Two divisors of $(n^2+1)/2$ summing up to $\delta n+\varepsilon$, for δ and ε even

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Previous Results

- In [1] Ayad gives sufficient conditions for a polynomial P to be indecomposable[1] in terms of its critical points and critical values and conjectures the following claim

There do not exist two divisors d_1, d_2 of $(p^2+1)/2$, greater than 1 such that

$$d_1 + d_2 = p + 1.$$

[1] Polynomial P is ***indecomposable*** if it can be expressed as the composition of two non-constant and non-linear polynomials.

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[1] Polynomial P is ***indecomposable*** if it can be expressed as the composition of two non-constant and non-linear polynomials.

Previous Results

- In [2], Ayad and Luca have proved that there does not exist an odd integer $n>1$ and two positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that

$$d_1+d_2=n+1$$

and they also proved that the primitive function of

$$\int (x-x_1)^{(p^2-1)/2}(x-x_2)^{(p^2-1)/2}dx,$$

$$p\in\mathbb{P}\backslash\{2\},\;\; x_1,x_2\in\mathbb{C},\;\; x_1\neq x_2$$

is indecomposable over $\mathbb{C}$ which was already proved in [1] by a different method

Previous Results

A Diophantine application of the previous claim that is also proved in [2] is

Corollary

Let $a<b, c<d, e$ fixed integers and let $p \leq q$ be odd primes. If Diophantine equation

$$\int_0^x ((t-a)(t-b))^{(p^2-1)/1} dt - \int_0^y ((s-c)(s-d))^{(q^2-1)/1} ds = e$$

has infinitely many solutions (x,y), then

$$p=q, \quad c-a=d-b=f, \quad e=\int_0^{-f} ((t-a)(t-b))^{(p^2-1)/2} dt.$$

Previous Results

In [3], Dujella and Luca have dealt with a more general issue, where $n+1$ was replaced with an arbitrary linear polynomial $\delta n+\varepsilon$, where $\delta > 0$ and ε are given integers.

Previous Results

- Since d_1, d_2 are divisors of a sum of two coprime squares we conclude
$$d_1 \equiv d_2 \equiv 1 \pmod 4$$
and
$$d_1 + d_2 = \delta n + \varepsilon,$$
then there are two cases: it is either $\delta \equiv \varepsilon \equiv 1 \pmod 2$, or $\delta \equiv \varepsilon + 2 \equiv 0$ or $2 \pmod 4$.
- In [3] authors have focused on the first case and we deal with the second case.

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then there are two cases: it is either $\delta \equiv \varepsilon \equiv 1 \pmod 2$, or $\delta \equiv \varepsilon + 2 \equiv 0 \text{ or } 2 \pmod 4$.
- In [3] authors have focused on the first case and we deal with the second case.

We completely solve cases when

- $\delta = 2$ and $\varepsilon \equiv 0 \pmod 4$,
- $\delta = 4$ and $\varepsilon \equiv 2 \pmod 4$,
- $\varepsilon = 0$ and $\delta \equiv 2 \pmod 4$.

The case $\delta = 2$ and $\varepsilon \equiv 0 \pmod 4$

Theorem

If $\varepsilon \equiv 0 \pmod 4$, then there exist infinitely many positive odd integers n with the property that there exists a pair of positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = 2n + \varepsilon$.

The case $\delta = 2$ and $\varepsilon \equiv 0 \pmod 4$

- Let n be an odd integer, $\varepsilon \equiv 0 \pmod 4$ and d_1, d_2 divisors of $(n^2+1)/2$ where
$$d_1 + d_2 = 2n + \varepsilon.$$

- Let $g = \gcd(d_1, d_2)$. We can write $d_1 = gd_1', \; d_2 = gd_2', \; d_1', d_2' \in \mathbb{N}$.
- Since d_1, d_2 are divisors of $(n^2+1)/2$, we can conclude $gd_1'd_2' = \mathrm{lcm}(d_1, d_2)$ divides $\frac{n^2+1}{2}$
- There exists a positive integer d such that
$$d_1 d_2 = \frac{g(n^2+1)}{2d}.$$

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The case $\delta = 2$ and $\varepsilon \equiv 0 \pmod 4$

From the identity

$$(d_2-d_1)^2=(d_1+d_2)^2-4d_1d_2,$$

we can easily obtain

$$(d_2-d_1)^2=(2n+\varepsilon)^2-4\frac{g(n^2+1)}{2d},$$

$$d(4d-2g)(d_2-d_1)^2=(4d-2g)^2n^2+4(4d-2g)d\varepsilon n+4d^2\varepsilon^2-8dg-2\varepsilon^2dg+4g^2. \tag{1}$$

For $X=(4d-2g)n+2d\varepsilon$, $Y=d_2-d_1$, the equation (1) becomes

$$X^2-d(4d-2g)Y^2=8dg+2\varepsilon^2dg-4g^2.$$

The case $\delta = 2$ and $\varepsilon \equiv 0 \pmod 4$

For $g = 1$ the previous equation becomes

$$X^2 - 2d(2d-1)Y^2 = 2d(4+\varepsilon^2) - 4. \tag{2}$$

The equation (2) is a Pellian equation. The right-hand side of (2) is greater than zero.

Our goal is to make the right-hand side of (2) a perfect square. That condition can be satisfied by taking $d = \frac{1}{8}\varