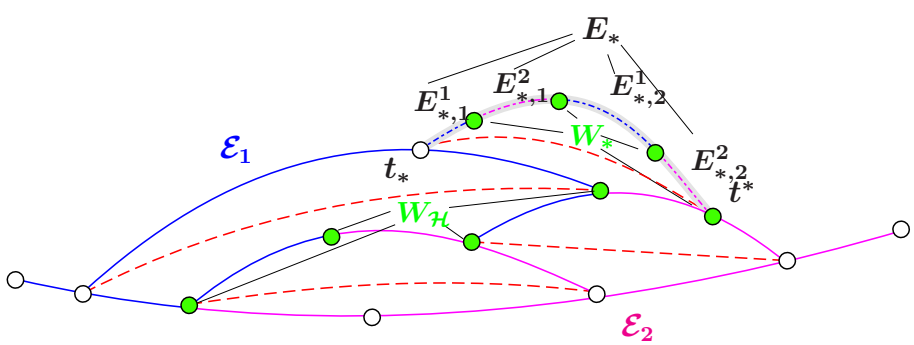


Figure 3.8: The new edge E^* contains points from W.

Let $E^j_{*,k}$, $k = 1, \ldots, n_j$ denotes the edges in $\mathcal{E}_j$ that cover the edge E_*. It is not hard

to see, since the process along X_{E^*} is Markov that

$$X_{E^j_{*,k}} \perp X_{R(\mathcal{G})\setminus R(E^j_{*,k})} | X_{\partial E^j_{*,k}}. \tag{3.1}$$

The endpoints of at least one of the sequences $(E^1_{*,k})$ or $(E^2_{*,k})$ will be only in W. Otherwise, t_* and t^* won't be in W, and they won't be both in $\mathcal{E}_1$. Under the assumption that t_* and t^* are in $\mathcal{E}_1 \cup W$, it follows that $(E^2_{*,k})$ has all its endpoints in W, and call that set W_*.

Let Y_1 be a bounded $\sigma(X(t) : \bar{t} \in R(\mathcal{E}_1) \setminus R(E_*))$, Y_2 a bounded $\sigma(X(t) : \bar{t} \in R(\mathcal{E}_2) \setminus R(E_*))$-measurable, and $Y^j_{*,k}$ a bounded $\sigma(X(t) : t \in E^j_{*,k})$-measurable random variable, for $j = 1, 2$, $k = 1, \ldots, n_j$,

$$Y_1' = \prod_{k=1}^{n_1} Y^1_{*,k}, \quad Y_2' = \prod_{k=1}^{n_1} Y^2_{*,k}.$$

First, we will show that Y_1' is independent of Y_2' given X_{W_*}. Let $A \in \sigma(X_{W_*})$. Using (3.1) we get

$$\begin{aligned}
\mathbb{E}(Y_1' Y_2' \mathbf{1}_A) &= \mathbb{E}(Y_1' \mathbb{E}(Y^1_{*,2} | X_{\partial E^1_{*,k}}) Y^2_{*,2} \ldots Y^{n_2}_{*,2} \mathbf{1}_A) \\
&= \mathbb{E}(Y_1' \mathbb{E}(Y^2_{*,1} | X_{\partial E^2_{*,1}}) \mathbb{E}(Y^2_{*,2} | X_{\partial E^2_{*,2}}) \ldots Y^{n_2}_{*,2} \mathbf{1}_A) \\
&\vdots \\
&= \mathbb{E}(Y_1' \mathbb{E}(Y^2_{*,1} | X_{\partial E^2_{*,1}}) \mathbb{E}(Y^2_{*,2} | X_{\partial E^2_{*,2}}) \ldots \mathbb{E}(Y^{n_2}_{*,2} | X_{\partial E^2_{*,n_2}}) \mathbf{1}_A)
\end{aligned}$$

Now, we condition everything under the expectation with respect to X_{W_*}:

$$\begin{aligned}
&= \mathbb{E}(\mathbb{E}[Y_1' \mathbb{E}(Y^2_{*,1} | X_{\partial E^2_{*,1}}) \mathbb{E}(Y^2_{*,2} | X_{\partial E^2_{*,2}}) \ldots \mathbb{E}(Y^{n_2}_{*,2} | X_{\partial E^2_{*,n_2}}) \mathbf{1}_A | X_{W_*}]) \\
&= \mathbb{E}(\mathbb{E}[Y_1' | X_{W_*}] \mathbb{E}(Y^2_{*,1} | X_{\partial E^2_{*,1}}) \mathbb{E}(Y^2_{*,2} | X_{\partial E^2_{*,2}}) \ldots \mathbb{E}(Y^{n_2}_{*,2} | X_{\partial E^2_{*,n_2}}) \mathbf{1}_A).
\end{aligned}$$

Using (3.1) again we get

$$\begin{aligned}
&= \mathbb{E}(\mathbb{E}[Y_1' | X_{W_*}] Y^2_{*,1} \mathbb{E}(Y^2_{*,2} | X_{\partial E^2_{*,2}}) \ldots \mathbb{E}(Y^{n_2}_{*,2} | X_{\partial E^2_{*,n_2}}) \mathbf{1}_A) \\
&\vdots \\
&= \mathbb{E}(\mathbb{E}[Y_1' | X_{W_*}] Y^2_{*,1} Y^2_{*,2} \ldots Y^{n_2}_{*,2} \mathbf{1}_A) = \mathbb{E}(\mathbb{E}[Y_1' | X_{W_*}] Y_2' \mathbf{1}_A).
\end{aligned}$$

Finally, conditioning everything under the expectation with respect to X_{W_*} we get

$$= \mathbb{E}(\mathbb{E}[Y_1'|X_{W_*}]\mathbb{E}(Y_2'|X_{W_*})\mathbf{1}_A),$$

and the claim follows.

Further, by Lemma 3.11, we have that

$$X_{\cup_{k=1}^{n_2} E^2_{*,k}} \perp X_{R(\mathcal{G})\setminus(\cup_{k=1}^{n_2} R(E^2_{*,k}))}|X_{W_*}. \tag{3.2}$$

Let $A_H \in \sigma(X_{W_\mathcal{H}})$ and $A_* \in \sigma(X_{W_*})$. Now, since $W_\mathcal{H}$ separates $\mathcal{G}$ into $\mathcal{E}_1' \cup \{E_*\}$ and $\mathcal{E}_2' = \mathcal{E}_2 \cap R(\mathcal{H})$ we have :

$$\mathbb{E}(Y_1Y_2Y_1'Y_2'\mathbf{1}_{A_H}\mathbf{1}_{A_*}) = \mathbb{E}(Y_1Y_2Y_1'\mathbb{E}(Y_2'|X_{W_*})\mathbf{1}_{A_H}\mathbf{1}_{A_*}) \tag{3.3}$$

$$=\mathbb{E}(Y_1\mathbb{E}(Y_2|X_{W_\mathcal{H}})Y_1'\mathbb{E}(Y_2'|X_{W_*})\mathbf{1}_{A_H}\mathbf{1}_{A_*}) \tag{3.4}$$

$$=\mathbb{E}(\mathbb{E}(Y_1Y_1'|X_W)\mathbb{E}(Y_2|X_{W_\mathcal{H}})\mathbb{E}(Y_2'|X_{W_*})\mathbf{1}_{A_H}\mathbf{1}_{A_*})$$

$$=\mathbb{E}(\mathbb{E}(Y_1Y_1'|X_W)Y_2\mathbb{E}(Y_2'|X_{W_*})\mathbf{1}_{A_H}\mathbf{1}_{A_*})$$

$$=\mathbb{E}(\mathbb{E}(Y_1Y_1'|X_W)Y_2Y_2'\mathbf{1}_{A_H}\mathbf{1}_{A_*})$$

$$=\mathbb{E}(\mathbb{E}(Y_1Y_1'|X_W)\mathbb{E}(Y_2Y_2'|X_W)\mathbf{1}_{A_H}\mathbf{1}_{A_*}).$$

To get (3.3) we use 3.2. In (3.4) we use the fact that $W_\mathcal{H}$ separates $\mathcal{E}_2'$ from the rest of $\mathcal{G}$, and then the property proven in the previous $\bullet$ case.

Now, by Monotone Class Theorem the claim follows.

■

The following corollary, gives us a connection to the Markov random fields and classical graphical models (see Appendix §A.4).

Corollary 3.13. *For a Markov family $\mathcal{M}$, let X be a natural $\mathcal{M}$-process on a TLG* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Let W be a finite set of points on $\mathcal{G}$ such that $\{t \in \mathcal{V} : d(t) \geq 3\} \subset W$, then $(X(t) : t \in W)$ is a random Markov field with a global Markov property. Further, X_W is a random*

Markov field indexed by the graph $G_W = (W, E_W)$ where E_W contains an edge between w_1 and w_2 if there is a time path π in $\mathcal{G}^\heartsuit$ between w_1 and w_2 such that $R(\pi) \cap R(W) = \{w_1, w_2\}$.

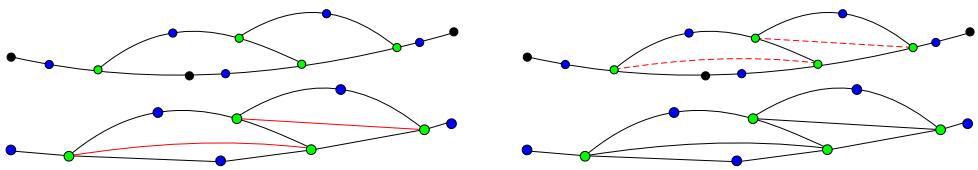


Figure 3.9: Getting the MRF: The radnom variables at green and blue points form a Markov random field, where the underlying graph is given on the last figure.

Proof. It is easy to see that $C \subset W$ separates graph E if and only if it separates $\mathcal{G}^\heartsuit$. Now it follows that $X_A \perp X_B | X_C$, since A and B are in two different components in $\mathcal{G}^\heartsuit$ separated by C. ■

The constructed process is edge-Markovian

Definition 3.14. *We say that the process X on a TLG $\mathcal{G} = (\mathcal{E}, \mathcal{V})$ is* ***edge-Markovian*** *if for each $E' = E_{j'k'} \in \mathcal{E}$ the process $(X(t) : t \in E')$ is independent of $(X(t) : t \in E, E \in \mathcal{E} \setminus \{E'\})$ given $X(t_{j'})$ and $X(t_{k'})$.*

Corollary 3.15. *Let X be a natural $\mathcal{M}$-process on a TLG* $\mathcal{G}$*. Let π be a time-path between t_* and t^* two points on $\mathcal{G}$ such that π (in the interior) doesn't contain a vertex of degree 3 or more. Then X_π and $X_{R(\mathcal{G}) \setminus R(\pi)}$ are independent given $X(t_*)$ and $X(t^*)$.*

Proof. Except the endpoints, the path, can't contain an edge in $\mathcal{E}^\heartsuit \setminus \mathcal{E}$. Therefore, endpoints t_* and t^* separate the graph $\mathcal{G}^\heartsuit$ with representations of components being $R(\pi)$ and $R(\mathcal{G}) \setminus R(\pi)$. The calim follows. ■

Theorem 3.16. *The process X defined on $\mathcal{G}$ defined in 2.3.1 for a Markov family $\mathcal{M}$ is an edge-Markovian process.*

3.3.3 Summary

Everything we proved so far, can be summarized in the following theorem.

Theorem 3.17. *For every TLG* $\mathcal{G}$ with finite vertex set $\mathcal{V}$ and every Markov family $\mathcal{M}$ there exists a hereditary spine-Markovian $\mathcal{M}$-process X on $\mathcal{G}$, and the distribution of such a process is unique. This process also has time-Markovian, cell-Markovian, moralized graph-Markovian and edge-Markovian properties. Further, if $\mathcal{G}$ can be constructed from a TLG* $\mathcal{H}$, then $(X(t) : t \in \mathcal{H})$ also has these properties.*

Corollary 3.18. *Let X be a natural $\mathcal{M}$-process on a TLG* $\mathcal{G}$, where $\mathcal{M}$ is a Markov family. Then for $\tau_1 \prec \tau_2$ the process $(X(t) : t \in \mathcal{G}[\tau_1, \tau_2])$ has time-Markovian, cell-Markovian, moralized graph-Markovian and edge-Markovian properties (induced by the structure of $\mathcal{G}[\tau_1, \tau_2]$).*

Proof. We can assume that τ_1 and τ_2 are vertices on $\mathcal{G}$. By Theorem 1.20 $\mathcal{G}[\tau_1, \tau_2]$ is a TLG*. Further,

$$\mathcal{M}(\mathcal{G}[\tau_1, \tau_2]) = \{\mu_\sigma \circ \pi^{-1}_{[\tau_1,\tau_2]} : \sigma \in P_{0\to 1}(\mathcal{G}), \tau_1, \tau_2 \in \sigma\}$$

satisfies (3T) properties. By Theorem 1.22 we can construct first construct a full time path σ containing τ_1 and τ_2, and then $\mathcal{G}[\tau_1, \tau_2]$, and after that the rest of $\mathcal{G}$. It is not hard to see that when we are done constructing $\mathcal{G}[\tau_1, \tau_2]$ in that TLG*-tower, the process restricted to $\mathcal{G}[\tau_1, \tau_2]$ will be a natural $\mathcal{M}(\mathcal{G}[\tau_1, \tau_2])$-process. ■

3.4 Homogeneous Markov family $\mathcal{M}_\mathcal{P}$

Let $\mathcal{P}$ be distribution of a continuous or RCLL Markov process on $[0, 1]$. Then we will call $\mathcal{M}_\mathcal{P} = \{\mu_\sigma = \mathcal{P} : \sigma \in P_{0\to 1}(\mathcal{G})\}$, a **homogeneous Markov family**. Note that for this family properties (T2) and (T3) are automatically satisfied.

Further, using Theorem 3.17. we have the following fact.

Theorem 3.19. *For every TLG* $\mathcal{G}$ with finite vertex set $\mathcal{V}$ and every Markov process $\mathcal{P}$ there exists a hereditary spine-Markovian $\mathcal{M}_{\mathcal{P}}$-process X on $\mathcal{G}$, and the distribution of such a process is unique. This process also has time-Markovian, cell-Markovian, moralized graph-Markovian, and edge-Markovian properties.*

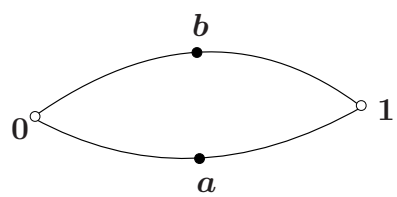
We will refer to the process X described in the Theorem as the **natural $\mathcal{P}$-process** on the TLG* $\mathcal{G}$.

3.4.1 The graph-Markovian property doesn't hold

In paper [7] it was claimed that for the the natural $\mathcal{P}$-process, on what they called NCC graphs, the graph-Markovian property holds.

However, the following example shows that this is not true.

In our simple model we look at a family of random variables $\{X_0, X_a, X_b, X_1\}$. Such that (X_0, X_a, X_1) is a Markov chain. (X_0, X_b, X_1) is also a Markov chain independent of the fist one given (X_0, X_1) and has the same distribution.



We suppose that the state space S is finite or countable.

We set for $x_0, x_a, x_b, x_1 \in S$

$$\mathbb{P}(X_0 = x_0, X_a = x_a, X_1 = x_1) = \mathbb{P}(X_0 = x_0, X_b = x_a, X_1 = x_1) = p^2_{x_1 x_a} p^1_{x_a x_0} p^0_{x_0},$$

with the usual assumptions on initial probabilities $(p^0_s)_{s\in S}$ and transition probabilities $(p^1_{ss'})_{s,s'\in S}$ and $(p^2_{ss'})_{s,s'\in S}$. Further from the assumption of independence given (X_0, X_1) we have

$$\mathbb{P}(X_a = x_a, X_b = x_b | X_0 = x_0, X_1 = x_1) =$$

$$\mathbb{P}(X_a = x_a | X_0 = x_0, X_1 = x_1)\mathbb{P}(X_b = x_b | X_0 = x_0, X_1 = x_1).$$

Our ultimate goal is to see does

$$\mathbb{P}(X_1 = x_1 | X_a = x_a, X_b = x_b, X_0 = x_0) \tag{$*$}$$

depend on x_0. We will first calculate

$$\begin{aligned}\mathbb{P}(X_1 = x_1, X_0 = x_0) &= \sum_{\alpha \in S} \mathbb{P}(X_1 = x_1, X_a = \alpha, X_0 = x_0) \\ &= \sum_{\alpha \in S} p^2_{x_1\alpha} p^1_{\alpha x_0} p^0_{x_0}.\end{aligned}$$

Next, using the definition of conditional probability and conditional independence we calculate

$$\begin{aligned}
&\mathbb{P}(X_1 = x_1, X_a = x_a, X_b = x_b, X_0 = x_0) \\
&= \mathbb{P}(X_a = x_a, X_b = x_b | X_1 = x_1, X_0 = x_0)\mathbb{P}(X_1 = x_1, X_0 = x_0) \\
&= \mathbb{P}(X_a = x_a | X_1 = x_1, X_0 = x_0)\mathbb{P}(X_b = x_b | X_1 = x_1, X_0 = x_0)\mathbb{P}(X_1 = x_1, X_0 = x_0) \\
&= \frac{\mathbb{P}(X_1 = x_1, X_a = x_a, X_0 = x_0)\mathbb{P}(X_1 = x_1, X_b = x_b, X_0 = x_0)}{\mathbb{P}(X_1 = x_1, X_0 = x_0)} \\
&= \frac{p^2_{x_1x_a} p^1_{x_ax_0} p^0_{x_0} p^2_{x_1x_b} p^1_{x_bx_0} p^0_{x_0}}{\sum_{\alpha \in S} p^2_{x_1\alpha} p^1_{\alpha x_0} p^0_{x_0}} \\
&= p^0_{x_0} \frac{p^2_{x_1x_a} p^1_{x_ax_0} p^2_{x_1x_b} p^1_{x_bx_0}}{\sum_{\alpha \in S} p^2_{x_1\alpha} p^1_{\alpha x_0}}.
\end{aligned}$$

To get $(*)$ we need to calculate

$$\begin{aligned}\mathbb{P}(X_a = x_a, X_b = x_b, X_0 = x_0) &= \sum_{\gamma_1 \in S} \mathbb{P}(X_1 = \gamma_1, X_a = x_a, X_b = x_b, X_0 = x_0) \\ &= p^0_{x_0} \sum_{\gamma_1 \in S} \frac{p^2_{\gamma_1 x_a} p^1_{x_a x_0} p^2_{\gamma_1 x_b} p^1_{x_b x_0}}{\sum_{\alpha \in S} p^2_{\gamma_1\alpha} p^1_{\alpha x_0}}\end{aligned}$$

Finally, we have

$$\begin{aligned}
&\mathbb{P}(X_1 = x_1 | X_a = x_a, X_b = x_b, X_0 = x_0) \\
&= \frac{\mathbb{P}(X_1 = x_1, X_a = x_a, X_b = x_b, X_0 = x_0)}{\mathbb{P}(X_a = x_a, X_b = x_b, X_0 = x_0)} \\
&= p^0_{x_0} \frac{p^2_{x_1 x_a} p^1_{x_a x_0} p^2_{x_1 x_b} p^1_{x_b x_0}}{\sum_{\alpha \in S} p^2_{x_1 \alpha} p^1_{\alpha x_0}} \left(p^0_{x_0} \sum_{\gamma_1 \in S} \frac{p^2_{\gamma_1 x_a} p^1_{x_a x_0} p^2_{\gamma_1 x_b} p^1_{x_b x_0}}{\sum_{\alpha \in S} p^2_{\gamma_1 \alpha} p^1_{\alpha x_0}} \right)^{-1} \\
&= \frac{p^2_{x_1 x_a} p^2_{x_1 x_b}}{\sum_{\alpha \in S} p^2_{x_1 \alpha} p^1_{\alpha x_0}} \left(\sum_{\gamma_1 \in S} \frac{p^2_{\gamma_1 x_a} p^2_{\gamma_1 x_b}}{\sum_{\alpha \in S} p^2_{\gamma_1 \alpha} p^1_{\alpha x_0}} \right)^{-1}.
\end{aligned}$$

The last shows that X_1 given X_a, X_b, X_0 depends on the value of X_0. If the graph-Markovian property holds this should not be so.

Simplifying our model to $S = \{0, 1\}$, and setting $p^0_0 = p^0_1 = 1/2$,and $p^1_{10} = p^2_{10} = 3/4$, and $p^1_{11} = p^2_{11} = 1/4$, we get that

$$\mathbb{P}(X_1 = 1 | X_a = 0, X_b = 1) = 1/2,$$

while

$$\mathbb{P}(X_1 = 1 | X_a = 0, X_b = 1, X_0 = 0) = 3/8.$$

Hence, the graph-Markovian property doesn't hold.

3.4.2 Construction problems on non-TLG TLG's*

Why the construction described in 2.3.1 (on page 48) can't work for all TLG's? As an example of Burdzy and Pal presented in [7] shows it may not be possible to construct such a process and have all the properties Markov processes on TLG*'s had.

Let's take a look at the example of a TLG that is not a TLG* given in Theorem 1.8.(i). $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{t_j = j/5 : j = 0, 1, \ldots, 5\}$ and

$$\mathcal{E} = \{E_{01}, E_{02}, E_{14}, E_{13}, E_{23}, E_{24}, E_{45}, E_{35}\}.$$

Let's take $\mathcal{P}$ to be Markov process on $[0, 1]$.

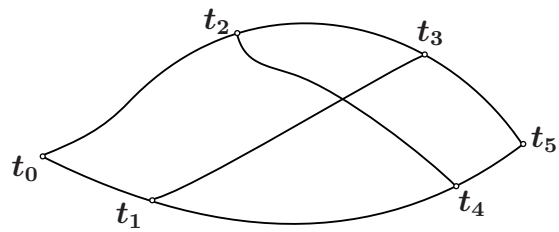


Figure 3.10: Example from Theorem 1.8.(i).

We will try to construct a process on $\mathcal{G}$, with a similar approach as in the construction of Markov processes on TLG* (see 2.3.1.)

Construction attempt

We first define the process on $\sigma(0,2,4,5)$ with distribution $\mathcal{P}$, we construct a $\mathcal{P}$-Markov bridge on $\sigma(2,3,5)$ between $(t_2, X(t_2))$ and $(t_5, X(t_5))$ which is independent of the rest of the process already defined given $X(t_2)$ and $X(t_5)$. Further, we construct a $\mathcal{P}$-Markov bridge between $(t_0, X(t_0))$ and $(t_5, X(t_5))$ on $\sigma(0,1,4)$ independent of the rest given $X(t_0)$ and $X(t_4)$. Finally, we construct a $\mathcal{P}$-Markov bridge between $(t_1, X(t_1))$ and $(t_3, X(t_3))$ on $\sigma(1,3)$ that is independent of everything already defined given $X(t_1)$ and $X(t_3)$.

The problem in this construction is in the last step. Since, at that time t_1 and t_3 are not connected, the process on the full time-path $\sigma(0,1,3,5)$ doesn't have to be $\mathcal{P}$-distributed.

We will prove this when $\mathcal{P}$ is Brownian motion on $[0,1]$. Then $X_{\sigma(2,3,5)}$ and $X_{\sigma(0,1,4)}$ are Brownian bridges. Using Theorem A.16. we can have

$$X(t_1) = \frac{t_1}{t_4}(X(t_4) - W(t_4)) + W(t_1), \tag{3.5}$$

$$X(t_3) = \frac{t_5 - t_3}{t_5 - t_2}(X(t_2) - B(t_2)) + B(t_3) + \frac{t_3 - t_2}{t_5 - t_2}(X(t_5) - B(t_5)), \tag{3.6}$$

where W, B, $X_{\sigma(0,2,4,5)}$ are independent Brownian motions on $[0,1]$. If $X_{\sigma(0,1,3,5)}$ is Brownian

motion on $[0,1]$ then $\mathbb{E}(X(t_1)X(t_3)) = t_1 = 1/5$, but in our case we get from (3.5) and (3.6):

$$\mathbb{E}(X(t_1)X(t_3)) = \frac{1}{3}.$$

Problems with cell-Markovian property

The other problem, that might occur, is that the processes we defined so far on TLG*'s have the cell-Markovian property (recall Definition 3.1.), while on this TLG there might not exist such a process.

Will show this, again, on the example when $\mathcal{P}$ is the distribution of Brownian motion on $[0,1]$.

Proposition 3.20. *If Z and Y be distributed as Brownian motion on $[s_1, s_2]$ such that $Z(s_j) = Y(s_j)$ for $j = 1, 2$ and Z and Y are independent given $Y(s_1)$ and $Y(s_2)$. Then for $\tau_1, \tau_2 \in [s_1, s_2]$ we have*

$$\mathbb{E}(Z(\tau_1)Y(\tau_2)) = s_1 + \frac{(\tau_1 - s_1)(\tau_2 - s_1)}{(s_1 - s_2)}.$$

Proof. We will use the representation given in Theorem A.16. Let Y be Brownian motion on $[s_1, s_2]$, and $(W(t) : t \geq 0)$ Brownian motion independent of Y. Then we can take Z to be

$$Z(t) = \frac{s_2 - t}{s_2 - s_1}(Y(s_1) - W(s_1)) + W(t) + \frac{t - s_1}{s_2 - s_1}(Y(s_2) - W(s_2)),$$

for $t \in [s_1, s_2]$. Now, we have

$$\begin{aligned}\mathbb{E}[Z(\tau_1)Y(\tau_2)] =& \frac{s_2 - \tau_1}{s_2 - s_1}\mathbb{E}[Y(\tau_2)(Y(s_1) - W(s_1))] + \mathbb{E}(Y(\tau_2)W(\tau_1)) \\ &+ \frac{\tau_1 - s_1}{s_2 - s_1}\mathbb{E}[Y(\tau_2)(Y(s_2) - W(s_2))] = \frac{s_2 - \tau_1}{s_2 - s_1}s_1 + \frac{\tau_1 - s_1}{s_2 - s_1}\tau_2.\end{aligned}$$

■

Theorem 3.21. *There doesn't exist a process X on $\mathcal{G}$ such that:*

- *X is cell-Markovian.*

- *For each full-time σ the process X_σ is distributed as Brownian motion on $[0,1]$.*

Proof. Assume otherwise. Note that cells $(\sigma(2,3,5), \sigma(2,4,5))$ and $(\sigma(1,4,5), \sigma(1,3,5))$ are simple.

Then $X_{\sigma(2,3,5)}$ and $X_{\sigma(2,4,5)}$ are distributed as Brownian motions on $[t_1, t_5]$, so using the cell-Markovian property of X, i.e. the fact that $X_{\sigma(2,3,5)}$ and $X_{\sigma(2,4,5)}$ are independent given $X(t_2)$ and $X(t_5)$ from Proposition 3.20. we have:

$$\mathbb{E}(X(t_3)X(t_4)) = \mathbb{E}(X_{\sigma(2,3,5)}(t_3)X_{\sigma(2,4,5)}(t_4)) = t_2 + \frac{(t_3 - t_2)(t_4 - t_2)}{(t_5 - t_2)} = \frac{8}{15}.$$

For $X_{\sigma(1,4,5)}$ and $X_{\sigma(1,3,5)}$ in a similar way we get:

$$\mathbb{E}(X(t_3)X(t_4)) = \mathbb{E}(X_{\sigma(1,3,5)}(t_3)X_{\sigma(1,4,5)}(t_4)) = t_1 + \frac{(t_3 - t_1)(t_4 - t_1)}{(t_5 - t_1)} = \frac{13}{10}.$$

This shows the claim. ■

3.5 Three simple examples

Through this section $\mathcal{G}$ is a graph consisting of one cell (see Figure 3.11):

$$\mathcal{G} = (\{0,1\}, \{E^1_{01}, E^2_{01}\}).$$

We define three functions $f_1, f_2, f_3 : [0,1] \to [0,1]$:

$$f_1(t) = t, \quad f_2(t) = t^2$$

$$f_3(t) = \begin{cases} 2x, & 0 \le x \le 1/3; \\ 1 - x, & 1/3 \le x \le 2/3; \\ 2x - 1, & 2/3 \le x \le 1. \end{cases}$$

Further, let $(B_t : t \in [0,1])$ be Brownian motion on $[0,1]$. For $k = 1,2,3$ we set μ_k to be the law of $(B_{f_k(t)} : t \in [0,1])$. Notice that μ_k for $k = 1,2,3$ are all laws of continuous

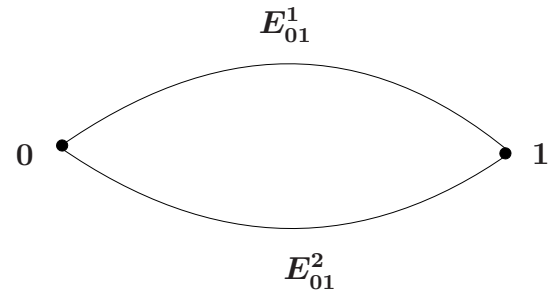


Figure 3.11: Graph $\mathcal{G}$

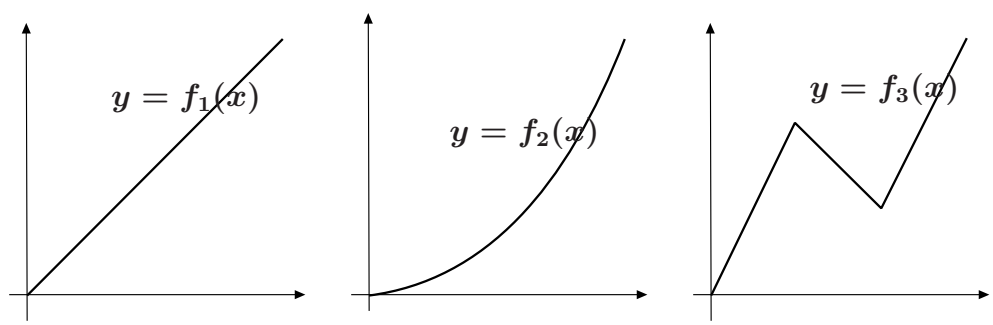


Figure 3.12: Graphs of f_1, f_2 and f_3

processes. Also μ_1 and μ_2 are laws of Markov processes, while μ_3 is not a law of a Markov process. Now we set

$$\mathcal{M}_1 := \{\mu_{E^1_{01}} = \mu_1, \mu_{E^2_{01}} = \mu_1\},$$

$$\mathcal{M}_2 := \{\mu_{E^1_{01}} = \mu_1, \mu_{E^2_{01}} = \mu_2\},$$

$$\mathcal{M}_3 := \{\mu_{E^1_{01}} = \mu_1, \mu_{E^2_{01}} = \mu_3\}.$$

Since $f_1(0) = f_2(0) = f_3(0) = 0$ and $f_1(1) = f_2(1) = f_3(1) = 1$, $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$ satisfy (3T) properties. Therefore we can construct a natural $\mathcal{M}_k$-process on $\mathcal{G}$ for each $k \in \{1, 2, 3\}$.

Now, note the following:

- $\mathcal{M}_1$ is a homogeneous Markov family.

- $\mathcal{M}_2$ is a Markov family that is not homogeneous.
- $\mathcal{M}_3$ is not a Markov family.

Chapter 4

FILTRATIONS, MARTINGALES AND STOPPING TIMES

Let's look at a simple example of process on a time-like graph.

Y a value two persons (1&2) are trying to estimate based on the information they are getting over time.

- The information they collect will be modeled as a filtration

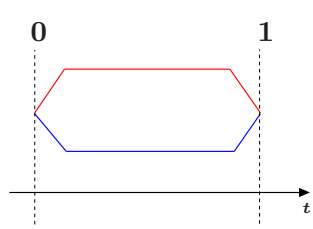
$$\{\mathcal{F}_t^1 : t \in [0,1]\} \quad \text{and} \quad \{\mathcal{F}_t^2 : t \in [0,1]\}.$$

- At $t = 0$ they start with the same information $\mathcal{F}_0^1 = \mathcal{F}_0^2$.

- At time $t = 1$ everything is known: $\mathcal{F}_1^1 = \mathcal{F}_1^2 = \mathcal{F} \supset \sigma(Y)$.

Set $X_t^1 = \mathbb{E}(Y|\mathcal{F}_t^1)$ and $X_t^2 = \mathbb{E}(Y|\mathcal{F}_t^2)$.

For a TLG $\mathcal{G} = (\{0,1\}, \{E_{01}^1, E_{01}^2\})$, we can define $X = (X(t) : t \in \mathcal{G})$ to be given by $X_{E_{01}^1} = X^1$ and $X_{E_{01}^2} = X^2$. In this way the process is well defined.



X will be a martingale indexed by a TLG, and we will talk more about martingales in this chapter. We will also show some results on the right-continuity of filtrations, define stopping times and prove the Optional Sampling Theorem for this class of processes.

4.1 Expanding the filtrations

The following will state some equivalent forms of the time-Markovian property.

Corollary 4.1. *Let X be a process on a TLG* $\mathcal{G}$. The following are equivalent: we have:*

(a) X is time-Markovian on $\mathcal{G}$

(b) For any point

$$\mathcal{F}_t = \sigma(X(u) : u \preceq t), \text{ and } \mathcal{G}_t = \sigma(X(t) : u \succeq t)$$

are conditionally independent given $X(t)$. ('$\preceq$' is the order induced by $\mathcal{G}$.)

(c) If $Y \in b\mathcal{G}_t$, then we have

$$\mathbb{E}(Y|\mathcal{F}_t) = \mathbb{E}(Y|X(t)). \tag{4.1}$$

The main result in this section will be to show under which conditions we can expand the σ-algebra $\mathcal{F}_t$ so that the relation (4.1) still holds. The main idea is to choose the filtration that is right continuous.

Definition 4.2. *Let X be a process on a TLG $\mathcal{G}$*

- *Set $\mathcal{F}_t^0 = \sigma(X(u) : u \preceq t)$ and $\mathcal{G}_t^0 = \sigma(X(u) : u \succeq t)$. If not specified otherwise $\mathcal{F}_t = \mathcal{F}_t^0$ and $\mathcal{G}_t = \mathcal{G}_t^0$.*

- *For each $\pi \in P_{0\to 1}(\mathcal{G})$ and $t \in [0,1]$ we define*

$$\mathcal{F}_{t+}^{\pi} := \bigcap_{t \prec s, s \in \pi} \mathcal{F}_s. \tag{4.2}$$

Definition 4.3. *For the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{G}$ a sub-σ-algebra of $\mathcal{F}$ we will denote*

$$\mathcal{N}^{\mathbb{P}} = \{A \subset \Omega : (\exists B \in \mathcal{G})(A \subset B)(\mathbb{P}(B) = 0)\}.$$

$$\mathcal{G}^{\mathbb{P}} = \sigma(\mathcal{G} \cup \mathcal{N}^{\mathbb{P}}). \tag{4.3}$$

Lemma 4.4. *For $\mathcal{G}^{\mathbb{P}}$ defined by* (4.3) *the following holds*

$$\mathcal{G}^{\mathbb{P}} = \{A \subset \Omega : (\exists B \in \mathcal{G})(A \triangle B \in \mathcal{N}^{\mathbb{P}})\}.$$

Lemma 4.5. *For the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{G}_1$ and $\mathcal{G}_2$ sub-σ-algebras of $\mathcal{F}$ the following are equivalent:*

(i) $\mathcal{G}_1^{\mathbb{P}} = \mathcal{G}_2^{\mathbb{P}}$;

(ii) For each $Y \in L^1(\Omega, \mathcal{F}, \mathbb{P})$

$$\mathbb{E}(Y|\mathcal{G}_1) = \mathbb{E}(Y|\mathcal{G}_2) \ a.s.$$

(iii) For each $A \in \mathcal{F}$

$$\mathbb{P}(A|\mathcal{G}_1) = \mathbb{P}(A|\mathcal{G}_2) \ a.s.$$

Proof. (i)⇒(ii): Let $A \in \mathcal{G}_1$. Since $\mathcal{G}_1 \subset \mathcal{G}_1^{\mathbb{P}} = \mathcal{G}_2^{\mathbb{P}}$, there exists $B \in \mathcal{G}_2$ and N_1, N_2 $\mathbb{P}$-null sets such that $A \cup N_1 = B \cup N_2$. Now, for $Y \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ we have

$$\begin{aligned}
\int_A \mathbb{E}(Y|\mathcal{G}_2)\, d\mathbb{P} &= \int_{A \cup N_1} \mathbb{E}(Y|\mathcal{G}_2)\, d\mathbb{P} = \int_{B \cup N_2} \mathbb{E}(Y|\mathcal{G}_2)\, d\mathbb{P} \\
&= \int_B \mathbb{E}(Y|\mathcal{G}_2)\, d\mathbb{P} = \int_B Y\, d\mathbb{P} = \int_{B \cup N_2} Y\, d\mathbb{P} \\
&= \int_{A \cup N_1} Y\, d\mathbb{P} = \int_A Y\, d\mathbb{P} = \int_A \mathbb{E}(Y|\mathcal{G}_1)\, d\mathbb{P}
\end{aligned}$$

Since this holds for all $A \in \mathcal{G}_1$ the claim follows.

(ii)⇒(iii): This is clear.

(iii)⇒(i): Let $A \in \mathcal{G}_1$, then

$$\mathbf{1}_A = \mathbb{P}(A|\mathcal{G}_1) = \mathbb{P}(A|\mathcal{G}_2) \quad a.s.$$

Since, $\mathbb{P}(A|\mathcal{G}_2)$ is $\mathcal{G}_2$-measurable, hence $\mathbf{1}_A$ is $\mathcal{G}_2^{\mathbb{P}}$ measurable. Therefore, $\mathcal{G}_1 \subset \mathcal{G}_2^{\mathbb{P}}$, and we have $\mathcal{G}_1^{\mathbb{P}} \subset \mathcal{G}_2^{\mathbb{P}}$. By symmetry $\mathcal{G}_2^{\mathbb{P}} \subset \mathcal{G}_1^{\mathbb{P}}$, and the claim follows. ■

Theorem 4.6. *Let $\mathcal{M}$ be the (3T)-family, and X a natural $\mathcal{M}$ process on a TLG* $\mathcal{G}$ such that for each $\pi \in P_{0\to 1}(\mathcal{G})$ the process X_π is Markov with respect to the $(\mathcal{F}^\pi_{t+} : t \in [0,1])$ (recall (4.2)). Then*

$$\{\mathcal{F}^{\mathbb{P}}_t : t \in \mathcal{G}\}$$

is a right-continuous filtration, that is

$$\mathcal{F}^{\mathbb{P}}_t = \bigcap_{t \prec s} \mathcal{F}^{\mathbb{P}}_s.$$

<u>*Proof.*</u> Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space on which X is defined. We pick $t \in \mathcal{G}$. Now, we pick a path π that contains t, and let $E_{k_1 k_2}$ be the edge that is contained in π such that $t_{k_1} \le t < t_{k_2}$. To prove that that at t the filtration is right continuous we will restrict our probability space to $(\Omega, \mathcal{F}', \mathbb{P}' = \mathbb{P}|_{\mathcal{F}'})$ where

$$\mathcal{F}' = \sigma(\mathcal{F}_{t_{k_2}} \cup \mathcal{G}_{t_{k_1}} \cup \mathcal{N}^{\mathbb{P}}).$$

.

With $F \in \mathcal{F}_t$ and $G \in \mathcal{G}_t$ we have

$$\mathbb{P}(F \cap G|\mathcal{F}^\pi_{t+}) = \mathbf{1}_F \mathbb{P}(G|\mathcal{F}^\pi_{t+}) = \mathbf{1}_F \mathbb{P}(G|X(t)) = \mathbf{1}_F \mathbb{P}(G|\mathcal{F}^\pi_t) = \mathbb{P}(F \cap G|\mathcal{F}^\pi_t).$$

Using the monotone class theorem we have that for all $A \in \mathcal{F}'$

$$\mathbb{P}(A|\mathcal{F}^\pi_t) = \mathbb{P}(A|\mathcal{F}^\pi_{t+}). \tag{4.4}$$

Since $\mathcal{N}^{\mathbb{P}'} = \mathcal{N}^{\mathbb{P}}$, we have by Lemma 4.5 (iii) that

$$\mathcal{F}^{\pi,\mathbb{P}}_t = \mathcal{F}^{\pi,\mathbb{P}}_{t+}.$$

Further, note that $\mathcal{F}^\pi_{t+} \subset \mathcal{F}^{\pi,\mathbb{P}}_t$.

Now, let

$$A \in \bigcap_{t<s} \mathcal{F}^{\pi,\mathbb{P}}_s = \bigcap_{n=1}^{\infty} \mathcal{F}^{\pi,\mathbb{P}}_{t+1/n}.$$

Hence, we have $A \in \mathcal{F}^{\pi,\mathbb{P}}_{t+1/n}$, then there exists $B_n \in \mathcal{F}^{\pi}_{t+1/n}$ such that $A \triangle B_n \in \mathcal{N}^{\mathbb{P}}$. Set

$$B := \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} B_m = \bigcap_{n=M}^{\infty} \bigcup_{m=n}^{\infty} B_m \in \mathcal{F}^{\pi}_{t+1/M},$$

hence $B \in \mathcal{F}^{\pi}_{t+}$, hence $B \in \mathcal{F}^{\pi,\mathbb{P}}_{t}$. Now, we can show that

$$B \setminus A \subset \left(\bigcup_{n=1}^{\infty} B_n \right) \setminus A = \bigcup_{n=1}^{\infty} (B_n \setminus A) \in \mathcal{N}^{\mathbb{P}}.$$

$$A \setminus B = A \cap B^c = A \cap \left(\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} B_m \right)^c = \bigcup_{n=1}^{\infty} A \cap \left(\bigcap_{m=n}^{\infty} B_m^c \right) \subset$$

$$\subset \bigcup_{n=1}^{\infty} A \cap B_m^c = \bigcup_{n=1}^{\infty} (A \setminus B_m) \in \mathcal{N}^{\mathbb{P}}.$$

This implies that $A \in \mathcal{F}^{\pi,\mathbb{P}}_{t}$, and the filtration $(\mathcal{F}^{\pi,\mathbb{P}}_{s} : s \in [0,1])$ is right-continuous at t, and to prove the claim we should note that $\mathcal{F}^{\pi}_{t} = \mathcal{F}_t$, hence this implies $(\mathcal{F}^{\mathbb{P}}_{s} : s \in \pi)$ is right-continuous at t, but since π is an arbitrary path that contains t the claim follows, since there is only finitely many such paths. Hence

$$\mathcal{F}^{\mathbb{P}}_{t} = \bigcap_{\pi : \pi \ni t} \mathcal{F}^{\pi,\mathbb{P}}_{t} = \bigcap_{\pi : \pi \ni t} \bigcap_{t < s} \mathcal{F}^{\pi,\mathbb{P}}_{s} = \bigcap_{t \prec s} \mathcal{F}^{\mathbb{P}}_{s}.$$

■

It turns out that the condition from the previous theorem is satisfied by the natural Brownian motion. Before we prove that we need the following lemma.

Lemma 4.7. *Let $\mathcal{G}$ be a TLG* and X the natural Brownian motion on $\mathcal{G}$. For $t \in \mathcal{G}$ we have that the processes*

$$(X(s) : s \preceq t) \quad \textit{and} \quad (X(s) - X(t) : s \succeq t)$$

are independent.

<u>*Proof.*</u> Pick a full time-path $\pi \in P_{0\to 1}(\mathcal{G})$ such that $t \in \pi$. We pick a TLG* tower $(\mathcal{G}_j)_{j=0}^{n}$ where $\mathcal{G}_0$ has the same representation as π and $\mathcal{G}_n = \mathcal{G}$.

Let $X^j := (X(s) : s \in \mathcal{G}_j)$ be the natural Brownian motion indexed by $\mathcal{G}_j$. By induction we will show that

$$(X^j(s) : s \in P^j(t)) \quad \textit{and} \quad (X^j(s) - X^j(t) : s \in F^j(t)) \tag{4.5}$$

It is clear that (4.5) holds for $j = 0$. Let's assume that it holds for $j = h \geq 0$. Let's show the claim for $j = h + 1$.

If a new edge not in $P^{h+1}(t)$ and not in $F^{h+1}(t)$ has been added to $\mathcal{G}_h$ to construct $\mathcal{G}_{h+1}$, then the processes in (4.5) are the same for $j = h$ and $j = h + 1$, and the claim follows.

If a new edge $E_{k_1k_2}$ in $F^{h+1}(t)$ has been added to $\mathcal{G}_h$ to obtain $\mathcal{G}_{h+1}$. Then since for $s \in E_{k_1k_2}$ we have

$$X^{h+1}(s) - X(t) = \frac{t_{k_2} - s}{t_{k_2} - t_{k_1}}(X^h(t_{k_2}) - X(t)) + \frac{s - t_{k_1}}{t_{k_2} - t_{k_1}}(X^h(t_{k_1}) - X(t)) + B^{br}_{k_1k_2}(s),$$

where $B^{br}_{k_1k_2}$ is a Brownian bridge independent of X^h. Hence, both $(X^h(s) - X(t) : s \in F^h(t))$ and $(X_{E_{k_1k_2}}(s) - X(t) : s \in E_{k_1k_2})$ are independent pointwise of $(X^h(s) : s \in P^j(t))$, and (4.5) follows for $j = h + 1$. ■

Theorem 4.8. *Let $\mathcal{G}$ be a TLG* and X the natural Brownian motion on $\mathcal{G}$. For $t \in \mathcal{G}$ and $\pi \in P_{0\to 1}(\mathcal{G})$ such that $t \in \pi$ we have that*

$$\mathcal{F}^{\pi}_{t+} \quad \textit{and} \quad (X_\pi(s) - X_\pi(t) : s \geq t))$$

are independent. (See Figure 4.1.)

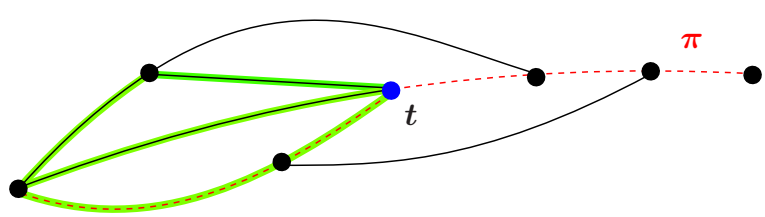


Figure 4.1: Illustration of Theorem 4.8.

Proof. Let $n \in \mathbb{N}$, and $A \in \mathcal{F}^{\pi}_{t+}$ and $t \prec s_1, s_2, \ldots, s_n \in \pi$. For small $\varepsilon > 0$ we know that $Y := \mathbf{1}_A \in b\mathcal{F}_{t+\varepsilon/2}$ and $\Delta^\varepsilon := (X_\pi(s_1) - X_\pi(t+\varepsilon), \ldots X_\pi(s_n) - X_\pi(t+\varepsilon))$ are independent. Now using the characteristic functions $\varphi_Y(t) = \mathbb{E}(\exp(itY))$ and $\varphi_{\Delta^\varepsilon}(\mathbf{t}) = \mathbb{E}(\exp(i\mathbf{t} \cdot \Delta^\varepsilon))$ we have

$$\varphi_{Y,\Delta^\varepsilon}(t, \mathbf{t}) = \varphi_Y(t)\varphi_{\Delta^\varepsilon}(\mathbf{t}) \tag{4.6}$$

Continuity of X gives us $\lim_{\varepsilon \downarrow 0}(Y, \Delta^\varepsilon) = (Y, \Delta^0)$ a.s. Hence, from (4.6) we have

$$\varphi_{Y,\Delta^0}(t, \mathbf{t}) = \varphi_Y(t)\varphi_{\Delta^0}(\mathbf{t}).$$

Therefore, $\mathbf{1}_A$ and $(X_\pi(s_1) - X_\pi(t), \ldots X_\pi(s_n) - X_\pi(t))$ are independent. ■

Corollary 4.9. *For the natural Brownian motion X on the TLG* $\mathcal{G}$ the following claims hold:*

(a) The filtration $(\mathcal{F}^{\mathbb{P}}_t : t \in \mathcal{G})$ is right continuous.

(b) $\mathcal{F}^{\mathbb{P}}_t$ and $(X(s) - X(t) : s \in F(t))$ are independent.

(c) For $t \prec s$ we have $\mathbb{E}(X(s)|\mathcal{F}^{\mathbb{P}}_t) = X(t)$.

(d) For $t \prec s$ and $Y \in b\mathcal{G}_s$ we have

$$\mathbb{E}(Y|\mathcal{F}^{\mathbb{P}}_t) = \mathbb{E}(Y|X(t)).$$

4.2 Markov martingales

Here we will show that under some conditions we can get a martingale property for the process defined on a TLG*.

Definition 4.10. *The Markov family of measures*

$$\mathcal{M} = \{\mu_\sigma : \sigma \in P_{0\to 1}(\mathcal{G})\}$$

will be called a ***Markov martingale family*** *if for each μ_σ-distributed process $(X_\sigma(t) \in [0,1])$, we have*

- $\mathbb{E}|X_\sigma(t)| < \infty$;
- $\mathbb{E}(X_\sigma(t)|(X_\sigma(u) : u \in [0, s])) = X_\sigma(s)$.

Theorem 4.11. *Let $\mathcal{M}$ be a Markov martingale family, and X an $\mathcal{M}$-process on a TLG* $\mathcal{G}$. Then we have*

$$\mathbb{E}(X(t)|(X(u) : u \preceq s)) = X(s), \tag{4.7}$$

for all points $s \preceq t$ in $\mathcal{G}$.

Proof. First from the time-Markovian property we have that

$$\mathbb{E}(\varphi_M(X(t))|(X(u) : u \preceq s)) = \mathbb{E}(\varphi_M(X_t)|X_s),$$

where $\varphi_M(x) = \begin{cases} x, & |x| < M, \\ M, & |x| \geq M. \end{cases}$. Using the dominated convergence theorem when $M \to \infty$ we have

$$\mathbb{E}(X(t)|(X(u) : u \preceq s)) = \mathbb{E}(X(t)|X(s)).$$

Now, we pick a full time-path σ such that t and s are on it, and we get

$$\mathbb{E}(X(t)|(X(u) : u \preceq s)) = \mathbb{E}(X_\sigma(t)|X_\sigma(s)) = X_\sigma(s) = X(s).$$

■

The following is a consequence of Lemma 4.5 (ii).

Corollary 4.12. *Let $\mathcal{M}$ be a Markov martingale family, and X an $\mathcal{M}$-process on a TLG* $\mathcal{G}$. Then we have*

$$\mathbb{E}(X(t)|\mathcal{F}_s^{\mathbb{P}}) = X(s), \tag{4.8}$$

for all points $s \preceq t$ in $\mathcal{G}$.

The equality (4.7) says that X defined in Theorem 4.11. is an example of a **martingale indexed by directed set $\mathcal{G}$**. These types of martingales have been investigated and there are a lot of results including the optional sampling theorem. We will talk more about this in section 4.3.

4.2.1 Example of glued diffusions

In this subsection we give an example of a general non-homogeneous Markov martingale family $\mathcal{M}$.

Definition 4.13. *The family of functions*

$$f_{\mathcal{G}} = \{f_\sigma : [0,1] \to \mathbb{R} : \sigma \in P_{0\to 1}(\mathcal{G})\}$$

is called consistent on the TLG $\mathcal{G}$ if for $\sigma_1, \sigma_2 \in P_{0\to 1}(\mathcal{G})$

$$f_{\sigma_1}|_T = f_{\sigma_2}|_T$$

where $T = \{t \ : \ t \in E, E \in \sigma_1 \,\&\, E \in \sigma_2\}$.

Theorem 4.14. *Let $F_{\mathcal{G}} = \{F_\sigma : [0,1] \to \mathbb{R} : \sigma \in P_{0\to 1}(\mathcal{G})\}$ be a consistent family of absolutely continuous functions. Then there exists a consistent family $f_{\mathcal{G}} = \{f_\sigma : [0,1] \to \mathbb{R} : \sigma \in P_{0\to 1}(\mathcal{G})\}$ of densities of $F_{\mathcal{G}}$, that is for all $\sigma \in P_{0\to 1}(\mathcal{G})$ and all $t \in [0,1]$*

$$F_\sigma(t) - F_\sigma(0) = \int_0^t f_\sigma(s)\, ds.$$

Proof. Let σ_1 and σ_2 be full time-paths. Then

$$T_{12} = \{t : t \in E, E \in \sigma_1 \& E \in \sigma_2\}$$

is a finite union of closed segments. For each $a < b$ such that $(a,b) \subset T_{12}$ we have

$$\int_a^b f_{\sigma_1}(s)\, ds = \int_a^b f_{\sigma_2}(s)\, ds,$$

so therefore $f_{\sigma_1} = f_{\sigma_2}$ λ-almost everywhere on T_{12}.

Assume $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, for each edge $E_{kj} \in \mathcal{E}$ choose some fixed full time-path σ^* containing E_{jk}. For each full time-path σ containing that edge we can fix f_σ on (t_j, t_k), to be some density of the function $t \mapsto F_{\sigma^*}(t) - F_{\sigma^*}(t_j)$ defined on (t_j, t_k).

Since there are only at most countably many vertices (in this case finitely many) the values at the vertices won't influence the values of the integrals, hence we can set the values at vertices to be any real numbers. Now, we have constructed a consistent family. ■

Let $\mathcal{G}$ be a TLG* and $V : R(\mathcal{G}) \to \mathbb{R}^+$ be a positive function, such that for each full time-path σ the restriction of V along $R(\sigma)$ $V_\sigma : [0,1] \to \mathbb{R}^+$ is an increasing continuous function.

From the theory of functions of bounded variation, we know that there exists a positive function f_σ in $L^1[0,1]$, such that

$$V_\sigma(t) = \int_0^t f_\sigma(s)\, ds,$$

for all $t \in [0,1]$. By Theorem 4.14. we can assume that

$$\{f_\sigma : [0,1] \to \mathbb{R} : \sigma \in P_{0\to 1}(\mathcal{G})\}$$

is a consistent family of densities.

For $\sigma \in P_{0\to 1}(\mathcal{G})$ let μ_σ be the distribution of the process $(N(t) : t \in [0,1])$, given by the formula

$$N_\sigma(t) := \int_0^t \sqrt{f_\sigma(s)}\, dB_s,$$

for $t \in [0,1]$. (This is an Ito integral with respect to the Brownian motion (B_t).) This is well defined since $\sqrt{f_\sigma} \in L^2[0,1]$.

Clearly, N_σ is a Markov process with zero expectation on $[0,1]$. The variance is

$$\mathbb{E}(N_\sigma^2(t)) = \mathbb{E}\left(\int_0^t \sqrt{f_\sigma(s)}\, dB_s\right)^2 = \int_0^t f_\sigma(s)\, ds = V_\sigma(t).$$

We will show that $\{\mu_\sigma : \sigma \in P_{0\to 1}(\mathcal{G})\}$ is a consistent family. Again, let σ_1 and σ_2 be two full time-paths, and T_{12} as before. Let $\tau_1 \le \tau_2$ be from T_{12}. We have

$$\begin{aligned}
\mathbb{E}(N_{\sigma_1}(\tau_1)N_{\sigma_1}(\tau_2)) &= \mathbb{E}(N_{\sigma_1}(\tau_1)[(N_{\sigma_1}(\tau_2) - N_{\sigma_1}(\tau_1)) + N_{\sigma_1}(\tau_1)]) \\
&= V_{\sigma_1}(\tau_1) = V_{\sigma_2}(\tau_1) \\
&= \mathbb{E}(N_{\sigma_2}(\tau_1)N_{\sigma_2}(\tau_2)).
\end{aligned}$$

Since, the covariance structure of the Gaussian processes N_{σ_1} and N_{σ_2} on T_{12} is the same, we have that the finite dimensional distributions on T_{12} are the same. Hence, by Kolmogorov's Existence Theorem we have that their distributions on T_{12} are the same. Therefore $\{\mu_\sigma : \sigma \in P_{0\to 1}(\mathcal{G})\}$ is a consistent Markov martingale family.

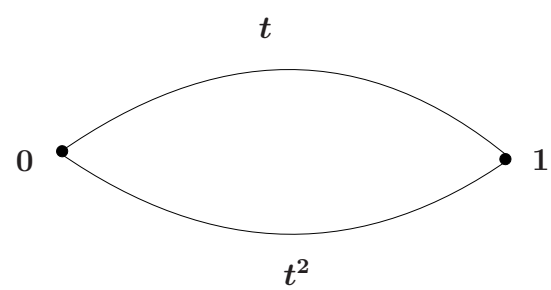


Figure 4.2: Graph $\mathcal{G}$

Let

$$\mathcal{G} = (\{0,1\}, \{E^1_{01}, E^2_{01}\}).$$

If we define V as

$$V(t) = \begin{cases} t & \text{for } t \in E^1_{01}, \\ t^2 & \text{for } t \in E^2_{01}; \end{cases}$$

Brownian motion runs along E^1_{01}, while $N(t) = \int_0^t s\, dB_s$ runs along E^2_{01}. (See Figure 4.2.)

Glued diffusions have several nice properties. Since along each path the distribution is inducing a martingale and a Markov process the whole process is a martingale and a Markov process indexed by the underlying TLG*.

Further, we have the following property which is a generalization of the Lemma 4.7.

Lemma 4.15. *Let X be a natural glued diffusion on a TLG* $\mathcal{G}$. Then for each $t \in \mathcal{G}$*

$$(X(s) : s \preceq t) \quad \textit{and} \quad (X(s) - X(t) : t \preceq s)$$

are independent.

Proof. Pick a full time-path $\pi \in P_{0\to 1}(\mathcal{G})$ such that $t \in \pi$. We pick a TLG* tower $(\mathcal{G}_j)_{j=0}^n$ where $\mathcal{G}_0$ has the same representation as π and $\mathcal{G}_n = \mathcal{G}$.

Let $X^j := (X(s) : s \in \mathcal{G}_j)$ be the natural natural glued diffusion indexed by $\mathcal{G}_j$. By induction we will show that

$$(X^j(s) : s \in P^j(t)) \quad \textit{and} \quad (X^j(s) - X^j(t) : s \in F^j(t)) \tag{4.9}$$

It is clear that (4.9) holds for $j = 0$. Let's assume that it holds for $j = h \geq 0$. Let's show the claim for $j = h + 1$.

If a new edge not in $P^{h+1}(t)$ and not in $F^{h+1}(t)$ has been added to $\mathcal{G}_h$ to construct $\mathcal{G}_{h+1}$, then the processes in (4.9) are the same for $j = h$ and $j = h + 1$, and the claim follows.

If a new edge $E_{k_1k_2}$ in $F^{h+1}(t)$ has been added to $\mathcal{G}_h$ to obtain $\mathcal{G}_{h+1}$. Then since for $s \in E_{k_1k_2}$ we have

$$X^{h+1}(s) - X(t) = \frac{V(t_{k_2}) - V(s)}{V(t_{k_2}) - V(t_{k_1})}(X^h(t_{k_2}) - X(t)) + \frac{V(s) - V(t_{k_1})}{V(t_{k_2}) - V(t_{k_1})}(X^h(t_{k_1}) - X(t)) + N_{0,0}^{t_{k_1}t_{k_2}}(s),$$

where $N_{0,0}^{t_{k_1}t_{k_2}}(s)$ (see Corollary A.17.) is a diffusion bridge independent of X^h. Hence, both $(X^h(s) - X(t) : s \in F^h(t))$ and $(X_{E_{k_1k_2}}(s) - X(t) : s \in E_{k_1k_2})$ are independent pointwise of $(X^h(s) : s \in P^j(t))$, and (4.9) follows for $j = h + 1$. ■

Theorem 4.16. *Let $\mathcal{G}$ be a TLG* and X the natural glued diffusion on $\mathcal{G}$. For $t \in \mathcal{G}$ and $\pi \in P_{0\to 1}(\mathcal{G})$ such that $t \in \pi$ we have that*

$$\mathcal{F}_{t+}^{\pi} \quad \textit{and} \quad (X_\pi(s) - X_\pi(t) : s \geq t)$$

are independent.

Proof. Let $n \in \mathbb{N}$, and $A \in \mathcal{F}_{\pi,t+}$ and $t \prec s_1, s_2, \ldots, s_n \in \pi$. For small $\varepsilon > 0$ we know that $Y := \mathbf{1}_A \in b\mathcal{F}_{t+\varepsilon/2}$ and $\Delta^\varepsilon := (X_\pi(s_1) - X_\pi(t+\varepsilon), \ldots, X_\pi(s_n) - X_\pi(t+\varepsilon))$ are independent. Now using the characteristic functions $\varphi_Y(t) = \mathbb{E}(\exp(itY))$ and $\varphi_{\Delta^\varepsilon}(\mathbf{t}) = \mathbb{E}(\exp(i\mathbf{t} \cdot \Delta^\varepsilon))$ we have

$$\varphi_{Y,\Delta^\varepsilon}(t, \mathbf{t}) = \varphi_Y(t)\varphi_{\Delta^\varepsilon}(\mathbf{t}) \tag{4.10}$$

Continuity of X gives us $\lim_{\varepsilon\downarrow 0}(Y,\Delta^\varepsilon) = (Y,\Delta^0)$ a.s. Hence, from (4.10) we have

$$\varphi_{Y,\Delta^0}(t,\mathbf{t}) = \varphi_Y(t)\varphi_{\Delta^0}(\mathbf{t}).$$

Therefore, $\mathbf{1}_A$ and $(X_\pi(s_1) - X_\pi(t), \ldots, X_\pi(s_n) - X_\pi(t))$ are independent. ■

Corollary 4.17. *For the natural glued diffusion X on the TLG* $\mathcal{G}$ the following claims hold:*

(a) The filtration $(\mathcal{F}_t^{\mathbb{P}} : t \in \mathcal{G})$ is right continuous.

(b) $\mathcal{F}_t^{\mathbb{P}}$ and $(X(s) - X(t) : s \in F(t))$ are independent.

(c) For $t \prec s$ we have $\mathbb{E}(X(s)|\mathcal{F}_t^{\mathbb{P}}) = X(t)$.

(d) For $t \prec s$ and $Y \in b\mathcal{G}_s$ we have

$$\mathbb{E}(Y|\mathcal{F}_t^{\mathbb{P}}) = \mathbb{E}(Y|X(t)).$$

4.3 Optional sampling theorem for martingales indexed by directed sets

In his paper [36] Kurtz defined stopping times for martingales on *directed sets*. The way they are defined, TLG's are directed sets. We will state some of the results obtained by Kurtz and apply them to the processes on TLG's.

Let $\mathcal{S}$ be a **directed set** with partial ordering denoted by $t \prec s$. That is, $\mathcal{S}$ is partially ordered and for $t_1, t_2 \in \mathcal{S}$ there exists $t_3 \in \mathcal{S}$ such that $t_1 \prec t_3$ and $t_2 \prec t_3$.
Remark. Note that TLG's satisfy this definition.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $(\mathcal{F}_t)_{t\in\mathcal{S}}$ be a filtration indexed by $\mathcal{S}$, that is

- $(\mathcal{F}_t)_{t\in\mathcal{S}}$ is a family of sub-σ-algebras of $\mathcal{F}$;
- $t \prec s$ implies $\mathcal{F}_t \subset \mathcal{F}_s$.

A stochastic process X indexed by $\mathcal{S}$ is a **martingale** with respect to $(\mathcal{F}_t)_{t\in\mathcal{S}}$ if

$$\mathbb{E}(X(t)|\mathcal{F}_s) = X(s),$$

for all $s \preceq t$.

A $\mathcal{S}$-valued random variable T is a **stopping time** if $(T \preceq t) \in \mathcal{F}_t$ for all $t \in \mathcal{S}$.

As usual we define

$$\mathcal{F}_T = \{A \in \mathcal{F} : A \cap (T \preceq t) \in \mathcal{F}_t, \forall t \in \mathcal{S}\}.$$

The following is the first form of the optional stopping theorem.

Lemma 4.18. *Let $X(t)$ be martingale and let $T_1 \preceq T_2$ be stopping times assuming countably many values. If there exists a sequence (t_m) in $\mathcal{S}$ such that*

$$\lim_{m\to\infty} \mathbb{P}(T_2 \preceq t_m) = 1, \tag{4.11}$$

and

$$\lim_{m\to\infty} \mathbb{E}(|X(t_m)|\mathbf{1}_{(T_2\preceq t_m)^c}) = 0, \tag{4.12}$$

and $\mathbb{E}(|X(T_2)|) < \infty$, then

$$\mathbb{E}(X(T_2)|\mathcal{F}_{T_1}) = X(T_1).$$

Remark. In a TLG with a finite number of vertices, we could pick the sequence $t_m = 1$. In that case conditions (4.11) and (4.12) would be automatically satisfied.

In order to extend the result of Lemma 4.18 to general stopping times we need to make some assumptions about the index set $\mathcal{S}$ and the process X. The assumption we make on $\mathcal{S}$ is that it is a topological lattice.

Recall the Definition 1.33. of a topological lattice from Section 1.6:

A Hausdorff space X with some order '$\leq$' is called a **topological lattice** if for $x_1, x_2 \in X$:

- there exists a unique element $x_1 \wedge x_2$ such that

$$\{x \in X : x \leq x_1\} \cap \{x \in X : x \leq x_2\} = \{x \in X : x \leq x_1 \wedge x_2\};$$

- there exists a unique element $x_1 \vee x_2$ such that

$$\{x \in X : x \geq x_1\} \cap \{x \in X : x \geq x_2\} = \{x \in X : x \geq x_1 \vee x_2\}.$$

and $x_1 \wedge x_2$ and $x_1 \vee x_2$ are continuous mappings of $X \times X$ (with product topology) onto X.

If $\mathcal{S}$ is a topological lattice, note that this implies that the sets of the form $[t_1, t_2] = \{t : t_1 \preceq t \preceq t_2\}$ (intervals) are closed, and hence Borel measurable.

Definition 4.19. *We will say that a* ***topological lattice $\mathcal{S}$ is separable from above*** *if there exists a* ***separating sequence*** $\{t_k\} \subset \mathcal{S}$, *such that all* $t \in \mathcal{S}$ *we have*

$$t = \lim_{n\to\infty} t^{(n)}$$

where

$$t^{(n)} := \min\{t_k : k \leq n, t_k \succeq t\}. \tag{4.13}$$

In Section 1.6 (see Theorem 1.34) we have shown that TLG* $\mathcal{G}$ is a topological lattice, and clearly we can set $\{t_k\}$ to be the set of points with rational times.

The following is the main result for the martingales on directed sets.

Theorem 4.20. *Let $\mathcal{S}$ be separable from above with separating set $\{t_k\}$, $\mathcal{F}_t = \bigcap_{n=1}^{\infty} \mathcal{F}_{t^{(n)}}$ for all t, and let $X(t)$ be a martingale satisfying*

$$\lim_{n\to\infty} X(t^{(n)}, \omega) = X(t, \omega),$$

for all (t, ω) for which the limit exists. Let $T_1 \preceq T_2$ be $\mathcal{S}$-valued stopping times. Suppose there exists a sequence (s_m) in $\{t_k\}$ such that

$$\lim_{m\to\infty} \mathbb{P}(T_2 \preceq s_m) = 1,$$

and

$$\lim_{m\to\infty} \mathbb{E}(|X(s_m)|\mathbf{1}_{(T_2\preceq s_m)^c}) = 0,$$

and that $\mathbb{E}(|X(T_1)|) < \infty$. *Then*

$$\mathbb{E}(X(T_2)|\mathcal{F}_{T_1}) = X(T_1).$$

The following theorem will translate the results we have into the ones of the process indexed by time-like graphs.

Theorem 4.21. *Let* $\mathcal{G}$ *be a* TLG^*.

(a) Let $X(t)$ *be a martingale with respect to the filtration* $(\mathcal{F}_t)_{t\in\mathcal{G}}$ *and let* $T_1 \preceq T_2$ *be stopping times assuming countably many values. If* $\mathbb{E}(|X(T_2)|) < \infty$ *then*

$$\mathbb{E}(X(T_2)|\mathcal{F}_{T_1}) = X(T_1).$$

(b) Let $X(t)$ *be a RCLL martingale with respect to the filtration* $(\mathcal{F}_t)_{t\in\mathcal{G}}$ *such that*

$$\mathcal{F}_t = \bigcap_{t\prec s} \mathcal{F}_s. \tag{4.14}$$

For stopping times $T_1 \preceq T_2$, *if* $\mathbb{E}(|X(T_2)|) < \infty$ *then*

$$\mathbb{E}(X(T_2)|\mathcal{F}_{T_1}) = X(T_1).$$

The key problem will be choosing a good filtration $(\mathcal{F}_t)_{t\in\mathcal{G}}$ such that the (4.14) is satisfied.

4.4 TLG - valued stopping times

Let's assume that $(\mathcal{F}_t : t \in \mathcal{G})$ is a right-continuous filtration and X is an RCLL process adapted to this filtration.

First, let's define two random times that we want to make stopping times.

If σ is a path in $\mathcal{G}$, then clearly

$$H_U^\sigma := \inf\{t \in \sigma : X_\sigma(t) \in U\},$$

where U is an opened set. This is a standard one-dimensional stopping time. A more interesting example is

$$T_U^\sigma := \inf\{t \in \sigma : (\exists \tau \preceq t)(X(\tau) \in U)\}.$$

It is not hard to see that $H_U^\sigma \preceq T_U^\sigma$.

Lemma 4.22. *T_U^σ is an $(\mathcal{F}_t)$ stopping time.*

Proof. Let $t \in \sigma$, then by right continuity we have

$$(T_U^\sigma < t) = \bigcup_{n=1}^{\infty} \bigcup_{s \prec t - \frac{1}{n}, s \in \mathbb{Q}} (X(s) \in U) \in \mathcal{F}_t.$$

Where $t - \frac{1}{n}$ is the point on σ with that time, and $s \prec t - 1/n$, $s \in \mathbb{Q}$ means the point on TLG $\mathcal{G}$ that is before $t - 1/n$ and has rational time. If $t \notin \sigma$ then there exists

$$t^\sigma = \max\{s \in \sigma \ : \ s \prec t\}.$$

Now, from the continuity of the filtration we have

$$(T_U^\sigma \prec t) = (T_U^\sigma \preceq t^\sigma) = \bigcap_{n=k}^{\infty} (T_U^\sigma \prec t^\sigma + 1/n) \in \mathcal{F}_{t^\sigma + 1/k},$$

for all $k \in \mathbb{N}$. Therefore, the right-continuity of the filtration implies

$$(T_U^\sigma \prec t) \in \mathcal{F}_{t^\sigma} \subset \mathcal{F}_t.$$

■

Let K be a compact set. We define T_K^σ and H_K^σ in the same way as we did T_U^σ and H_U^σ. We know from classical Markov processes that H_K^σ is a stopping time.

Lemma 4.23. *T_K^σ is an $(\mathcal{F}_t)$ stopping time.*

Proof. Define $U_n = \{x : d(x, K) < 1/n\}$. It is clear that $K = \bigcap_{n=1}^{\infty} \overline{U_n} = \bigcap_{n=1}^{\infty} U_n$, and also it is clear that $T^{\sigma}_{U_n} \leq T^{\sigma}_{U_{n+1}} \leq T^{\sigma}_{K}$. Set $T := \sup_n T^{\sigma}_{U_n} = \lim_{n\to\infty} T^{\sigma}_{U_n}$. If $T \geq 1$ then clearly $T^{\sigma}_{K} = T$, on the event $T < 1$ we have

$$\lim_{n\to\infty} X(T^{\sigma}_{U_n}) = X(T),$$

but then $X(T) \in \overline{U_n}$, and hence

$$X(T) \in K.$$

Therefore, $T^{\sigma}_{K} \leq T$, and this implies $T^{\sigma}_{K} = T$. But we know that T is an $(\mathcal{F}_t)$ stopping time, and hence so is T^{σ}_{K}. ■

Here are is a general result about stopping times.

Proposition 4.24. *(a) If S and T are TLG* valued stopping times, so is $S \vee T$.*

(b) If (T_n) is a sequence of stopping times then $\vee_{n=1}^{\infty} T_n$ is also a stopping time.

Proof. We have

$$(\vee_{n=1}^{\infty} T_n \preceq t) = \bigcap_{n=1}^{\infty} (T_n \preceq t),$$

and the claim follows. The case (a) is proved similarly. ■

On the other hand, unlike in the classical case, the **minimum of two stopping times is not a stopping time**. The following example will illustrate that. Let $\mathcal{G}$ be a TLG* like in Figure 4.3. where $t_0 = 0$, $t_1 = 1/3$, $t_2 = 1/2$, $t_3 = 1$ and σ_1 is the bottom time-path, σ_2 the middle time-path, and σ_3 the upper time-path. Let $B^{br}_{a,b}$ represent the Brownian bridge starting at a and ending at b, and set

$$X_{\sigma_1}(t) \stackrel{d}{=} \begin{cases} B^{br}_{1/2,2/3}(t) & t \in [1/2, 2/3] \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad X_{\sigma_3}(t) \stackrel{d}{=} \begin{cases} B^{br}_{2/3,3/4}(t) & t \in [2/3, 3/4] \\ 0 & \text{otherwise} \end{cases},$$

and let $X_{\sigma_2} \stackrel{d}{=} 0$ (it can be any other Markov process consistent with the distributions of X_{σ_1} and X_{σ_3}).

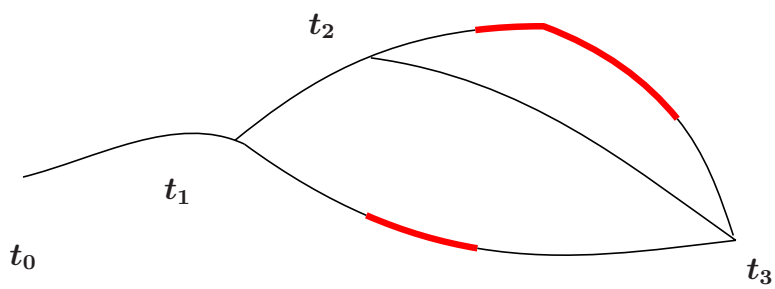


Figure 4.3: The red part of the graph is the part where Brownian bridges are defined, everywhere else we set the process to be 0.

Let $U = (1, \infty)$, and $T_1 = H_U^{\sigma_1}$ and $T_3 = H_U^{\sigma_3}$. It is clear that $T_3 \wedge T_1$ equals t_1 with probability greater than 0. But the event

$$(T_3 \wedge T_1 = t_1)$$

depends on events that happen after time 1/3, and it will not be contained in $\mathcal{F}_{t_1}$.

4.5 A simple coupling and branching process

In this section we will describe a simple coupling and branching process.

We are reconstructing the movement of two persons/objects, and we have the following information

- 2 persons moving around;
- (time $t_0 = 0$) started at the **same time from point** $A \in \mathbb{R}^2$;
- (time $t_3 = 1$) stooped at the **same time in point** $B \in \mathbb{R}^2$;
- we have an *additional information* that **from time** $t_1 = 1/3$ **to time** $t_2 = 2/3$ they were moving **together**.

Note, that we only know that the two persons were together in **time** interval $[1/3, 2/3]$, but we don't know anything about the **locations** they visited together!

We will model this as a process on a TLG. Let, $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be given by

$$\mathcal{V} = \{t_0, t_1, t_2, t_3\}, \quad \mathcal{E} = \{E^1_{01}, E^2_{01}, E_{12}, E^1_{23}, E^2_{23}\}.$$

Figure 4.4: The blue path is indexing the movement of the first person, and red path the movement of the second person.

Let $\sigma_j = (E^j_{01}, E_{12}, E^j_{23})$ for $j = 1, 2$. (σ_1 is the blue path, and σ_2 is the red path on Figure 4.4.)

Now we set $\mathcal{P}$ to be the the distribution of the (two-dimensional) Brownian bridge from A to B on $[0, 1]$ with variance σ^2. We define X on $\mathcal{G}$ to be a natural $\mathcal{M}_{\mathcal{P}}$-process.

Note, that X_{σ_1} and X_{σ_2} are Brownian bridges from A to B with variance σ, and $X_{\sigma_1}|_{[t_1,t_2]} = X_{\sigma_2}|_{[t_1,t_2]}$. Figure 4.5 shows a simulation of such a process.

Further, for this model we can calculate the expectations

$$\mathbb{E}(X_{\sigma_j}(t)) = (tx_a + (1-t)x_b, ty_a + (1-t)y_b), \quad j = 1, 2.$$

Also, it is not hard to calculate the covariance structure. The two processes have a known covariance structure

$$\mathbf{Cov}(X^l_{\sigma_j}(\tau_1), X^l_{\sigma_j}(\tau_2)) = \sigma^2 \tau_1 (1 - \tau_2), \quad j, l = 1, 2, \ \tau_1 \leq \tau_2.$$

Since the all full-time paths have the same distribution we have that for $\tau_1 \leq t_2$, and $t_1 \leq \tau_2$

$$\mathbf{Cov}(X^l_{\sigma_j}(\tau_1), X^l_{\sigma_i}(\tau_2)) = \mathbf{Cov}(X^l_{\sigma_j}(\tau_1), X^l_{\sigma_j}(\tau_2)), \quad i \neq j.$$

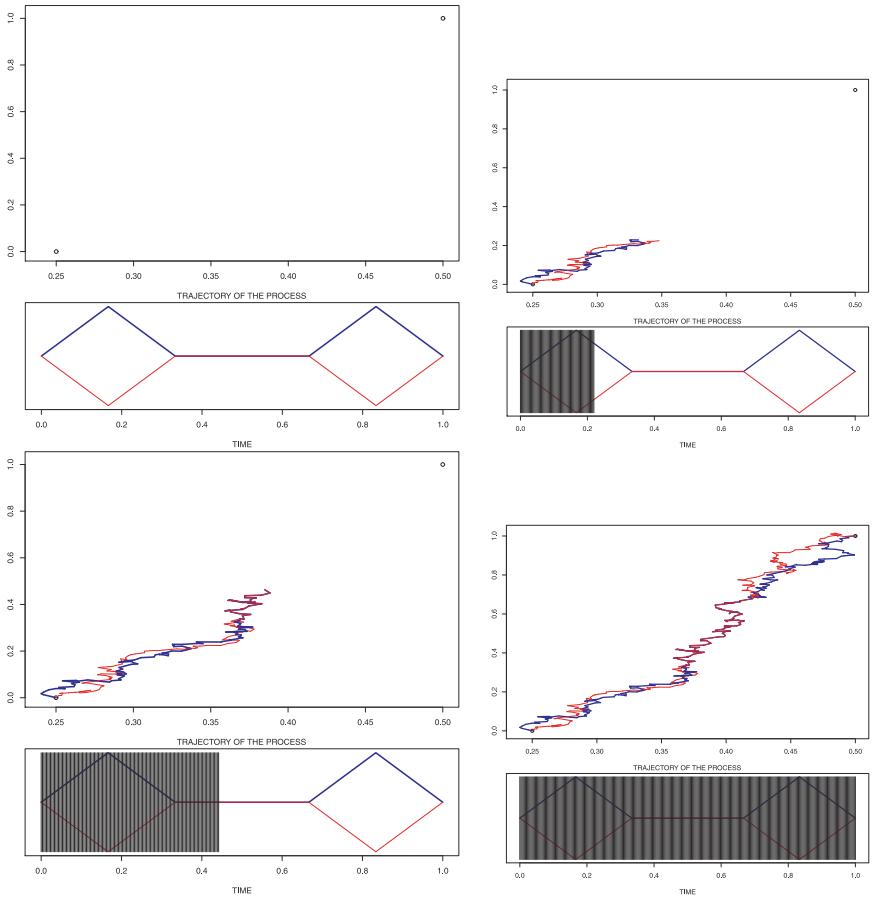


Figure 4.5: Simulation of the simple coupling and branching process with $\sigma^2 = 0.005$.

The last case is when $\tau_1 \leq \tau_2$ are on different sides of a cell:

$$\mathbf{Cov}(X^l_{\sigma_j}(\tau_1), X^l_{\sigma_i}(\tau_2)) = \begin{cases} \sigma^2 \frac{\tau_1 \tau_2 (1-t_1)}{t_1} & \tau_1, \tau_2 \in [t_0, t_1], \\ \sigma^2 \frac{(1-\tau_1)(1-\tau_2) t_2}{1-t_2} & \tau_1, \tau_2 \in [t_2, t_3]. \end{cases}$$

Part II

NATURAL BROWNIAN MOTION AND THE STOCHASTIC HEAT EQUATION

In this part we study what happens (in some special cases) when the the process is indexed by a time-like graph whose representation is dense in (a subset of) the t-x plane.

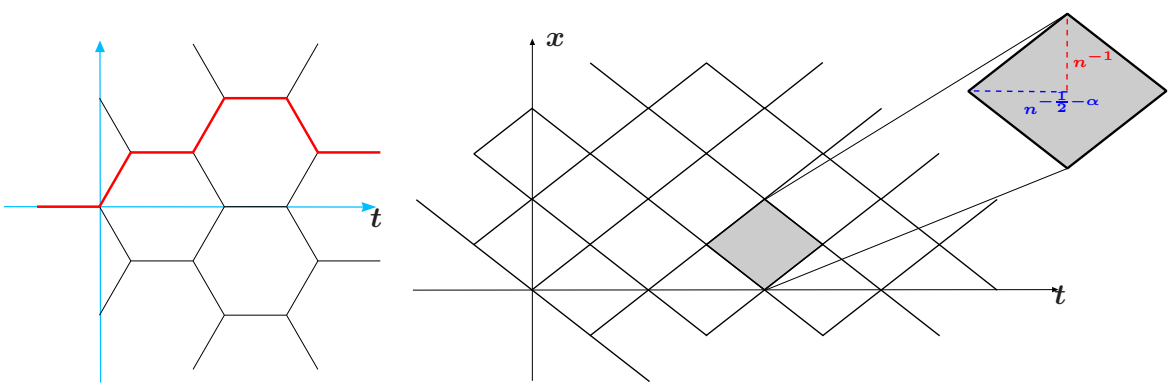


Figure 4.6: Index set: Honeycomb graph and α-rhombus grid

We will restrict our process to be a natural two-sided Brownian motion indexed by the graph whose representation is a rhombus grid. Burdzy and Pal studied the same process indexed by a honeycomb graph and found that (under certain scaling) when the mesh size goes to zero, the covariance structure is non-trivial (see Theorem 6.1. in [7]). (See Figure 4.6.)

The images in Figure 4.8 show what happens when ratio of the half-diagonals is $n^{-1/2-\alpha} : n^{-1}$ for $\alpha > 0$. It turns out, in this case, the process in the limit only depends on the time coordinate (t) and not on the space coordinate (x).

For the limit case $\alpha = 0$, however, the simulation (see Figure 4.9) indicates that the

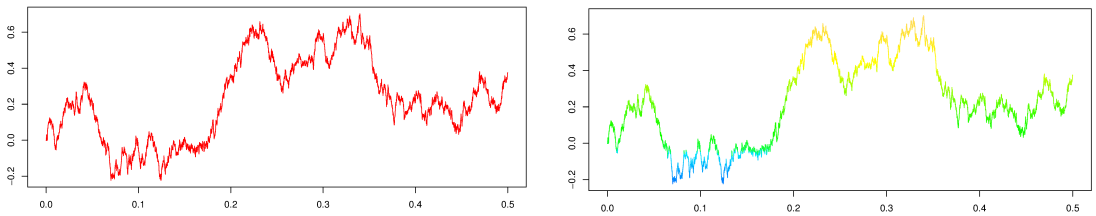


Figure 4.7: We use topographical colors to represent values of the Brownian motion

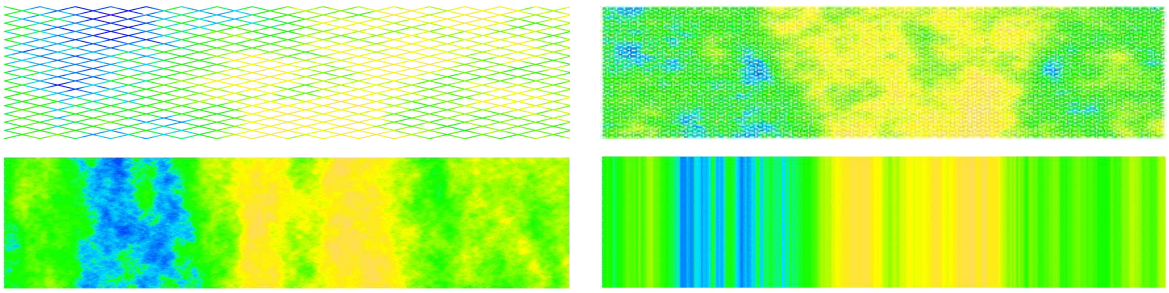


Figure 4.8: Scaled simulation of the process when the ratio is $1/n : 1/n$, for $n = 32, 128, 512, \infty$.

structure of the process in the plane is more complex. It turns out that the process in the limit is the stochastic heat equation.

In this part we prove that these are the limits.

First, we introduce some results about maximums of Gaussian processes in Chapter 5. Then in Chapter 6, we prove some general results about the (stochastic) heat equation, its approximation by Euler's method and the connections to the random walk.

After developing those tools, in Chapter 7. we prove the claims stated in this introduction in Theorem 7.8.

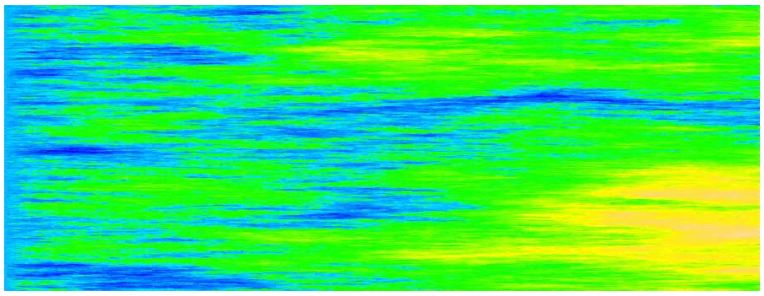


Figure 4.9: Simulation of a natural Brownian motion indexed by a rhombus grid where the ratio is $n^{-1/2} : n^{-1}$ and $n = 1024$.

Chapter 5

MAXIMUMS OF GAUSSIAN PROCESSES

In this section we will review the

- bounds for the second moment of the maximum of a finite sequence of independent Brownian bridges
- bounds for the second moment of the maximum of a finite sequence of (not necessarily independent) normal random variables;
- concentration of the maximum of Gaussian random element in $C(K)$ for some compact set K.

5.1 Sequence of Brownian bridges

For $k = 1, 2, \ldots$ we will denote $(B_k^{br}(t) : t \in [0,1])$ a Brownian bridge starting and ending at 0. (See Definition A.16.) We are interested in getting some estimation on moments of

$$M_n := \sup\{|B_k^{br}(t)| : t \in [0,1], k = 1, 2, \ldots, n\}.$$

In order to do this, we will estimate the moments of

$$M_n^+ := \sup\{B_k^{br}(t) : t \in [0,1], k = 1, 2, \ldots, n\}$$

$$M_n^- := \inf\{B_k^{br}(t) : t \in [0,1], k = 1, 2, \ldots, n\} = -\sup\{-B_k^{br}(t) : t \in [0,1], k = 1, 2, \ldots, n\}.$$

Since, $-B_k^{br}$ has the same distribution as B_k^{br}, it follows that M_n^+ has the same distribution as $-M_n^-$.

Now, since $B_k^{br}(0) = B_k^{br}(1) = 0$, $M_n^+ > 0$ and $M_n^- < 0$. Further, it is clear that

$$M_n = \max\{M_n^+, -M_n^-\}.$$

So if we find, a bound on moments of M_n^+ we will be able to find a bound on the moments of M_n.

From classical results on boundary crossing probabilities for Brownian motion (see [33, Karatzas, Shreve], page 262-265), we have the following Lemma:

Lemma 5.1. *If $(W_t : t \geq 0)$ is a Brownian motion starting at 0, then*

$$\mathbb{P}(\max_{0\leq t\leq T} W_t \geq \beta | W_T = a) = e^{-2\beta(\beta-a)/T} \tag{5.1}$$

for $T > 0$ and $\beta > \max\{0, a\}$.

From the last Lemma we get what we need to calculate $\mathbb{E}(M_n^{+2})$.

Proposition 5.2. *(a) For the Brownian bridge B_k^{br} we have*

$$\mathbb{P}(\max_{t\in[0,1]} B_k^{br}(t) > \beta) = e^{-2\beta^2}.$$

(b) If (B_k^{br}) are independent Brownian bridges, the following equality holds:

$$4\mathbb{E}(M_n^{+2}) = \frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{n}. \tag{5.2}$$

Proof. (a) This follows from (5.1) when we set $T = 1$, and $a = 0$. (b) For this we first note that, the independence of the sequence $(B_k^{br})_{k=1}^n$ implies

$$\begin{aligned}
\mathbb{P}(M_n^+ > \beta) &= 1 - P(M_n^+ \leq \beta) = 1 - \mathbb{P}(\bigcap_{k=1}^n (\max_{t\in[0,1]} B_k^{br}(t) \leq \beta)) \\
&= 1 - \prod_{k=1}^n \mathbb{P}(\max_{t\in[0,1]} B_k^{br}(t) \leq \beta) = 1 - \prod_{k=1}^n (1 - \mathbb{P}(\max_{t\in[0,1]} B_k^{br}(t) > \beta)) \\
&\overset{(5.1)}{=} 1 - (1 - e^{-2\beta^2})^n.
\end{aligned}$$

Now, we get

$$\mathbb{E}(M_n^{+2}) = \int_0^\infty \beta \mathbb{P}(M_n^+ > \beta)\, d\beta = \int_0^\infty \beta(1-(1-e^{-2\beta^2})^n)\, d\beta \tag{5.3}$$

Now, we will use some simple algebra,

$$\begin{aligned}\beta(1-(1-e^{-2\beta^2})^n) &= \beta e^{-2\beta^2}\frac{1-(1-e^{-2\beta^2})^n}{1-(1-e^{-2\beta^2})} \\ &= \beta e^{-2\beta^2}(1+(1-e^{-2\beta^2})+\ldots+(1-e^{-2\beta^2})^{n-1}).\end{aligned}$$

Using Tonelli's Theorem the equality (5.3) becomes

$$\mathbb{E}(M_n^{+2}) = \int_0^\infty \sum_{k=0}^{n-1} \beta e^{-2\beta^2}(1-e^{-2\beta^2})^k = \sum_{k=0}^{n-1}\int_0^\infty \beta e^{-2\beta^2}(1-e^{-2\beta^2})^k = \frac{1}{4}\sum_{k=0}^{n-1}\frac{1}{k+1},$$

since the expression under the last integral is the derivative of

$$-\frac{(1-e^{-2\beta^2})^{k+1}}{4(k+1)}.$$

■

Recall, that for the harmonic sequence

$$H_n = \sum_{k=1}^{n}\frac{1}{k}$$

we have,

$$\ln n \le H_n \le \ln(n+1). \tag{5.4}$$

Corollary 5.3. *The following inequalities hold for M_n the maximum of n independent Brownian bridges*

$$\mathbb{E}(M_n) \le \sqrt{\ln(n+1)}. \tag{5.5}$$

$$\mathbb{E}(M_n^2) \le \frac{1}{2}\ln(n+1) \tag{5.6}$$

Proof. From (5.2), using (5.4) we get

$$\mathbb{E}(M_n^+) \leq \sqrt{E(M_n^{+2})} \leq \frac{1}{2}\sqrt{\ln(n+1)}.$$

Now, using the fact that $\max\{a, b\} \leq a + b$ for $a, b \geq 0$ we have, since M_n^+ and $-M_n^-$ have the same distribution we get $\mathbb{E}(M_n) \leq \mathbb{E}(M_n^+) + \mathbb{E}(-M_n^-) = 2\mathbb{E}(M_n^+)$. The inequality (5.5) follows. The inequality (5.6) follows in the similar way. ■

5.2 Sequence of normal variables

When we have several normal random variables, what can we say about the expectation of the maximum of these random variables?

Let $X_k \sim N(0, \sigma_k^2)$ for $k = 1, \ldots, n$, we are interested in the upper bounds for

$$\mathbb{E}(\max\{|X_1|, \ldots, |X_n|\}),$$

and

$$\mathbb{E}(\max\{|X_1|^2, \ldots, |X_n|^2\}).$$

Using the result we got for the Brownian bridge we can get the upper bound.

Proposition 5.4. *For $(X_k)_{k=1}^n$ independent we have*

$$\mathbb{E}(\max\{|X_1|, \ldots, |X_n|\}) \leq 2 \max_{1\leq k\leq n} \sigma_k \sqrt{\ln(n+1)} \tag{5.7}$$

$$\mathbb{E}(\max\{|X_1|^2, \ldots, |X_n|^2\}) \leq 2 \left(\max_{1\leq k\leq n} \sigma_k^2\right) \ln(n+1) \tag{5.8}$$

Proof. Let $\sigma_{max} = \max_{1\leq k\leq n} \sigma_j > 0$ and $X'_k := \frac{1}{2\sigma_{max}} X_k$. Now X'_k are normal random variables with expectation 0 and variance at most 1/4. Hence, for each $k = 1, \ldots, n$ there exists a time t_k such that the variance of $B_k^{br}(t_k)$ is the same as that of X'_k, where $(B_k^{br})_{k=1}^n$ is a sequence of independent Brownian bridges starting and ending at 0. Hence, X'_k has

the same distribution as $B_k^{br}(t_k)$, so the distribution of $\max\{|X_1'|, \ldots, |X_n'|\}$ is the same as of $\max\{|B_1^{br}(t_1)|, \ldots, |B_n^{br}(t_n)|\}$ and this is less than $M_n = \sup\{|B_k^{br}(t)| : t \in [0,1], k = 1, \ldots, n\}$. So by (5.5), we have

$$\mathbb{E}(\max\{|X_1'|, \ldots, |X_n'|\}) \leq \mathbb{E}(M_n) \leq \sqrt{\ln(n+1)}.$$

Multiplying this with $2\sigma_{max}$ we get (5.7). Similar argument using inequality (5.6) will give (5.8) ■

Now, we will deal with the case when $(X_k)_{k=1}^n$ are not necessarily independent. We will do this with the help of a lemma that is due to Šidák (see [44]).

Lemma 5.5. *(a) (Šidák 1967.) For positive numbers $c_1, c_2, \ldots, c_n$*

$$\mathbb{P}(|X_1| \leq c_1, |X_2| \leq c_2, \ldots |X_n| \leq c_n) \geq \mathbb{P}(|X_1| \leq c_1)\mathbb{P}(|X_2| \leq c_2) \ldots \mathbb{P}(|X_n| \leq c_n).$$

(b) Let $Y_1, \ldots, Y_n$ be independent random variables, such that for each $k = 1, 2, \ldots, n$ Y_k and X_k have the same distribution, then

$$\mathbb{E}(\max\{|X_1|, \ldots, |X_n|\}^p) \leq \mathbb{E}(\max\{|Y_1|, \ldots, |Y_n|\}^p)$$

for all $p \geq 1$.

The proof of part (a) of this Lemma can be found in [44] or [40]. Part (b) is a direct consequence of part (a).

As a consequence of Lemma 5.5 (part (b) for $p = 1$) and Proposition 5.4 we get the following theorem.

Theorem 5.6. *For $(X_k \sim N(0, \sigma_k))_{k=1}^n$ (possibly correlated) we have*

$$\mathbb{E}(\max\{|X_1|, \ldots, |X_n|\}) \leq 2 \max_{1 \leq k \leq n} \sigma_k \sqrt{\ln(n+1)} \tag{5.9}$$

$$\mathbb{E}(\max\{|X_1|^2, \ldots, |X_n|^2\}) \leq 2 \left(\max_{1 \leq k \leq n} \sigma_k^2 \right) \ln(n+1) \tag{5.10}$$

A much general result (of the same order) can be found in the paper by Chatterjee in [9].

5.3 Some concentration and convergence results

We will shortly state some concentration results taken from Chapter 3.1. of Talagrand's book [48], and apply it to the convergence of Gaussian processes.

Let B be a Banach space, and D some countable subset of the unit ball of the dual space B' such that

$$\|x\| = \sup_{f \in D} |f(x)|,$$

for all $x \in B$.

We say that X is a **Gaussian random variable** in B if $f(X)$ is measurable for every $f \in D$ and if every finite linear combination

$$\sum_i \alpha_i f_i(X),$$

where $\alpha_i \in \mathbb{R}$ and $f_i \in D$, is Gaussian.

Let X be a Gaussian, $M = M(X)$ be the **median** of $\|X\|$, that is M has the property that

$$\mathbb{P}(\|X\| \geq M) \geq 1/2 \quad \text{and} \quad \mathbb{P}(\|X\| \leq M) \geq 1/2.$$

Further, set the **supremum of weak deviations** to be

$$\sigma = \sigma(X) = \sup_{f \in D} \mathbb{E}[f(X)^2]^{1/2}.$$

The following result is a Lemma 3.1. from [48].

Lemma 5.7. *Let X be a Gaussian with median $M = M(X)$ and the supremum of weak deviations $\sigma = \sigma(X)$, then*

$$\mathbb{P}(|\|X\| - M| > t) \leq \exp(-t^2/2\sigma^2).$$

Corollary 5.8. *Let X be a Gaussian with $\mathbb{E}[\|X\|^2] < \infty$, then*

$$\mathbb{P}(\|X\| > t) \leq 4 \exp\left(\frac{-t^2}{2\mathbb{E}[\|X\|^2]}\right). \tag{5.11}$$

Proof. Follows from Lemma 5.7 and the fact that $\sigma^2 \leq \mathbb{E}[\|X\|^2]$ and $M^2 \leq \mathbb{E}[\|X\|^2]$. ■

Let $Y = (Y_1, Y_2, \ldots, Y_d)$ be a Gaussian vector with expectation vector 0, then since $B = \mathbb{R}^d$ is the Banach space with usual norm, and the set of projections $D = \{\pi_k : k \in \{1, 2, \ldots, d\}\}$ is a subset of B' we have that

$$\mathbb{P}(\max_{1\leq k\leq n} |Y_k| > t) \leq 4 \exp\left(\frac{-t^2}{2\mathbb{E}[\max_{1\leq k\leq n} |Y_k|^2]}\right). \tag{5.12}$$

It will not always be easy to get an estimate for $\mathbb{E}[\max_{1\leq k\leq n} |Y_k|^2]$, but when we do the inequality (5.12) will tell us a lot.

Theorem 5.9. *Let Y^k be a sequence of Gaussian vectors (not necessarily of the same dimension) with expectation vector 0 on the same probability space, such that*

$$\mathbb{E}[\|Y^k\|_\infty^2] \leq \frac{C}{k^\alpha}, \tag{5.13}$$

for some $C > 0$ and $\alpha > 0$. Then

$$\|Y^k\|_\infty \to 0 \quad a.s.$$

Proof. It is easy to show that for sufficiently large k we have

$$\exp\left(-\frac{t^2 k^\alpha}{2C}\right) \leq \frac{1}{k^2}.$$

Now, using (5.12) and (5.12) the previous inequality for sufficiently large k gives

$$\mathbb{P}(\|Y^k\|_\infty > t) \leq \frac{1}{k^2},$$

and hence

$$\sum_{k=1}^{\infty} \mathbb{P}(\|Y^k\|_\infty > t) < \infty.$$

Since this holds for all $t > 0$, $\|Y^k\|_\infty \to 0$ a.s. ■

Corollary 5.10. *Let Y^k be a sequence of Gaussian vectors (not necessarily of the same dimension) with expectation vector 0 on the same probability space, such that*

$$\mathbb{E}[\|Y^k\|_\infty^2] \leq \frac{C}{k^\alpha}, \tag{5.14}$$

for some $C > 0$ and $\alpha > 0$. Then for $0 < \beta < \alpha/2$

$$k^\beta \|Y^k\|_\infty \to 0 \quad a.s.$$

Proof. We see that for $Z^k := k^\beta Y^k$ we have

$$\mathbb{E}[\|Z^k\|_\infty^2] \leq \frac{C}{k^{\alpha-2\beta}}.$$

Hence, since $\alpha - 2\beta > 0$ by Theorem 5.9 we have $\|Z^k\|_\infty \to 0$ a.s. ■

We can get similar results for continuous Gaussian fields.

Theorem 5.11. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $K \subset \mathbb{R}^n$ be a compact set, and $X_n : K \times \Omega \to \mathbb{R}$ have the following properties:*

(1) For each $x \in K$ $X_n(x)$ is a Gaussian random variable.

(2) For each $\omega \in \Omega$ $x \mapsto X_n(x, \omega)$ is a continuous function.

Then if

$$\mathbb{E}[\|X_n\|_\infty^2] \leq \frac{C}{n^\alpha},$$

we have

$$\|X_n\|_\infty \to 0 \quad a.s. \tag{5.15}$$

Further, for $0 < \beta < \alpha/2$ we have

$$n^\beta \|X_n\|_\infty \to 0 \quad a.s. \tag{5.16}$$

Proof. Let $B = C(K)$ with the usual $\|\cdot\|_\infty$ supremum norm, and set $D = \{\pi_q \,:\, q \in \mathbb{Q}^n \cap K\}$. Now it is clear that X_n is a Gaussian random variable in B, and since X_n is continuous we have $\|X_n\|_\infty = \sup_{q\in\mathbb{Q}^n\cap K} |\pi_q(X_n)|$, we have from (5.11) that

$$\mathbb{P}(\|X_n\|_\infty > t) \leq \exp\left(-\frac{t^2 k^\alpha}{2C}\right).$$

Using the same technique as in proof of Theorem 5.9 we have the desired results. ■

Chapter 6

RANDOM WALK AND STOCHASTIC HEAT EQUATION REVIEWED

6.1 Modification of the Local Limit Theorem

In the rest of this chapter (S_n) will denote the **simple random walk**, where $S_n = X_1 + X_2 + \ldots + X_n$, (X_k) are i.i.d. and $\mathbb{P}(X_1 = \pm 1) = 1/2$.

First we introduce some notation. For a simple random walk (S_n) we set

$$p_n^k(x) = \mathbb{P}(\frac{S_k}{\sqrt{n}} = x), \quad \text{for} \quad x \in \mathcal{L}_n^k := \{(k+2z)/\sqrt{n} : z \in \mathbb{Z}\},$$

and

$$\rho_n^k(x) = \frac{1}{\sigma_n^k \sqrt{2\pi}} \exp\left(-\frac{x^2}{2(\sigma_n^k)^2}\right),$$

where $(\sigma_n^k)^2 = \frac{n}{k}$. The main result of this section is Theorem 6.3 which gives the bound on the difference of p_n^k and ρ_n^k.

We will need the following two lemmas. The first lemma is a consequence of the inversion formula for characteristic functions. (See [15].)

Lemma 6.1. *If Y is a random variable with $\mathbb{P}(Y \in a + \theta\mathbb{Z}) = 1$, and $\psi(t) = \mathbb{E}(e^{itY})$ is its characteristic function, then*

$$\mathbb{P}(Y = x) = \frac{1}{2\pi/\theta} \int_{-\pi/\theta}^{\pi/\theta} e^{-itx} \psi(t)\, dt.$$

The second lemma is a consequence of the Stirling formula.

Lemma 6.2. *For $k \in \mathbb{N}$ set*

$$I_k := \int_0^{\pi/2} \cos^k(x)\, dx,$$

there exists a $C > 0$ such that

$$\left|\sqrt{k}I_k - \sqrt{\frac{\pi}{2}}\right| \leq \frac{C}{k}. \tag{6.1}$$

Proof. From integration by parts we have

$$I_k = \frac{k-1}{k} I_{k-2},$$

and further we can calculate $I_1 = 1$ and $I_2 = \frac{\pi}{4}$. Now, this recursion gives us

$$I_{2k+1} = \frac{2k}{2k+1} \cdot \frac{2k-2}{2k-1} \cdots \frac{2}{3} \cdot I_1 = \frac{2^{2k}(k!)^2}{(2k+1)!},$$

$$I_{2k} = \frac{2k-1}{2k} \cdot \frac{2k-3}{2k-2} \cdots \frac{3}{4} \cdot I_2 = \frac{(2k)!}{2^{2k}(k!)} \cdot \frac{\pi}{2}.$$

Using Stirling's Formula (see for example Gamelin [22] page 368), i.e. the fact that

$$n! = \left(\frac{n}{e}\right)^n \sqrt{2n\pi} \exp\left(\frac{1}{12n} + O(\frac{1}{n^3})\right),$$

we have

$$2^k\sqrt{2k+1}(k!)^2 = \frac{(2k)^{2k+1}}{e^{2k}} \pi\sqrt{2k+1} \exp\left(\frac{1}{6k} + O(\frac{1}{8k^3})\right),$$

$$(2k+1)! = \left(\frac{2k+1}{e}\right)^{2k+1} \sqrt{2\pi}\sqrt{2k+1} \exp\left(\frac{1}{12(2k+1)} + O(\frac{1}{8k^3})\right),$$

and therefore

$$\sqrt{2k+1}I_{2k+1} = \left(1 - \frac{1}{2k+1}\right)^{2k+1} e\sqrt{\frac{\pi}{2}} \exp(\frac{1}{6k} - \frac{1}{12(2k+1)} + O(\frac{1}{k^3})).$$

Now,

$$\begin{aligned}
&(2k+1)|\sqrt{2k+1}I_{2k+1} - \sqrt{\pi/2}| \\
=&(2k+1)\sqrt{\frac{\pi}{2}}\left|\left(1 - \frac{1}{2k+1}\right)^{2k+1} e(1 + \frac{1}{6k} - \frac{1}{12(2k+1)} + O(\frac{1}{k^2})) - 1\right| \\
\leq&(2k+1)\sqrt{\frac{\pi}{2}}e\left|\left(1 - \frac{1}{2k+1}\right)^{2k+1} - e^{-1}\right| + \sqrt{\frac{\pi}{2}}e\left|\frac{2k+1}{6k} - \frac{1}{12} + O(\frac{1}{k})\right|
\end{aligned} \tag{6.2}$$

The second absolute value is clearly bounded. For the first absolute value we use the well-known fact that if $|u|, |z| \leq 1$ then for $m \in \mathbb{N}$ we have $|u^m - z^m| \leq m|u - z|$. So, by setting $m = 2k+1$, $u = 1 - (2k+1)^{-1}$ and $z = e^{-(2k+1)^{-1}}$ we have

$$\left|\left(1 - \frac{1}{2k+1}\right)^{2k+1} - e^{-1}\right| \leq (2k+1)\left|1 - \frac{1}{2k+1} - e^{-1/(2k+1)}\right|$$

$$\leq (2k+1)\frac{1}{2(2k+1)^2} = \frac{1}{2(2k+1)},$$

where the last inequality follows from the Taylor's Theorem. Hence, the first absolute value in (6.2) is also bounded.

Using the same methods we get the same result for (I_{2k}). ■

Theorem 6.3. *There exists a $C > 0$ such that for any $\beta(n)$ we have*

$$\sup_{\beta(n)\leq k} \sup_{x\in\mathcal{L}_n^k} \left|\frac{n^{1/2}}{2} p_n^k(x) - \rho_n^k(x)\right| \leq \frac{C}{\pi}\sqrt{\frac{n}{\beta(n)^3}},$$

for all n.

Proof. Using Lemma 6.1. for $\theta = 2/\sqrt{n}$ and function

$$\psi_k(t) = \mathbb{E}[\exp(\frac{itS_k}{\sqrt{n}})] = \varphi^k(\frac{t}{\sqrt{n}}),$$

we have that

$$\frac{n^{1/2}}{2} p_n^k(x) = \frac{1}{2\pi}\int_{-\pi\sqrt{n}/2}^{\pi\sqrt{n}/2} e^{-itx}\varphi^k(t/\sqrt{n})\,dt. \tag{6.3}$$

The inversion formula gives that

$$\rho_n^k(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itx}\exp(-(\sigma_n^k)^2 t^2/2)\,dt. \tag{6.4}$$

From (6.3) and (6.3) we have

$$\left|\frac{n^{1/2}}{2} p_n^k(x) - \rho_n^k(x)\right| \leq \frac{1}{2\pi}\int_{-\pi\sqrt{n}/2}^{\pi\sqrt{n}/2} |\varphi^k(t/\sqrt{n}) - \exp(-(\sigma_n^k)^2 t^2/2)|\,dt$$
$$+ \frac{1}{\pi}\int_{\pi\sqrt{n}/2}^{\infty} \exp(-(\sigma_n^k)^2 t^2/2)\,dt.$$

First, note that the right side the inequality doesn't depend on x. Now by substituting $u = t\sqrt{k/n}$ in both integrals, we get that the right side of the inequality is

$$\frac{\sqrt{n/k}}{\pi}\left[\frac{1}{2}\int_{-\pi\sqrt{k}/2}^{\pi\sqrt{k}/2}|\varphi^k(u/\sqrt{k}) - \exp(-u^2/2)|\,du + \int_{\pi\sqrt{k}/2}^{\infty}\exp(-u^2/2)\,du\right]. \tag{6.5}$$

For the first integral in (6.5), first note that $\varphi(t) = \mathbb{E}(e^{itX_1}) = \cos t$, so since the function under the integral is even we have

$$\frac{1}{2}\int_{-\pi\sqrt{k}/2}^{\pi\sqrt{k}/2}|\varphi^k(u/\sqrt{k}) - \exp(-u^2/2)|\,du = \int_{0}^{\pi\sqrt{k}/2}|\varphi^k(u/\sqrt{k}) - \exp(-u^2/2)|\,du.$$

Further, it is not hard to show that $e^{-\frac{x^2}{2}} \geq \cos x$ for $x \in [-\pi/2, \pi/2]$. So, $e^{-\frac{x^2}{2k}} \geq \cos(x/\sqrt{k})$ for x in the bounds of the integral, and therefore

$$\int_{0}^{\pi\sqrt{k}/2}|\varphi^k(u/\sqrt{k}) - \exp(-u^2/2)|\,du = \int_{0}^{\pi\sqrt{k}/2}\exp(-u^2/2) - \varphi^k(u/\sqrt{k})\,du,$$

and now right-side of (6.5) becomes

$$\begin{aligned}
&\frac{\sqrt{n/k}}{\pi}\left[\int_{0}^{\infty}\exp(-u^2/2)\,du - \int_{0}^{\pi\sqrt{k}/2}\varphi^k(u/\sqrt{k})\,du\right]\\
=&\frac{\sqrt{n/k}}{\pi}\left[\sqrt{\frac{\pi}{2}} - \int_{0}^{\pi\sqrt{k}/2}\cos^k(u/\sqrt{k})\,du\right]\\
=&\frac{\sqrt{n/k}}{\pi}\underbrace{\left[\sqrt{\frac{\pi}{2}} - \sqrt{k}\int_{0}^{\pi/2}\cos^k(u)\,du\right]}_{\overset{(6.1)}{\leq}\frac{C}{k}}.
\end{aligned}$$

From Lemma 6.2 we have that (6.5) is less than

$$\frac{C}{\pi}\sqrt{\frac{n}{k^3}}$$

■

Corollary 6.4. *If* $\lim_{n\to\infty}\frac{n}{\beta(n)^3} = 0$, *then*

$$\lim_{n\to\infty}\sup_{\beta(n)\leq k}\sup_{x\in\mathcal{L}_n^k}\left|\frac{n^{1/2}}{2}p_n^k(x) - \rho_n^k(x)\right| = 0.$$

Specially, in the case when $\beta(n) = n$, we have

$$\sup_{x \in \mathcal{L}_n^n} \left| \frac{n^{1/2}}{2} p_n^n(x) - \rho_n^n(x) \right| \leq \frac{C}{\pi n} \to 0,$$

as $n \to \infty$.

6.2 Approximations of the classical heat equation solution

In this section we will review the one-dimensional heat equation (mostly classical results that can be found in books that deal with connections to PDEs like Karatzas and Shreve [33], and some books on classical PDEs like Folland [18]) and develop more general results that will later help us.

In this section we use the usual space-time (x-t) coordinate system. We are considering the classical initial value problem

$$\begin{cases} \partial_t w = \frac{1}{2} \partial_{xx} w & \text{on} \quad \mathbb{R} \times (0, \infty), \\ w(0, x) = f(x) & \text{for} \quad x \in \mathbb{R}. \end{cases} \tag{6.6}$$

If we assume that $f : \mathbb{R} \to \mathbb{R}$ is a Borel measurable function satisfying

$$\int_{-\infty}^{\infty} e^{-ax^2} |f(x)| \, dx < \infty \tag{6.7}$$

for some $a > 0$. Then the solution exists.

Theorem 6.5. *If the condition* (6.7) *is satisfied, then*

$$w(t, x) := \mathbb{E}(f(x + W_t)) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} f(y) \exp\left(\frac{-(y - x)^2}{2t} \right) dy, \tag{6.8}$$

for $0 < t < \frac{1}{2a}$ and $x \in \mathbb{R}$ is the solution to the initial value (6.6)*. This solution has derivatives of all orders. Furthermore, if f is continuous at x, then*

$$\lim_{(t,y) \to (0,x)} w(t, y) = f(x). \tag{6.9}$$

Proof. This follows from the fact that the so called Gaussian kernel

$$K_t(x) := \frac{1}{\sqrt{2\pi t}} \exp\left(\frac{-x^2}{2t}\right),$$

satisfies the heat equation. (This can be checked by a direct calculation.) The rest follows from the dominated convergence theorem. ■

The main question that will be of interest to us is: if f satisfies (6.7) and it is continuous, for a simple random walk (S_n) starting from zero is

$$\mathbb{E}(f\left(\frac{S_{\lfloor nt \rfloor}}{\sqrt{n}} + x\right)) \to w(t,x), \tag{6.10}$$

where u is given by (6.8) and how strong is this convergence.

It is clear from the definition of convergence in distribution and the Donsker's theorem that this convergence holds if f is bounded. We will show that this holds for a much wider set of functions.

Lemma 6.6. *(a)* (Hoeffding's Inequality) *For $y \geq 0$ we have*

$$\mathbb{P}\left(\left|\frac{S_n}{\sqrt{n}}\right| \geq y\right) \leq 2e^{-y^2/2},$$

where (S_n) is a simple random walk.

(b) If $\tau > 0$ then for all $t \leq \tau$

$$\mathbb{P}\left(\left|\frac{S_{\lfloor nt \rfloor}}{\sqrt{n}}\right| \geq y\right) \leq 2e^{-y^2/(2\tau)}$$

for all $y \geq 0$.

Proof. (a) This is a well known inequality. For the proof see, for example, [29] or [8]. (b) For $\lfloor nt \rfloor = 0$ the claim is clear. Otherwise, we have

$$\mathbb{P}\left(\left|\frac{S_{\lfloor nt \rfloor}}{\sqrt{n}}\right| \geq y\right) = \mathbb{P}\left(\left|\frac{S_{\lfloor nt \rfloor}}{\sqrt{\lfloor nt \rfloor}}\right| \geq \frac{y\sqrt{n}}{\sqrt{\lfloor nt \rfloor}}\right).$$

Now, since

$$\frac{y\sqrt{n}}{\sqrt{\lfloor nt\rfloor}} \geq \frac{y\sqrt{n}}{\sqrt{nt}} = \frac{y}{\sqrt{t}} \geq \frac{y}{\sqrt{\tau}},$$

we have

$$\mathbb{P}\left(\left|\frac{S_{\lfloor nt\rfloor}}{\sqrt{\lfloor nt\rfloor}}\right| \geq \frac{y\sqrt{n}}{\sqrt{\lfloor nt\rfloor}}\right) \leq \mathbb{P}\left(\left|\frac{S_{\lfloor nt\rfloor}}{\sqrt{\lfloor nt\rfloor}}\right| \geq \frac{y}{\sqrt{\tau}}\right),$$

and the claim follows from part (a). ■

Define B^n to be the linear interpolation of $t \mapsto \dfrac{S_{\lfloor nt\rfloor}}{\sqrt{\lfloor nt\rfloor}}$, that is

$$B^n(t) := \frac{S_{\lfloor nt\rfloor}}{\sqrt{n}} + (nt - \lfloor nt\rfloor)\left(\frac{S_{\lfloor nt\rfloor+1}}{\sqrt{n}} - \frac{S_{\lfloor nt\rfloor}}{\sqrt{n}}\right).$$

Lemma 6.7. *For any f continuous, $a < b$ real numbers, and $\varepsilon > 0$ we have*

$$\mathbb{E}(f(\frac{S_{\lfloor nt\rfloor}}{\sqrt{n}} + x)g^{\varepsilon}_{a,b}(\frac{S_{\lfloor nt\rfloor}}{\sqrt{n}})) \to \mathbb{E}(f(W_t + x)g^{\varepsilon}_{a,b}(W_t))$$

uniformly on compact sets in (t, x), where

$$g^{\varepsilon}_{a,b}(x) = \begin{cases} 1 & x \in [a, b], \\ \frac{x-a+\varepsilon}{\varepsilon} & x \in [a-\varepsilon, a], \\ \frac{x-b-\varepsilon}{-\varepsilon} & x \in [b, b+\varepsilon], \\ 0 & x \notin [a-\varepsilon, b+\varepsilon]. \end{cases}$$

Proof. Let $K \subset \mathbb{R}^+ \times \mathbb{R}$ be a compact set and define $K_t := \pi_t(K)$ and $K_x := \pi_x(K)$. They are also compact. Hence, the function $h : \mathbb{R} \times K_x \to \mathbb{R}$ given by $h(u, x) := f(u + x)g^{\varepsilon}_{a,b}(u)$ is a continuous function supported on a compact set (which is a subset of $[a-\varepsilon, b+\varepsilon] \times K_x$). Now, since K_t is compact, there exists $T > 0$ such that $K_t \subset [0, T]$. By Donsker's Theorem we know that $B^n \xrightarrow{d} W$ on $[0, T]$, hence by Skorohod's Representation Theorem there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ with random elements $\tilde{B}^n \overset{d}{=} B^n$ and $\tilde{W} \overset{d}{=} W$ such that

$$\|\tilde{B}^n(\omega) - \tilde{W}(\omega)\| = \sup_{t\in[0,T]} |\tilde{B}^n(t)(\omega) - \tilde{W}(t)(\omega)| \to 0,$$

for all $\omega \in \tilde{\Omega}$. Note that if we define

$$\tilde{S}_t^n := \tilde{B}^n_{\lfloor nt \rfloor/n},$$

$\tilde{S}^n$ has the same distribution as $S_{\lfloor nt \rfloor}/\sqrt{n}$. Further, it is clear that

$$\|\tilde{S}^n - B^n\| \leq \frac{1}{\sqrt{n}}.$$

Therefore $\|\tilde{S}^n - \tilde{W}^n\| \to 0$. Now since $h \in C_c(\mathbb{R}^2)$ it is uniformly continuous function, and therefore

$$\sup_{(t,x)\in K} |h(\tilde{S}_t^n, x) - h(\tilde{W}(t), x)| \to 0,$$

Now

$$\begin{aligned}
&|\mathbb{E}(f(\frac{S_{\lfloor nt \rfloor}}{\sqrt{n}} + x) g^{\varepsilon}_{a,b}(\frac{S_{\lfloor nt \rfloor}}{\sqrt{n}})) - \mathbb{E}(f(W_t + x) g^{\varepsilon}_{a,b}(W_t))| \\
&= |\tilde{\mathbb{E}}[f(\tilde{S}_t^n + x) g^{\varepsilon}_{a,b}(\tilde{S}_t^n)) - f(\tilde{W}_t + x) g^{\varepsilon}_{a,b}(\tilde{W}_t))]| \\
&\leq \tilde{\mathbb{E}}\left[\sup_{(t,x)\in K} |h(\tilde{S}_t^n, x) - h(\tilde{W}(t), x)|\right].
\end{aligned}$$

The convergence follows from the dominated convergence theorem. ■

In order to get (6.10) we have to make some *mild* assumptions on f.

Lemma 6.8. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that there exist $C > 0$ and a locally integrable $g : \mathbb{R}^+ \to \mathbb{R}^+$ with the property*

$$|f(x)|^2 \leq C + \int_0^{|x|} g(y)\, dy, \tag{6.11}$$

for all $x \in \mathbb{R}$, and

$$\int_0^{\infty} g(y) e^{-y^2/(2\tau)}\, dy < \infty, \tag{6.12}$$

for some $\tau > 0$. Then there exists $M > 0$ (that depends on C, g and τ) such that for all $t \leq \tau$ and all n

$$\mathbb{E}[|f(S_{\lfloor nt \rfloor}/\sqrt{n})|^2] < M. \tag{6.13}$$

Further,

$$\int_{-\infty}^{\infty} |f(x)| e^{-x^2/(2\tau)}\, dx < \infty. \tag{6.14}$$

<u>*Proof.*</u> We set $G(x) := \int_0^{|x|} g(y)\, dy$. Now,

$$\begin{aligned}
\mathbb{E}[|f(S_{\lfloor nt \rfloor}/\sqrt{n})|^2] &\le C + \mathbb{E}(G(|S_{\lfloor nt \rfloor}/\sqrt{n}|)) \\
&\le C + \mathbb{E}(\int_0^{\infty} g(y) \mathbf{1}_{(y \le |S_{\lfloor nt \rfloor}/\sqrt{n}|)}\, dy) \\
&= C + \int_0^{\infty} g(y) \mathbb{E}(\mathbf{1}_{(y \le |S_{\lfloor nt \rfloor}/\sqrt{n}|)}\, dy \\
&= C + \int_0^{\infty} g(y) \mathbb{P}(|S_{\lfloor nt \rfloor}/\sqrt{n}| \ge y)\, dy \\
&= C + \int_0^{\infty} g(y) \mathbb{P}(|S_{\lfloor nt \rfloor}/\sqrt{n}| \ge y)\, dy \\
&= C + 2\int_0^{\infty} g(y) e^{-y^2/(2\tau)}\, dy =: M.
\end{aligned}$$

For (6.14) we first show a similar results using the same arguments. Let $X \sim N(0, \tau)$. Then $\mathbb{P}(|X| > x) \le 2e^{-x^2/(2\tau)}$. Now,

$$\begin{aligned}
\int_{-\infty}^{\infty} |f(x)|^2 e^{-x^2/(2\tau)}\, dx &= \sqrt{2\pi \cdot \tau}\, \mathbb{E}(|f(X)|^2) \\
&\le \sqrt{2\pi\tau}(C + \mathbb{E}(G(X))) \\
&= C\sqrt{2\pi\tau} + \sqrt{2\pi\tau}\, \mathbb{E}(G(X)) \\
&= C\sqrt{2\pi\tau} + \sqrt{2\pi\tau} \int_0^{\infty} g(y) \mathbb{P}(|X| \ge y)\, dy \\
&= C\sqrt{2\pi\tau} + 2\sqrt{2\pi\tau} \int_0^{\infty} g(y) e^{-y^2/(2\tau)}\, dy \\
&= M\sqrt{2\pi\tau}.
\end{aligned}$$

Now, it is clear from Cauchy-Schwarz inequality we have that

$$\begin{aligned}
\int_{-\infty}^{\infty} |f(x)| e^{-x^2/(2\tau)}\, dx &\le \left(\int_{-\infty}^{\infty} e^{-x^2/(2\tau)}\, dx \right)^{1/2} \left(\int_{-\infty}^{\infty} |f(x)|^2 e^{-x^2/(2\tau)}\, dx \right)^{1/2} \\
&= \sqrt{2\pi\tau M}.
\end{aligned}$$

■

Remark. The conditions (6.11) and (6.12) given by the previous lemma are satisfied by a wide family of functions. For instance, if for $\alpha \geq 1$ we have

$$\limsup_{|y|\to\infty} \frac{|f(y)|}{|y|^\alpha} =: L < \infty.$$

Then there exists a $C > 0$ such that

$$|f(y)| \leq C + L|y|^\alpha,$$

for all $y \in \mathbb{R}$. Now, the function $g(y) := \alpha y^{\alpha-1}$ satisfies (6.11). Further, since the normal distribution has all the α-moments for $\alpha \geq 1$, g satisfies (6.12).

Theorem 6.9. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and $a < b$ finite real numbers, such that there exist $C > 0$ and a locally integrable $g : \mathbb{R}^+ \to \mathbb{R}^+$ with the property*

$$\sup_{r\in[a,b]} (f(x+r))^2 \leq C + \int_0^{|x|} g(y)\, dy,$$

for all $x \in \mathbb{R}$, and

$$\int_0^\infty g(y) e^{-y^2/(2\tau)}\, dy < \infty,$$

for some $\tau > 0$. Then for all $r \in [a, b]$

$$\mathbb{E}(f(\frac{S_{\lfloor nt \rfloor}}{\sqrt{n}} + r)) \to w(t, r), \tag{6.15}$$

as $n \to \infty$ where $t < \tau$ and w is the solution to the initial value problem (6.6) *given by*

$$w(t, x) = \mathbb{E}(f(x + W_t)).$$

Further, the convergence in (6.15) *is uniform on* $[0, \tau) \times [a, b]$

Proof. From Lemma 6.8. and Theorem 6.5. we know that $w(\cdot, \cdot)$ is the solution to (6.6).

From Lemma 6.8. applied on the function $f(\cdot + r)$ we know that for each $K > 0$

$$\begin{aligned}
|\mathbb{E}[f(\frac{S_{\lfloor nt \rfloor}}{\sqrt{n}} + r)(1 - g^{\varepsilon}_{-K,K}(\frac{S_{\lfloor nt \rfloor}}{\sqrt{n}}))]| &\leq \sqrt{\mathbb{E}(f(\frac{S_{\lfloor nt \rfloor}}{\sqrt{n}} + r)^2)\mathbb{E}((1 - g^{\varepsilon}_{-K,K}(\frac{S_{\lfloor nt \rfloor}}{\sqrt{n}}))^2} \\
&\leq \sqrt{\mathbb{E}(f(\frac{S_{\lfloor nt \rfloor}}{\sqrt{n}} + r)^2)\mathbb{E}(\mathbf{1}_{(|S_{\lfloor nt \rfloor}/\sqrt{n}| > K)})} \\
&\leq \sqrt{M\mathbb{P}(|S_{\lfloor nt \rfloor}/\sqrt{n}| > K)} \\
&\leq \sqrt{2M \exp(-\frac{K^2}{2\tau})}.
\end{aligned}$$

Pick $\varepsilon > 0$, then there exists $K > 0$ such that

$$|\mathbb{E}[f(\frac{S_{\lfloor nt \rfloor}}{\sqrt{n}} + r)(1 - g^{\varepsilon}_{-K,K}(\frac{S_{\lfloor nt \rfloor}}{\sqrt{n}}))]| < \varepsilon/3$$

for all n and in the same way

$$|\mathbb{E}[f(x + W_t)(1 - g^{\varepsilon}_{-K,K}(W_t)]| < \varepsilon/3$$

. Finally, using Lemma 6.7, $a = -K$, $b = -K$ we have that for sufficiently large n

$$|\mathbb{E}(f(\frac{S_{\lfloor nt \rfloor}}{\sqrt{n}} + r)) - w(t, r)| < \varepsilon.$$

■

6.2.1 The case when $\alpha > 0$

In this subsection we will show that for $\alpha > 0$

$$\mathbb{E}(f\left(\frac{S_{\lfloor nt \rfloor}}{n^{1/2+\alpha}} + x\right)) \to f(x), \tag{6.16}$$

and uniformly for (t, x) over a compact set.

Lemma 6.10. *If $\tau > 0$ then for all $t \leq \tau$*

$$\mathbb{P}\left(\left|\frac{S_{\lfloor nt \rfloor}}{n^{1/2+\alpha}}\right| \geq y\right) \leq 2e^{-y^2/(2\tau)}$$

for all $y \geq 0$.

Proof. It is not hard to show that

$$\left\{\left|\frac{S_{\lfloor nt\rfloor}}{n^{1/2+\alpha}}\right| \geq y\right\} \subset \left\{\left|\frac{S_{\lfloor nt\rfloor}}{n^{1/2}}\right| \geq y\right\},$$

and the claim now follows from the result of Lemma 6.6. part (b). ■

Using exactly the same argumentation we get a version of Lemma 6.8:

Lemma 6.11. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that there exist $C > 0$ and a locally integrable $g : \mathbb{R}^+ \to \mathbb{R}^+$ with the property*

$$|f(x)|^2 \leq C + \int_0^{|x|} g(y)\, dy, \tag{6.17}$$

for all $x \in \mathbb{R}$, and

$$\int_0^\infty g(y) e^{-y^2/(2\tau)}\, dy < \infty, \tag{6.18}$$

for some $\tau > 0$. Then there exists $M > 0$ (that depends on C, g and τ) such that for all $t \leq \tau$ and all n

$$\mathbb{E}[|f(S_{\lfloor nt\rfloor}/n^{1/2+\alpha})|^2] < M. \tag{6.19}$$

Now, under similar conditions as in Theorem 6.9, we have:

Theorem 6.12. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and $a < b$ finite real numbers, such that there exist $C > 0$ and a locally integrable $g : \mathbb{R}^+ \to \mathbb{R}^+$ with the property*

$$\sup_{r\in[a,b]} (f(x+r))^2 \leq C + \int_0^{|x|} g(y)\, dy,$$

for all $x \in \mathbb{R}$, and

$$\int_0^\infty g(y) e^{-y^2/(2\tau)}\, dy < \infty,$$

for some $\tau > 0$. Then for all $r \in [a, b]$

$$\mathbb{E}(f(\frac{S_{\lfloor nt\rfloor}}{n^{1/2+\alpha}} + r)) \to f(r), \tag{6.20}$$

as $n \to \infty$ where $t < \tau$ Further, the convergence in (6.20) is uniform on $[0, \tau) \times [a, b]$.

Proof. Let $\varepsilon > 0$. The function f on $[a-1, b+1]$ is uniformly continuous, and hence there exists $\delta \in (0,1)$ such that for all $y, y' \in [a-1, b+1]$ if $|y - y'| < \delta$ then $|f(y) - f(y')| < \varepsilon$. Now,

$$|\mathbb{E}(f(\frac{S_{\lfloor nt \rfloor}}{n^{1/2+\alpha}} + r)) - f(r)| \leq \mathbb{E}\left| f(\frac{S_{\lfloor nt \rfloor}}{n^{1/2+\alpha}} + r) - f(r) \right|$$

$$\leq \mathbb{E}\left[\underbrace{\left| f(\frac{S_{\lfloor nt \rfloor}}{n^{1/2+\alpha}} + r) - f(r) \right|}_{\leq \varepsilon} \mathbf{1}(|S_{\lfloor nt \rfloor}/n^{1/2+\alpha}| < \delta) \right]$$

$$+\mathbb{E}\left[\left| f(\frac{S_{\lfloor nt \rfloor}}{n^{1/2+\alpha}} + r) - f(r) \right| \mathbf{1}(|S_{\lfloor nt \rfloor}/n^{1/2+\alpha}| \geq \delta) \right],$$

by uniform continuity of f, and triangle inequality we get

$$\leq \varepsilon + \mathbb{E}\left[\left| f(\frac{S_{\lfloor nt \rfloor}}{n^{1/2+\alpha}} + r) \right| \mathbf{1}(|\frac{S_{\lfloor nt \rfloor}}{n^{1/2+\alpha}}| \geq \delta) \right] + |f(r)| \mathbb{P}(|\frac{S_{\lfloor nt \rfloor}}{n^{1/2+\alpha}}| \geq \delta).$$

By Cauchy-Schwarz we get

$$\leq \varepsilon + \mathbb{E}\left[\left| f(\frac{S_{\lfloor nt \rfloor}}{n^{1/2+\alpha}} + r) \right|^2 \right]^{1/2} \mathbb{P}(|\frac{S_{\lfloor nt \rfloor}}{n^{1/2}}| \geq n^{\alpha}\delta)^{1/2} + \left(\max_{y \in [a,b]} |f(y)| \right) \mathbb{P}(|\frac{S_{\lfloor nt \rfloor}}{n^{1/2}}| \geq n^{\alpha}\delta).$$

Using Lemma 6.11 for the function $f(\cdot + r)$ on the expectation, and Lemma 6.6. part (b) on the probabilities, we get

$$\leq \varepsilon + M^{1/2}\sqrt{2}\exp(\frac{-n^{2\alpha}\delta^2}{4}) + \left(\max_{y \in [a,b]} |f(y)| \right) \cdot 2\exp(\frac{-n^{2\alpha}\delta^2}{2}).$$

Note that the bound doesn't depend on t or r, and we have

$$\limsup_{n \to \infty} \sup_{t,r} |\mathbb{E}(f(\frac{S_{\lfloor nt \rfloor}}{n^{1/2+\alpha}} + r)) - f(r)| \leq \varepsilon.$$

Since, $\varepsilon > 0$ is arbitrary the claim follows. ■

6.2.2 Summary

Theorem 6.13. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and $a < b$ finite real numbers, such that there exist $C > 0$ and a locally integrable $g : \mathbb{R}^+ \to \mathbb{R}^+$ with the property*

$$\sup_{r \in [a,b]} (f(x+r))^2 \leq C + \int_0^{|x|} g(y)\, dy,$$

for all $x \in \mathbb{R}$*, and*

$$\int_0^\infty g(y)e^{-y^2/(2\tau)}\,dy < \infty,$$

for some $\tau > 0$*. Then for all* $r \in [a,b]$

$$\mathbb{E}\left[f\left(\frac{S_{\lfloor nt\rfloor}}{n^{1/2+\alpha}} + r\right)\right] \to w_\alpha(t,r), \tag{6.21}$$

as $n \to \infty$ *where* $t < \tau$ *and* w_α *is the solution to the initial value problem given by*

$$\begin{cases} \partial_t w_\alpha = \begin{cases} 0, & \alpha > 0 \\ \frac{1}{2}\partial_{xx} w_\alpha & \alpha = 0 \end{cases} \\ w_\alpha(0,x) = f(x) \end{cases}.$$

Further, the convergence in (6.21) *is uniform on* $[0,\tau) \times [a,b]$*.*

6.3 Euler method for the stochastic heat equation

Let u be the solution to the heat equation

$$\partial_t u = \beta \partial_{xx} u + f. \tag{6.22}$$

Now, we discretize this equation at the point (t,x)

$$u_t(t,x) \approx \frac{u(t+\Delta t,x) - u(t,x)}{\Delta t},$$
$$u_{xx}(t,x) \approx \frac{u(t,x+\Delta x) - 2u(t,x) + u(t,x-\Delta x)}{(\Delta x)^2},$$

where Δt and Δx are small and positive. So equation (6.22) becomes

$$u(t+\Delta t,x) \approx \frac{\beta\Delta t}{(\Delta x)^2}(u(t,x+\Delta x) + u(t,x-\Delta x)) + (1 - 2\frac{\beta\Delta t}{(\Delta x)^2})u(t,x) + \Delta t f(t,x).$$

Now, if we set $t_k = k\Delta t$, $x_k = k\Delta x$, for $k = 0,1,2,\ldots$, and if we replace $u(t_j,x_k)$ by U_k^j and $f(t_j,x_k)$ by f_k^j, we get

$$U_k^{j+1} = s(U_{k+1}^j + U_{k-1}^j) + (1-2s)U_k^j + (\Delta t)f_k^j,$$

where $s = \frac{\beta\Delta t}{(\Delta x)^2}$.

Further, if the equation (6.22) has

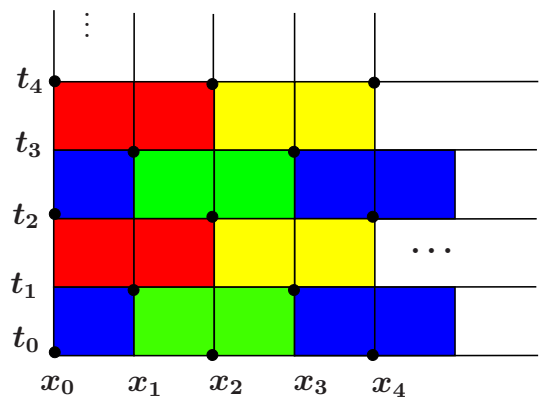


Figure 6.1: Euler method

(a) initial condition $u(x,0) = g(x)$;

(b) 0-boundary condition $u(0,t) = 0$;

then we set

(a) initial condition $U_k^0 = g(x_k)$;

(b) 0-boundary condition $U_0^j = 0$.

The given scheme is called the **explicit Euler method** for the one dimensional heat equation. It is stable if $s \leq 1/2$. (See [26].)

We are interested for the Euler method in the case of the stochastic heat equation

$$\partial_t v = \frac{1}{2}\partial_{xx} v + \mathbb{W},$$

with initial and boundary value conditions $v(0,x) = 0$ and $v(t,0) = 0$. We will look at the method when $\Delta x = n^{-1/2}$ and $\Delta t = 1/n$. In this case $s = \frac{1}{2}$ and Euler method looks like this

$$V_k^{j+1} = \frac{1}{2}(V_{k+1}^j + V_{k-1}^j) + \frac{\sqrt{n}}{2}\mathbb{W}(R_{jk}), \quad k \geq 1, j \geq 0 \tag{6.23}$$

where $R_{jk} = [x_{k-1}, x_{k+1}] \times [t_j, t_{j+1}]$. We will work with the case when $V_0^j = 0$ and $V_k^0 = 0$. (Initial and boundary value conditions are 0.)

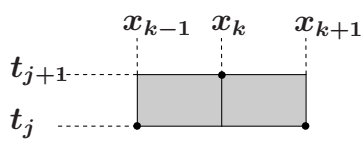


Figure 6.2: Rectangle R_{jk}

It is not hard to get the following result.

Lemma 6.14. *The solution to the difference equation* (6.23) *with initial and boundary condition* 0 *is given by*

$$V_k^j$$

$$= \frac{\sqrt{n}}{2} \sum_{j'=1}^{j} \sum_{k'=1}^{k+j} \left(\mathbb{P}(\frac{1}{\sqrt{n}} S_{n(t_j - t_{j'})} + x_k = x_{k'}) - \mathbb{P}(\frac{1}{\sqrt{n}} S_{n(t_j - t_{j'})} + x_k = -x_{k'}) \right) \mathbb{W}(R_{j'-1,k'}) \tag{6.24}$$

$$= \frac{\sqrt{n}}{2} \sum_{j'=1}^{j} \sum_{k'=1}^{k+j} \left(\mathbb{P}(S_{j-j'} = k' - k) - \mathbb{P}(S_{j-j'} = -k - k') \right) \mathbb{W}(R_{j'-1,k'}). \tag{6.25}$$

Remark. Note that $j - j'$ and $k - k'$ have to be of both either even or odd. Otherwise, $\mathbb{P}(S_{j-j'} = k' - k) = \mathbb{P}(S_{j-j'} = -k - k') = 0$. So in the upper formula the only use the rectangles $R_{j'-1,k'}$ where $j - j' \equiv k - k' \pmod 2$.

Having in mind this remark we will focus on lattice points that are in the same class as $(0, 0)$:

$$\mathcal{L}_0^n = \{(t_j, x_k) : k, j \geq 0,\ j \equiv k \pmod 2\}$$

(Points in $\mathcal{L}_0^n$ are marked with $\bullet$ on Figure 6.1.) Note that if $(t_j, x_k) \in \mathcal{L}_0^n$ then if $(t_{j'}, x_{k'}) \notin \mathcal{L}_0^n$

then

$$\mathbb{P}(\frac{1}{\sqrt{n}}S_{n(t_j-t_{j'})}+x_k=x_{k'})-\mathbb{P}(\frac{1}{\sqrt{n}}S_{n(t_j-t_{j'})}+x_k=-x_{k'})=0.$$

Also, note that if $|k'-k|>j-j'$ then the previous equality also holds.

Our aim is to show that for a compact set K when $n\to\infty$to discover the rate of convergence to 0 of

$$\sup\{|V_k^j-v(t_j,x_k)|:(t_j,x_k)\in K\cap\mathcal{L}_0^n\},$$

where v is the solution to the stochastic heat equation with 0-boundary and 0-initial condition. Recall (see §A.6), that the solution to the homogeneous stochastic heat equation with 0 initial and boundary conditions is

$$v(t,x)=\int_0^t\frac{1}{\sqrt{2\pi(t-s)}}\int_{\mathbb{R}^+}\left(e^{-\frac{|x-y|^2}{2(t-s)}}-e^{-\frac{|x+y|^2}{2(t-s)}}\right)\mathbb{W}(ds,dy).$$

Note that for $(t_j,x_k)\in\mathcal{L}_0^n$, $j,k>0$ we have

$$V_k^j=\sum_{j'=1}^{j}\sum_{(t_{j'},x_{k'})\in\mathcal{L}_0^n}\frac{\sqrt{n}}{2}\left(\mathbb{P}(\frac{1}{\sqrt{n}}S_{n(t_j-t_{j'})}+x_k=x_{k'})-\mathbb{P}(\frac{1}{\sqrt{n}}S_{n(t_j-t_{j'})}+x_k=-x_{k'})\right)\mathbb{W}(R_{j'-1,k'})$$

where $R_{j,0}=[t_j,t_{j+1}]\times[x_0,x_1]$. (Notice, that in the case when $k'=0$, the expression in the sum is equal to 0.)

Note that (see Figure 6.1)

$$\bigcup_{j'=1}^{j}\bigcup_{(t_{j'},x_{k'})\in\mathcal{L}_0^n}R_{j'-1,k'}=[0,t_j]\times[0,\infty],$$

therefore

$$\begin{aligned}&V_j^k-v(t_j,x_k)\\=&\sum_{j'=1}^{j}\sum_{(t_{j'},x_{k'})\in\mathcal{L}_0^n}\\&\frac{\sqrt{n}}{2}\left(\mathbb{P}(\frac{1}{\sqrt{n}}S_{n(t_j-t_{j'})}+x_k=x_{k'})-\mathbb{P}(\frac{1}{\sqrt{n}}S_{n(t_j-t_{j'})}+x_k=-x_{k'})\right)\mathbb{W}(R_{j'-1,k'})-\\&\int_{R_{j'-1,k'}}\frac{1}{\sqrt{2\pi(t_j-s)}}\left(e^{-\frac{|x_k-y|^2}{2(t_j-s)}}-e^{-\frac{|x_k+y|^2}{2(t_j-s)}}\right)\mathbb{W}(ds,dy).\end{aligned}$$

Since $\mathbb{E}(V_j^k - v(t_j, x_k)) = 0$, for the variance we have

$$\begin{aligned}
&\mathbb{E}[(V_j^k - v(t_j, x_k))^2] \\
=&\sum_{j'=1}^{j} \sum_{(t_{j'}, x_{k'}) \in \mathcal{L}_0^n} \\
&\int_{R_{j'-1,k'}} \left[\frac{\mathbb{P}(\frac{1}{\sqrt{n}} S_{n(t_j - t_{j'})} + x_k = x_{k'}) - \mathbb{P}(\frac{1}{\sqrt{n}} S_{n(t_j - t_{j'})} + x_k = -x_{k'})}{2/\sqrt{n}} \right. \\
&\left. - \frac{1}{\sqrt{2\pi(t_j - s)}} \left(e^{-\frac{|x_k - y|^2}{2(t_j - s)}} - e^{-\frac{|x_k + y|^2}{2(t_j - s)}} \right) \right]^2 dy\, ds
\end{aligned} \tag{6.26}$$

6.3.1 Convergence of the Euler Method

The main result of this subsection is the following,

Theorem 6.15. *There exists $\gamma > 0$ and $\Gamma > 0$ such that*

$$\mathbb{E}[(v(t_j, x_k) - V_k^j)^2] \leq \frac{\Gamma}{n^\gamma}.$$

This rate will help us to prove the convergence of the Euler method.

Recall from (6.26) that

$$\begin{aligned}
&\mathbb{E}[(V_j^k - v(t_j, x_k))^2] \\
=&\sum_{j'=1}^{j} \sum_{(t_{j'}, x_{k'}) \in \mathcal{L}_0^n} \\
&\int_{R_{j'-1,k'}} \left[\frac{\sqrt{n}}{2} \left(\mathbb{P}\left(\frac{S_{j-j'}}{\sqrt{n}} = x_{k'} - x_k \right) - \mathbb{P}\left(\frac{S_{j-j'}}{\sqrt{n}} = -x_{k'} - x_k \right) \right) \right. \\
&\left. - \frac{1}{\sqrt{2\pi(t_j - s)}} \left(e^{-\frac{|x_k - y|^2}{2(t_j - s)}} - e^{-\frac{|x_k + y|^2}{2(t_j - s)}} \right) \right]^2 dy\, ds
\end{aligned}$$

We will decompose our sum over j's into two sums:

$$\sum_{j'=1}^{j} = \sum_{j - j' < n^{1/3+\alpha}} + \sum_{j - j' \geq n^{1/3+\alpha}} \tag{6.27}$$

where $\alpha > 0$ is some small positive number to be determined later.

$$\sum_{j-j'<n^{1/3+\alpha}} \sum_{(t_{j'},x_{k'})\in\mathcal{L}_0^n} \int_{R_{j'-1,k'}} \left[\frac{\sqrt{n}}{2} \left(\mathbb{P}\left(\frac{S_{j-j'}}{\sqrt{n}} = x_{k'} - x_k \right) - \mathbb{P}\left(\frac{S_{j-j'}}{\sqrt{n}} = -x_{k'} - x_k \right) \right) - \frac{1}{\sqrt{2\pi(t_j - s)}} \left(e^{-\frac{|x_k - y|^2}{2(t_j - s)}} - e^{-\frac{|x_k + y|^2}{2(t_j - s)}} \right) \right]^2 dy\, ds. \tag{6.28}$$

From the inequality $(a+b+c+d)^2 \leq 4a^2 + 4b^2 + 4c^2 + 4d^2$, so we have that the sum (6.28) is less than:

$$\sum_{j-j'<n^{1/3+\alpha}} \sum_{(t_{j'},x_{k'})\in\mathcal{L}_0^n} \left[\frac{2}{\sqrt{n}} \left(\mathbb{P}\left(\frac{S_{j-j'}}{\sqrt{n}} = x_{k'} - x_k \right)^2 + \mathbb{P}\left(\frac{S_{j-j'}}{\sqrt{n}} = -x_{k'} - x_k \right)^2 \right) + \int_{R_{j'-1,k'}} \frac{2}{\pi(t_j - s)} \left(e^{-\frac{|x_k - y|^2}{(t_j - s)}} + e^{-\frac{|x_k + y|^2}{(t_j - s)}} \right) \right] dy\, ds. \tag{6.29}$$

It is not hard to see that using sub-additivity we have

$$\sum_{(t_{j'},x_{k'})\in\mathcal{L}_0^n} \mathbb{P}\left(\frac{S_{j-j'}}{\sqrt{n}} = \pm x_{k'} - x_k \right)^2 \leq \sum_{(t_{j'},x_{k'})\in\mathcal{L}_0^n} \mathbb{P}\left(\frac{S_{j-j'}}{\sqrt{n}} = \pm x_{k'} - x_k \right) \leq 1.$$

Therefore

$$\sum_{0\leq j-j'<n^{1/3+\alpha}} \sum_{(t_{j'},x_{k'})\in\mathcal{L}_0^n} \left[\frac{2}{\sqrt{n}} \left(\mathbb{P}\left(\frac{S_{j-j'}}{\sqrt{n}} = x_{k'} - x_k \right)^2 + \mathbb{P}\left(\frac{S_{j-j'}}{\sqrt{n}} = -x_{k'} - x_k \right)^2 \right) \right] \leq \frac{4}{\sqrt{n}} n^{1/3+\alpha} = 4n^{\alpha-1/6}. \tag{6.30}$$

Further

$$\sum_{j-j'<n^{1/3+\alpha}} \sum_{(t_{j'},x_{k'})\in\mathcal{L}_0^n} \int_{R_{j'-1,k'}} \frac{2}{\pi(t_j-s)} e^{-\frac{|x_k\pm y|^2}{(t_j-s)}}\,dy\,ds$$
$$= \int_{t_{j*}}^{t_j}\int_0^\infty \frac{2}{\pi(t_j-s)} e^{-\frac{|x_k\pm y|^2}{(t_j-s)}}\,dy\,ds \le \int_{t_{j*}}^{t_j}\int_{-\infty}^\infty \frac{2}{\pi(t_j-s)} e^{-\frac{|x_k\pm y|^2}{(t_j-s)}}\,dy\,ds$$
$$= \int_{t_{j*}}^{t_j} \frac{2}{\sqrt{\pi(t_j-s)}}\,ds = \frac{4}{\sqrt{\pi}}\sqrt{t_j-t_{j*}} = \frac{8}{\sqrt{\pi}}\sqrt{\frac{j-j^*}{n}}, \tag{6.31}$$

where $j^*+1 = \min\{j' : j'\ge 1,\ 0\le j-j'\le n^{1/3+\alpha}\}$, so the sum (6.31) is less than

$$\frac{8}{\sqrt{\pi}}\sqrt{\frac{n^{1/3+\alpha}+1}{n}} = \frac{8}{\sqrt{\pi}}\sqrt{n^{\alpha-2/3}+n^{-1}}. \tag{6.32}$$

Hence, from (6.30) and (6.32) the sum (6.28) is bounded by

$$4n^{\alpha-1/6} + \frac{8}{\sqrt{\pi}}\sqrt{n^{\alpha-2/3}+n^{-1}}. \tag{6.33}$$

In order to estimate $\sum_{j-j'\ge n^{\alpha+1/3}}$ - part of the sum (6.27) we first need to do some estimates on the gradient of the function

$$F(t,x) = \frac{1}{\sqrt{2\pi(t_j-t)}}\exp\left(-\frac{(x_j-x)^2}{2(t_j-t)}\right)$$

for $(t,x)\in[0,t_j)\times\mathbb{R}$.

Lemma 6.16. *(a) For fixed $t<t_j$ we have*

$$\sup_{x\in\mathbb{R}} \|\nabla_{t,x}F(t,x)\|^2 \le \max\left\{\frac{1}{8\pi(t_j-t)^3}, \frac{1}{2e(t_j-t)^2}, \frac{(1-(t_j-t))e^{-3+4(t_j-t)}}{2\pi(t_j-t)^3}\right\}. \tag{6.34}$$

(b) For $A>0$ there exists a constant C_A (depending only on A) such that for $0\le t<t_j\le A$ we have

$$\sup_{x\in\mathbb{R}} \|\nabla_{t,x}F(t,x)\| \le \frac{C_A}{(t_j-t)^{3/2}}.$$

Proof. (a) By doing taking derivatives we have

$$D^2(t,x) = \|\nabla_{t,x}F(t,x)\|^2 = (\partial_t F(t,x))^2 + (\partial_x F(t,x))^2 =$$

$$= \exp(-\frac{(x-x_k)^2}{t_j-t}) \left[\frac{1}{8\pi} \left(\frac{1}{(t_j-t)^{3/2}} - \frac{(x-x_k)^2}{(t_j-t)^{5/2}} \right)^2 + \frac{1}{2\pi} \cdot \frac{(x-x_k)^2}{(t_j-t)^3} \right]$$

It is clear that $\lim_{x\to\pm\infty} D^2(t,x) = 0$, so there exists a maximum, and it is obtained at the zeros of $\partial_x(D^2(t,x)) =$

$$= -\frac{(x-x_k)((x-x_k)^2-(t_j-t))((x-x_k)^2-(3(t_j-t)-4(t_j-t)^2)}{4\pi(t_j-t)^6} \exp\left(-\frac{(x-x_k)^2}{t_j-t}\right).$$

If we set $x - x_k = 0$ we get $D^2(t,x) = (8\pi(t_j-t)^3)^{-1}$; for $(x-x_k)^2 = (t_j-t)$ we have $D^2(t,x) = (2e(t_j-t)^2)^{-1}$; for $(x-x_k)^2 = 3(t_j-t) - 4(t_j-t)^2$ (note that this may not be solvable) we have $D^2(t,x) = \frac{(1-(t_j-t))e^{-3+4(t_j-t)}}{2\pi(t_j-t)^3}$. If we can solve the equation in the last case then we have an equality in (6.34), otherwise we have an inequality.

(b) Since $(t,t_j) \mapsto (1-(t_j-t))e^{-3+4(t_j-t)}$ obtains a maximum M_A on the compact set $[0,A]^2$, we have

$$(t_j-t)^3 \sup_{x\in\mathbb{R}} \|\nabla_{t,x} F(t,x)\|^2 \le \max\{\frac{1}{8\pi}, \frac{\overbrace{t_j-t}^{\le 2A}}{2e}, M_A\}.$$

■

Corollary 6.17. *For $0 \le t < t_j \le A$ and $0 < L \le U$ we have*

$$\sup_{L\le t_j-t\le U} \sup_{x\in\mathbb{R}} \|\nabla_{t,x} F(t,x)\| \le \frac{C_A}{L^{3/2}}.$$

We now have everything we need to estimate $\sum_{j-j'\ge n^{1/3+\alpha}}$ - part of the sum:

$$\sum_{j-j'\ge n^{1/3+\alpha}} \sum_{(t_{j'},x_{k'})\in\mathcal{L}_0^n}$$
$$\int_{R_{j'-1,k'}} \left[\frac{\sqrt{n}}{2} \left(\mathbb{P}\left(\frac{S_{j-j'}}{\sqrt{n}} = x_{k'} - x_k \right) - \mathbb{P}\left(\frac{S_{j-j'}}{\sqrt{n}} = -x_{k'} - x_k \right) \right) \right.$$
$$\left. - \frac{1}{\sqrt{2\pi(t_j-s)}} \left(e^{-\frac{|x_k-y|^2}{2(t_j-s)}} - e^{-\frac{|x_k+y|^2}{2(t_j-s)}} \right) \right]^2 dy\, ds \tag{6.35}$$

We first give an upper bound for

$$\left|\frac{\sqrt{n}}{2}\left(\mathbb{P}\left(\frac{S_{j-j'}}{\sqrt{n}}=x_{k'}-x_k\right)-\mathbb{P}\left(\frac{S_{j-j'}}{\sqrt{n}}=-x_{k'}-x_k\right)\right)\right.$$
$$\left.-\frac{1}{\sqrt{2\pi(t_j-s)}}\left(e^{-\frac{|x_k-y|^2}{2(t_j-s)}}-e^{-\frac{|x_k+y|^2}{2(t_j-s)}}\right)\right| \tag{6.36}$$

where $(s,y)\in R_{j'-1,k'}$. By triangle inequality, expression (6.36) is less or equal to

$$\left|\frac{\sqrt{n}}{2}\mathbb{P}\left(\frac{S_{j-j'}}{\sqrt{n}}=x_{k'}-x_k\right)-\frac{1}{\sqrt{2\pi(t_j-t_{j'})}}e^{-\frac{|x_k-x_{k'}|^2}{2(t_j-t_{j'})}}\right|$$
$$+\left|-\frac{\sqrt{n}}{2}\mathbb{P}\left(\frac{S_{j-j'}}{\sqrt{n}}=-x_{k'}-x_k\right)+\frac{1}{\sqrt{2\pi(t_j-t_{j'})}}e^{-\frac{|x_k+x_{k'}|^2}{2(t_j-t_{j'})}}\right|$$
$$+\left|\frac{1}{\sqrt{2\pi(t_j-t_{j'})}}e^{-\frac{|x_k-x_{k'}|^2}{2(t_j-t_{j'})}}-\frac{1}{\sqrt{2\pi(t_j-s)}}e^{-\frac{|x_k-y|^2}{2(t_j-s)}}\right|$$
$$+\left|-\frac{1}{\sqrt{2\pi(t_j-t_{j'})}}e^{-\frac{|x_k+x_{k'}|^2}{2(t_j-t_{j'})}}+\frac{1}{\sqrt{2\pi(t_j-s)}}e^{-\frac{|x_k+y|^2}{2(t_j-s)}}\right|. \tag{6.37}$$

The first two terms in (6.37), by Theorem 6.3, are less than $\frac{C}{n^{3\alpha}}$. By mean-value theorem, the last two terms in (6.37) are less than

$$\sup_{(s,y)\in R_{j'-1,k'}}\|\nabla_{t,x}F(s,y)\|\sqrt{(t_j-s)^2+(x_k-y)^2},$$

and by the definition of $R_{j'-1,k'}=[t_{j'-1},t_{j'}]\times[x_{(k'-1)\wedge 0},x_{k'+1}]$, this is less than

$$\left(\sup_{t_j-t_{j'}\le t_j-s\le t_j-t_{j'-1}}\|\nabla_{t,x}F(s,y)\|\right)\sqrt{(t_j-s)^2+(x_k-y)^2}\le\frac{C_A}{(t_j-t_{j'})^{3/2}}\sqrt{\frac{1}{n^2}+\frac{1}{n}}$$

$$=\frac{C_A}{(j-j')^{3/2}}\sqrt{n+n^2}.$$

Now, for $j - j' \geq n^{1/3+\alpha}$ we have

$$\begin{aligned}
&\sum_{(t_{j'},x_{k'})\in\mathcal{L}_0^n} \int_{R_{j'-1,k'}} \left[\frac{\sqrt{n}}{2} \left(\mathbb{P}\left(\frac{S_{j-j'}}{\sqrt{n}} = x_{k'} - x_k \right) - \mathbb{P}\left(\frac{S_{j-j'}}{\sqrt{n}} = -x_{k'} - x_k \right) \right) \right. \\
&\left. - \frac{1}{\sqrt{2\pi(t_j - s)}} \left(e^{-\frac{|x_k - y|^2}{2(t_j - s)}} - e^{-\frac{|x_k + y|^2}{2(t_j - s)}} \right) \right]^2 dy\, ds \\
\leq &\left(\frac{C_A}{(j-j')^{3/2}} \sqrt{n + n^2} + \frac{C}{n^{3\alpha}} \right) \sum_{(t_{j'},x_{k'})\in\mathcal{L}_0^n} \\
&\int_{R_{j'-1,k'}} \left| \frac{\sqrt{n}}{2} \left(\mathbb{P}\left(\frac{S_{j-j'}}{\sqrt{n}} = x_{k'} - x_k \right) - \mathbb{P}\left(\frac{S_{j-j'}}{\sqrt{n}} = -x_{k'} - x_k \right) \right) \right. \\
&\left. - \frac{1}{\sqrt{2\pi(t_j - s)}} \left(e^{-\frac{|x_k - y|^2}{2(t_j - s)}} - e^{-\frac{|x_k + y|^2}{2(t_j - s)}} \right) \right| dy\, ds
\end{aligned} \tag{6.38}$$

First note, that we have

$$\begin{aligned}
&\sum_{(t_{j'},x_{k'})\in\mathcal{L}_0^n} \int_{R_{j'-1,k'}} \left| \frac{\sqrt{n}}{2} \mathbb{P}\left(\frac{S_{j-j'}}{\sqrt{n}} = \pm x_{k'} - x_k \right) \right| \\
\leq &\frac{1}{n} \sum_{(t_{j'},x_{k'})\in\mathcal{L}_0^n} \mathbb{P}\left(\frac{S_{j-j'}}{\sqrt{n}} = \pm x_{k'} - x_k \right) \leq \frac{1}{n}.
\end{aligned} \tag{6.39}$$

Further,

$$\begin{aligned}
&\sum_{(t_{j'},x_{k'})\in\mathcal{L}_0^n} \int_{R_{j'-1,k'}} \left| \frac{1}{\sqrt{2\pi(t_j - s)}} \left(e^{-\frac{|x_k - y|^2}{2(t_j - s)}} - e^{-\frac{|x_k + y|^2}{2(t_j - s)}} \right) \right| dy\, ds \\
\leq &\int_{t_{j'-1}}^{t_{j'}} \int_0^\infty \left| \frac{1}{\sqrt{2\pi(t_j - s)}} \left(e^{-\frac{|x_k - y|^2}{2(t_j - s)}} - e^{-\frac{|x_k + y|^2}{2(t_j - s)}} \right) \right| dy\, ds \\
\leq &\int_{t_{j'-1}}^{t_{j'}} \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi(t_j - s)}} e^{-\frac{|x_k - y|^2}{2(t_j - s)}}\, dy\, ds \\
= &t_{j'} - t_{j'-1} = \frac{1}{n}.
\end{aligned} \tag{6.40}$$

Using triangle inequality the expression (6.38) is bounded by

$$\left(\frac{C_A}{(j-j')^{3/2}} \sqrt{n + n^2} + \frac{C}{n^{3\alpha}} \right) \frac{3}{n}.$$

Finally, the $\sum_{j-j'\geq n^{\alpha+1/3}}$-part of the sum is less than

$$\sum_{j-j'\geq n^{\alpha+1/3}} \left(\frac{3C_A}{(j-j')^{3/2}} \sqrt{\frac{1}{n}+1} + \frac{3C}{n^{3\alpha+1}} \right).$$

Since the sum goes over j' with the property $n^{\alpha+1/3} \leq j-j' \leq \lfloor nA \rfloor \leq nA$ we have

$$\leq \int_{n^{\alpha+1/3}-1}^{\infty} \frac{1}{h^{3/2}}\, dh + \frac{3A}{n^{3\alpha}} =$$

$$(n^{\alpha+1/3}-1)^{-1/2} + \frac{3A}{n^{3\alpha}}. \tag{6.41}$$

Now, from (6.33) and the last bound we have that

$$\mathbb{E}[(V_j^k - v(t_j,x_k))^2] \leq 4n^{\alpha-1/6} + \frac{8}{\sqrt{\pi}}\sqrt{n^{\alpha-2/3}+n^{-1}} + (n^{\alpha+1/3}-1)^{-1/2} + \frac{3}{n^{3\alpha}}$$

Set $\gamma := \min\{1/6-\alpha, 3\alpha\}$, where $\alpha > 0$ such that $\gamma > 0$. Then there exists $\Gamma > 0$ (that depends on γ) such that

$$\mathbb{E}[(V_j^k - v(t_j,x_k))^2] \leq \frac{\Gamma}{n^{\gamma}}.$$

This discussion proves Theorem 6.15.

Now from Theorem 5.6 (inequality (5.10)) we know that

$$\mathbb{E}[\sup_{(t_j,x_k)\in\mathcal{L}_0^n\cap K} |V_j^k - v(t_j,x_k)|^2] \leq 2\frac{\Gamma}{n^{\gamma}} \ln(n^{3/2}AB+1). \tag{6.42}$$

since $|\mathcal{L}_0^n \cap K| \leq \lfloor nA \rfloor \lfloor \sqrt{n}B \rfloor \leq n^{3/2}AB$.

The following shows th convergence of the Euler method.

Corollary 6.18. *Using the same notation as before we have,*

$$\sup_{(t_j,x_k)\in\mathcal{L}_0^n\cap K} |V_j^k - v(t_j,x_k)| \to 0 \quad as \;\; n\to\infty \quad a.s. \tag{6.43}$$

Further for $\beta < \gamma/2$,

$$n^{\beta} \sup_{(t_j,x_k)\in\mathcal{L}_0^n\cap K} |V_j^k - v(t_j,x_k)| \to 0 \quad as \;\; n\to\infty \quad a.s.$$

Proof. Since $\{V_j^k - v(t_j, x_k) : (t_j, x_k) \in \mathcal{L}_0^n \cap K\}$ is a family of Gaussian random variables, by Theorem 5.9 the inequality (6.42) implies (6.43).

The second inequality follows from the fact that for every $\varepsilon \in (0, \gamma)$ there exists $C > 0$ such that

$$\frac{\Gamma}{n^\gamma} \ln(n^{3/2} AB) \leq \frac{C}{n^{\gamma - \varepsilon}}.$$

By using Corollary 5.10. we get the desired result. ■

6.4 Convergence of interpolation of the Euler method

We know the values V_j^k at $(t_j, x_k) \in \mathcal{L}_0^n$ and we want to approximate the solutions to the heat equation on the rest of the plane.

We are doing the interpolation in the following way:

- We do a linear interpolation between points (t_j, x_k) and (t_{j+1}, x_{k+1}) for all $(t_j, x_k) \in \mathcal{L}_0^n$.
- We do a linear interpolation between points (t_j, x_k) and (t_{j-1}, x_{k+1}) whenever (t_j, x_k), $(t_{j-1}, x_{k+1}) \in \mathcal{L}_0^n$.
- We set all values on x and y axis to be 0.
- Finally, each point (t, x) is linearly approximated by the values (t, x_-) and (t, x_+) the closest points previously defined with respect to the x-coordinate.

In this way we obtain the approximation $V_n(t, x)$ of the stochastic heat equation on $\mathbb{R}_+^2$, and we want to show convergence to u on compact sets, where

$$v(t, x) = \int_0^t \frac{1}{\sqrt{2\pi(t-s)}} \int_{\mathbb{R}^+} \left(e^{-\frac{|x-y|^2}{2(t-s)}} - e^{-\frac{|x+y|^2}{2(t-s)}} \right) \mathbb{W}(ds, dy). \tag{6.44}$$

We will show that for a compact set $K \subset \mathbb{R}_+^2$ we have

$$\sup_{(t,x) \in K} |V_n(t, x) - v(t, x)| \to 0.$$

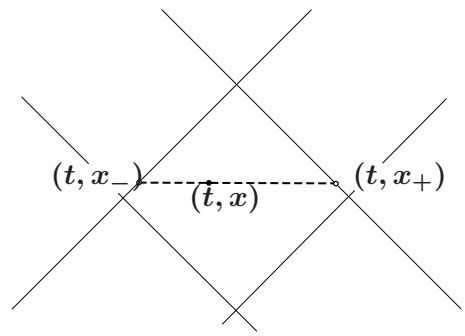


Figure 6.3: Interpolation

Theorem 6.19. *For a compact set $K \subset \mathbb{R}^2_+$ we have*

$$\lim_{n\to\infty} \sup_{(t,x)\in K} |V_n(t,x) - v(t,x)| = 0 \quad a.s.$$

Proof. Pick K, and then pick $K_{ab} = [0,a] \times [0,b]$, such that

$$\sup\{x : (t,x) \in K\} < a,$$

and

$$\sup\{t : (t,x) \in K\} < b.$$

For large n, the points $\mathcal{L}_0^n \cap K_{ab}$ will be enough to calculate the value of V_n for all points on K.

V_n at point (t,x) can be written as a convex combination of the values of the four points in $\mathcal{L}_0^n$ that make the rhombus in which the point is. Therefore

$$V_n(t,x) = \alpha_1 V_j^k + \alpha_2 V_j^{k+2} + \alpha_3 V_{j+1}^{k+1} + \alpha_4 V_{j-1}^{k+1},$$

where $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1$ (one or more of α_i's will be 0).

Now,

$$\begin{aligned} V_n(t,x) - v(t,x) &= \alpha_1(V_j^k - v(t_j,x_k)) + \alpha_2(V_j^{k+2} - v(t_j,x_{k+2})) + \alpha_3(V_{j+1}^{k+1} - v(t_j,x_{k+2})) \\ &+ \alpha_4(V_{j-1}^{k+1} - v(t_{j-1},x_{k+1})) \\ &+ \alpha_1(v(t_j,x_k) - v(t,x)) + \alpha_2(v(t_j,x_{k+2}) - v(t,x)) \\ &+ \alpha_3(v(t_j,x_{k+2}) - v(t,x)) + \alpha_4(v(t_{j-1},x_{k+1}) - v(t,x)). \end{aligned}$$

Not that v is a continuous function, and K_{ab} a compact set. Therefore, u is uniformly continuous on K_{ab}. The distance between (t,x) and points (t_j,x_k), (t_{j-1},x_{k+1}), (t_{j+1},x_{k+1}) and (t_j,x_{k+2}) goes to 0 uniformly. So, by uniform continuity for any $\varepsilon > 0$ we have $|v(t_*,x_*) - v(t,x)| < \varepsilon$, when the distance between (t_*,x_*) and (t,x) is less than some δ.

Hence, for a large n

$$\sup_{(t,x)\in K} |V_n(t,x) - v(t,x)| \leq \sup_{\mathcal{L}_0^n \cap K_{ab}} |V_j^k - v(t_j,x_k)| + \varepsilon.$$

When $n \to \infty$ we have, by (6.43)

$$\limsup_{n\to\infty} \sup_{(t,x)\in K} |V_n(t,x) - v(t,x)| \leq \varepsilon.$$

Finally, since ε is an arbitrary positive number, the claim follows. ■

We have shown that the Euler method converges uniformly on compact subsets $\mathbb{R}_+^2$ to the stochastic heat equation.

Proposition 6.20. *For each n let (E_{jk}^n) be an i.i.d. sequence of $N(0, \frac{1}{\sqrt{2n}})$, and*

$$Y_k^{j+1} = \frac{1}{2}(Y_{k+1}^j + Y_{k-1}^j) + E_{jk}, \quad j \geq 0, k \geq 1,$$

with $Y_k^0 = 0$ and $Y_0^j = 0$. Then the described interpolation $\widetilde{Y}_n(t,x)$ converges in distribution to the solution of the stochastic heat equation

$$\begin{cases} v_t = \frac{1}{2} v_{xx} + \mathbb{W} \\ v(0,\cdot) = 0, v(\cdot,0) = 0 \end{cases}$$

6.4.1 Euler method with weaker noise

We finish the study of Euler method by looking at the case when the noise is weak, so that in the limit it has no effect. What happens if E^n_{jk} would be distributed as $N(0, \frac{1}{\sqrt{2}n^{1/2+\alpha}})$ in Proposition 6.20? If we have noise with slightly lower variance, would we still have convergence. It turns out we would and that convergence would be to 0.

Lemma 6.21. *Let $\alpha > 0$. For each n let $(E^n_{jk} : k \equiv j+1 \ (mod\ 2))$ be an i.i.d. sequence of $N(0, \frac{1}{\sqrt{2}n^{1/2+\alpha}})$, and*

$$Y^{j+1}_k = \frac{1}{2}(Y^j_{k+1} + Y^j_{k-1}) + E_{jk}, \quad j \geq 0, k \geq 1, \tag{6.45}$$

with $Y^0_k = 0$ and $Y^j_0 = 0$. Then for all $0 < a < 1 + 2\alpha$,$b > 0$ and $A, B > 0$ there exist $\Gamma > 0$ and $\gamma > 0$

such that

$$\mathbb{E}\left[\sup_{j \leq An^a, k \leq Bn^b} |Y^j_k|^2\right] \leq \frac{\Gamma}{n^\gamma}. \tag{6.46}$$

Proof. It can be shown that

$$Y^j_k = \sum_{j'=1}^{j} \sum_{k'=1}^{k+j} \left(\mathbb{P}(S_{j-j'} = k' - k) - \mathbb{P}(S_{j-j'} = -k - k')\right) E^n_{j'-1,k'}$$

is a solution to (6.45), where (S_j) is a simple random walk. From the fact that E^n_{jk} are i.i.d. we have

$$\mathbb{E}(Y^j_k)^2 = \sum_{j'=1}^{j} \sum_{k'=1}^{k+j} \left(\mathbb{P}(S_{j-j'} = k' - k) - \mathbb{P}(S_{j-j'} = -k - k')\right)^2 \frac{1}{\sqrt{2}n^{1/2+\alpha}}.$$

Since $|k'-k| < |k'+k|$, we have $0 \leq \mathbb{P}(S_{j-j'} = k'-k) - \mathbb{P}(S_{j-j'} = -k-k') \leq \mathbb{P}(S_{j-j'} = k'-k)$, and so

$$\mathbb{E}(Y^j_k)^2 \leq \sum_{j'=1}^{j} \sum_{k'=1}^{k+j} \mathbb{P}(S_{j-j'} = k' - k)^2 \frac{1}{\sqrt{2}n^{1/2+\alpha}}.$$

Now, it follows from the properties of the random walk that

$$\mathbb{P}(S_{j-j'} = k' - k)^2 = \mathbb{P}(S_{2(j-j')} - S_{(j-j')} = -(k'-k))\mathbb{P}(S_{j-j'} = k' - k) =$$

$$= \mathbb{P}(S_{2(j-j')} - S_{(j-j')} = -(k'-k), S_{j-j'} = k' - k) = \mathbb{P}(S_{2(j-j')} = 0, S_{j-j'} = k' - k).$$

Furthermore,

$$\sum_{k'=1}^{k+j} \mathbb{P}(S_{j-j'} = k' - k)^2 = \sum_{k'=1}^{k+j} \mathbb{P}(S_{2(j-j')} = 0, S_{j-j'} = k' - k)$$

$$\leq \sum_{k'} \mathbb{P}(S_{2(j-j')} = 0, S_{j-j'} = k' - k) = \mathbb{P}(S_{2(j-j')} = 0).$$

If $j \leq An^a$ by Stirling's formula we have

$$\mathbb{E}(Y_k^j)^2 \leq \frac{1}{\sqrt{2}n^{1/2+\alpha}} \sum_{j'=1}^{An^a} \mathbb{P}(S_{2(j-j')} = 0) \sim \frac{1}{\sqrt{2}n^{1/2+\alpha}} \sum_{j'=1}^{An^a} \frac{1}{\sqrt{\pi j}}.$$

The last sum can be bounded by $1 + \int_1^{An^a} \frac{1}{\sqrt{t}}\, dt = 2\sqrt{An^a}$. Hence, there exists $C > 0$ such that

$$\mathbb{E}(Y_k^j)^2 \leq C \frac{2\sqrt{An^a}}{\sqrt{2}n^{1/2+\alpha}} = \frac{C\sqrt{2A}}{n^{1/2+\alpha-a/2}}.$$

Now, by (5.10) we have

$$\mathbb{E}\left[\sup_{j \leq An^a, k \leq Bn^b} |Y_{jk}|^2\right] \leq \frac{C\sqrt{2A}}{n^{1/2+\alpha-a/2}} \ln(ABn^{ab} + 1).$$

Now, for any $\gamma \in (0, 1/2 + \alpha - a/2)$ there exists Γ such that (6.46) holds. ■

Proposition 6.22. *Let Y_k^j be as in the previous Lemma. $t_j = jn^{-1}$ and $x_k = \frac{k}{n^{1/2+\alpha}}$, and set $\mathcal{L}_0^n = \{(t_j, x_k) : k \equiv j \ (mod\ 2)\}$, if we define $V_n(t_j, x_k)$ for $(t_j, x_k) \in \mathcal{L}_0^n$ to have a value Y_k^j, and do the interpolation described in §6.4, V_n converges in distribution to 0.*

Proof. Let K be a compact set, there exists $A > 0, B > 0$ such that $K \subset [0, A] \times [0, B]$, the value $\max_{(t,x)\in K} V_n(t, x)$ is obtained at some point $\mathcal{L}_0^n \cap [0, A] \times [0, B]$. Now, from Lemma 6.21. and Theorem 5.11. the claim follows. ■

6.5 Euler method with initial value condition and no external noise

We saw what happens with Euler scheme when the external source is replaced by noise.

In this section we shortly comment what happens with the Euler scheme for the heat equation with no external force and a non-zero initial value function:

$$W_k^{j+1} = \frac{1}{2}(W_{k+1}^j + W_{k-1}^j), \quad k \geq 1, j \geq 0 \tag{6.47}$$

We will work with the case when $W_0^j = 0$ and $W_k^0 = g(x_k)$.

In order to simplify our work we will set $W_{-k}^0 := -W_k^0$. In this way we don't have to think about the boundary condition $W_0^j = 0$, because it is easy to show that the scheme

$$\begin{cases} W_k^{j+1} = \frac{1}{2}(W_{k+1}^j + W_{k-1}^j), & k \in \mathbb{Z}, j \geq 0 \\ W_k^0 = \tilde{g}(x_k) & k \in \mathbb{Z}. \end{cases} \tag{6.48}$$

Where $\tilde{g}(x) = g(x)$ for $x \geq 0$ and $\tilde{g}(x) = -g(-x)$ for $x \leq 0$.

With this setup the following lemma holds.

Lemma 6.23. *For the scheme given by (6.48) we have*

$$W_k^j = \mathbb{E}\left[\tilde{g}\left(\frac{S_j}{\sqrt{n}} + \frac{k}{\sqrt{n}}\right)\right].$$

Under some mild properties on $\tilde{g}$ we have that the Euler method converges to the heat equation

$$\begin{cases} \partial_t w = \partial_{xx} w & \text{on } \mathbb{R} \times (0, \infty), \\ w(x, 0) = \tilde{g}(x) & x \in \mathbb{R}. \end{cases} \tag{6.49}$$

It is well-known that u restricted to $\mathbb{R}^+ \times (0, \infty)$ is the solution to

$$\begin{cases} \partial_t w = \partial_{xx} w & \text{on } \mathbb{R} \times (0, \infty), \\ w(x, 0) = g(x) & x \in \mathbb{R}, \\ w(0, t) = 0 & t \geq 0. \end{cases} \tag{6.50}$$

Lemma 6.24. *If $g : \mathbb{R}^+ \to \mathbb{R}$ is continuous and we have*

$$\limsup_{x\to\infty} \frac{|g(x)|}{|x|} < \infty, \tag{6.51}$$

then for a compact set $K \subset \mathbb{R}^+ \times [0,\infty)$ we have

$$\lim_{n\to\infty} \sup_{(x_k,t_j)\in\mathcal{L}_0^n\cap K} |w(x_k,t_j) - W_k^j| = 0,$$

where W_k^j is the solution to (6.47) *and w to* (6.50)*.*

Proof. First, note that there exists $L, C > 0$ such that

$$|g(x)|^2 \le L|x|^2 + C.$$

Further we have $|\tilde{g}(x)|^2 \le L|x|^2 + C$, and now for any $a < b$ we have

$$\sup_{r\in[a,b]} |\tilde{g}(x+r)|^2 \le 2L\max\{|a|^2,|b|^2\} + 2L|x|^2 + C = 2L\max\{|a|^2,|b|^2\} + C + \int_0^{|x|} 4Ly\,dy,$$

for all x. Also, we have

$$\int_0^\infty Lye^{-\frac{y^2}{2\tau}}\,dy < \infty$$

for all $\tau > 0$. For a compact set $K \subset \mathbb{R}\times[0,\infty)$,there exists $\tau > 0$ and $a < b$ such that $K \subset [a,b]\times[0,\tau)$, hence Theorem 6.13 implies that

$$\lim_{n\to\infty} \sup_{(x_k,t_j)\in\mathcal{L}_0^n\cap K} |W_k^j - w(x_j,t_j)| = 0.$$

Where W_k^j is the solution to (6.48) and w to (6.50).

It is now clear that the result follows. ■

We can interpolate $\{W_k^j \ : \ (x_k,t_j)\in\mathcal{L}_0^n\}$ as in sections 6.4 and 7.2 to obtain a function $W_n(x,t)$. The following can be shown using the same techniques.

Theorem 6.25. *Let W_n be the interpolation described in 6.4 of the Euler method, and w the solution to the equation* (6.50) *where g satisfies* (6.51)*. Then we have*

$$\lim_{n\to\infty} W_n = w,$$

uniformly on compact sets.

Corollary 6.26. *If we set $g(x) = B(x)$ where $(B(x) : x \geq 0)$ is Brownian motion, then for W_n be the described interpolation of the Euler method, and w the solution to the equation* (6.50) *we get*

$$\lim_{n\to\infty} W_n = w,$$

uniformly on compact sets almost surely.

Proof. Follows from the strong law of large numbers for the Brownian motion since

$$\lim_{x\to\infty} \frac{|B(x)|}{|x|} = 0 \ a.s.$$

Now we can apply Lemma 6.24, and all the results after. ■

Chapter 7

LIMIT OF THE NATURAL BROWNIAN MOTION ON A RHOMBUS GRID

7.1 Natural Brownian motion on a rhombus grid

We will investigate the process on a rhombus grid where the ratio of diagonals depends on n. A natural question is what happens when $n \to \infty$.

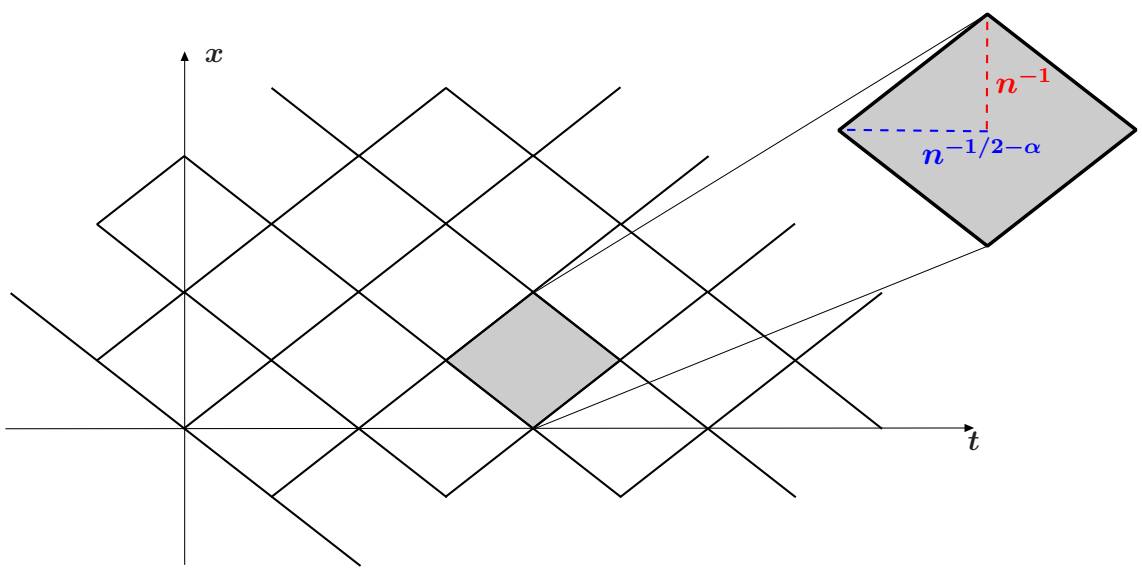


Figure 7.1: α-rhombus grid

Definition 7.1. *We will call the TLG* $\mathcal{G}$ the (α, n)-rhombus grid if the plane is divided into congruent rhombuses, diagonals of which are parallel to the x and t axis, the length of the half-diagonal parallel to x is $\frac{1}{n}$ and the length of the half-diagonal parallel to t is $\frac{1}{n^{1/2+\alpha}}$, and there is a rhombus that has vertex $(0, 0)$. (See Figure 7.1.)*

Remark. In our representation the vertices are represented by the set, where $t_j = jn^{-1/2-\alpha}$ and $x_k = k/n$:

$$\mathcal{L}_0^{\alpha,n} = \{(t_j, x_k) : k, j \geq 0,\ j \equiv k \pmod 2\}$$

Using Theorem 2.24. we construct a natural two-sided Brownian motion $X_{\alpha,n}$ on this grid. Recall, that if $(W_1(t) : t \geq 0)$ and $(W_2(t) : t \geq 0)$ are two independent Brownian motions, then

$$B(t) := \begin{cases} W_1(t), & t \geq 0 \\ W_2(-t), & t < 0 \end{cases}$$

is a two-sided Brownian motion. It is not hard to check that this is a Markov process on $T = \mathbb{R}$. Further, covariance of this process is

$$C_B(t, s) = \frac{1}{2}(|t| + |s| - |t - s|).$$

The following result will be useful.

Lemma 7.2. *The processes* $(X_{\alpha,n}(\bar{t}) : t \geq 0)$ *and* $(X_{\alpha,n}(\bar{t}) : t \leq 0)$ *are independent.*

Proof. Let $Y^+_{\alpha,n}$ be a natural $\mathcal{P}_+$-Markovian process on a (α, n)-grid, where $\mathcal{P}_+$ is the distribution of

$$B^0_+(t) := \begin{cases} W_1(t), & t \geq 0 \\ 0, & t < 0 \end{cases}.$$

In the same way we can construct $Y^-_{\alpha,n}$ as a natural $\mathcal{P}_-$-Markovian process on a (α, n)-grid, where $\mathcal{P}_-$ is the distribution of

$$B^0_-(t) := \begin{cases} 0, & t \geq 0 \\ W_2(-t), & t < 0 \end{cases}.$$

We can construct $Y^+_{\alpha,n}$ and $Y^-_{\alpha,n}$ such that they are independent and on the same space and using the same TLG*-towers. Then, it is not hard to see that $Y^+_{\alpha,n} + Y^-_{\alpha,n}$ in each member of a TLG*-tower has the same distribution as $X_{\alpha,n}$ on this TLG*. Therefore, the distribution of

$X_{\alpha,n}$ and $Y^+_{\alpha,n}+Y^-_{\alpha,n}$ are the same by Theorem 2.24. Furthermore, $((X_{\alpha,n}(\bar{t}) : t \geq 0), (X_{\alpha,n}(\bar{t}) : t \leq 0))$ are distributed as $(Y^+_{\alpha,n}, Y^-_{\alpha,n})$. ■

Due to the last lemma, we can focus on what happens with the process $X_{\alpha,n}(x,t)$ for $t \geq 0$.

The final distribution of the process, by Theorem 2.24. doesn't depend on the way we construct the process. We fix a construction that we will refer to.

For our construction we need:

- two-sided Brownian motion $(B(t) : t \in \mathbb{R})$;
- for $j \neq -1$, $k \in \mathbb{Z}$: $(B^{br}_{jk}(t) : t \in [t_j, t_{j+2}])$ be a collection of Brownian bridges $(n \in \mathbb{N})$;
- for $j = -1$, $k \in \mathbb{Z}$: $(B^{br}_{jk-}(t) : t \in [t_j, t_{j+1}])$, $(B^{br}_{jk+}(t) : t \in [t_{j+1}, t_{j+2}])$ be a collection of Brownian bridges $(n \in \mathbb{N})$;

all of these things are independent.

Step 0 We run the two-sided Brownian motion on the time-path σ that is going through $(-\infty, \infty) \times [0, \frac{1}{n}]$ (this will be our spine), that is we define $X_\sigma(t) = B(t)$. (See Figure 7.2.)

Step 1 Now if we defined the process at points (t_j, x_k) and (t_{j+2}, x_k) then we define the process on the time-path π_+ (if $x_k > 0$) $(t_j, x_k) - (t_{j+1}, x_{k+1}) - (t_{j+2}, x_k)$ or time-path π_- $(t_j, x_k) - (t_{j-1}, x_{k-1}) - (t_{j+2}, x_k)$ (if $x_k \leq 0$) by setting $X_{\pi_\pm}$ to be:

- if $j = -1$ two Brownian bridges (on the intervals $[t_j, t_{j+1}]$ and $[t_{j+1}, t_{j+2}]$, where the value of the process at t_j is $X_{\alpha,n}(t_j, x_k)$, t_{j+1} is 0 and t_{j+2} is $X_{\alpha,n}(t_{j+2}, x_k)$;
- a Brownian bridge at times t_j and t_{j+2} between values $X_{\alpha,n}(t_j, x_k)$ and $X_{\alpha,n}(t_{j+2}, x_k)$.

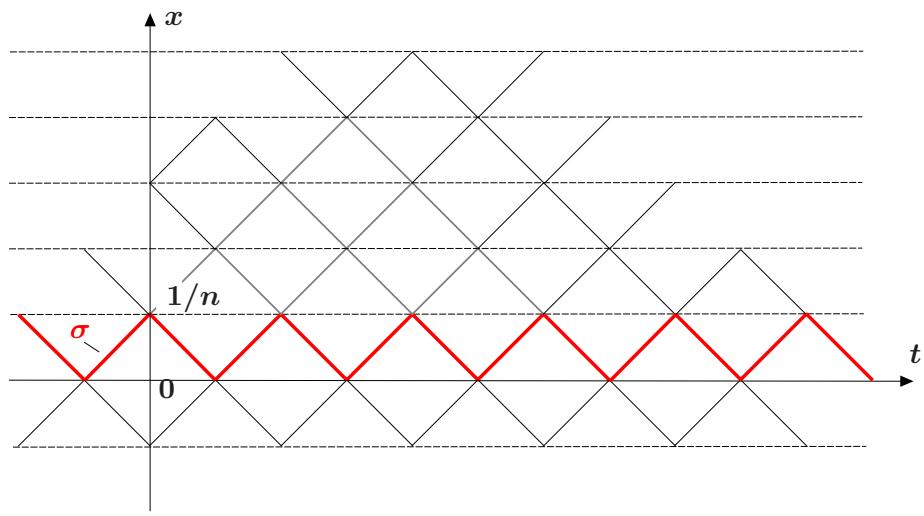
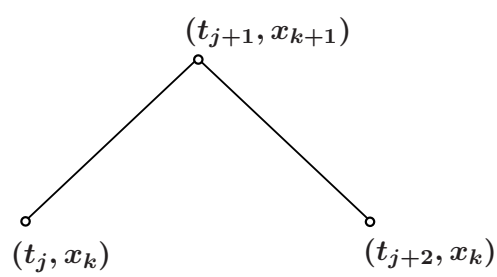


Figure 7.2:



Specially, if the path if $t_j \geq 0$ then $X_{\pi_\pm}$ will be of the form

$$X_{\pi_\pm}(t) := \frac{t_{j+2} - t}{t_{j+2} - t_j} X_{\alpha,n}(t_j, x_k) + \frac{t - t_j}{t_{j+2} - t_j} X_{\alpha,n}(t_{j+2}, x_k) + B^{br}_{jk}(t) \tag{7.1}$$

where B^{br}_{jk} is a Brownian bridge that has value 0 at times t_j and t_{j+2}, and independent of the other Brownian bridges.

If $j = -1$ ($t_{j+1} = 0$), then $X_{\pi_\pm}$ on $[t_{j+1}, t_{j+2}]$ is equal

$$X_{\pi_\pm}(t) := \frac{t}{t_{j+2}} X_{\alpha,n}(t_{j+2}, x_k) + B^{br}_{jk+}(t) \tag{7.2}$$

where $B^{br\pm}_{jk}(t)$ is a Brownian bridge with value 0 at times t_{j+1} and t_{j+2}.

Step 2 We repeat **Step 1** in such a way that at every point in $\mathcal{L}_0^{\alpha,n}$ the process will be eventually defined.

*Remark.*Along every path from $-\infty$ to $+\infty$ we have a two-sided Brownian motion.

Having in mind Lemma 7.2, we will focus our attention to the process $X_{\alpha,n}$ defined in the first quadrant. The convergence of the process in other quadrants can be shown in a similar way.

The most important thing to note from the construction of the process, that if from the equation (7.1) is that when we set $t = t_{j+1}$ we get:

$$X_{\alpha,n}(t_{j+1}, x_{k+1}) = \frac{1}{2}X_{\alpha,n}(t_j, x_k) + \frac{1}{2}X_{\alpha,n}(t_{j+2}, x_k) + E_{j+1,k+1}, \tag{7.3}$$

where

$$E_{j+1,k+1} = B_{jk}^{br}(t_{j+1}) \stackrel{d}{=} N(0, 2^{-1/2}n^{-1/2-\alpha}),$$

for $j \geq 0, k \geq 1$ such that $(t_j, x_k) \in \mathcal{L}^{\alpha,n}$. This is a form of the discrete stochastic heat equation (see [38]) with random external source.

We discussed the convergence of these equations in Chapter 6 (see §6.3), that is what happens when $n \to \infty$.

7.1.1 Interpolation

Now our process is defined on the representation of the whole (α, n)-rhombus grid, and we will extend the definition of the process on the whole plane (see Figure 7.3):

- $X_{\alpha,n}(0, x) = 0$ (the process on the x-axis is 0);

- by interpolation we will extend the definition of our process on the whole plane:

$$X_{\alpha,n}(t, x) := \frac{t_+ - t}{t_+ - t_-}X_{\alpha,n}(t_-, x) + \frac{t - t_-}{t_+ - t_-}X_{\alpha,n}(t_+, x),$$

  where (t_+, x) and (t_-, x) are points on the representation of the graph or on the x-axis that are the closest to (t, x).

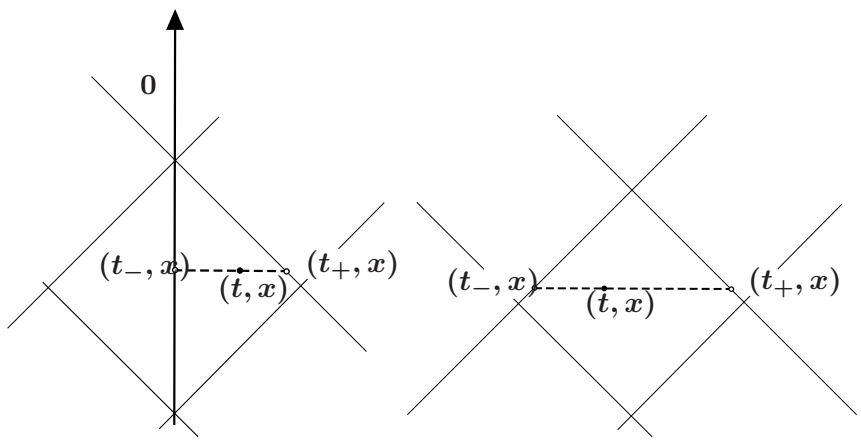


Figure 7.3: Interpolation of the process

In further text we will denote the interpolated process as $(Y_{\alpha,n}(t,x) : t \in \mathbb{R}, x \in \mathbb{R})$. Note that this is a continuous Gaussian process on $\mathbb{R}^2$.

We will study two cases $\alpha = 0$ and $\alpha > 0$, and how does $Y_{\alpha,n}$ behaves as $n \to \infty$.

7.2 Network of Brownian bridges

The final result that we need to show that the Brownian motion on a rhombus grid converges is the fact that a network of Brownian bridges will converge to 0 on compact sets.

Theorem 7.3. *Let K be a compact subset of $\mathbb{R}^2_+$, then*

$$Z_n(K) = \sup\{\max |B^{br}_{jk(+)}| : (t_j, x_k) \in K\} \xrightarrow{L^2} 0.$$

Proof. Pick K, and then pick $K_{ab} = [0,a] \times [0,b]$, such that

$$\sup\{x : (t,x) \in K\} < a,$$

and

$$\sup\{t : (t,x) \in K\} < b.$$

For each point in $\mathcal{L}_0^n$ there are at most 2 bridges going out of this point (in the direction of time). Hence on K_{ab} there are at most $an^{1/2+\alpha} \cdot bn = abn^{3/2+\alpha}$ such bridges. The bridges on K_{ab} define the Z_n on K. We have at most $abn^{3/2+\alpha}$ on intervals of length $[0, n^{-\frac{1}{2}-\alpha}]$. Hence, by Corollary 5.3 (inequality (5.6)) and the fact that for $(B^{br}(t) : t \in [0,1])$ Brownian bridge on $[0,1]$ $(n^{-\frac{1}{4}-\alpha/2} B^{br}(n^{1/2+\alpha}t) \ : \ t \in [0, n^{-1/2-\alpha}])$ is the Brownian bridge on $[0, n^{-1/2-\alpha}]$, the second moment of their maximum is bounded by

$$\frac{1}{2n^{\frac{1}{2}+\alpha}} \ln(abn^{3/2+\alpha} + 1). \tag{7.4}$$

Since the maximum is obtained in the points where the Brownian bridges have been defined, the claim follows. ■

From the rate of convergence in (7.4), and Theorem 5.11 we get the following result.

Corollary 7.4. *Let K be a compact subset of $\mathbb{R}_+^2$, then*

$$\lim_{n\to\infty} Z_n(K) = 0 \quad a.s. \tag{7.5}$$

Further, for $\beta < \frac{1}{4} + \alpha/2$ we have

$$\lim_{n\to\infty} n^{\beta} Z_n(K) = 0$$

.

7.3 The main result

The process that we will be more interested is the interpolation (similar to the one described in §7.1.1) between values of $X_{\alpha,n}$ at points in $\mathcal{L}^{\alpha,n} \cap \mathbb{R}_+^2$:

- $X_{\alpha,n}(0,x) = 0$ (the process on the x-axis is 0);
- (t_j, x_k) and $(t_{j+1}, x_{k\pm1})$ we interpolate between the values at these points;

- by interpolation we will extend the definition of our process on the whole plane:

$$X_{\alpha,n}(t,x) := \frac{t_+ - t}{t_+ - t_-} X_{\alpha,n}(t_-,x) + \frac{t - t_-}{t_+ - t_-} X_{\alpha,n}(t_+,x), \tag{7.6}$$

  where (t_+, x) and (t_-, x) are points on the representation of the graph or on the x-axis that are the closest to (t, x).

We will call this process $\widetilde{Y}_{\alpha,n}$.

From the construction of $Y_{\alpha,n}$ and $\widetilde{Y}_{\alpha,n}$ it is not hard to see that for each rhombus $\Diamond$ the value

$$\max_{(t,x)\in\Diamond} |\widetilde{Y}_{\alpha,n}(t,x) - Y_{\alpha,n}(t,x)|,$$

due to linear interpolation, is obtained on $\partial\Diamond$. That means we can focus on the process $|\widetilde{Y}_{\alpha,n}(t,x) - Y_{\alpha,n}(t,x)|$ on the representation of the (α, n)-rhombus grid.

Lemma 7.5. *The process on the path $(t_j, x_k) - (t_{j+1}, x_{k\pm1}) - (t_{j+1}, x_{k+1})$ is bounded by*

$$|\widetilde{Y}_{\alpha,n}(t,x) - Y_{\alpha,n}(t,x)| \le 2 \max_{t\in[t_j,t_{j+2}]} |B^{br}_{jk}(t)|$$

Proof. From (7.6) and (7.3) we have:

$$Y_{\alpha,n}(t,x) = \frac{t_{j+2} - t}{t_{j+2} - t_j} X_{\alpha,n}(t_j, x_k) + \frac{t - t_j}{t_{j+2} - t_j} X_{\alpha,n}(t_{j+2}, x_k) + B^{br}_{jk}(t)$$

$$\widetilde{Y}_{\alpha,n}(t,x) = \frac{t_{j+2} - t}{t_{j+2} - t_j} X_{\alpha,n}(t_j, x_k) + \frac{t - t_j}{t_{j+2} - t_j} X_{\alpha,n}(t_{j+2}, x_k) + \alpha(t) E_{j+1,k+1},$$

where $|\alpha(t)| < 1$ obtained by interpolation between values at points (t_j, x_k) and (t_{j+1}, x_{k+1}) (if $t \in [t_j, t_{j+1}]$) or (t_{j+1}, x_{k+1}) or (t_{j+2}, x_k) (for $t \in [t_{j+1}, t_{j+2}]$). Since $E_{j+1,k+1} = B_{jk}(t_{j+1})$, the claim follows. ■

Lemma 7.6. *For a compact set $K \subset \mathbb{R}^2_+$ we have*

$$\sup_{(t,x)\in K} |\widetilde{Y}_{\alpha,n}(t,x) - Y_{\alpha,n}(t,x)| \to 0, \ a.s.$$

Proof. There exists a compact set $\hat{K}$ such that all the rhombi whose interior intersects K, are contained in $\hat{K}$. Now,

$$\sup_{(t,x)\in K} |\widetilde{Y}_{\alpha,n}(t,x) - Y_{\alpha,n}(t,x)| \leq Z_n(\hat{K}),$$

and by Corollary 7.4. the claim follows. ■

Proposition 7.7. *The process* $(\widetilde{Y}_{\alpha,n}(t,x) : (t,x) \in \mathbb{R}^2_+)$ *converges to* u*, where* u *is the solution to the stochastic heat equation*

$$\partial_x u = \begin{cases} \frac{1}{2}\partial_{tt} u + \mathbb{W} & \alpha = 0, \\ 0 & \alpha > 0, \end{cases} \qquad \text{on} \quad \mathbb{R}^2_+,$$

$$u(0,t) = B(t) \qquad \text{for} \quad t \in \mathbb{R}.$$

$$u(x,0) = 0 \qquad \text{for} \quad x \in \mathbb{R}.$$

Proof. We will write the process $\widetilde{Y}_{\alpha,n} = \widetilde{Y}^1_{\alpha,n} + \widetilde{Y}^2_{\alpha,n}$, where

$$\begin{array}{cc} \widetilde{Y}^1_{\alpha,n}(t_{j+1}, x_{k+1}) = & \widetilde{Y}^2_{\alpha,n}(t_{j+1}, x_{k+1}) = \\ \frac{1}{2}\widetilde{Y}^1_{\alpha,n}(t_j, x_k) + \frac{1}{2}\widetilde{Y}^1_{\alpha,n}(t_{j+2}, x_k) + E_{j+1,k+1} & \frac{1}{2}\widetilde{Y}^2_{\alpha,n}(t_j, x_k) + \frac{1}{2}\widetilde{Y}^2_{\alpha,n}(t_{j+2}, x_k) \\ \widetilde{Y}^1_{\alpha,n}(0, x_k) = 0, \widetilde{Y}^1_{\alpha,n}(t_j, 0) = 0 & \widetilde{Y}^2_{\alpha,n}(0, x_k) = 0, \widetilde{Y}^2_{\alpha,n}(t_j, 0) = B(t_j) \end{array}$$

Now, by Propositions 6.20. and 6.22. $\widetilde{Y}^1 \xrightarrow{d} u^1$ where

$$\begin{cases} u^1_x = \begin{cases} \frac{1}{2}u^1_{tt} + \mathbb{W} & \alpha = 0 \\ 0 & \alpha > 0 \end{cases} \\ u^1(0,\cdot) = 0, u^1(\cdot,0) = 0 \end{cases}.$$

By Corollary 6.26. $\widetilde{Y}^2 \xrightarrow{d} u^2$ where

$$\begin{cases} u^2_x = \begin{cases} \frac{1}{2}u^2_{tt} & \alpha = 0 \\ 0 & \alpha > 0 \end{cases} \\ u^2(0,\cdot) = 0, u^2(\cdot,0) = B(\cdot) \end{cases}.$$

Since $u = u^1 + u^2$, the claim follows. ■

Therefore by previous results we have the following theorem.

Theorem 7.8. *$Y_{\alpha,n}$ the interpolated natural two-sided Brownian motion on the (α, n)-rhombus lattice converges in distribution to u as $n \to \infty$, where u is the solution to following stochastic heat equation*

$$\begin{aligned} \partial_x u = \begin{cases} \frac{1}{2}\partial_{tt} u + \mathbb{W} & \alpha = 0, \\ 0 & \alpha > 0, \end{cases} & \quad \text{on} \quad (\mathbb{R} \setminus \{0\})^2, \\ u(0,t) = B(t) & \quad \text{for} \quad t \in \mathbb{R}. \\ u(x,0) = 0 & \quad \text{for} \quad x \in \mathbb{R}. \end{aligned} \tag{7.7}$$

and $t \mapsto B(t)$ is a two-sided Brownian motion independent of $(\mathbb{W}(A) : A \in \mathcal{B}(\mathbb{R}^2))$.

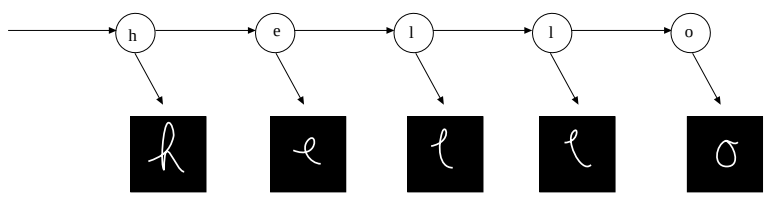
Proof. We will show the claim on $\mathbb{R}^2_+$, the other quadrants are shown in the same way. From $Y_{\alpha,n} = \widetilde{Y}_{\alpha,n} + (Y_{\alpha,n} - \widetilde{Y}_{\alpha,n})$, Proposition 7.7. and Lemma 7.6. we have $Y_{\alpha,n} \xrightarrow{d} u + 0 = u$. ■

Part III

PROCESSES ON GENERAL AND RANDOM TIME-LIKE GRAPHS

The TLG's defined so far (see Chapter 1) have only one beginning and one end (usually denoted by 0 and 1).

In applications and theory of classical graphical models an important role belongs to processes indexed by trees. This includes one of the most widely used models - hidden Markov model.



Hidden Markov model in image (letter) analysis.

Hidden Markov model is also a collection of Markov processes combined together. (See for example §6.2.3.1 in [35].)

Another model that is of wide interest is the branching Brownian motion. This is a similar model, but underlying graph is a random tree.

Having this in mind, it is natural to ask can we have more than one beginning and more than one end. Could we define a process indexed by a TLG with a structure of a tree? We will show that this can be so in Chapter 9, and that there is a natural embedding into the existing family of TLG's. This embedding will help us define processes on a generalized family of TLG*'s in Chapter 9.

Later, in Chapter 10. we will be able to randomize the underlying graph, and see how it is connected to the branching Markov processes.

Chapter 8

NON-SIMPLE TLG'S

8.1 New definitions

The TLG's defined in Chapter 1, from now on, we will call **simple TLG's**.

Definition 8.1. *A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ will be called a* ***time-like graph (TLG)*** *if its sets of vertices $\mathcal{V}$ and edges $\mathcal{E}$ satisfy the following properties.*

(i) Let $A, B > 0$. The set $\mathcal{V}$ contains at least two elements, $\mathcal{V} = \{t_0, t_1, \ldots, t_N\}$, where for $k = 1, 2, \ldots, N-1$,

$$A \leq t_k \leq t_{k+1} \leq B.$$

(ii) An edge between t_j and t_k will be denoted E_{jk}. We assume that there is no edge between t_j and t_k if $t_j = t_k$. E_{jk} indicates that $t_j < t_k$. (We use E^1_{jk}, E^2_{jk},... if there is more than one edge connecting t_k and t_j.)

(iii) We assume that all vertices have a finite non-zero degree.

We will call $\mathcal{G}$ the ***unit*** *TLG if $A = 0$ and $B = 1$.*

Remarks.

(a) Notice that in the new definition there are no longer unique vertices with times A and B.

(b) We dropped part (iv) of the original definition and added an assumption in (iii) that all vertices are of non-zero degree.

(c) Notice, that this definition no longer guaranties that the graph is connected. (See Figure 8.1.)

Again, as in Chapter 1, we will restrict our attention to unit TLG's and prove all the claims for them.

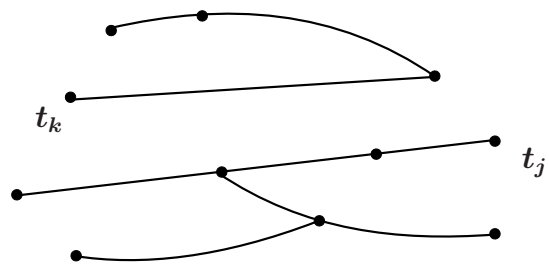


Figure 8.1: TLG $\mathcal{G}$ with entrance vertex t_k and exit vertex t_j

Definition 8.2. *(a) A vertex t_k that is not connected to any other vertex with time less than t_k will be called an* ***entrance (vertex)****. We will denote the set of entrance vertices by $En(\mathcal{G})$.*

(b) A vertex t_j that is not connected to any other vertex with time greater than t_k will be called an ***exit (vertex)****. We will denote the set of exit vertices by $Ex(\mathcal{G})$.*

The definitions of the **representation**, **time-paths**, **(simple) cells** remain the same as before (see Definitions 1.2, 1.3, 1.4).

We no longer have the full-time path as in the case of simple TLG's but instead we define the full path.

Definition 8.3. *A time-path σ is called a* ***full path*** *if it starts with an entrance vertex and ends with an exit vertex. We denote the set of full paths by $P(\mathcal{G})$, while the full paths starting at $t_k \in En(\mathcal{G})$ and ending at $t_j \in Ex(\mathcal{G})$ we will denote by $P_{t_k \to t_j}(\mathcal{G})$.*

Remark. Note that it can be $P_{t_k \to t_j}(\mathcal{G}) = \emptyset$ (see Figure 8.1.) and further

$$P(\mathcal{G}) = \bigcup_{t_k \in En(\mathcal{G})} \bigcup_{t_j \in Ex(\mathcal{G})} P_{t_k \to t_j}(\mathcal{G}).$$

8.2 Embedding TLG's into simple TLG's

Although it seems that TLG's are much more general objects than simple TLG's, there is a natural embedding that will enable us to use most of the results that we had for simple TLG's. As a result we will be able to construct processes under similar conditions as we did on simple TLG's.

Minimal embedding

The first embedding will use the minimal number of edges to embed the (unit) TLG into a simple TLG.

Procedure is the following:

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a TLG.

- Set $t_{-\infty} = -1$, $t_{\infty} = 2$.
- For all $t_k \in En(\mathcal{G})$ we denote $E_{-\infty k}$ and edge between $t_{-\infty}$ and t_k, and for all $t_j \in Ex(\mathcal{G})$ we denote $E_{j\infty}$ and edge between t_j and t_{∞}.
- Set

$$\mathcal{V}^{\#} = \mathcal{V} \cup \{t_{-\infty}, t_{\infty}\},$$

  and

$$\mathcal{E}^{\#} = \mathcal{E} \cup \{E_{-\infty k} : t_k \in En(\mathcal{G})\} \cup \{E_{j\infty} : t_j \in Ex(\mathcal{G})\}.$$

The transformation that defines the embedding has some nice properties.

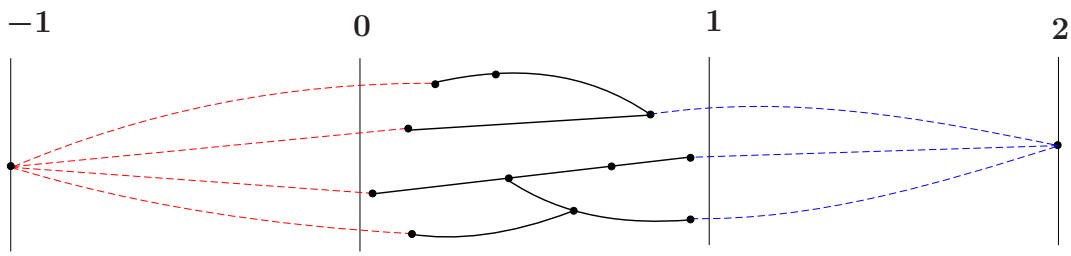


Figure 8.2: Minimal embedding of the TLG $\mathcal{G}$ (from Figure 8.1) into a simple TLG.

Proposition 8.4. *The following claims hold:*

(i) $\mathcal{G}'$ *is a simple TLG.*

(ii) $\mathcal{G} \mapsto \mathcal{G}'$ *is an injective map.*

(iii) $\mathcal{G} \mapsto \mathcal{G}'$ *preserves the connectedness by time-paths.*

(iv) $\mathcal{G} \mapsto \mathcal{G}'$ *preserves the order induced by* $\mathcal{G}$ *and* $\mathcal{G}'$*, i.e.*

$$t \overset{\mathcal{G}}{\prec} s \quad \Leftrightarrow \quad t' \overset{\mathcal{G}'}{\prec} s'$$

.

Proof. (i) Follows form Definition 1.1. (ii) If we have $\mathcal{G}'$, we can delete the edges connected to $t_{-\infty}$ and $t_{+\infty}$ and get $\mathcal{G}$. (iii) From the definition of the mapping it is clear that if t and s are connected by time-path then t' and s' will also be connected. If t and s are not connected by a time-path neither will t' and s' be connected by a time path, since all the new edges added include vertices 0 and 1 in $\mathcal{G}'$. (iv) This follows from (iii) and the the fact that time remains the same. ■

Maximal embedding

The embedding will add a number of edges to embed the TLG into a simple TLG.

Procedure is the following:

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a TLG.

- Set $t_{-\infty} = -1$, $t_{\infty} = 2$.
- For all $t_k \in \mathcal{V}$ we denote $E_{-\infty k}$ and edge between $t_{-\infty}$ and t_k, and $E_{k\infty}$ an edge between t_k and t_{∞}.
- Set
$$\mathcal{V}^{\#} = \mathcal{V} \cup \{t_{-\infty}, t_{\infty}\},$$
and
$$\mathcal{E}^{\#} = \mathcal{E} \cup \{E_{-\infty k}, E_{k\infty} : t_k \in \mathcal{V}\}$$

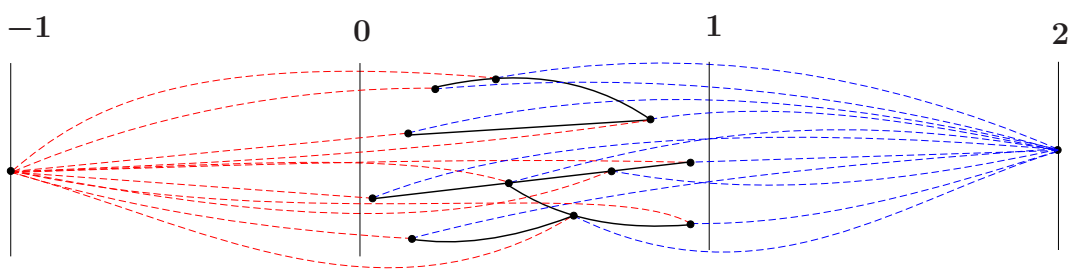


Figure 8.3: Maximal embedding of the TLG $\mathcal{G}$ (from Figure 8.1) into a simple TLG.

The transformation that defines the embedding has some nice properties.

Proposition 8.5. *The following claims hold:*

(i) $\mathcal{G}''$ *is a simple TLG.*

(ii) $\mathcal{G} \mapsto \mathcal{G}''$ *is an injective map.*

(iii) $\mathcal{G} \mapsto \mathcal{G}''$ *preserves the connectedness by time-paths.*

(iv) $\mathcal{G} \mapsto \mathcal{G}''$ *preserves the order induced by* $\mathcal{G}$ *and* $\mathcal{G}'$*, i.e.*

$$t \overset{\mathcal{G}}{\prec} s \quad \Leftrightarrow \quad t'' \overset{\mathcal{G}''}{\prec} s''$$

.

Remark on the embeddings

We will use both embeddings of a TLG $\mathcal{G}$ *for several reasons. It is easier to draw and see properties of* $\mathcal{G}'$ *than of* $\mathcal{G}''$*. On the other hand, for the many of the proofs that we have to do* $\mathcal{G}''$ *will be much better to use.*

Proof. The proof is similar to the proof of Proposition 8.4. ■

8.3 TLG** family

As we have already seen in §3.4.2 we might have problems to define a process with natural properties on some TLG's. In this section we introduce the family TLG**, similar to the family TLG* that we had defined for simple TLG's.

We will describe the family of TLG graphs that is generated from a minimal graph by adding vertices, adding edges between vertices connected by a time-path and adding edges between a new vertex and a vertex already on the graph.

Definition 8.6. *The* ***TLG**-family*** *is given in the following inductive way.*

(i) The minimal graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$*, with* $\mathcal{V} = \{t_0, t_N\}$ $(t_0 < t_N)$ *and* $\mathcal{E} = \{E_{0N}\}$ *is a TLG**.*

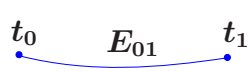


Figure 8.4: A minimal graph

(ii) Let $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ *be a TLG*, where* $\mathcal{V}_1 = \{t_0, t_2, \ldots, t_N\}$*.*

(1) If $\tau_k \in [0,1]$ (not a vertex), and for some $E_{k_1k_2} \in \mathcal{E}$ and $t_{k_1} < \tau_k < t_{k_2}$ then set

$$\mathcal{V}_2 := \mathcal{V}_1 \cup \{\tau_k\} \quad \text{and} \quad \mathcal{E}_2 := \mathcal{E}_1 \cup \{E_{k_1k}, E_{kk_2}\} \setminus \{E_{k_1k_2}\}.$$

$\mathcal{G}_2 := (\mathcal{V}_2, \mathcal{E}_2)$ is also a TLG**.

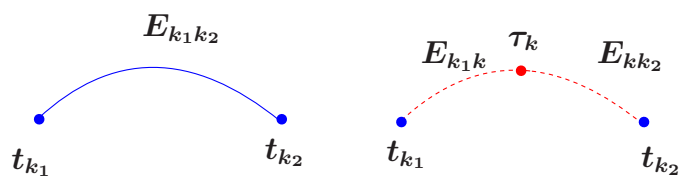


Figure 8.5: Adding a vertex

(2) If $\tau_k \in [0,1]$ (not a vertex), and for some $\tau_k < t_{k_2}$ then set

$$\mathcal{V}_2 := \mathcal{V}_1 \cup \{\tau_k\} \quad \text{and} \quad \mathcal{E}_2 := \mathcal{E}_1 \cup \{E_{kk_2}\}.$$

$\mathcal{G}_2 := (\mathcal{V}_2, \mathcal{E}_2)$ is also a TLG**.

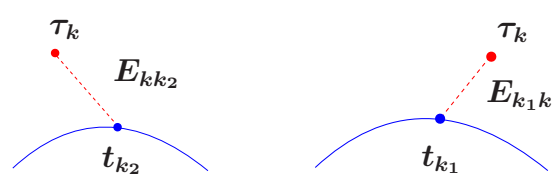


Figure 8.6: Adding the edge and the vertex

(3) If $\tau_k \in [0,1]$ (not a vertex), and for some $t_{k_1} < \tau_k$ then set

$$\mathcal{V}_2 := \mathcal{V}_1 \cup \{\tau_k\} \quad \text{and} \quad \mathcal{E}_2 := \mathcal{E}_1 \cup \{E_{k_1k}\}.$$

$\mathcal{G}_2 := (\mathcal{V}_2, \mathcal{E}_2)$ is also a TLG**.

(4) Let $t_j, t_k \in \mathcal{V}_1$ such that $t_j < t_k$, and assume that there exists a time-path $\sigma(j, \ldots, k)$ between these vertices. Then set

$$\mathcal{V}_2 := \mathcal{V}_1 \quad \text{and} \quad \mathcal{E}_2 := \mathcal{E}_1 \cup \{E^*_{jk}\}.$$

$\mathcal{G}_2 := (\mathcal{V}_2, \mathcal{E}_2)$ is also a TLG**. (E^*_{jk} is an new edge (not in $\mathcal{E}_1$).)

Figure 8.7: Adding the edge E^*_{jk}

(iii) If $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ and $\mathcal{G}_2 = (\mathcal{V}_2, \mathcal{E}_2)$ are two disjoint TLG** their union is also a a TLG**.

(iv) We will say that $(\mathcal{G}_j)_{1\leq j\leq k}$ is a ***tower of TLG**'s*** *if for $j > 1$, $\mathcal{G}_j$ is constructed from $\mathcal{G}_{j-1}$ as in (ii).*

Remarks. (1) Applying only the procedure in (ii) will clearly give us a a connected TLG**. (2) Any connected component of a TLG** can be obtained only by using step (ii). (3) It can be easily seen that a TLG** is a TLG. (4) It is clear that a TLG* is a TLG**.

Lemma 8.7. *Let $\mathcal{G}$ be a TLG. If $\mathcal{G}''$ is a TLG* then $\mathcal{G}$ is a TLG**.*

Proof. If $\mathcal{G}''$ is a TLG*, then there exists a TLG*-tower $(\mathcal{H}_j)_{j=0}^n$ such that $\mathcal{H}_n = \mathcal{G}''$ and $\mathcal{H}_0$ contains an edge in $\mathcal{G}$. Now, we construct a TLG**-tower $(\mathcal{G}_j)_{j=0}^{m_n}$ from the tower $(\mathcal{H})_{j=0}^n$. Let $\mathcal{G}_0$ be the minimal graph that is contained in $R(\mathcal{H}_0)$. If $\mathcal{H}_j$ is obtained by $\mathcal{H}_{j-1}$ by

- adding a new vertex, then let $\mathcal{G}_j$ be obtained from $\mathcal{G}_{j-1}$ by adding a new vertex (step (1));
- adding a new edge contained in $R(\mathcal{G})$, then let $\mathcal{G}_j$ be obtained from $\mathcal{G}_{j-1}$ by adding a new edge (step (4));
- adding a new edge partially contained in $R(\mathcal{G})$, then let $\mathcal{G}_j$ be obtained from $\mathcal{G}_{j-1}$ by adding a new edge with a new vertex (steps (2) or (3));
- adding a new edge not contained in $R(\mathcal{G})$, then let $\mathcal{G}_j = \mathcal{G}_{j-1}$.

Let's assume that $(\mathcal{G}_j)$ doesn't have repeating TLG's. In order to show that it is a TLG**-tower, we need to check that each time we add an edge (step (4)) the two endpoints are connected. This is clear from the fact that one endpoint of all other edges in $\mathcal{H}_j$ not in $\mathcal{G}_j$ is in the set $\{t_{-\infty}, t_{+\infty}\}$. So the two points on $\mathcal{G}_j$ are connected by a time path in $\mathcal{H}_j$ only if they are connected by a time path in $\mathcal{G}_j$. ■

Theorem 8.8. *Let $\mathcal{G}$ be a TLG, then $\mathcal{G}$ is a TLG** if and only if its embedding $\mathcal{G}''$ is a TLG*.*

Proof. Let n be the number of vertices and edges of $\mathcal{G}$. For $n = 3$ we have a minimal graph and the claim is clear. Let's assume that the claim holds for $n \geq 3$, and show that the claim is true for $n + 1$.

($\Rightarrow$): Let $\mathcal{G}^{\#}$ be a TLG** such that we can construct $\mathcal{G}$ using steps (1)$-$(4) from Definition 8.6. Then $\mathcal{G}''$ can be constructed from $(\mathcal{G}^{\#})''$ in several steps from Definition 1.5.

($\Leftarrow$): See Lemma 8.7. ■

From the previous proof we get the following fact.

Corollary 8.9. *If $(\mathcal{H}_j)$ is a TLG**-tower, then $(\mathcal{H}''_j)$ is a subsequence of a TLG*-tower.*

Theorem 8.10. *The following statements are equivalent:*

(a) $\mathcal{G}$ is a TLG**.

(b) $\mathcal{G}'$ is a TLG*.

(c) $\mathcal{G}''$ is a TLG*.

Proof. (a) $\Leftrightarrow$ (c): Follows from Theorem 8.8.

(b) $\Rightarrow$ (c): Every t'_k in $\mathcal{G}'$ is connected to $t_{-\infty}$ and t_{∞} by a time-path. Therefore, we can add an edge to $\mathcal{G}'$ between $t_{-\infty}$ and t'_k, and an edge between t'_k and t_{∞}. Hence, we can construct $\mathcal{G}''$ from $\mathcal{G}'$ by adding edges. Hence, $\mathcal{G}''$ is a TLG*.

(c) $\Rightarrow$ (b): Let t_k be a vertex that is not an entrance, then the if we remove the edge $E_{-\infty,k}$ from $\mathcal{G}''$ we get a TLG* by Corollary 1.13. The same holds if t_k is not an exit for the edge $E_{k,\infty}$. Doing this until all such edges are removed gives us $\mathcal{G}'$, that will, by repeated use of Corollary 1.13, be a TLG*. ■

Corollary 8.11. *If $\mathcal{G}''$ is a TLG*, then there is a TLG*-tower $(\mathcal{H}_j)_{j=1}^n$ such that $\mathcal{H}_1 = \mathcal{G}'$ and $\mathcal{H}_n = \mathcal{G}''$.*

The order '$\preceq$' between the points is defined in the same way as in Chapter 1. See Definition 1.14.

Lemma 8.12. *For points t_1 and t_2 on a TLG** $\mathcal{G}$*

- *there exists a point $t_1 \wedge t_2$ on $\mathcal{G} \cup \{-1\}$ such that*

$$\{t \in \mathcal{G} : t \preceq t_1\} \cap \{t \in \mathcal{G} : t \preceq t_2\} = \{t \in \mathcal{G} : t \preceq t_1 \wedge t_2\};$$

- *there exists a point $t_1 \vee t_2$ on $\mathcal{G} \cup \{2\}$ such that*

$$\{t \in \mathcal{G} : t \succeq t_1\} \cap \{t \in \mathcal{G} : t \succeq t_2\} = \{t \in \mathcal{G} : t \succeq t_1 \vee t_2\};$$

in the sense that if we have an empty set on one side we define $t_1 \wedge t_2 = -1$ in the first case, and $t_1 \vee t_2 = 2$ in the second case.

<u>*Proof.*</u> By Proposition 8.4, we will have $t_1' \wedge t_2' = (t_1 \wedge t_2)'$ and $(t_1 \vee t_2)' = t_1' \vee t_2'$. Since $\mathcal{G}$ is a TLG*, $t_1' \wedge t_2'$ and $t_1' \vee t_2'$ exists and can obtain one of the values in $\mathcal{V}' \cup \{0, 1\}$. Since, the transformation is injective so are $t_1 \wedge t_2$ and $t_1 \vee t_2$. ■

We know from Theorem 1.6, that all planar simple TLG's are TLG*. Unfortunately, the same is not true for TLG**'s.

Proposition 8.13. *The following statements hold:*

(a) If $\mathcal{G}$ is a planar TLG its embedding $\mathcal{G}'$ doesn't have to be a planar TLG.

(b) If $\mathcal{G}$ is a planar TLG** its embedding $\mathcal{G}'$ doesn't have to be a planar TLG*.

(c) There exists a planar TLG that is not a TLG**.

Proof. (a) See Figure 8.8.

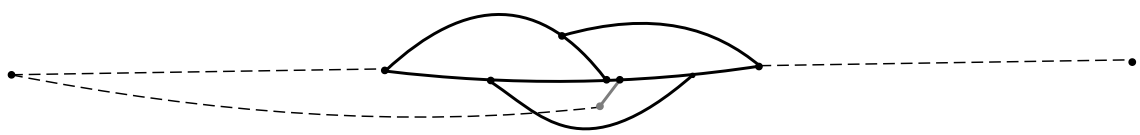


Figure 8.8: TLG $\mathcal{G}$ is planar (full lines), but its embedding $\mathcal{G}'$ is not.

(b) The black part of the graph $\mathcal{G}$ (in Figure 8.8) is a (planar) simple TLG, so it is a TLG*. Hence, we can first construct the black part, and then add the gray vertex and the gray edge connecting it to the rest of the graph. So, $\mathcal{G}$ is a TLG*.

(c) See the graph in Figure 8.9. This is not a TLG**, because $t_3 \wedge t_4$ is not defined, and by Lemma 8.12 this should be defined in the case of a TLG**.

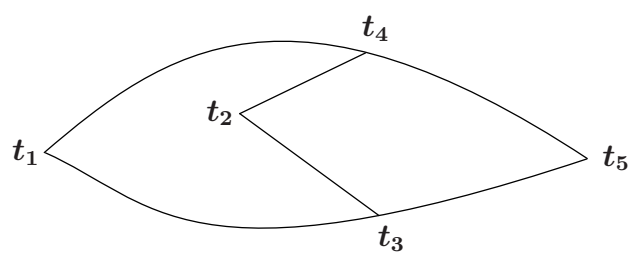


Figure 8.9: A planar TLG that is not a TLG**.

■

There are two important cases of planar TLG's that are planar TLG**'s, and we will encounter with them in the future.

Proposition 8.14. *(a) A planar TLG $\mathcal{G}$ such that all vertices in $En(\mathcal{G})$ and all vertices in $Ex(\mathcal{G})$ have the same time component is a TLG**.*

(b) TLG that has the structure of a tree is also a TLG**

Proof. (a) The proof follows from the fact that $\mathcal{G}'$ the embedding of $\mathcal{G}$ is a planar TLG, therefore a TLG*, and by Theorem 8.8 $\mathcal{G}$ is a TLG**. (b) Follows by induction on the number of edges. ■

Chapter 9

PROCESSES ON NON-SIMPLE TLG'S

9.1 Processes on TLG**

Idea of the construction is the similar to the one that we had in the case of simple TLG's (as described in Section 2.2):

- We take a family $\mathcal{M}$ of measures μ_σ on full paths $P(\mathcal{G})$ with certain properties.
- Using these properties we create a (3T)-family $\mathcal{M}'$ of measures $\mu_{\sigma'}$ on full-time paths of the embedding $\mathcal{G}'$.
- We create a natural $\mathcal{M}'$-process on $\mathcal{G}'$, and from that process we create the process on X on $\mathcal{G}$.

We could do the same approach for $\mathcal{G}''$, and we will briefly discuss it.

We need the version of the consistent family of measures along full paths.

Definition 9.1. *Let $\mathcal{G}$ be a TLG, for a family of distributions*

$$\mathcal{M} = \{\mu_\sigma : \sigma \in H \subset P(\mathcal{G})\}$$

where if σ is a full path from t_k to t_j then μ is a distribution of a stochastic process on $[t_k, t_j]$, we say that it is ***consistent*** *if for $\sigma_1, \sigma_2 \in H$*

$$\mu_{\sigma_1} \circ \pi_T^{-1} = \mu_{\sigma_2} \circ \pi_T^{-1},$$

where $T = \{t : t \in E, E \in \sigma_1 \ \& \ E \in \sigma_2\}$.

We also need a notion of the half-cell that didn't exist for simple TLG's.

Definition 9.2. *Let $\mathcal{G} = (\mathcal{E}, \mathcal{V})$ be a TLG.*

(a) We say that time paths σ_1 and σ_2 in $\mathcal{G}$ starting at $t_{k_1} \in En(\mathcal{G})$ and respectively at $t_{k_2} \in En(\mathcal{G})$ and both ending at t_m which is their only common vertex, form a ***right half-cell*** *(σ_1, σ_2).*

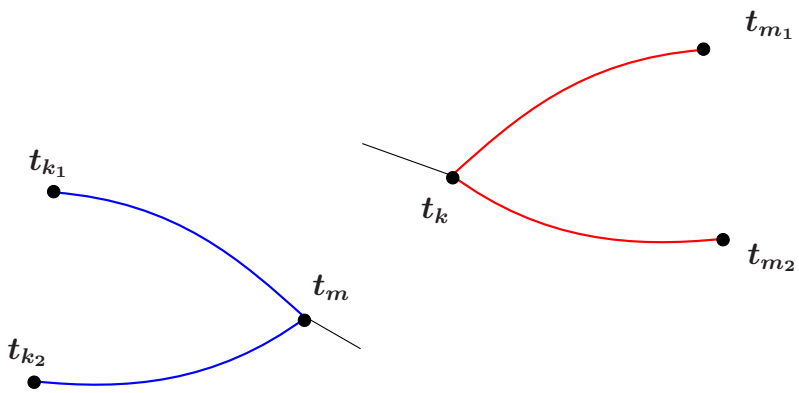


Figure 9.1: Right and left half-cells.

(b) We say that time paths σ_1 and σ_2 in $\mathcal{G}$ both starting at t_k which is their only common vertex, and ending at $t_{m_1} \in Ex(\mathcal{G})$ and respectively $t_{m_2} \in Ex(\mathcal{G})$, form a ***left half-cell*** *(σ_1, σ_2).*

(c) A half-cell (σ_1, σ_2) is called ***simple*** *if there is no time-path connecting vertex on σ_1 and a vertex on σ_2 (both must be different from the connecting vertex t_m).*

Remarks. Note that a half-cell in $\mathcal{G}$ will be embedded into a cell in the embedding $\mathcal{G}'$.

9.1.1 Conditions

We will show that an $\mathcal{M}$-process exists if the following conditions are satisfied:

T1' $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a TLG**.

T2' $\mathcal{M}$ is a consistent family of measures that induce continuous or RCLL processes.

T3' Let π be a full-path in $P(\mathcal{G})$ and $t_k \in \mathcal{V}$ a vertex on that path. Then $(X_\pi(t) : t \leq t_k)$ and $(X_\pi(t) : t \geq t_k)$ are independent given $X(t_k)$.

Definition 9.3. *The family $\mathcal{M} = \mathcal{M}(\mathcal{G}) = \{\mu_\sigma : \sigma \in P(\mathcal{G})\}$ satisfying properties (T1'), (T2') and (T3') is called the* ***(3T') family****.*

Proposition 9.4. *If $\mathcal{G}$ is a TLG* and $\mathcal{M}$ a (3T') family, then $\mathcal{M}$ is a (3T) family on $\mathcal{G}$.*

Proof. In this case we only need to check the (T3) property of $\mathcal{M}$. Let π be a path that contains t_* and t^* endpoints of a simple cell. Let $A_* \in \sigma(X_\pi(t) : t \leq t_*)$ $A_*^* \in \sigma(X_\pi(t) : t_* \leq t \leq t^*)$ and $A^* \in \sigma(X_\pi(t) : t^* \leq t)$, while $B_* \in \sigma(X(t_*))$ and $B^* \in \sigma(X(t^*))$. Now we have

$$
\begin{aligned}
&\mathbb{E}(\mathbb{P}(A_* \cap A_*^* \cap A^* | X(t_1), X(t_2))\mathbf{1}_{B_*}\mathbf{1}_{B^*}) \\
&= \mathbb{E}(\mathbb{E}(\mathbf{1}_{A_*}\mathbf{1}_{A_*^*}\mathbf{1}_{A^*} | X(t_1), X(t_2))\mathbf{1}_{B_*}\mathbf{1}_{B^*}) = \mathbb{E}(\mathbf{1}_{A_*}\mathbf{1}_{A_*^*}\mathbf{1}_{A^*}\mathbf{1}_{B_*}\mathbf{1}_{B^*}) \\
&= \mathbb{E}(\mathbb{E}(\mathbf{1}_{A_*} | X(t_*))\mathbf{1}_{A_*^*}\mathbf{1}_{A^*}\mathbf{1}_{B_*}\mathbf{1}_{B^*}) = \mathbb{E}(\mathbb{E}(\mathbf{1}_{A_*} | X(t_*))\mathbf{1}_{A_*^*}\mathbb{E}(\mathbf{1}_{A^*} | X(t^*))\mathbf{1}_{B_*}\mathbf{1}_{B^*}) \\
&= \mathbb{E}(\mathbb{E}(\mathbf{1}_{A_*} | X(t_*))\mathbb{E}(\mathbf{1}_{A_*^*} | X(t_*), X(t^*))\mathbb{E}(\mathbf{1}_{A^*} | X(t^*))\mathbf{1}_{B_*}\mathbf{1}_{B^*}) \\
&= \mathbb{E}(\mathbf{1}_{A_*}\mathbb{P}(A_*^* | X(t_*), X(t^*))\mathbf{1}_{A^*}\mathbf{1}_{B_*}\mathbf{1}_{B^*}) \\
&= \mathbb{E}(\mathbb{P}(A_*^* | X(t_*), X(t^*))\mathbb{E}(\mathbf{1}_{A_*}\mathbf{1}_{A^*} | X(t_*), X(t^*))\mathbf{1}_{B_*}\mathbf{1}_{B^*}) \\
&= \mathbb{E}(\mathbb{P}(A_*^* | X(t_*), X(t^*))\mathbb{P}(A_* \cap A^* | X(t_*), X(t^*))\mathbf{1}_{B_*}\mathbf{1}_{B^*}).
\end{aligned}
$$

The claim now follows from the Monotone Class Theorem. ■

Remark. The converse of of the statement of the previous proposition is not true. Take for example a non-Markovian process on the graph $\mathcal{G} = (\{t_0 = 1, t_1 = 1/2, t_2 = 1\}, \{E_{01}, E_{12}\})$, such that $X(0)$ and $X(1)$ are not independent given $X(1/2)$.

9.1.2 Construction

Let $\mathcal{M}$ be a (3T') family on a TLG** $\mathcal{G}$.

Let $\mathcal{G}''$ be the embedding of $\mathcal{G}$ into simple TLG's. Now for each time-path σ in $\mathcal{G}$ there exists a full-time path σ' in $\mathcal{G}''$ that corresponds to σ.

If σ starts at t_k and ends at t_j, then we can define a process $(Y_\sigma(t) : t \in [t_k, t_j])$ whose distribution is μ_σ. We will define $Y_{\sigma'}$ by interpolating Y_σ on the whole interval $[0, 1]$ (see Figure 9.2 for illustration):

$$Y_{\sigma'}(t) = \begin{cases} \dfrac{1+t}{1+t_k} Y_\sigma(t_k) & \text{if } t \le t_k \\ Y_\sigma(t) & \text{if } t \in [t_k, t_j] \\ \dfrac{2-t}{2-t_j} Y_\sigma(t_j) & \text{if } t \ge t_j \end{cases} \tag{9.1}$$

Note that if Y_σ is continuous or RCLL so is $Y_{\sigma'}$.

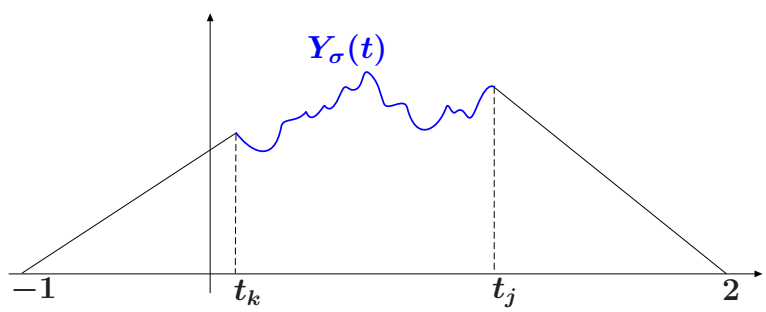


Figure 9.2: Construction of $Y_{\sigma'}$.

Now, we define $\mu_{\sigma'}$ to be the distribution of $(Y_{\sigma'}(t) : t \in [0, 1])$, and set

$$\mathcal{M}' = \mathcal{M}'(\mathcal{G}') = \{\mu_{\sigma'} : \sigma' \in P_{0\to 1}(\mathcal{G}')\},$$

or

$$\mathcal{M}'' = \mathcal{M}'(\mathcal{G}'') = \{\mu_{\sigma'} : \sigma' \in P_{0\to 1}(\mathcal{G}')\},$$

Theorem 9.5. *$\mathcal{M}'$ is a (3T')-family on $\mathcal{G}'$ and $\mathcal{M}''$ is a (3T')-family on $\mathcal{G}''$.*

Proof. The proof is similar $\mathcal{M}'$ and $\mathcal{M}''$. Properties (T1') and (T2') are clearly satisfied. While the property (T3') follows from the construction and the (T3') property of $\mathcal{M}$. ■

Corollary 9.6. *$\mathcal{M}'$ is a (3T) family on $\mathcal{G}'$ and $\mathcal{M}''$ is a (3T) family on $\mathcal{G}''$.*

Proof. By Theorem 8.10 both $\mathcal{G}'$ and $\mathcal{G}''$ are TLG*'s. The claim follows from Proposition 9.4. ■

We can construct the process on a TLG** similar to the construction of processes on TLG*'s (see Definition 2.12).

Definition 9.7. *Let $(\mathcal{G}_l)_{0\leq l\leq n}$ be a TLG**-tower where $\mathcal{G}_0$ is a minimal graph $\mathcal{V}_0 = \{t_0, t_N\}$, $\mathcal{E}_0 = \{E_{0N}\}$ and $\mathcal{G}_n = \mathcal{G}$. Further let $\mathcal{M}$ be a family of distributions satisfying (3T') conditions.*

- *On $\mathcal{G}_0$ we define a process X^0 with $\mu_{E_{0n}}$ distribution.*
- *If we have already defined X^l on $\mathcal{G}_l$ (for some $l < n$), then we define X^{l+1} on $\mathcal{G}_{l+1}$ in the following way depending how we constructed $\mathcal{G}_{l+1}$ from $\mathcal{G}_l$ (recall part (ii) of Definition 8.6.).*

  (1) In the construction a new vertex $\tau_l \in [0,1] \setminus \mathcal{V}_l$ was added to graph $\mathcal{G}_l$, by subdividing some E_{jk} such that $t_j < \tau_l < t_k$, into E_{jl} and E_{lk} to get G_{l+1}. In this case, the two graphs $\mathcal{G}_l$ and $\mathcal{G}_{l+1}$ have a common representation, $R(\mathcal{G}_l) = R(\mathcal{G}_{l+1})$, and we can define X^{l+1} on $\mathcal{G}_{l+1}$ to have the same values on this representation as X^l.

  (2) In the construction a new vertex τ_k and a new edge between the vertex $t_j < \tau_k$ in $\mathcal{V}_l$ and τ_k, was added to get $\mathcal{G}_{l+1}$. So, $\mathcal{G}_{l+1}$ has a new edge $E^* = E^*_{jk}$ and a new vertex τ_k. Let $Z_j = X^l(t_j)$.

  Now we pick a full-time path σ that contains E^*. Now we define $\mu_*(\cdot|x)$ to be the conditional probability of the process with the distribution $\mu_\sigma \circ \pi^{-1}_{[t_j,\tau_k]}$ conditioned to have the value x at t_j. So we construct the process X^{l+1} in such a way that X^{l+1} on $R(\mathcal{G}_l)$ is equal to X^l and $X^{l+1}_{E^*}$ is the process with distribution $\mu_*(\cdot|Z_j)$ and otherwise independent of X^l given Z_j.

(3) In the construction a new vertex τ_k and a new edge between the vertex $t_m > \tau_k$ in $\mathcal{V}_l$ and τ_k, was added to get $\mathcal{G}_{l+1}$. So, $\mathcal{G}_{l+1}$ has a new edge $E^* = E^*_{jk}$ and a new vertex τ_m. Let $Z_m = X^l(t_m)$.

Now we pick a full-time path σ that contains E^*. Now we define $\mu_*(\cdot|y)$ to be the conditional probability of the process with the distribution $\mu_\sigma \circ \pi^{-1}_{[\tau_k,t_m]}$ conditioned to have the value y at t_m. So we construct the process X^{l+1} in such a way that X^{l+1} on $R(\mathcal{G}_l)$ is equal to X^l and $X^{l+1}_{E^*}$ is the process with distribution $\mu_*(\cdot|Z_m)$ and otherwise independent of X^l given Z_m.

(4) In the construction a new edge between two vertices $t_j < t_k$ in $\mathcal{V}_l$ that are connected by a time path in $\mathcal{G}_l$, was added to get $\mathcal{G}_{l+1}$. So, $\mathcal{G}_{l+1}$ has a new edge $E^* = E^*_{jk}$. Let $Z_j = X^l(t_j)$ and $Z_k = X^l(t_k)$.

Now we pick a full-time path σ that contains E^*. Now we define $\mu_*(\cdot|x,y)$ to be the conditional probability of the process with the distribution $\mu_\sigma \circ \pi^{-1}_{[t_j,t_k]}$ conditioned to have values x at t_j and y at t_k. So we construct the process X^{l+1} in such a way that X^{l+1} on $R(\mathcal{G}_l)$ is equal to X^l and $X^{l+1}_{E^*}$ is the process with distribution $\mu_*(\cdot|Z_j, Z_k)$ and otherwise independent of X^l given Z_j and Z_k.

Since n is finite this procedure will end and we will have a process $X = X^n$ defined on $\mathcal{G}$.

We define:

- X on $\mathcal{G}$ to be defined as in Definition 9.7;
- X' to be a natural $\mathcal{M}'$-process on $\mathcal{G}'$ (in the sense of Definition 2.21);
- X'' to be a natural $\mathcal{M}''$-process on $\mathcal{G}''$ (in the sense of Definition 2.21).

Theorem 9.8. *The following processes have the same distribution on $\mathcal{G}$:*

(a) $(X(t) : t \in \mathcal{G})$;

(b) $(X'(t) : t \in \mathcal{G})$;

(c) $(X''(t) : t \in \mathcal{G})$.

Proof. To show that X'' and X' have the same distribution on $\mathcal{G}$ we will show that they have the same distribution on $\mathcal{G}'$.It is known by Corollary 8.11. that there exists a TLG*-tower that starts with $\mathcal{G}'$ and ends with $\mathcal{G}''$. Now, by Lemma 2.14, $\mathcal{M}''(\mathcal{G}')$ is a (3T)-family, and by definition it coincides with $\mathcal{M}'$. Since, the construction of a process on TLG* doesn't depend on the order (Theorem 2.20), we can first construct X'' on $\mathcal{G}'$ and later on the rest of $\mathcal{G}''$. Hence, X'' on $\mathcal{G}'$ will be a natural $\mathcal{M}'$-process, so $(X'(t) : t \in \mathcal{G}')$ and $(X''(t) : t \in \mathcal{G}')$ have the same distribution. Therefore, the restriction of these two processes to $\mathcal{G}$ is also the same.

It remains to show that X and $X''_{\mathcal{G}}$ have the same distribution. Let $(\mathcal{H})_{j=0}^{n}$ be a TLG**-tower, we will show that the construction of X on $\mathcal{G}$ can embedded int the construction of X'' on $\mathcal{G}''$. For $j = 0$ it is clear that $X_{\mathcal{H}_0}$ and $X''_{\mathcal{H}''_0}$ have the same distribution on $\mathcal{H}_0$. Let's assume for $j \geq 0$ $X_{\mathcal{H}_j}$ and $X''_{\mathcal{H}''_j}$ have the same distribution on $\mathcal{H}_j$, and prove it for $j+1$. We have the following cases to consider:

- A new vertex has been added to $\mathcal{H}_j$ to obtain $\mathcal{H}_{j+1}$. In this case the claim follows clearly.

- A new vertex τ and an edge E_* between that vertex and existing one has been added. In this case the claim follows from the fact that in $\mathcal{H}''_j$ we are adding:

  - a new edge E''_* between -1 or 2 and an vertex t_l on $\mathcal{H}_j$;
  - a vertex τ on that edge;
  - we are adding an edge between τ and between the vertex -1 or 2 to which it is not connected.

  Since, the $X''(-1) = X''(2) = 0$ the distribution of the process on E''_* whose representation is in $R(\mathcal{H}_{j+1})$ is given and depends only on the value $X(t_l)$. Hence, the claim follows.

- A new edge has been added to $\mathcal{H}_j$ to obtain $\mathcal{H}_{j+1}$. In this case the claim follows from the fact that the distribution of the process on the new edge is given and depends only on the values of the process on $\mathcal{H}_j$ (for both X and X'').

■

Theorem 9.9. *For a (3T')-family $\mathcal{M}$ the constructed process X on a TLG** $\mathcal{G}$ will always have the same distribution.*

Proof. By Theorem 9.8. we can embed the constructed process into a natural $\mathcal{M}'$ process on $\mathcal{G}'$ and this process has a unique distribution. ■

9.2 Properties of constructed processes

We know, from Chapters 2 and 3, that for the process X' many interesting properties hold. Many of these properties have their natural analogous for the process X.

We will show that for X the following properties hold:

- X is an $\mathcal{M}$-process;
- X is a spine-Markovian process;
- X is a hereditary spine-Markovian process;
- X is a cell-Markovian process.

Additionally if $\mathcal{M}$ is a Markov family of measures we have

- X is moralized graph-Markovian;
- X is time-Markovian;
- X is edge-Markovian.

All these properties are (slightly generalized) versions of the properties we had defined for simple TLG's.

9.2.1 X is an $\mathcal{M}$-process

It is easy to see $X_\sigma \sim \mu_\sigma$, that is, X is a $\mathcal{M}$-process. (This is the same as defined in Subsection 2.3, on page 47.)

9.2.2 X is a spine-Markovian process

We will first define the spine-Markovian property.

Definition 9.10. *Let σ be any full path in the TLG $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Let $\mathcal{G}_-$ be a subgraph (not necessarily a TLG) of $\mathcal{G}$ whose representation is a connected component of $R(\mathcal{G}) \setminus R(\sigma)$. Let W be the set of vertices - **roots** connecting $\mathcal{G}_-$ to σ and let $\mathcal{G}_+$ denote the graph represented by $R(\mathcal{G}) \setminus R(\mathcal{G}_-)$.*

*We say that the process X on a TLG $\mathcal{G}$ is **spine-Markovian** if for each such σ and $\mathcal{G}_-$ the processes $(X(t) : t \in \mathcal{G}_-)$ and $(X(t) : t \in \mathcal{G}_+)$ given $(X(t) : t \in W)$ are independent.*

Proposition 9.11. *The constructed process X is a **spine-Markovian** process on $\mathcal{G}$.*

Proof. Let σ be the full path, and σ' the corresponding full-time path in the embedding $\mathcal{G}'$. If $\mathcal{G}_-$ is as in the definition, this is a connected graph and is a connected component of $R(\mathcal{G}') \setminus R(\sigma')$. We set $\mathcal{G}_+$ and $\mathcal{G}'_+$ to be graphs that have the representation, respectively $R(\mathcal{G}) \setminus R(\mathcal{G}_-)$ and $R(\mathcal{G}') \setminus R(\mathcal{G}_-)$. The roots W' of $\mathcal{G}'$ include all the roots W of $\mathcal{G}$ and maybe -1 and 2. Since, $X'(-1) = X'(2) = 0$, we have $\sigma(X_W) = \sigma(X'_{W'}) = \sigma(X'_{W' \setminus \{-1,2\}})$. Therefore, since X' is spine Markovian, and $\sigma(X(t) : t \in \mathcal{G}_-) \subset \sigma(X'(t) : t \in \mathcal{G}_-)$, $\sigma(X(t) : t \in \mathcal{G}_+) \subset \sigma(X'(t) : t \in \mathcal{G}'_+)$ the spine-Markovian property for X follows. ■

9.2.3 Hereditary spine-Markovian property

Recall, Definition 2.6. of $S^*(\mathcal{G})$.

Definition 9.12. *For a TLG** $\mathcal{G}$ we define $S^{**}(\mathcal{G})$ to be the set of all TLG**'s $\mathcal{H}$ such that there exists a TLG**-tower $(\mathcal{K}_k)_{k=0}^n$ that starts with $\mathcal{K}_0 = \mathcal{H}$ and ends with $\mathcal{K}_n = \mathcal{G}$.*

Definition 9.13. *The process $(X(t) : t \in \mathcal{G})$ has a* ***hereditary spine-Markovian property*** *if $(X(t) : t \in \mathcal{H})$ is a spine-Markovian process for each $\mathcal{H} \in S^{**}(\mathcal{G})$.*

Corollary 9.14. *Let $\mathcal{G}$ be a TLG** and $\mathcal{G}''$ its embedding, if TLG** $\mathcal{H}$ is in $S^{**}(\mathcal{G})$ then $\mathcal{H}''$ is in $S^{*}(\mathcal{G}'')$.*

Proof. Let $(\mathcal{K}_l)_{l=1}^m$ TLG**-tower such that $\mathcal{K}_1 = \mathcal{H}$, and $\mathcal{K}_m = \mathcal{G}$. By Corollary 8.9. $\mathcal{K}_1''$, $\mathcal{K}_2''$, $\ldots \mathcal{K}_m''$ are one after another in a TLG*-tower, and the claim follows. ■

Proposition 9.15. *The constructed process X is hereditary spine-Markovian.*

Proof. Let $\mathcal{H} \in S^{**}(\mathcal{G})$. Then $\mathcal{H}''$ is in $S^{*}(\mathcal{G}'')$.

By Theorem 2.17., X' on $\mathcal{G}''$ is hereditary spine-Markovian, X' is spine Markovian on $\mathcal{H}''$, and therefore, X is spine Markovian on $\mathcal{H}$. ■

9.2.4 Cell-Markovian property

A cell will remain truly simple, as in Definition 1.41. We need to extend our definition to half-cells.

Definition 9.16. *(a) A right half-cell (σ_1, σ_2) ending at t_m is called* ***truly simple*** *if there is no path $\{t \in \mathcal{G} : t \prec t_m\}$ that starts on on one side of the cell and ends on the other.*

(b) A left half-cell (σ_1, σ_2) starting at t_k is called ***truly simple*** *if there is no path $\{t \in \mathcal{G} : t_k \prec t\}$ that starts on on one side of the cell and ends on the other.*

Lemma 9.17. *A truly simple half-cell in $\mathcal{G}$ is a part of a truly simple cell in $\mathcal{G}'$.*

Proof. We will prove the claim for the right half-cell, the proof for the left half-cell is similar. Let σ_j' be the path consisting including $t_{-\infty}$ and σ_j, for $j = 1, 2$. Now, sigma (σ_1, σ_2) is a cell. If there exists a path in $\mathcal{G}[t_{-\infty}, t_m] \setminus \{t_{-\infty}, t_m\}$ connecting vertices on σ_1 and σ_2, then these vertices are in $\mathcal{G}$. Further, since the path can't go through $t_{-\infty}$, the path it self is in $\mathcal{G}$. Hence, (σ_1, σ_2) is not a truly simple half-cell. ■

Definition 9.18. *We will say that a process X on a TLG $\mathcal{G}$ is* ***cell-Markovian*** *if for*

(a) any truly simple cell (σ_1, σ_2) starting at t_* and ending at t^* the processes X_{σ_1} and X_{σ_2} are conditionally independent, given the values $X(t_*)$ and $X(t^*)$;

(b) any truly simple right half-cell (σ_1, σ_2) ending at t^* the processes X_{σ_1} and X_{σ_2} are conditionally independent, given the value of $X(t^*)$;

(c) any truly simple left half-cell (σ_1, σ_2) starting at t_* the processes X_{σ_1} and X_{σ_2} are conditionally independent, given the value $X(t_*)$.

Definition 9.19. *We will say that a process X on a TLG $\mathcal{G}$ is* ***strong cell-Markovian*** *if it is cell-Markovian and for*

(a) any truly simple cell (σ_1, σ_2) starting at t_* and ending at t^* the processes $(X(t) : t_* \preceq t \preceq t^*)$ and $(X(t) : t^* \preceq t$ or $t \preceq t^*)$ are independent, given the values $X(t_*)$ and $X(t^*)$;

(b) any truly simple right half-cell (σ_1, σ_2) ending at t^* the processes $(X(t) : t \prec t^*)$ and $(X(t) : t^* \preceq t)$ are independent, given the value $X(t^*)$;

(c) any truly simple left-cell (σ_1, σ_2) starting at t_* the processes $(X(t) : t^* \prec t)$ and $(X(t) : t \preceq t^*)$ are independent, given the value $X(t_*)$.

Proposition 9.20. *The constructed process X on $\mathcal{G}$ is strong cell-Markovian.*

Proof. A simple cell in $\mathcal{G}$ is clearly a simple cell in $\mathcal{G}'$, and by Lemma 9.17. a truly simple half-cell is a part of a truly simple cell in $\mathcal{G}'$. By Theorem 3.4. X' (on $\mathcal{G}'$) is strong cell-Markovian (in the sense of the Definition 3.2.), and all the claims now follow.

■

9.2.5 Distribution uniqueness

Proposition 9.21. *A hereditary spine-Markovian $\mathcal{M}$-process (satisfying (3T') properties) on a TLG** $\mathcal{G}$ has a unique distribution.*

Proof. This is a consequence of the unique distribution of $\mathcal{M}'$-process on a TLG* $\mathcal{G}'$. (See Theorem 2.20.) ■

9.3 Properties for Markov family $\mathcal{M}$

Again, $\mathcal{M}$ is called a Markov family, if all the measures in $\mathcal{M}$ are distributions of Markov processes.

Lemma 9.22. *If $\mathcal{M}$ is a Markov family, so is $\mathcal{M}'$ and $\mathcal{M}''$.*

Proof. For $\sigma \in P(\mathcal{G})$ if μ_σ is the distribution of a Markov process Y_σ, then the process $Y_{\sigma'}$ is also a Markov process, and hence $\mu_{\sigma'}$ is a distribution of a Markov process. ■

9.3.1 Moralized graph-Markovian property

The definition of moralized graph-Markovian property is the same as in Definition 3.10.

Lemma 9.23. *The constructed process X on $\mathcal{G}$ for a Markov family $\mathcal{M}$ is a moralized graph-Markovian process.*

Proof. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two components of $\mathcal{G}$ connected through points W, and let W separate $\mathcal{E}_1$ and $\mathcal{E}_2$ in $(\mathcal{G})^\heartsuit$. $\mathcal{G}'$ we will get new edges connecting $t_{-\infty}$ and $t_{+\infty}$, so all the new cells (that are not in $\mathcal{G}$) will have one endpoint in $\{t_{-\infty}, t_{+\infty}\}$. If $\mathcal{E}_1$ and $\mathcal{E}_2$ were separated by W in $\mathcal{G}^\heartsuit$, they will be separated in $(\mathcal{G}')^\heartsuit$ by $W \cup \{t_{-\infty}, t_{+\infty}\}$. Since $X(t_{-\infty}) = X(t_{+\infty}) = 0$, $\sigma(X_{W\cup\{t_{-\infty},t_{+\infty}\}}) = \sigma(X_W)$. Now, by Theorem 3.12, $X'_{\mathcal{E}_1} = X_{\mathcal{E}_1}$ and $X'_{\mathcal{E}_2} = X_{\mathcal{E}_2}$ are independent given $\sigma(X_{W\cup\{t_{-\infty},t_{+\infty}\}})$. ■

9.3.2 Time-Markovian property

The definition of time-Markovian property is the same as in Section 3.2. (see Definition 3.7.).

Lemma 9.24. *The constructed process X on $\mathcal{G}$ for a Markov family $\mathcal{M}$ is a time-Markovian process.*

Proof. Let t be a point in $\mathcal{G}$. By construction of X we have that

$$\mathcal{F}_t = \sigma\{X(u) : u \in \mathcal{G}, u \preceq t\} \subset \mathcal{F}'_t = \sigma\{X'(u) : u \in \mathcal{G}', u \preceq t\},$$

$$\mathcal{H}_t = \sigma\{X(u) : u \in \mathcal{G}, u \succeq t\} \subset \mathcal{H}'_t = \sigma\{X'(u) : u \in \mathcal{G}', u \succeq t\}.$$

(Actually equalities hold in both expressions.) Since $\mathcal{M}'$ is a Markov family, X' is a time-Markovian process. Therefore, $\mathcal{F}'_t$ and $\mathcal{H}'_t$ are independent given $X'(t) = X(t)$, but then also $\mathcal{F}_t$ and $\mathcal{H}_t$ are independent given $X(t)$. ■

9.3.3 Edge-Markovian property

The definition of edge-Markovian processes remains the same (see Definition 3.14.).

Proposition 9.25. *The constructed process X on $\mathcal{G}$ for a Markov family $\mathcal{M}$ is an edge-Markovian process.*

Proof. E be an arbitrary edge in $\mathcal{G}$. Since $\mathcal{M}'$ is a Markov family, X' is edge Markovian, so since $\sigma(X'_E) = \sigma(X_E)$ and $\sigma(X(t) : t \in \mathcal{G}, t \notin E) \subset \sigma(X'(t) : t \in \mathcal{G}', t \notin E')$ are independent given the values at the endpoints of E, X_E is independent of $(X(t) : t \in \mathcal{G}, t \notin E)$ given the values at the endpoints of E. ■

9.4 Processes on time-like trees

Among all graphs trees have a special place. Processes on trees have been widely studied and used. For examples see Markov chains indexed trees ([4, Benjamini, Peres]), branching Markov processes (where the underlying tree is random), hidden Markov models, ...

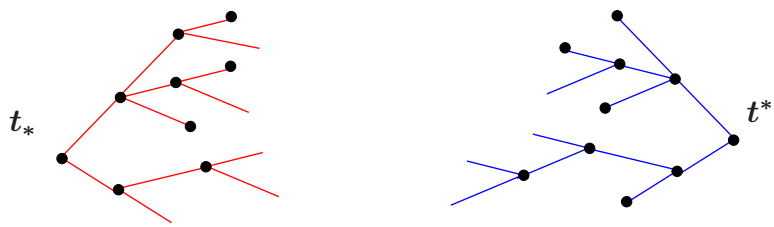


Figure 9.3: Forward and backward trees.

In this section we will look at the properties processes on trees have. We start by defining time-like trees.

Definition 9.26. *(a) A* ***time-like tree*** *(TLT) is a TLG with no cells.*

(b) A ***forward time-like tree*** $\mathcal{T}$ *is a TLT with exactly one entrance. The entrance vertex we will call the* ***root*** *of the forward time-like tree.*

(c) A ***backward time-like tree*** $\mathcal{T}$ *is a TLT with exactly one exit . The exit vertex we will call the* ***root*** *of the backward time-like tree.*

Remark. Since all vertices are connected to the root, it is clear that $\mathcal{T}$ is connected as a graph.

We know from Theorem 8.14 (b) that a time-like tree is a TLG**, and further the following holds:

Lemma 9.27. *Let* $\mathcal{T}$ *be a TLT, and* $\mathcal{T}'$ *be a TLG that is a connected sub-graph of* $\mathcal{T}$*. Then* $\mathcal{T}'$ *is a TLT, and* $\mathcal{T}'$ *and* $\mathcal{T}$ *are elements of the TLG****-tower.*

Proof. Let n be the difference between the number of edges $\mathcal{T}$ and $\mathcal{T}'$ have. For $n = 0$ the claim is clear. Let's assume the claim holds for $n \geq 0$ and prove it for $n + 1$. Pick a leaf t_m on $\mathcal{T}$ not in $\mathcal{T}'$, and an edge E that that is connected to it. Now, let $\mathcal{T}''$ be $\mathcal{T}$ without t_m and E. $\mathcal{T}''$ is a TLT, and further we can construct since the difference between the edges of

$\mathcal{T}''$ and $\mathcal{T}'$ is n, we can construct $\mathcal{T}''$ from $\mathcal{T}'$. Hence, they are in some TLG**-tower. It is clear that $\mathcal{T}''$ and $\mathcal{T}$ are in some TLG**-tower. The claim now follows. ■

For a (3T') family $\mathcal{M}$ on $\mathcal{T}$ we can construct a natural $\mathcal{M}$-process on $\mathcal{T}$. By changing time to each vertex from t_k into $\tilde{t} = 1 - t$ we can transform a backward graph into a forward graph, and in the same way transform the process on a backward time-like tree into a process on a forward time-like tree. Everything we prove for processes on forward TLT's will in a similar way hold for backward TLT's.

Theorem 9.28. *If $\mathcal{M}$ is a (3T') family on a TLT $\mathcal{T}$ and $t_k \in \mathcal{V}$ is a vertex and X a natural $\mathcal{M}$-process on $\mathcal{T}$ then*

(a) the closures of connected components of $R(\mathcal{T}) \setminus R(t_k)$ are representations of several time-like trees $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_m$;

(b) the processes $X_{\mathcal{T}_1}, \ldots, X_{\mathcal{T}_m}$ are independent given the value of $X(t_k)$.

Proof. (a) Each of the components is a TLG without any cells. Hence, every component is a TLT.

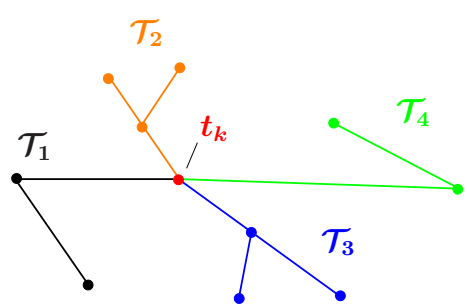


Figure 9.4:

(b) For each $\mathcal{T}_l$ there is a full path such with no edges in $\mathcal{T}_l$. Using the spine-Markovian property we know that $X_{\mathcal{T}_l}$ is independent of the process on the rest of the graph given

the value of $X(t_k)$ (since t_k is the only root). Now, applying this fact several times for $A_l \in \sigma(X_{\mathcal{T}_l})$ for $l = 1, \ldots, m$, we have

$$\begin{aligned}
&\mathbb{E}(\mathbb{P}(A_1 \cap \ldots \cap A_m | X(t_k))\mathbf{1}_B) = \mathbb{E}(\mathbf{1}_{A_1} \ldots \mathbf{1}_{A_m}\mathbf{1}_B) \\
&= \mathbb{E}(\mathbb{E}(\mathbf{1}_{A_1}|X(t_k)) \ldots \mathbf{1}_{A_m}\mathbf{1}_B) = \ldots = \mathbb{E}(\mathbb{E}(\mathbf{1}_{A_1}|X(t_k)) \ldots \mathbb{E}(\mathbf{1}_{A_m}|X(t_k))\mathbf{1}_B) \\
&= \mathbb{E}(\mathbb{P}(A_1|X(t_k)) \ldots \mathbb{P}(A_m|X(t_k))\mathbf{1}_B).
\end{aligned}$$

for arbitrary $B \in \sigma(X(t_k))$. Hence the claim follows. ■

Corollary 9.29. *If $\mathcal{M}$ is a (3T') family on a forward TLT $\mathcal{T}$ and t_k is a vertex of degree at least 3, then the natural $\mathcal{M}$-process X on $\mathcal{T}$ will have the property that given process X on $\mathcal{T}^+_{t_k} = \{s \in \mathcal{T} : t_k \preceq s\}$ is independent of the process on the rest of $\mathcal{T}$ given $X(t_k)$.*

The graph-Markovian property was introduced in Definition 3.6, and it was shown in Subsection 3.4.1. that this property doesn't have to hold on TLG*'s. This property was replaced by the moralized graph-Markovian property on TLG*'s (see Definition 3.10), and in Theorem 9.23 it was shown to also hold for natural $\mathcal{M}$-processes on TLG**'s when $\mathcal{M}$ is a Markov family.

Theorem 9.30. *If $\mathcal{M}$ is a (3T') Markov family on a TLT $\mathcal{T}$ the process will have the graph-Markovian property.*

Proof. By Theorem 9.23, we know that every natural $\mathcal{M}$ process on $\mathcal{T}$ is a moralized graph-Markovian process. Since $\mathcal{T}$ has no cells, the claim follows. ■

Corollary 9.31. *If $\mathcal{M}$ is a (3T') Markov family on a TLT $\mathcal{T}$ and $\tau^t_1, \ldots, \tau^t_n$ are all the points on $\mathcal{T}$ with time t, then the natural $\mathcal{M}$-process X on $\mathcal{T}$ will have the property that*

$$\mathcal{F}^t_{\leftarrow} = \sigma(X(s) : s \leq t) \quad \textit{and} \quad \mathcal{F}^t_{\rightarrow} = \sigma(X(s) : s \geq t)$$

are independent given $X(\tau^t_1)$, ..., $X(\tau^t_n)$.

Proof. The points τ^t_1, ..., τ^t_m separate the graph into two parts $\{s \in \mathcal{G} : s \leq t\}$ and $\{s \in \mathcal{G} : s \geq t\}$, and the claim follows by graph-Markovian property. ■

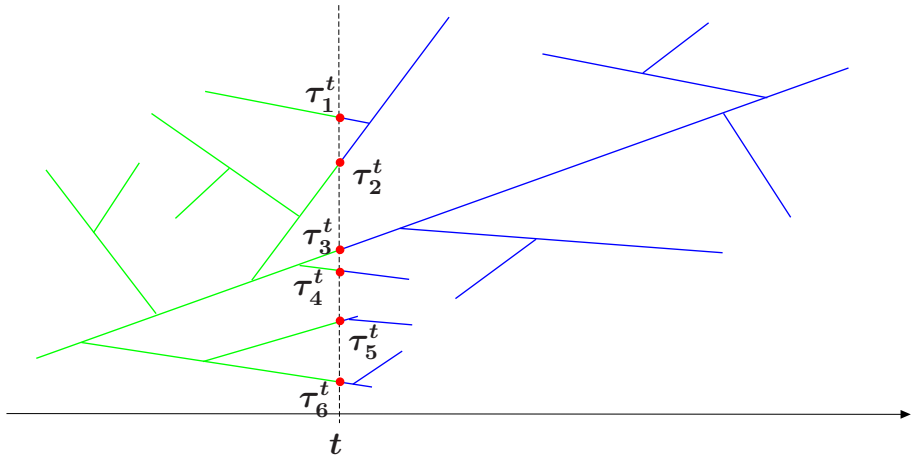


Figure 9.5: The process before and after time t.

Remark. The previous corollary states that the process $(\tilde{X}(t) = (X(s) : s \in R(\mathcal{G}) \cap (\{t\} \times \mathbb{R}^2)) : t \geq 0)$ is a Markov process.

The following lemma states that the spine-Markovian property and hereditary spine-Markovian properties are equivalent on time-like trees. (Note that we didn't have this result for TLG*'s.)

Lemma 9.32. *If $\mathcal{T}$ is a time-like tree, and X a process indexed by $\mathcal{T}$ then the following claims are equivalent:*

(a) X is a spine-Markovian process;

(b) X is a hereditary spine-Markovian process.

Proof. Clearly (b) implies (a). Now, let's prove that (a) implies (b). Let $(\mathcal{G}_k)_{k=1}^n$ be a TLG** tower leading towards the construction of $\mathcal{T}$. Note that since each $\mathcal{G}_k$ is a connected subgraph of $\mathcal{T}$, it is also a tree.

If we pick a π' full-path in $\mathcal{G}_k$, then there is a full-path π in $\mathcal{T}$ such that $R(\pi') \subset R(\pi)$. (See Figure 9.6.) But the representation of roots of $\mathcal{T}$ with respect to π will contain the

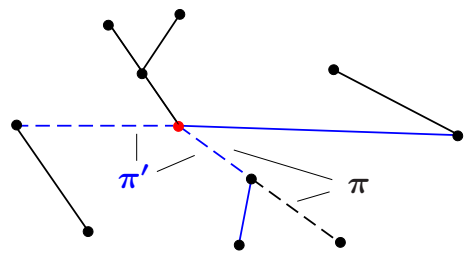


Figure 9.6: $\mathcal{G}_k$, the spine π' and the root $\bullet$.

representations of roots of $\mathcal{G}_k$ with respect to π'. Since the roots decompose the graph into disjoint components the claim follows. ■

Chapter 10

GALTON-WATSON TIME-LIKE TREES AND THE BRANCHING MARKOV PROCESSES

10.1 TLG's with an infinite number of vertices

We will allow t_0 and t_N to take values in $\mathbb{R}$.

Definition 10.1. *(i) Suppose that the vertex set of a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is infinite. We will call $\mathcal{G}$ a time-like graph (TLG) if it satisfies the following conditions.*

(a) There is a sequence of TLG's $\mathcal{G}_n = (\mathcal{V}_n, \mathcal{E}_n)$ with finite vertex set $\mathcal{V}_n$, $n \geq 1$, and for some representations of $\mathcal{G}_n$'s and G we have

$$\bigcup_{n=1}^{\infty} R(\mathcal{G}_n) = R(\mathcal{G}).$$

(b) The graph $\mathcal{G}$ is locally finite, i.e. it has a representation $R(\mathcal{G})$ such that for any compact $K \subset \mathbb{R}^3$ a finite number of edges intersects K.

(ii) A TLG $\mathcal{G}$ with infinite vertex set will be called an TLG** if it satisfies the following conditions.

(a) We can choose a sequence of TLG**'s $\mathcal{G}_n$ in (i). (In the sense of the Definition 8.6.(iv), i.e. $(\mathcal{G}_j)_{1 \leq j \leq n}$ is a tower of TLG**'s for all n.)

(b) Let $\mathcal{V}_n = \{t_{0,n}, t_{1,n}, \ldots, t_{N_n,n}\}$. The initial vertices $t_{0,n} \in \mathcal{V}_n$ and $t_{N_n,n} \in \mathcal{V}_n$ are the same for all $\mathcal{G}_n$, i.e. for all $1 \leq m \leq n$

$$t_{0,n} \leq t_{0,m} \quad \text{and} \quad t_{N_n,n} \geq t_{N_m,m}.$$

The following lemma will be useful for the construction of processes. (It is a version of the Lemma 1.50. for TLG**'s.)

Lemma 10.2. *Let $(\mathcal{G}_n)$ and $(\mathcal{G}'_n)$ be two TLG**-towers that lead to the construction of $\mathcal{G}$. Let $\mathcal{H}$ be a sub-graph (not necessarily a TLG*) of some $\mathcal{G}_{n_0}$. Then there exists $\mathcal{G}'_{n_1}$ such that $R(\mathcal{H}) \subset R(\mathcal{G}'_{n_1})$ and all the vertices of $\mathcal{H}$ are contained in $\mathcal{G}'_{n_1}$.*

Proof. Since $\mathcal{G}$ is locally finite, there are finitely many vertices with representation on $R(\mathcal{H})$, also these vertices are of finite degree. For each such vertex v, by same argument, there has to be $\mathcal{G}'_{n_v}$ such v in $\mathcal{G}'_{n_v}$ has that degree. Now if n_1 is the maximum of n_v over each such vertex v the claim follows. ■

The definition of (forward/backward) time-like trees is the same as in Definition 9.26.

Proposition 10.3. *Time-like tree $\mathcal{T}$ with infinite number of vertices is a TLG**.*

Proof. Pick a vertex t_k, and let K_n be a set of compact sets such that

$$\bigcup_{n=1}^{\infty} K_n = \mathbb{R}^3.$$

It is clear that the connected component of $R(\mathcal{T}) \cap K_n$ that contains t_k is a tree, and we set $\mathcal{T}_n$ to be the time-like tree such that $R(\mathcal{T}_{n-1}) \subset R(\mathcal{T}_n) \subset R(\mathcal{T}) \cap K_n$ and the number of $\mathcal{T}_n$ is as large as possible. By Lemma 9.27, $\mathcal{T}_n$ can be constructed from $\mathcal{T}_{n-1}$. So $(\mathcal{T}_n)$ is a subsequence of some TLG**-tower $(\mathcal{H}_n)$. ■

10.2 Galton – Watson time-like tree

We will encode a continuous version of Galton-Watson process into a (forward) time-like tree. The idea is to use the setup in the Crump - Mode - Jagers model (see Section A.7.).

Let $I = \{\emptyset\} \cup \bigcup_{n=1}^{\infty} \mathbb{N}^n$, and we interpret that $(x, j) \in I$, $j \in \mathbb{N}$ is a child of $x \in I$.

First, lets make some assumptions:

- Let $(\lambda_x : x \in I)$ be a collection of exponential random variables with parameter V. (Lifetime of an individual.)

- Let $(R_x : x \in I)$ be a collection of random variables with distribution given by the generating function
$$\Phi(s) = \sum_{k=0}^{\infty} a_k s^k, \quad \Phi(1) = 1.$$

- $(\lambda_x, R_x)_{x \in I}$ is an i.i.d. sequence.

In our model at the end of its lifetime, the individual gets divided into nonnegative number of new individuals (0, 1, 2, ...), so we define the reproduction function to be

$$\xi_x(t) = R_x \mathbf{1}_{(t \geq \lambda_x)}.$$

Recall, that we defined with τ_x the birth time of x, with $\tau_\emptyset = 0$, $\tau_{(x',i)} = \tau_{x'} + \inf\{u : \xi_{x'}(u) \geq i\}$.

We define the Galton - Watson tree in the following way:

- Let $\mathcal{V}_0 = \{t_{-1} = 0, t_0 = \lambda_\emptyset\}$, and $\mathcal{E}_1 = \{E_{-1,0}\}$. Set $\mathcal{G}_0 = (\mathcal{V}_0, \mathcal{E}_0)$.

- Define $l^+_{Ex}(\mathcal{G}_k)$ all the x labels of exit vertices in $\mathcal{G}_k$ such that $\xi_x > 0$.

- Now set

$$\mathcal{V}_k := \mathcal{V}_{k-1} \cup \bigcup_{x' \in l^+_{Ex}(\mathcal{G}_{k-1})} \{t_{(x',j)} := \tau_{(x',j)} : j = 1 \dots \xi_{x'}\},$$
$$\mathcal{E}_k := \mathcal{E}_{k-1} \cup \bigcup_{x' \in l^+_{Ex}(\mathcal{G}_{k-1})} \{E_{x',(x',j)} : j = 1 \dots \xi_{x'}\},$$

$$\mathcal{G}_k := (\mathcal{V}_k, \mathcal{E}_k). \tag{10.1}$$

- Now, set

$$\mathcal{V} := \bigcup_{k=1}^{\infty} \mathcal{V}_k, \quad \mathcal{E} = \bigcup_{k=1}^{\infty} \mathcal{E}_k.$$

We define $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ as the **Galton − Watson time-like tree**.

Lemma 10.4. *For all $t \geq 0$ we have $\mathbb{E}(\xi_x(t)) \leq \mathbb{E}(R_x)$.*

Therefore, if $\mathbb{E}(R_x) < \infty$, almost surely for all $t \geq 0$ the number of vertices from $\mathcal{V}$ with time at most t is finite.

Proof. Since $\xi_x \leq R_x$ the first claim follows. For the second claim, first note $\mathbb{E}(\xi(t)) < \infty$. Further $(0 \geq \lambda_x) = (0 = \lambda_x)$, and this is a set of probability 0, hence $\mathbb{E}(\xi(0)) = 0 < 1$. Therefore, by Theorem A.35. the set of vertices with time label at most t is finite. ■

Theorem 10.5. *If $\mathbb{E}(R_x) < \infty$, the Galton - Watson time-like tree is a TLG**. Specially, it is a forward time-like tree.*

Proof. It is clear that $(\mathcal{G}_k)$ from (10.1) is the TLG**-tower that leads towards the construction of $\mathcal{G}$. Further, any representation is locally finite, since any compact set K will have a finite time component, i.e. it will be contained in $[0, T] \times \mathbb{R}^2$, and by Lemma 10.4. it can contain finitely many points finitely many edges. We know by Lemma 10.4 that the number of vertices whose representation is in $[0, T] \times \mathbb{R}^2$ is finite a.s., and also since R_x is finite a.s. we have that number of edges intersecting K is finite. ■

10.3 Processes on TLG**'s with infinite number of vertices

10.3.1 Construction

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ a TLG** such that $\mathcal{V}$ is infinite. According to the definition, there exists a tower of TLG**'s $\mathcal{G}_n = (\mathcal{V}_n, \mathcal{E}_n)$, $n \geq 1$, such that $\mathcal{V}_n$ is finite, where $\mathcal{V} = \bigcup_{n \geq 1} \mathcal{V}_n$.

Let

$$\mathcal{M} = \{\mu_\sigma : \sigma \in P(\mathcal{G})\} \tag{10.2}$$

be a family of distributions of processes along full-time paths in $\mathcal{G}$ satisfying conditions (T'1)-(T'3) given in Subsection 9.1.1.

Since

$$\mathcal{M}(\mathcal{G}_n) = \{\mu_\sigma : \sigma \in P(\mathcal{G}_n)\}$$

is well-defined, and we can show similarly as in Lemma 2.14 that $\mathcal{M}(\mathcal{G}_n)$ satisfies (T'1)-(T'3), we can define a hereditary spine-Markovian process X^n on $\mathcal{G}_n$, such that for each $\sigma \in P(\mathcal{G}_n)$ the process X^n_σ has the distribution μ_σ. Further, the restriction of this process to $\mathcal{G}_k$ $(k \leq n)$ has the same distribution as the $\mathcal{M}(\mathcal{G}_k)$-process X^k defined on $\mathcal{G}_k$ in the similar manner.

Now, Kolomogorov's consistency theorem shows, that there exists a process X on $\mathcal{G}$ such that the restriction of X to any $\mathcal{G}_k$ has same distribution as X^k. Note, that since each $\sigma \in P(\mathcal{G})$ is in some of the $\mathcal{G}_k$'s we have X_σ has the distribution μ_σ.

10.3.2 Uniqueness of distribution

Using a similar approach as in §2.4.2 we will get that the distribution of the process X doesn't depend on the choice of the TLG**-tower $(\mathcal{G}_n)$.

Lemma 10.6. *Let $\mathcal{G}$ be a TLG** with infinitely many vertices, $(\mathcal{G}^1_j)$ and $(\mathcal{G}^2_j)$ two TLG**-towers that construct $\mathcal{G}$ and X^1 and X^2 the natural $\mathcal{M}$-processes constructed using these two towers. The distribution of the processes X^1 and X^2 restricted on $\mathcal{G}^1_k$ is the same for all k.*

Proof. We first prove the claim when the vertices of $\mathcal{G}$ have only real values. By Lemma 10.2. we can choose k_1, and l_1 in such that

$$R(\mathcal{G}^1_k) \subset R(\mathcal{G}^2_{l_1}) \subset R(\mathcal{G}^1_{k_1}),$$

where $\mathcal{V}_{\mathcal{G}^1_k} \subset \mathcal{V}_{\mathcal{G}^2_{l_1}} \subset \mathcal{V}_{\mathcal{G}^1_{k_1}}$. Now, we look at the embeddings $(\mathcal{G}^1_k)''$, $(\mathcal{G}^2_{l_1})''$ and $(\mathcal{G}^1_{k_1})''$. We will have the same relationships, and by Lemma 2.22, we know that $(\mathcal{G}^1_k)''$ and $(\mathcal{G}^1_{l_1})''$ are in some TLG*-tower. Now, by Theorem 2.20. and Theorem 9.8. the result follows. ■

Theorem 10.7. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a TLG**'s with infinitely many vertices in $\mathcal{V}$, and let X^1 and X^2 be two $\mathcal{M}$-processes constructed using the TLG**-towers $(\mathcal{G}_n^1)$ and $(\mathcal{G}_n^2)$, then X^1 and X^2 have the same distribution.*

Proof. Let $t_1, \ldots, t_m$ be the points on $\mathcal{G}$ with finite time. Then, by Lemma 10.2., there exists $\mathcal{G}_k^1$ that contains all of these points. By Lemma 10.6 it follows, that X^1 and X^2 have the same distribution on $\mathcal{G}_k^1$. Specially, $(X^1(t_1), \ldots, X^1(t_m))$ and $(X^2(t_1), \ldots, X^2(t_m))$ have the same distribution. Now, by Kolomogorov's Consistency Theorem the claim follows. ■

Corollary 10.8. *The distribution of the process X on $\mathcal{G}$ doesn't depend on the choice of the TLG**-tower $(\mathcal{G}_j)$ that constructs $\mathcal{G}$.*

Definition 10.9. *We call the constructed process X the natural $\mathcal{M}$-process on the TLG** $\mathcal{G}$.*

10.4 Natural $\mathcal{P}$-Markov process

First, let's define the natural $\mathcal{P}$-Markov process.

Definition 10.10. *Let $\mathcal{G}$ be a TLG** and $\mathcal{P}$ a distribution of a Markov process on $[0, \infty)$, then* ***natural $\mathcal{P}$-Markov process*** *on $\mathcal{G}$ is a stochastic process X indexed by $\mathcal{G}$ such that the distribution of X along each path π from any point t_j to any other point t_k is distributed as a $\mathcal{P}$-Markov process along $[t_j, t_k]$, and satisfies (3T') conditions. This induces a (3T') family $\mathcal{M}_{\mathcal{P}}$, and the natural $\mathcal{P}$-Markov process on $\mathcal{G}$ is the natural $\mathcal{M}_{\mathcal{P}}$-process on $\mathcal{G}$ (see Definition 10.9.)*

The following was shown in Section 10.3.

Theorem 10.11. *For any distribution $\mathcal{P}$ of a Markov process on $[0, \infty)$ and any TLG** $\mathcal{G}$ whose time components are all greater or equal to 0, there exists a natural $\mathcal{P}$-Markov process.*

10.5 Branching $\mathcal{P}$-Markov process

Idea of this section is to construct a natural $\mathcal{P}$-Markov process on a random Galton - Watson tree, where $\mathcal{P}$ is a distribution of an RCLL or continuous process. We will also show its connection to the branching $\mathcal{P}$-Markov process. Specially, to show that in the case when $\mathcal{P}$ is the distribution of the Brownian motion, that we have the branching Brownian motion.

Basically, we first construct a Galton – Watson tree, and then on that tree we construct the $\mathcal{P}$-Markov process indexed by it.

- Based on the construction in Section 10.2 construct a Galton-Watson time-like tree $\mathcal{T}$.
- Construct a natural $\mathcal{P}$-Markov process on $\mathcal{T}$ whose values are independent of $\mathcal{T}$.

Note, that the probability space on which we live can be written as

$$\left[\prod_{x\in I}(\mathbb{R}\times\mathbb{N}_0,\mathcal{B}(\mathbb{R})\times\mathcal{P}(\mathbb{N}_0))\right]\times\left[\prod_{x\in I}(D[0,\infty),\mathcal{B}(D[0,\infty)))\right]$$

This is a product of countably many Borel spaces, and therefore it is a Borel space. The first part of the product encodes the tree, while the second part is used to construct the process on the tree.

Construction of the tree

As discussed in Section 10.2. the sequence $(\lambda_x,\xi_x)_{x\in I}$ encodes the whole tree, and from there we can get the time τ_x of birth of each individual $x\in I$. (Recall, that λ_x is the lifetime of x and ξ_x is the number of children.)

If $\tau_x=\infty$ then x was never born. Since the sequence was i.i.d. we can construct a probability measure on

$$(\Omega_{\mathcal{T}},\mathcal{F}_{\mathcal{T}})=\prod_{x\in I}(\mathbb{R}\times\mathbb{N}_0,\mathcal{B}(\mathbb{R})\times\mathcal{P}(\mathbb{N})).$$

We know that $\mathcal{T}$ is a time-like tree a.s.

Construction of the process

We will construct a probability on the space

$$(\Omega, \mathcal{F}) = (\Omega_{\mathcal{T}}, \mathcal{F}_{\mathcal{T}}) \times \prod_{x \in I} (D[0,\infty), \mathcal{B}(D[0,\infty))).$$

For each element $((\lambda_x, \xi_x)_{x\in I}, (f_x)_{x\in I})$:

- $(\lambda_x, \xi_x)_{x\in I}$ is distributed as Galton-Watson time-like tree
- $f_x|_{[\tau_x, \tau_x+\lambda_x)}$ represents the space position of x during its lifetime
- $f_x|_{\mathbb{R}\setminus[\tau_x,\tau_x+\lambda_x)} = \Delta$ for all $h \geq 0$ (represents cemetary).
- If $\tau_{(x,j)} < \infty$ then $f_{(x,j)}(\tau_{(x,j)}) = f_x((\tau_x + \lambda_x)^-)$ almost surely for all $x \in I$ and $j \in \mathbb{N}$ (last position of the parent, is the first position of the child).

Specially, if $\tau_x = \infty$ then

- $f_x(h) = \Delta$ for all $h \geq 0$ (never born, remains on cemetary).

Let's make some assumptions on the distribution $\mathcal{P}$ and introduce some notation. Let $(X(t) : t \geq 0)$ be a $\mathcal{P}$-distributed process:

- by $\mathcal{P}_\tau^x$ we are denoting the distribution of the process $(X(\tau + t) : t \geq 0)$ conditioned on the event $X_\tau = x$.

We will assume the following on $(\mathcal{P}_\tau^x \ : \ \tau \geq 0, x \in \mathbb{R})$ for all $A \in \mathcal{B}(D[0,\infty))$ the map

$$(\tau, x) \mapsto \mathcal{P}_\tau^x(A)$$

is a measurable function. This clearly holds in the case of many time-homogeneuos Markov process (e.g. Brownian motion or Levy processes).

We do the following construction, based on first child - next sibling idea from computer science.

```
A_0 = {∅};
k = 0;
loop
    k = k + 1;
    for x ∈ A_{k-1} do
        add to A_k first child and next sibling of x;
    end
endif
```

Algorithm 4: First child - next sibling search of the plane tree

We now order the I in a sequence (x_n), such that we first all the elements of A_0 appear, then of all the elements of A_1 appear, then of A_2 ...

Now $(\Omega_{\mathcal{T}}, \mathcal{F}_{\mathcal{T}})$, $(\Omega_{x_1}, \mathcal{F}_{x_1})$, $(\Omega_{x_2}, \mathcal{F}_{x_2})$, ... is a sequence of measurable spaces, and we have the following probability measures on them:

- On $(\Omega_{\mathcal{T}}, \mathcal{F}_{\mathcal{T}})$ we define $\mathbb{P}_{\mathcal{T}}$ as explained in the previous subsection;

- On $(\Omega_{x_1}, \mathcal{F}_{x_1})$ we define $\mathbb{P}_{x_1}^{\lambda_{x_1}}$ as the distribution of the process $(Y(t) : t \geq 0)$ where

$$Y(t) = \begin{cases} X(t), & t < \lambda_{x_1}; \\ \Delta, & t \geq \lambda_{x_1}; \end{cases}$$

  where the distribution of $(X(t) : t \geq 0)$ is $\mathcal{P}$.

- On $(\Omega_{x_j}, \mathcal{F}_{x_j})$ we define $\mathbb{P}_{x_j}^{\tau_{x_j}, \lambda_{x_j}, f_{x_{j'}}}$ to be the distribution of the process $(Y(t) : t \geq 0)$ given by

$$Y(t) = \begin{cases} \Delta & t < \tau_{x_j} \\ X(t), & \tau_{x_j} \leq t < \tau_{x_j} + \lambda_{x_j}; \\ \Delta, & t \geq \tau_{x_j} + \lambda_{x_j}; \end{cases}$$

  where $(X(t) : t \geq 0)$ is distributed as $\mathcal{P}_{\tau_{x_j}}^{f_{x_{j'}}(\tau_{x_j})}$ where $f_{x_{j'}} \in \Omega_{x_{j'}}$, $x_{j'}$ is the parent of x_j, and we can show that τ_{x_j} is a measurable function on $(\Omega_{\mathcal{T}}, \mathcal{F}_{\mathcal{T}})$. Therefore, since

$j' < j$, for $A \in \mathcal{F}_{x_j}$

$$\mathbb{P}_{x_j}^{\tau_{x_j},\lambda_{x_j},f_{x_{j'}}}(A)$$

is $\prod_{k=1}^{j-1}(\Omega_{x_j}, \mathcal{F}_{x_j})$-measurable.

Now we can define a product probability on $(\Omega, \mathcal{F})$ using Theorem A.9.

Now, for $\omega = (\omega_{\mathcal{T}}, (f_{x_j})) \in \Omega$. $\mathcal{T}(\omega)$ is represented by $\omega_{\mathcal{T}}$, and for $E_{x_{j'}x_j}$ an edge in $\mathcal{T}(\omega)$ we define

$$X_{E_{x_{j'}x_j}}(\omega)(t) = f_{x_j}(t)$$

for $\tau_{x_{j'}} \leq t < \tau_{x_j}$.

Properties of the construction

Theorem 10.12. *The probability measure is well-defined, that is $\mathbb{P}$ doesn't depend on the choice of (x_n) as along as:*

(1) $x_0 = \emptyset$;

(2) $\{x_n : n \in \mathbb{N}_0\} = \{x : x \in I\}$;

(3) For each $j \geq 1$ there exists $j' < j$ such that $x_{j'}$ is a parent of x_j.

Proof. Conditioned on $(\mathcal{T} = T)$ the constructed process can be mapped into a construction of a natural $\mathcal{P}$-process on a TLT T. The distribution of the process by Corollary 10.8 doesn't depend on the constrcution, hence the probability measure is well-defined. ■

Corollary 10.13. *The distribution of constructed process conditioned that the underlying tree $\mathcal{T} = T$ is a natural $\mathcal{P}$-process on T.*

Theorem 10.14. *If $\mathcal{P}$ is a distribution of a Markov process, for constructed process $(\mathcal{T}, X)$ the process*

$$Y(t) = \{X(\tau) : \bar{\tau} \in R(\mathcal{T}) \cap (\{t\} \times \mathbb{R}^2)\} \tag{10.3}$$

is a Branching $\mathcal{P}$-Markov process.

Proof. Follows from stated in § ■

Corollary 10.15. *If $\mathcal{P}$ is a distribution of Brownian motion, then the process given by* (10.3) *is the branching Brownian motion.*

OPEN QUESTIONS AND APPENDIX

Chapter 11

OPEN QUESTIONS

In this chapter we will state some open problems that could be of interest for further research.

11.1 Construction of process on all TLG's

As it was pointed out by Burdzy and Pal in [7] (and in §3.4.2 of this paper), it is not possible to construct a natural Markov process on every TLG.

Theorem 3.21. shows that a Brownian motion with the cell-Markovian property indexed by the TLG $\mathcal{G}$ given on the first image of the Figure 11.1 does not exist.

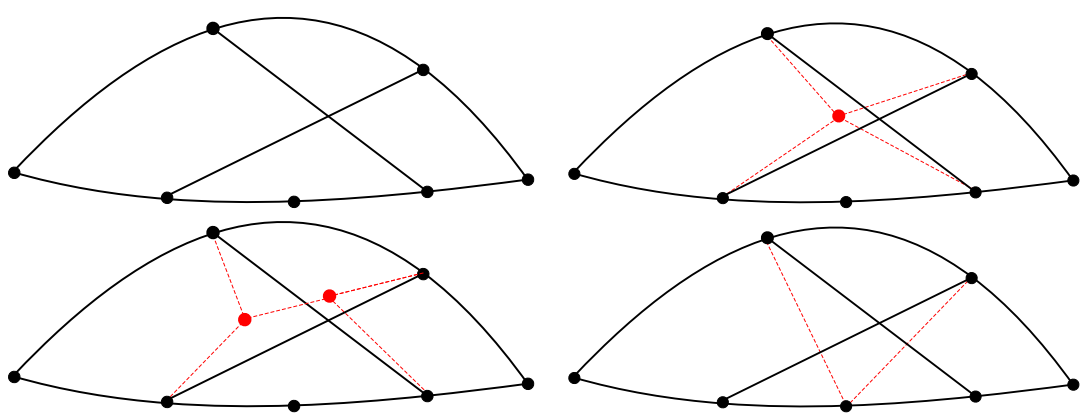


Figure 11.1: Example from Theorem 3.21. and different embeddings into a TLG*.

We know, by discussion in Section 2.3, that it is possible to construct a Markov process on a TLG*. We could try to embed $\mathcal{G}$ into some TLG* $\mathcal{H}$, define a natural Brownian motion X on $\mathcal{H}$ and then restrict X to $\mathcal{G}$ (i.e. set $X_{\mathcal{G}} = (X(t) : t \in \mathcal{G})$).

It is possible to embed any TLG into a TLG*.

Theorem 11.1. *Let $\mathcal{G} = (\mathcal{G}, \mathcal{V})$ be a (unit) TLG, then there exists a TLG* $\mathcal{H}$ that is a sup-graph of $\mathcal{G}$.*

Proof. Let $\tau_1, \ldots, \tau_m$ be times of vertices of $\mathcal{V}$. Now, we construct $\mathcal{V}_{\mathcal{H}}$ that contains $\mathcal{V}$ and vertices $t^*_{1/2}, t^*_{3/2}, \ldots, t^*_{m+1/2}$ with times $\tau_{1/2} = -1$, $\tau_{3/2} = \frac{\tau_1+\tau_2}{2}$, $\ldots$, $\tau_{m-1/2} = \frac{\tau_{m-1}+\tau_m}{2}$, $\tau_{m+1/2} = 2$. Now, we set $\mathcal{E}^0_{\mathcal{H}}$ is constructed in such a way that $t_k \in \mathcal{V}$ with time τ_j the edge

- $E_{j-1/2,k}$ between $t^*_{j-1/2}$ and t_k is in $\mathcal{E}^0_{\mathcal{H}}$;
- $E_{k,j+1/2}$ between t_k and $t^*_{j+1/2}$ is in $\mathcal{E}^0_{\mathcal{H}}$.

It is not hard to see that $\mathcal{H}_0 = (\mathcal{V}_{\mathcal{H}}, \mathcal{E}^0_{\mathcal{H}})$ is a planar simple TLG, therefore by Theorem 1.6. a TLG*.

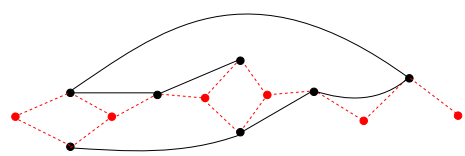


Figure 11.2: $\mathcal{H}_0$ (induced by dashed edges) is planar.

Further, every two vertices t_j and t_k are connected by a time-path in $\mathcal{H}_0$. Hence, we can add one by one edge from $\mathcal{E}$ to $\mathcal{H}_0$, and $\mathcal{H} = (\mathcal{V}_{\mathcal{H}}, \mathcal{E}^0_{\mathcal{H}} \cup \mathcal{E})$ is a TLG*. ■

It is clear that the distribution of a Brownian motion on $\mathcal{G}$ will depend on the embedding $\mathcal{H}$.

- For a given (simple) TLG $\mathcal{G}$, under what conditions on the distributions along time-paths can we construct a process on $\mathcal{G}$?
- Are there examples of distributions of (Markov) processes along time-paths for which this is not possible?

- Is there a way of getting the uniqueness of distribution of X on $\mathcal{G}$?

- What properties will the constructed process have?

11.2 Reconstruction of TLG's based on the process

As we saw in the previous section, the fact that the underlying graph is not a TLG* or TLG** does not have to prevent us from defining a process on it.

It could be that a part of the graph and a part of the process on that graph is hidden from us.

Suppose X is a natural $\mathcal{M}$-process on a TLG* $\mathcal{H}$ where $\mathcal{M}$ is a family of distributions of Gaussian Markov processes. Let $\mathcal{G}$ be a TLG such that $R(\mathcal{G}) \subset R(\mathcal{H})$.

- If we know how the graph $\mathcal{G}$ looks like and we know the distribution of $X_{\mathcal{G}} = (X(t) : t \in \mathcal{G})$, how much can we say about $\mathcal{H}$?

- What if we don't know the distribution of the process X on the whole $\mathcal{G}$, but only on the part of it?

- Could we use any of this on the branching Markov process (specially on branching Brownian motion)?

In classical graphical models problems of hidden (latent) variables have been studied (see Chapter 20. in [35] or §17.4 in [28]). One of the strong tools in solving the problems could be the moralized graph-Markovian property, which enables us to project a process on a TLG into Markov random field (MRF) . We could use some of the properties of MRF's to detect hidden parts of the graph.

There is an interesting criteria for finding edges in a Gaussian MRF. Before we state that we will need the following lemma.

Lemma 11.2. *If $X = (X_1, \ldots, X_n)$ is a Gaussian random vector with positive definite covariance matrix Σ, then $X_i \perp X_j | (X_k : k \in \{1, 2, \ldots, n\} \setminus \{i, j\})$ if and only if $\Sigma^{-1}_{ij} = 0$.*

Proposition 11.3. *Let $G = (V, E)$ be a undirected graph and $X = (X_v : v \in V)$ a Gaussian Markov random field. Let K be the positive definite covariance matrix of X. If $\{u, v\} \notin E$ then $K^{-1}_{u,v} = 0$.*

For proof of these claims see Chapter 7. in [35] or Chapter 5. in [37]. With these results we can show the following.

Proposition 11.4. *Let X be a natural $\mathcal{M}$-process on a unit TLG* $\mathcal{H}$, where $\mathcal{M}$ is a family of Gaussian Markov processes. Assume we know the distribution of $X_{R(\mathcal{G})}$ where $\mathcal{G} = (\{0, 1\}, \{E^1_{01}, E^2_{01}\})$. If $R(\mathcal{G})$ is a representation of a truly simple cell in $\mathcal{H}$ then for the covariance matrix $K(t_1, t_2)$ of the vector*

$$X = (X(0), X(t_1), X(t_2), X(1))$$

we have $K(t_1, t_2)^{-1}_{2,3} = 0$ for every point $t_1 \in E^1_{01}$ and $t_2 \in E^2_{01}$ with times in the interval $(0, 1)$.

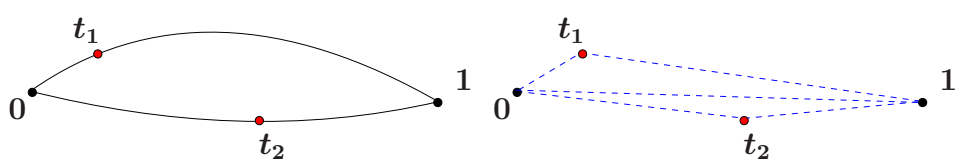


Figure 11.3: $\mathcal{G}$ and the induced graphical model

Proof. If $\mathcal{G}$ is a representation of truly simple cell, then by the Corollary 3.18, the strong cell-Markovian property (Theorem 3.4) and moralized graph-Markovian property (Theorem 3.12.) we know that X can be represented as a graphical model. In this representation there will be no edge between t_1 and t_2, and by Proposition 11.3. the claim follows. ■

Making some natural conditions on the distributions on the family $\mathcal{M}$ and using the variable elimination algorithm (see Chapter 9. in [35]) in for MRF's we could try to get the converse of the statement.

- If $\mathcal{G}$ is not the representation of a truly simple which paths can we detect?

11.3 Strong Markov property, parametrization, evolution over time,...

In Chapter 4 we defined stopping times and proved the Optional Sampling Theorem. We also proved the time-Markovian property, and the the following question naturally follows.

- Do we have a version of the strong Markov property for a natural $\mathcal{M}$-process, where $\mathcal{M}$ is a Markov family?

Parametrizng the process in suitable way and calculating probabilities is always a challenge.

- Is there a convenient way to parametrize the family along time-paths of a TLG $\mathcal{G}$?
- Is there a procedure how to calculate finite dimensional distributions of the process on the TLG $\mathcal{G}$?
- Is there a procedure how to calculate finite dimensional distributions conditioned that we know some values of the process on the TLG $\mathcal{G}$?

We could evolve the process on a graph $\mathcal{G}$ over time, and maybe even make the graph evolve over time.

- Could we define a process $(X^\tau : \tau \geq 0)$ such that $X^\tau = (X^\tau(t) : t \in \mathcal{G})$ is a process indexed by a TLG $\mathcal{G}$?
- Could we define a process $(X^\tau : \tau \geq 0)$ such that $X^\tau = (X^\tau(t) : t \in \mathcal{G}(\tau))$ is a process indexed by a TLG $\mathcal{G}(\tau)$?

We saw one way to randomize the underlying graph in Chapter 10, we could try to randomize the underlying graph in a different way.

- Let $\mathcal{G}$ be infinite TLG, suppose we run site or bond percolation on $\mathcal{G}$, and then on the connected component we define a Markov process. What properties will the process have?

Appendix A

INDEPENDENCE AND PROCESSES

A.1 Conditional independence and expectations

The results in this section are taken from Section 21.5. in [20].

We will often use conditional independence, so we need to define it.

Definition A.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{F}_1$, $\mathcal{F}_2$, and $\mathcal{G}$ sub-σ-fields of $\mathcal{F}$. The σ-fields $\mathcal{F}_1$ and $\mathcal{F}_2$ are* ***conditionally independent*** *given $\mathcal{G}$ if*

$$\mathbb{P}(A_1 \cap A_2|\mathcal{G}) = \mathbb{P}(A_1|\mathcal{G})\mathbb{P}(A_2|\mathcal{G}) \quad a.s.$$

for all $A_1 \in \mathcal{F}_1$ and $A_2 \in \mathcal{F}_2$.

Proposition A.2. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{F}_1$, $\mathcal{F}_2$, and $\mathcal{G}$ sub-σ-fields of $\mathcal{F}$, and suppose that $\mathcal{F}_2 \subset \mathcal{G}$. Then $\mathcal{F}_1$ and $\mathcal{F}_2$ are conditionally independent.*

Proposition A.3. *Let $\mathcal{G}$, $\mathcal{H}$, and $\mathcal{K}$ be σ-fields of events in a probability space. If $\mathcal{G}$ and $\mathcal{H}$ are conditionally independent given $\mathcal{K}$, then $\mathcal{G}$ and $\sigma(\mathcal{H}, \mathcal{K})$ are conditionally independent given $\mathcal{K}$.*

Proposition A.4. *Let $\mathcal{G}$ and $\mathcal{H}$ be two σ-fields of events in a probability space, and let $\mathcal{G}_1$ and $\mathcal{H}_1$ be sub-σ fields of $\mathcal{G}$ and $\mathcal{H}$, receptively. Suppose that $\mathcal{G}$ and $\mathcal{H}$ are independent. Then $\mathcal{G}$ and $\mathcal{H}$ are conditionally independent given $\sigma(\mathcal{G}_1, \mathcal{H}_1)$.*

Conditional expectations

Proposition A.5. *Let X be $(\Psi, \mathcal{H})$-valued random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and suppose that a conditional distribution Z of X given $\mathcal{G}$ exists where $\mathcal{G}$ is a sub-σ-field of $\mathcal{F}$. Let f denote a $\overline{\mathbb{R}}$-valued function on $(\Psi, \mathcal{H})$. Then*

$$\mathbb{E}(f(X)|\mathcal{G}) = \int_{\Psi} f(x)Z(dx) \quad a.s.$$

Proposition A.6. *For $i = 1, 2$, let X_i be a $(\Psi_i, \mathcal{H}_i)$-valued random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$, such that X_2 is measurable with respect to $\mathcal{G}$. Suppose that each $(\Psi_i, \mathcal{H}_i)$ is a Borel space. Let f be a measurable $\overline{\mathbb{R}}$-valued function defined on $(\Psi_1, \mathcal{H}_1) \times (\Psi_2, \mathcal{H}_2)$. If Q_1 is the distribution of X_1, then*

$$\mathbb{E}(f(X_1, X_2)|\mathcal{G})(\omega) = \int_{\Psi_1} f(x, X_2(\omega)) Q_1(dx|\mathcal{G})(\omega) \quad a.s.$$

in the sense that the set of ω such that one side exist but the other does not is a null event.

A.2 Construction of a conditional sequence

Lemma A.7. *Let $(\Psi_0, \mathcal{G}_0)$ and $(\Psi_1, \mathcal{G}_1)$ be two measurable spaces, let R_0 denote the probability measure on $(\Psi_0, \mathcal{G}_0)$, and let $x_0 \mapsto R_1(x_0, \cdot)$ be a random distribution on $(\Psi_1, \mathcal{G}_1)$ whose domain is the probability space $(\Psi_0, \mathcal{G}_0, R_0)$. Then there is a unique distribution Q on $(\Psi_0 \times \Psi_1, \mathcal{G}_0 \times \mathcal{G}_1)$ such that if $X = (X_0, X_1)$ is any valued $\Psi_0 \times \Psi_1$-valued random variable having distribution Q, then R_0 is the distribution of X_0 and R_1 is a conditional distribution of X_1 given $\sigma(X_0)$. Moreover Q is given by*

$$Q(A) = \int_{\Psi_0} \int_{\Psi_1} \mathbf{1}_A(x_0, x_1) R_1(x_0, dx_1) R_0(dx_0)$$

for $A \in \mathcal{G}_0 \times \mathcal{G}_1$.

Theorem A.8. (Conditional Fubini) *Let $(\Psi_0, \mathcal{G}_0)$ and $(\Psi_1, \mathcal{G}_1)$ be two measurable spaces and let*

$$(\Omega, \mathcal{F}) = (\Psi_0, \mathcal{G}_0) \times (\Psi_1, \mathcal{G}_1).$$

Let R_0, R_1, and Q be as in Lemma A.7. If f is and $\bar{\mathbb{R}}$-valued measurable function defined on $(\Omega, \mathcal{F}, Q)$ whose integral with respect to Q exists, then the function

$$x_0 \mapsto \int_{\Psi_1} f(x_0, x_1) R_1(x_0, dx_1)$$

is an R_0-almost surely defined $\mathcal{G}_0$-measurable function, and

$$\int_\Omega f \, dQ = \int_{\Psi_0} \int_{\Psi_1} f(x_0, x_1) R_1(x_0, dx_1) R_0(dx_0).$$

Theorem A.9. *Let $(\Psi_n, \mathcal{G}_n)_{n\geq 0}$ be a sequence of measurable spaces. Let R_0 be a probability measure on $\mathcal{G}_0$, and for each $n \geq 0$, let R_{n+1} be a measurable function from $(\Psi_0, \mathcal{G}_0) \times \ldots \times (\Psi_n, \mathcal{G}_n)$ to the measurable space of probability measures on $(\Psi_{n+1}, \mathcal{G}_{n+1})$. Then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a random sequence $(X_k : k = 0, \ldots)$ defined on the space such that the distribution of X_0 is R_0, and for $n \geq 0$, conditional distribution of X_{n+1} given $\sigma(X_0, \ldots, X_n)$ is given by*

$$\omega \mapsto R_{n+1}(X_0(\omega), X_1(\omega), \ldots, X_n(\omega), \cdot).$$

The distribution of X is uniquely determined by the relations

$$\mathbb{P}((X_0, \ldots, X_n) \in A_n) = \int_{\Psi_0} \ldots \int_{\Psi_n} \mathbf{1}_A(x_0, \ldots, x_n) R_n((x_0, \ldots, x_{n-1}), dx_n) \ldots R_0(dx_0),$$

$n \in \mathbb{N}_0$ and $A_n \in \mathcal{G}_0 \times \ldots \times \mathcal{G}_n$.

A.3 Markov and Brownian bridges

The best way to describe a Markov bridge (Y_t) is as a Markov process on the time interval $[s, u]$ conditioned that we know the value of the process at times s and u.

In oder to construct such a process we need to see what is happening with a Markov process when we condition it on the outside of that interval. Here we will prove a slight generalization of the result stated in [1]. In this section we are working on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ until we extend it later. The proof of the following theorem can be found in [1].

Theorem A.10. *(*Two-sided Markovian property*)Let $(X_t)_{t\in T}$ be a Markov process with respect to the filtration $(\mathcal{F}_t)_{t\in T}$, and let $\mathcal{G}_t = \sigma\{X_u : u \geq t\}$. For $s < u$ in T and $T' \subset T \cap [s, u]$, if Y is a bounded $\sigma\{X_t : t \in T'\}$-measurable random variable then*

$$\mathbb{E}(Y|X_s, X_u) = \mathbb{E}(Y|\mathcal{F}_s \vee \mathcal{G}_u) \quad a.s.$$

Corollary A.11. *Let $(X_t)_{t\in T}$ be a Markov process with respect to the filtration $(\mathcal{F}_t)_{t\in T}$, and let $\mathcal{G}_t = \sigma\{X_u : u \geq t\}$. For $s < u$ and $t \in [s, u]$ in T, if f is a bounded $\mathbb{R}$-valued measurable function on the state space, then*

$$\mathbb{E}(f(X_t)|X_s, X_u) = \mathbb{E}(f(X_t)|\mathcal{F}_s \vee \mathcal{G}_u) \quad a.s.$$

If $T \subset \mathbb{R}$ be a closed finite interval, and $(X_t)_{t\in T}$ is RCLL (or continuous) process with real values. Then X can be viewed as a random map into a Borel space $(\Sigma, \mathcal{S})$ consisting of all $x \in \mathbb{R}^T$, such that $t \mapsto x_t$ is RCLL (or continuous) with the usual Skorohod (or uniform) topology. (See [5] for more on this.) Under those conditions, since the space of RCLL functions on a compact set is a Borel space, we can define a conditional probability $\mu(\omega, \cdot)$ for $\omega \in \Omega$ such that

$$\mu(\omega, H) = \mathbb{P}(X^{-1}(H)|X_u, X_s)(\omega), \tag{A.1}$$

where H is an element in the σ-algebra of that Borel space, for $\mathbb{P}$-almost all $\omega \in \Omega$.

Specially, since coordinate projection $\pi_t : \mathbb{R}^T \to \mathbb{R}$ are measurable, μ we get the conditional distribution for each X_u:

$$\mu(\omega, \pi_u^{-1}(A)) = \mathbb{P}(X_t \in A|X_u, X_s).$$

A property of this random measure.

Proposition A.12. *For $u \in \{s, t\}$, we have*

$$\mu(\cdot, \pi_u^{-1}(A)) = \delta_{X_u}(A).$$

Proof. Since $\mathbf{1}_A(X_u)$ is $\mathcal{F}_s \vee \mathcal{G}_t$-measurable, from Corollary A.11 we have

$$\mu(\cdot, \pi_u^{-1}(A)) = \mathbb{P}(X_u \in A|X_s, X_t) = \mathbb{E}(\mathbf{1}_A(X_u)|X_s, X_t) =$$
$$= \mathbb{E}(\mathbf{1}_A(X_u)|\mathcal{F}_s \vee \mathcal{G}_t) = \mathbf{1}_A(X_u) = \delta_{X_u}(A).$$

■

$\mathbb{P}$-almost all $\omega \in \Omega$ the measure $\mu(\omega, \cdot)$ on $(\Sigma, \mathcal{S})$ defines a random map Y such that $Y_u = X_u(\omega)$ $\mu(\omega, \cdot)$-a.s. for $u \in [0, s] \cup [t, \infty)$.

The construction

We will focus on RCLL (or continuous) Markov process $(X(t) : t \in [0,1])$ with distribution D.

Definition A.13. *For times $t_1 < t_2$ in $[0,1]$ we say that a process $(Y(t) : t \in [t_1, t_2])$ is a* ***Markov bridge*** *between (t_1, y_{t_1}) and (t_2, y_{t_2}) on some probability space if :*

- *$Y_{t_1} = y_{t_1}$ and $Y_{t_2} = y_{t_2}$;*
- *The distribution of $(Y_t : t \in [t_1, t_2])$ is the same as $(X(t) : t \in [t_1, t_2])$ given $(X(t_1) = y_{t_2}, X(t_2) = y_{t_2})$.*

Theorem A.14. *A Markov bridge between (t_1, X_{t_1}) and (t_2, X_{t_2}) exists, for D-almost all values of (X_{t_1}, X_{t_2}).*

Proof. The process $(X_t : t \in [t_1, t_2])$ is still Markov and RCLL (or continuous). Now, from the previous discussion (see (A.1)) and since the space of RCLL functions on a compact set is a Borel space, there exists R such that

$$R(X(t_1), X(t_2))(\cdot) = \mathbb{P}(X \in \cdot | X(t_1), X(t_2)).$$

Now the measure $B \mapsto R(X(t_1), X(t_2))(B)$ defines a process Y on $D[t_1, t_2]$ (or $C[0,1]$). From the Proposition A.12. we get that $Y(t_1) = X(t_1)$ and $Y(t_2) = X(t_2)$ $R(X(t_1), X(t_2))$ - a.s. ■

Often we will have a probability space a Markov process X and maybe some other process Y on that space, and we will need to extend that process to get construct an additional Markov bridge of the process X.

Theorem A.15. *Let $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0)$ be a probability space, $(X : t \in [0,1])$ a RCLL (or continuous) Markov process with distribution D, and Z some other random element on that space. Assume Q is the law of the Markov process on $[t_1, t_2]$, where $D \circ \pi^{-1}_{t_1,t_2} = Q \circ \pi^{-1}_{t_1,t_2}$. Then for all $t_1 < t_2$ in $[0,1]$ there exist a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a process $(\hat{X}(t) : t \in [0,1])$, random element $\hat{Z}$, and a Q - Markov bridge $(Y(t) : t \in [t_1, t_2])$ between $(t_1, \hat{X}(t_1))$ and $(t_2, \hat{X}(t_2))$ such that:*

- *The joint distribution of (X, Z) is the same as of $(\hat{X}, \hat{Z})$;*

- *$(\hat{X}, \hat{Z})$ and Y are conditionally independent given $(\hat{X}(t_1), \hat{X}(t_2))$.*

The process $\tilde{X}$ given by $\tilde{X}(t) = \hat{X}(t)$ for $t \in [0, t_1] \cup [t_2, 1]$ and $\tilde{X}(t) = Y(t)$ for $t \in (t_1, t_2)$ is a Markov process. Further, if D on $[t_1, t_2]$ is distributed as Q then $\tilde{X}$ has the same distribution (D) as X.

Proof. We construct a Markov bridge and the space $(\Omega, \mathcal{F}, \mathbb{P})$ using Lemma A.7. Let's prove that the process $\tilde{X}$ is Markov. Pick $u \in [0, 1]$ and with A_{lk} we denote a set in $\sigma(X_t : t \in [l, k])$.

If $u \in (t_1, t_2)$, and let $B_u \in \sigma(\tilde{X}_u)$ then when we condition on $\tilde{X}(t_1)$ and $\tilde{X}(t_2)$ from the construction we have

$$\begin{aligned}
&\mathbb{E}(\mathbf{1}_{A_{0t_1}} \mathbf{1}_{A_{t_1 u}} \mathbf{1}_{A_{ut_2}} \mathbf{1}_{A_{t_2 1}} \mathbf{1}_{B_u}) \\
&= \mathbb{E}(\mathbb{E}(\mathbf{1}_{A_{0t_1}} \mathbf{1}_{A_{t_1 u}} \mathbf{1}_{A_{ut_2}} \mathbf{1}_{A_{t_2 1}} \mathbf{1}_{B_u} | \tilde{X}(t_1), \tilde{X}(t_2))) \\
&= \mathbb{E}(\mathbf{1}_{A_{0t_1}} \mathbf{1}_{A_{t_2 1}} \mathbb{E}(\mathbf{1}_{A_{t_1 u}} \mathbf{1}_{B_u} \mathbf{1}_{A_{ut_2}} | \tilde{X}(t_1), \tilde{X}(t_2)))
\end{aligned}$$

Now using the Markov property of the process $\hat{X}$ we have

$$\begin{aligned}
&= \mathbb{E}(\mathbb{E}(\mathbf{1}_{A_{0t_1}} \mathbf{1}_{A_{t_2 1}} \mathbb{E}(\mathbf{1}_{A_{t_1 u}} \mathbf{1}_{B_u} \mathbf{1}_{A_{ut_2}} | \tilde{X}(t_1), \tilde{X}(t_2)) | X(t_1))) \\
&= \mathbb{E}(\mathbb{E}(\mathbf{1}_{A_{0t_1}} | X(t_1)) \mathbf{1}_{A_{t_2 1}} \mathbb{E}(\mathbf{1}_{A_{t_1 u}} \mathbf{1}_{B_u} \mathbf{1}_{A_{ut_2}} | \tilde{X}(t_1), \tilde{X}(t_2))) \\
&= \mathbb{E}(\mathbb{E}(\mathbb{E}(\mathbf{1}_{A_{0t_1}} | X(t_1)) \mathbf{1}_{A_{t_2 1}} \mathbb{E}(\mathbf{1}_{A_{t_1 u}} \mathbf{1}_{B_u} \mathbf{1}_{A_{ut_2}} | \tilde{X}(t_1), \tilde{X}(t_2)) | X(t_2))) \\
&= \mathbb{E}(\mathbb{E}(\mathbf{1}_{A_{0t_1}} | X(t_1)) \mathbb{E}(\mathbf{1}_{A_{t_2 1}} | X(t_2)) \mathbb{E}(\mathbf{1}_{A_{t_1 u}} \mathbf{1}_{B_u} \mathbf{1}_{A_{ut_2}} | \tilde{X}(t_1), \tilde{X}(t_2)))
\end{aligned}$$

Now, again using the properties of the conditional expectation we have

$$\begin{aligned}
&= \mathbb{E}(\mathbb{E}(\mathbb{E}(\mathbf{1}_{A_{0t_1}} | X(t_1)) \mathbb{E}(\mathbf{1}_{A_{t_2 1}} | X(t_2)) \mathbf{1}_{A_{t_1 u}} \mathbf{1}_{B_u} \mathbf{1}_{A_{ut_2}} | \tilde{X}(t_1), \tilde{X}(t_2))) \\
&= \mathbb{E}(\mathbb{E}(\mathbf{1}_{A_{0t_1}} | X(t_1)) \mathbb{E}(\mathbf{1}_{A_{t_2 1}} | X(t_2)) \mathbf{1}_{A_{t_1 u}} \mathbf{1}_{B_u} \mathbf{1}_{A_{ut_2}}).
\end{aligned}$$

Since $(\tilde{X}(t) : t \in [t_1, t_2])$ is a Q-Markov process, conditioning on $\tilde{X}(u)$ we get

$$\begin{aligned}
&= \mathbb{E}(\mathbb{E}(\mathbb{E}(\mathbf{1}_{A_{0t_1}} | X(t_1)) \mathbb{E}(\mathbf{1}_{A_{t_2 1}} | X(t_2)) \mathbf{1}_{A_{t_1 u}} \mathbf{1}_{B_u} \mathbf{1}_{A_{ut_2}} | \tilde{X}(u))) \\
&= \mathbb{E}(\mathbb{E}(\mathbb{E}(\mathbf{1}_{A_{0t_1}} | X(t_1)) \mathbb{E}(\mathbf{1}_{A_{t_2 1}} | X(t_2)) \mathbf{1}_{A_{t_1 u}} \mathbf{1}_{A_{ut_2}} | \tilde{X}(u)) \mathbf{1}_{B_u}) \\
&= \mathbb{E}(\mathbb{E}(\mathbb{E}(\mathbf{1}_{A_{0t_1}} | X(t_1)) \mathbf{1}_{A_{t_1 u}} | \tilde{X}(u)) \mathbb{E}(\mathbb{E}(\mathbf{1}_{A_{t_2 1}} | X(t_2)) \mathbf{1}_{A_{ut_2}} | \tilde{X}(u)) \mathbf{1}_{B_u}) \\
&= \mathbb{E}(\mathbb{E}(\mathbf{1}_{A_{0t_1}} | X(t_1)) \mathbf{1}_{A_{t_1 u}} \mathbb{E}(\mathbb{E}(\mathbf{1}_{A_{t_2 1}} | X(t_2)) \mathbf{1}_{A_{ut_2}} | \tilde{X}(u)) \mathbf{1}_{B_u})
\end{aligned}$$

We again condition on $X(t_1)$ and $X(t_2)$ and we get

$$= \mathbb{E}(\mathbb{E}(\mathbb{E}(\mathbf{1}_{A_{0t_1}}|X(t_1))\mathbf{1}_{A_{t_1u}}\mathbb{E}(\mathbb{E}(\mathbf{1}_{A_{t_21}}|X(t_2))\mathbf{1}_{A_{ut_2}}|\tilde{X}(u))\mathbf{1}_{B_u}|X(t_1),X(t_2)))$$
$$= \mathbb{E}(\mathbb{E}(\mathbf{1}_{A_{0t_1}}|X(t_1))\mathbb{E}(\mathbf{1}_{A_{t_1u}}\mathbb{E}(\mathbb{E}(\mathbf{1}_{A_{t_21}}|X(t_2))\mathbf{1}_{A_{ut_2}}|\tilde{X}(u))\mathbf{1}_{B_u}|X(t_1),X(t_2)))$$

Now, using Markov property of the process $\hat{X}$, and later the construction we get

$$= \mathbb{E}(\mathbf{1}_{A_{0t_1}}\mathbb{E}(\mathbf{1}_{A_{t_1u}}\mathbb{E}(\mathbb{E}(\mathbf{1}_{A_{t_21}}|X(t_2))\mathbf{1}_{A_{ut_2}}|\tilde{X}(u))\mathbf{1}_{B_u}|X(t_1),X(t_2)))$$
$$= \mathbb{E}(\mathbf{1}_{A_{0t_1}}\mathbf{1}_{A_{t_1u}}\mathbb{E}(\mathbb{E}(\mathbf{1}_{A_{t_21}}|X(t_2))\mathbf{1}_{A_{ut_2}}|\tilde{X}(u))\mathbf{1}_{B_u})$$

Now we again condition everything on $\tilde{X}(u)$ and using properties of the conditional expectation we get:

$$= \mathbb{E}(\mathbb{E}(\mathbf{1}_{A_{0t_1}}\mathbf{1}_{A_{t_1u}}|\tilde{X}(u))\mathbb{E}(\mathbb{E}(\mathbf{1}_{A_{t_21}}|X(t_2))\mathbf{1}_{A_{ut_2}}|\tilde{X}(u))\mathbf{1}_{B_u})$$
$$= \mathbb{E}(\mathbb{E}(\mathbb{E}(\mathbf{1}_{A_{0t_1}}\mathbf{1}_{A_{t_1u}}|\tilde{X}(u))\mathbb{E}(\mathbf{1}_{A_{t_21}}|X(t_2))\mathbf{1}_{A_{ut_2}}\mathbf{1}_{B_u}|\tilde{X}(u)))$$
$$= \mathbb{E}(\mathbb{E}(\mathbf{1}_{A_{0t_1}}\mathbf{1}_{A_{t_1u}}|\tilde{X}(u))\mathbb{E}(\mathbf{1}_{A_{t_21}}|X(t_2))\mathbf{1}_{A_{ut_2}}\mathbf{1}_{B_u})$$

Again conditioning on $X(t_1)$ and $X(t_2)$, and using Markov property of $\hat{X}$, and the construction we get

$$= \mathbb{E}(\mathbb{E}(\mathbb{E}(\mathbf{1}_{A_{0t_1}}\mathbf{1}_{A_{t_1u}}|\tilde{X}(u))\mathbf{1}_{A_{ut_2}}\mathbf{1}_{B_u}|X(t_1),X(t_2))\mathbb{E}(\mathbf{1}_{A_{t_21}}|X(t_2)))$$
$$= \mathbb{E}(\mathbb{E}(\mathbb{E}(\mathbf{1}_{A_{0t_1}}\mathbf{1}_{A_{t_1u}}|\tilde{X}(u))\mathbf{1}_{A_{ut_2}}\mathbf{1}_{B_u}|X(t_1),X(t_2))\mathbf{1}_{A_{t_21}})$$
$$= \mathbb{E}(\mathbb{E}(\mathbf{1}_{A_{0t_1}}\mathbf{1}_{A_{t_1u}}|\tilde{X}(u))\mathbf{1}_{A_{ut_2}}\mathbf{1}_{B_u}\mathbf{1}_{A_{t_21}})$$

Finally, conditioning on $\tilde{X}(u)$ we get

$$= \mathbb{E}(\mathbb{E}[\mathbb{E}(\mathbf{1}_{A_{0t_1}}\mathbf{1}_{A_{t_1u}}|\tilde{X}(u))\mathbf{1}_{A_{ut_2}}\mathbf{1}_{B_u}\mathbf{1}_{A_{t_21}}|\tilde{X}(u)])$$
$$= \mathbb{E}(\mathbb{E}(\mathbf{1}_{A_{0t_1}}\mathbf{1}_{A_{t_1u}}|\tilde{X}(u))\mathbb{E}[\mathbf{1}_{A_{ut_2}}\mathbf{1}_{A_{t_21}}|\tilde{X}(u)]\mathbf{1}_{B_u}).$$

This proves, using monotone class theorem that $(\tilde{X}(t) : t \leq u)$ and $(\tilde{X}(t) : t \geq u)$ are conditionally independent given $\tilde{X}(u)$.

When $u \in [0, t_1] \cup [t_2, 1]$ this can be shown in a similar way. ■

Brownian bridge

Brownian bridges are Markov bridges when the given Markov process is Brownian motion.

The following representation holds.

Theorem A.16. *For $0 < t_1 < t_2$ the process $(B^{br}(t) : t \geq 0)$ given by*

$$B^{br}(t) = \frac{t_2 - t}{t_2 - t_1}(x_1 - W_{t_1}) + W_t + \frac{t - t_1}{t_2 - t_1}(x_2 - W_{t_2}),$$

where $(W_t : t \geq 0)$ is Brownian motion has the same distribution as a Brownian bridge conditioned at times t_1 and t_2 to have values x_1 and x_2.

Corollary A.17. *Let $(N(t) : t \in [0, T])$ be given for each t by the Ito integral*

$$N(t) = \int_0^t f(s)\, dB_s.$$

For $0 \leq t_1 < t_2 \leq T$ the distribution of the process N conditioned at times t_1 and t_2 to have values x_1 and x_2 is the same as that of

$$N^{t_1,t_2}_{x_1,x_2}(t) = \frac{V(t_2) - V(t)}{V(t_2) - V(t_1)}(x_1 - W_{V(t_1)}) + W_{V(t)} + \frac{V(t) - V(t_1)}{V(t_2) - V(t_1)}(x_2 - W_{V(t_2)}),$$

where $(W_t : t \geq 0)$ is Brownian motion and $V(t) = \int_0^t (f(s))^2\, ds$.

A.4 Markov random fields

Let $G = (V, E)$ be a simple undirected graph, where V is a finite set of vertices and E is a set of edges. We are looking a process $(X_v : v \in V)$.

Definition A.18. *The process $(X_v : v \in V)$ has a*

(a) ***pairwise Markov property*** *if for all $v, u \in V$ such that $\{u, v\} \notin E$ we have*

$$X_v \perp X_u | X_{V \setminus \{v,u\}};$$

(b) ***local Markov property*** *if for all $v \in V$*

$$X_v \perp X_{V \setminus \{v\}} | X_{\{u : \{u,v\} \in E\}});$$

(c) ***global Markov property*** *if for every A, B and C subsets of V such that C separates A and B, we have*

$$X_A \perp X_B | X_C.$$

Definition A.19. *We say that the process $(X_v : v \in V)$ is a* ***Markov random field*** *(MRF) if it satisfies one of the three properties (a), (b) or (c) in Definition A.18.*

Lemma A.20. *The global Markov property implies local Markov property, and the local Markov property implies the pairwise Markov property.*

If the random vector $(X_v : v \in V)$ has a positive density then we have several interesting results. (For more details see [35].)

Theorem A.21. *Let $X = (X_v : v \in V)$ have a positive density function f. Then global, local, and pairwise Markov properties are equivalent.*

The following theorem was proven in an unpublished paper by Hammeresley and Clifford. There have been several proofs published obtained in different ways, see for example [25, Grimmett (1973)] or [11, Clifford (1990)].

Theorem A.22. *(Hammeresley-Clifford, 1971) Let $X = (X_v : v \in V)$ be a continuous or discrete random vector with a positive density function f. X is a Markov random field if and only if f is of the form*

$$f(x) = \frac{1}{Z} \prod_{C \in \mathcal{C}(G)} \phi_C(x_C),$$

where $\mathcal{C}(G)$ is the set of all maximal cliques in G.

A.5 White noise

In this section we define the one dimensional white noise on $\mathbb{R}^n$. This is a mean-zero Gaussian process indexed by Borel σ-algebra on $\mathbb{R}^n$ ($\mathcal{B}(\mathbb{R}^n)$), i.e.

$$(\mathbb{W}(A) \,:\, A \in \mathcal{B}(\mathbb{R}^n)),$$

with the covariance function

$$\Sigma(A, B) = \lambda(A \cap B), \tag{A.2}$$

where λ is the Lebesgue measure, and $A, B \in \mathcal{B}(\mathbb{R}^n)$.

Lemma A.23. *The function* $\Sigma : \mathcal{B}(\mathbb{R}^n) \times \mathcal{B}(\mathbb{R}^n) \to \mathbb{R}$ *given by* (A.2) *is symmetric and positive definite.*

By Kolmogorov's Consistency Theorem, the process $\mathbb{W}$ exists, and has the following properties:

Theorem A.24. *Let* $\mathbb{W} = (\mathbb{W}(A) : A \in \mathcal{B}(\mathbb{R}^n))$ *be the white noise on* $\mathbb{R}^n$.

(a) For all disjoint $A, B \in \mathcal{B}(\mathbb{R}^n)$, $\mathbb{W}(A)$ *and* $\mathbb{W}(B)$ *are independent.*

(b) For all $A, B \in \mathcal{B}(\mathbb{R}^n)$, $\mathbb{W}(A \cup B) = \mathbb{W}(A) + \mathbb{W}(B) - \mathbb{W}(A \cap B)$ *a.s.*

(c) If $A_1, A_2, \ldots \in \mathcal{B}(\mathbb{R}^n)$ *are disjoint and* $\sum_{i=1}^{\infty} \lambda(A_i) < \infty$, *then a.s.*

$$\mathbb{W}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{W}(A_i).$$

Although $\mathbb{W}$ is not a measure, it has enough properties (see details in Khoshnevisan) that for $h \in L^2(\lambda)$ we can define the **Wiener integral**

$$W(h) = \int h(s)\mathbb{W}(ds).$$

The stochastic process $(W(h) : h \in L^2(\lambda))$ is called the isonormal process.

Theorem A.25. *The isonormal process* $(W(h) : h \in L^2(\lambda))$ *is a mean zero Gaussian process indexed by* $L^2(\lambda)$ *such that for all* $h_1, h_2 \in L^2(\lambda)$,

$$\mathbb{E}(W(h_1)W(h_2)) = \int h_1 h_2 \, d\lambda.$$

Moreover, for every $\alpha, \beta \in \mathbb{R}$ *and* $f, g \in L^2(\lambda)$

$$W(\alpha f + \beta g) = \alpha W(f) + \beta W(g), \quad a.s.$$

A.6 The stochastic heat equation

The usual heat equation is the initial value problem

$$\begin{aligned} \partial_t u &= c\partial_{xx} u + f \qquad \text{on} \quad (0,\infty) \times \mathbb{R}, \\ u(0,x) &= g(x) \qquad \text{for} \quad x \in \mathbb{R}. \end{aligned} \tag{A.3}$$

Under mild assumptions (see [18, Folland]) it is well known that the following is a solution to $(A.3)$:

$$u(t,x) = \frac{1}{2\sqrt{\pi ct}} \int_{\mathbb{R}} e^{-\frac{|x-y|^2}{4ct}} g(y)\, dy + \int_0^t \frac{1}{2\sqrt{\pi c(t-s)}} \int_{\mathbb{R}} e^{-\frac{|x-y|^2}{4c(t-s)}} f(s,y)\, dy\, ds \tag{A.4}$$

The idea of the stochastic heat equation is to replace the external force f, with random noise, in our case the white noise $\mathbb{W}$. So the stochastic heat equation will be given by

$$\begin{aligned} \partial_t u &= c\partial_{xx} u + \sigma \mathbb{W} \qquad \text{on} \quad (0,\infty)\times\mathbb{R}, \\ u(0,x) &= g(x) \qquad \text{for} \quad x \in \mathbb{R}. \end{aligned} \tag{A.5}$$

where $\sigma : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is a *nice* function. The so called *mild* solution to $(A.5)$ is

$$u(t,x) = \frac{1}{2\sqrt{\pi ct}} \int_{\mathbb{R}} e^{-\frac{|x-y|^2}{4ct}} g(y)\, dy + \int_0^t \frac{1}{2\sqrt{\pi c(t-s)}} \int_{\mathbb{R}} e^{-\frac{|x-y|^2}{4c(t-s)}} \sigma(s,y)\mathbb{W}(ds,dy). \tag{A.6}$$

We will state the results adapted from [47] (see the appendix of the paper). The case that will interest us is the case when we have a boundary condition, and $c = \frac{1}{2}$ and $\sigma = 1$ are constants:

$$\begin{aligned} \partial_t u &= \tfrac{1}{2}\partial_{xx} u + \mathbb{W} \qquad \text{on} \quad (0,\infty)\times\mathbb{R}^+, \\ u(0,x) &= g(x) \qquad \text{for} \quad x \in \mathbb{R}^+. \\ u(t,0) &= 0 \qquad \text{for} \quad t \geq 0. \end{aligned} \tag{A.7}$$

We need to define precisely what the solution of this equation is, and when it is unique (and in what sense). The following definition and results have been taken from [21, Section 3 & 4], where more general result were obtained and by modification of results from [47].

First we will define a space of $C_{tem}(\mathbb{R}^+)$, and we will require that for all $t \geq 0$ the function $u(t,\cdot) \in C_{tem}(\mathbb{R}^+)$.

Definition A.26. *We denote by $C_{tem}(\mathbb{R}^+)$ the family of all continuous functions $f : \mathbb{R}^+ \to \mathbb{R}$ satisfying*

$$\|f\|_{(-\lambda)} = \sup_{x\in\mathbb{R}^+} |e^{-\lambda|x|} f(x)| < \infty,$$

for all $\lambda > 0$.

Definition A.27. *(a) We call a random function $\{u = u(t,x), t \geq 0, x \in \mathbb{R}^+\}$ a* ***weak solution*** *of the SPDE (A.7) with an initial value $u_0 \in C_{tem}(\mathbb{R}^+)$ if it is $(\mathcal{F}_t)$ adapted and has the following two conditions:*

- $u \in C([0,\infty), C_{tem}(\mathbb{R}^+))$, *a.s.*
- *For every* $\varphi \in C_c^\infty(\mathbb{R}^+)$ *such that* $\varphi(0) = 0$, *the following is satisfied:*

$$\int_{\mathbb{R}^+} u(t,x)\varphi(x)\,dx = \int_{\mathbb{R}^+} u_0(x)\varphi(x)\,dx + \\ + \frac{1}{2}\int_0^t \int_{\mathbb{R}^+} u(s,x)\varphi''(x)\,dx\,ds + \int_0^t \int_{\mathbb{R}^+} \varphi(x)\mathbb{W}(ds,dx)$$

(b) *We call* u *under the same assumptions a* ***mild solution*** *if the following holds*

$$u(t,x) = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}^+} \left(e^{-\frac{|x-y|^2}{2t}} - e^{-\frac{|x+y|^2}{2t}} \right) g(y)\,dy + \\ + \int_0^t \frac{1}{\sqrt{2\pi(t-s)}} \int_{\mathbb{R}^+} \left(e^{-\frac{|x-y|^2}{2(t-s)}} - e^{-\frac{|x+y|^2}{2(t-s)}} \right) \mathbb{W}(ds,dy). \quad \text{(A.8)}$$

(u is a C_{tem}-version of the integral on the right.)

(c) *We say that the pathwise uniqueness of the weak solution of the SPDE (A.7) holds if for arbitrary two weak solutions* $u^{(1)}$ *and* $u^{(2)}$ *of the SPDE (A.7) with the respect to the same filtration* $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ *and the same noise* $\mathbb{W}$ *we have*

$$\bigcap_{t\geq 0} \{u^{(1)}(t,\cdot) \neq u^{(2)}(t,\cdot)\} \subset N,$$

where $N \in \mathcal{F}$ *such that* $\mathbb{P}(N) = 0$.

In order to show that a C_{tem}-version of (A.8) exists we will need the following results.

Lemma A.28. *If* $\phi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ *is in* L^2, *for each* $p > 0$ *there exists a constant* $C_p > 0$ *such that*

$$\mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}^+} \phi(s,x)\mathbb{W}(ds,dx) \right)^{2p} \right] \leq C_p \left(\int_0^t \int_{\mathbb{R}^+} \phi(s,x)^2 ds\,dx \right)^p \quad \text{(A.9)}$$

Lemma A.29. *(i) There exists a constant* $C > 0$ *such that*

$$\int_0^{t\vee t'} \int_{\mathbb{R}} (G(t-s,x,y) - G(t'-s,x',y))^2\,ds\,dy \leq C(|t-t'|^{1/2} + |x-x'|)$$

for $t, t' \geq 0$ *and* $x, x' \in \mathbb{R}$, *where* $G(t,x,y) = (2\pi t)^{-1/2}\exp(-(x-y)^2/(2t))$ *for* $t > 0$ *and* $G(t,x,y) = 0$ *if* $t \leq 0$.

(ii) For every $\lambda \in \mathbb{R}$ *and* $T > 0$

$$\sup_{0\leq t\leq T} \sup_{x\in\mathbb{R}} e^{-\lambda|x|} \int_{\mathbb{R}} G(t,x,y)e^{\lambda|y|}dy < \infty.$$

Theorem A.30. *If $g \in C_{tem}$ a.s., the following claims are true:*

(a) The SPDE (A.7) has a at most one pathwise unique weak solution.

(b) If u is a mild solution to the SPDE (A.7) then it is also a weak solution.

Lemma A.31. *Brownian motion is in C_{tem} a.s.*

Proof. The claim follows from the strong law of large numbers for the Brownian motion, that is if $\lambda > 0$ then

$$\lim_{x\to\infty} e^{-\lambda x} W_x = \lim_{x\to\infty} (xe^{-\lambda x}) \frac{W_x}{x} = 0 \cdot 0 = 0.$$

■

The following result is Lemma 4.4. from [21].

Theorem A.32. *The Wiener measure is an invariant measure for the SPDE (A.7), i.e. if g is Brownian motion, then for each $x \in \mathbb{R}$ the process $t \mapsto u(x,t)$ is also Brownian motion.*

A.7 Crump - Mode - Jagers trees

Here we present an introduction to Crump -Mode - Jagers model which we will later mention in the context of time-like trees. We will use the notation given by Dawson in [13].

First some notation. We define $I = \{\emptyset\} \cup \bigcup_{n=1}^{\infty} \mathbb{N}^n$. Given $u = (u_1, \ldots, u_m), v = (v_1, \ldots, v_n) \in I$ we denote the composition by $uv := (u_1, \ldots, u_m, v_1, \ldots, v_n)$.

Definition A.33. *A **plane rooted tree** T with root $\emptyset$ is a subset of I such that:*

1. *$\emptyset \in T$,*
2. *If $v = uw \in T$ for some $u \in I$ and $w \in I$, then $u \in T$.*
3. *For every $u \in T$, there exists a number $k_u(T) \geq 0$, such that $uj \in T$ if and only if $1 \leq j \leq k_u(T)$.*

Set $\mathbb{T}$ to be the ***set of all plane rooted trees****. For $u \in T$ define the* ***level of the vertex*** *to be $|u| = |(u_1, \ldots, u_m)| = m$.*

A plane tree T can be given a structure of a graph in which $uw \in T$ is **descendant** of u. Specially, $(u)(j) \in T$ is the **child** of u.

Consider the following process: For each individual $x \in I$

- We denote his **birth time** τ_x.
- **Lifetime** λ_x.
- Point process ξ_x denoting **reproduction** function. ($\xi_x(t)$ is the number of offsprings produced by individual x born at 0 during $[0, t]$.)
- Assume that the pairs (λ_x, ξ_x) are i.i.d.
- Assume $\mathbb{P}(\xi_x(\lambda_x, \infty) = 0) = 1$. (Offsprings can't be produced after x is no longer alive.)

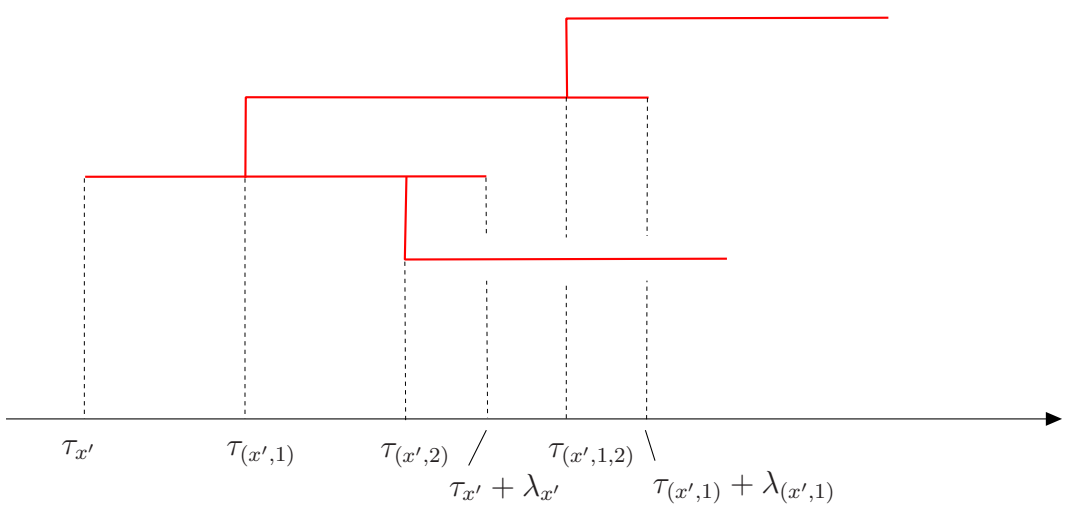


Figure A.1: Crump-Mode-Jagrers tree

The probability space that we are working in is

$$(\Omega, \mathcal{F}, \mathbb{P}) = \prod_{x \in I} (\Omega_x, \mathcal{F}_x, \mathbb{P}_x),$$

where each $(\Omega_x, \mathcal{F}_x, \mathbb{P}_x)$ supports (λ_x, ξ_x).

We can determine the birth times $\{\tau_x : x \in I\}$ as follows,

$$\tau_\emptyset = 0,$$

$$\tau_{(x',i)} = \tau_{x'} + \inf\{u : \xi_{x'}(u) \geq i\}.$$

The natural question that one may many individuals were born in the the time period $[0, t]$. Is that number even finite? We will introduce some results on this.

Set $\mu(t) := \mathbb{E}(\xi(t))$, and we define

$$T_t = \sum_{x \in I} \mathbf{1}_{(\tau_x \leq t)},$$

to be the number of individuals born up to time t. The following two results are form [32] (Theorem 6.2.1. and Theorem 6.2.2. pages 126-127).

Theorem A.34. *If $\mu(0) > 1$, then for all $t \geq 0$, $\mathbb{P}(T_t = \infty) > 0$.*

Theorem A.35. *If $\mu(0) < 1$ and $\mu(t)$ is finite for some $t > 0$, then*

$$\mathbb{P}(\forall t : T_t < \infty) = 1.$$

A.8 Branching Markov processes and branching Brownian motion

The following is a definition given in [16] of the branching Brownian motion.

Definition A.36. ***Branching Brownian motion*** *has three ingredients:*

- THE SPATIAL MOTION: *During its lifetime, each individual in the population moves around in $\mathbb{R}^d$ (independently of all other individuals) according to a Brownian motion.*
- THE BRANCHING RATE V: *Each individual has an exponentially distributed lifetime with parameter V.*

- THE BRANCHING MECHANISM Φ: *When it dies, and individual leaves behind (at the location where it died) a random number of offsprings with probability generating function* $\Phi(s) = \sum_{k=0}^{\infty} p_k s^k$. *Conditional on their time and place of birth, offsprings evolve independently of each other (in the same way as their parent).*

We could have defined any Markov process on any Polish space E to evolve in the same way, and in that case this would be the **branching Markov process**.

For more details on the definition of the branching Markov process see [30] and [31].

BIBLIOGRAPHY

[1] Al-Hussaini A.N., Elliot R.J., *Markov bridges and enlarged filtartions*, The Canadian Journal of Statistics, Vol. 17, No. 3, 1989, pp 329-332.

[2] Bass R.F., *The measurability of hitting times*, Electron. Comm. Probab. **15** (2010) 99-105

[3] Bass R.F., *Correction to: "The measurability of hitting times"*, Electron. Comm. Probab. **16** (2011) 189-191

[4] Benjamini I., Peres Y., *Markov chains indexed by trees*, Ann. Probab. **22** (1994), 219-243

[5] Billingsley P., *Convergence of Probability Measures*, John Wiley & Sons, New York, 1968.

[6] Blumenthal R.M., Getoor R.K., *Markov Processes and Potential Theory*, Dover, 1996.

[7] Burdzy, K., Pal S., *Markov processes on time-like graphs*, Ann. Probab. **39** (2011)

[8] Boucheron S., Lugosi G., Massart P., *Concentration Inequalities : A Nonasymptotic Theory of Independence*, Oxford University Press, 2013

[9] Chatterjee S., *Chaos, concentration, and multiple valleys*, arXiv:0810.4221v2

[10] Chaumont L., Uribe Bravo G., *Markovian bridges: weak continuity and pathwise construction*, Ann. Probab. **39** (2011)

[11] Clifford P., *Markov Random Fields in Statistics*, Disorder in physical systems (1990): 19-32.

[12] Davie A. M., Gaines J. G., *Convergence of Numerical Schemes for The Solution of Parabolic Stochastic Partial Differential Equations*, Mathematics of Computation, Vol. **70**, No. **223**, pp. 121-134, AMS, 2000.

[13] Dawson D., *Stochastic Population Systems*, Lecture notes from Summer School in Probability at PIMS - UBC 2009
`http://www.math.ubc.ca/~db5d/SummerSchool09/LectureNotes.html`

[14] Dawson D., *Measure-valued Markov processes*, Ecole d'Eté de Probabilités de Saint-Flour XXI-1991, Lecture Notes in Mathematics, Springer, 1993.

[15] Durrett R., *Probability: Theory and Examples*, Brooks/Cole, 2005.

[16] Etheridge A. M., *An Introduction to Superprocesses*, AMS, 2000.

[17] Evans L. C., *Partial Differential Equations*, Graduate Studies in Mathematics Vol. **19** (1997), AMS

[18] Folland G. B., *Introduction to Partial Differential Equations*, Princeton University Press, 1995.

[19] Fontes L. R. G., Isopi M., Newman C. M., Ravishankar K., *The Brownian Web: Characterization and Convergence*, The Annals of Probability , Vol. **32**, No. **4** (Oct., 2004), pp. 2857-2883

[20] Fristedt, B., Gray L., *A Modern Approach to Probability Theory*, Birkhäuser, 1997.

[21] Funaki T., Xie B., *A stochastic heat equation with the distributions of Lévy processes as its invariant measures*, Stochastic Processes and their Applications **119** (2009) 307-326

[22] Gamelin T. W., *Complex Analysis*, Springer, 2001.

[23] Gordon Y., Litvak A., Schüt C., Werner E., *Minima of sequences of Gaussian random variables*, C. R. Acad. Sci. Paris, Ser **I 340** (2005)

[24] Gordon Y., Litvak A., Schüt C., Werner E., *On the minimum of several random variables*, Proceedings of the American Mathematical Society, Vol 134; No. 12, (2006)

[25] Grimmett G.R., *A theorem about random fields*, Bull. London Math. Soc. **5** (1973), 81-84

[26] Gyöngy I., *Lattice Approximations for Stochastic Quasi-Linear Parabolic Partial Differential Equation driven by Space-Time White Noise II*, Potential Analysis **11**, 1-37, 1999.

[27] Hairer M., *An Introduction to Stochastic PDEs*, `www.hairer.org/notes/SPDEs.pdf`

[28] Hastie T., Tibshirani R., Friedman J., *The Elements of Statistical Learning: Data Mining, Inference, and Prediction*, Springer 2009.

[29] Hoeffding W., *Inequalities for Sums of Bounded Random Varaibles*, Journal of American Statistical Association, Vol. **58**, No. **301** (1963) pp. 13-30

[30] Ikeda N., Nagasawa M., Watanabe S., *Branching Markov processes I*, J. Math. Kyoto. Univ. **8**-2 (1968) 233-278

[31] Ikeda N., Nagasawa M., Watanabe S., *Branching Markov processes II*, J. Math. Kyoto. Univ. **8**-3 (1968) 365-410

[32] Jagers P., *Branching Processes with Biological Applications*, Wiley, 1975.

[33] Karatzas I., Shereve S.E., *Brownian Motion and Stochastic Calculus*, Second Edition, Springer, 1998.

[34] Khoshnevisan D., *Multiparameter Processes : An introduction to random fields*, Springer, 2002.

[35] Koller D., Friedman N., *Probabilistic Graphical Models : Principles and Techniques*, MIT Press, 2009.

[36] Kurtz T.G., *The Optional Sampling Theorem for Martingales Indexed by Directed Sets*, Ann. Probab. Volume **8**, Number **4** (1980), 675-681

[37] Lauritzen S., *Graphical Models*, Oxford Statistical Series **17**, Oxford University Press, New York, 1996.

[38] Lawler G. F., *Random Walk and the Heat Equation*, AMS, 2010.

[39] Ledoux M., Talagrand M., *Probability in Banach Spaces*, Springer, 2011.

[40] Li W.V., Shao Q-M., *Gaussian Processes: Inequalities, Small Ball Probabilities and Applications*, Stochastic processes: theory and methods, 2001

[41] Morrison J.M., Wise G.L., *Continuity of Sigma Algebras*, Statistics & Probability Letters **6** (1987) 55-60

[42] Perkins E., *Dawson - Watanabe Superprocesses and Measure Valued Diffusions*, Ecole d'Eté de Probabilités de Saint-Flour XXIX-1999, Lecture Notes in Mathematics, Springer, 2002.

[43] Sadeghi K., Lauritzen S., *Markov Properties for Mixed Graphs*, arXiv:1109.5909v2

[44] Šidák Z., *Rectangular Confidence Regions for the Means of Multivariate Normal Distributions*, Journal of the American Statistical Association, Vol. **62.**, No. **318** (1967)

[45] Studený M., *Probabilistic Conditional Independence Structures*, Springer, 2005.

[46] Sun R., Swart J. M., *The Brownian net*, Ann. Probab. Volume **36**, Number **3** (2008), 1153-1208.

[47] Shiga T., *Two contrasting properties of solutions for one-dimensional stochastic partial differential equations*, Can. J. Math. Vol **46**(2), 1994, pp. 415-437

[48] Talagrand M., *The generic chaining. Upper and lower bounds of stochastic processes.* Springer-Verlag, Berlin 2005.

INDEX

$\mathcal{M}$-process, *see* Process indexed by a TLG

Banach space, 116
  dual space, 116
Branching Brownian motion, 204, **226**
Branching Markov process, 200–204, **227**
Branching Markov processes, 188
Brownian bridge, 154–160, 219
  maximum, 111–114, 157

Cell collapse transformation, *see* Time-like graph (TLG)
Cell-Markovian property, 66–68, 84
  strong, 66
Conditional expectation, 212–213
Conditional Fubini's Theorem, 213
Conditional independence, **212**, 212
Consistent family of distributions along time-paths, *see* Process indexed by a TLG
Covariance matrix, 209
Crump - Mode - Jagers trees, 195, 224–226
  Galton-Watson, *see* Galton-Watson time-like tree

Directed set, 99
Donsker's Theorem, 125

Edge-Markovian property, 77–78, 188
Euler method, *see* Heat equation

Filtration, 88
  right-continuous, 90
Firs child - next sibling, 201
Full time-path, *see* Time-like graph (TLG)

Galton-Watson time-like tree, 195–197
Gaussian random variable (in Banach space), 116
  concentration inequality, 116
Global Markov property, 69, **220**
Graph-Markovian property, **69**
  doesn't hold, 79–81
  for time-like trees, 191
  moralized, **72**, 71–77, 187, 208

Harmonic sequence, 113
Heat equation, 124–161
  discrete, 156
  Euler method, 134–151
  solution, 124
  stochastic, 134–146, 156, 221–224
    mild solution, 223
    weak solution, 222
Hidden Markov model, 162
Hidden Markov models, 188

Hoeffding's Inequality, 125

Interpolation, 144
Interval TLG, *see* Time-like graph (TLG)

Local Limit Theorem, 120–124
Local Markov property, 219

Markov bridge, **216**, 214–219
Markov chains indexed by trees, 188
Markov random field (MRF), 71, 76, 208, **220**
  Gaussian, 208
  Hammeresley-Clifford Theorem, 220
Martingale indexed by directed set, 99–102
Martingales
  indexed by a TLG, 87
Maximal inequality for normal random variables, 114
Moralization, 71
Moralized graph-Markovian property, *see* Graph-Markovian property

Pairwise Markov property, 219
Percolation, 211
Plane rooted tree, 224
Process index by a TLG
  $\mathcal{M}$-process
    natural, **59**
Process indexed by a TLG, **41**
  $\mathcal{M}$-process, 47, 49–51, 177, 184
  cell-Markovian property, 185
  consistent family of distributions along time-paths, 45, 176
  construction, 47–49, 177–183
    (3T') conditions, 178
    (3T) conditions, 47
  construction problems, 81–84
  spine-Markovian property, 52–58, 184
    for time-like trees, 192
    hereditary, 57–58, 184
  uniqueness of distribution, 58–59, 187
  with infinite number of vertices, 59–63, 197–199
    construction, 59, 197
    uniqueness of distribution, 60, 198

Šidák's inequality, 115
Simple random walk, *see also* Local Limit Theorem, **120**, 123, 125–133, 135
Skorohod's Representation Theorem, 126
Spine of a TLG*, *see* TLG* family
Spine-Markovian property, 42–45, *see also* Process indexed by a TLG, 66
  hereditary, 45
Stingy algorithm, *see* TLG* family
Stirling formula, 120
Stochastic heat equation, *see* Heat equation
Stopping times, 100, 102–105

TLG - valued, 102

Strong Markov property, 210

Time-like graph (TLG), **8**, 7–40, **164**, 164–175

cell, 10

cell collapse transformation, 30

embedding, 166–169

maximal, 167

minimal, 166

entrance (vertex), 165

exit (vertex), 165

full path, 165

full-time path, 18

half-cell, 176

truly simple, 185

Interval TLG, 21

order, 173

order induced by a TLG $\preceq$, 19

planar, 9, 173

point, 19, 173

representation, 8

simple, **164**

time-path, 9

full-time path, 9

topology, 23–26

Convergence of points, 25

metrizability, 25

tree, 175

truly simple cell, 32

with infinitely many vertices, 39, 194–195

Time-like tree (TLT), **189**, 188–193

backward, 189

forward, 189

Galton-Watson, *see* Galton-Watson time-like tree

Time-Markovian property, **69**, 70–71, 188

Time-path, *see* Time-like graph (TLG)

TLG* family, 10–16

consistent representation, 17

spine, 18

stingy algorithm, 35

TLG*-tower, 11

topological lattice, 26

TLG*-tower, *see* TLG* family

TLG** family, **169**, 169–175

Topological lattice, *see also* TLG* family, 26, 28

separable from above, 101

Truly simple cell, *see* Time-like graph (TLG)

White noise, **220**, 220–221

Wiener integral, **221**

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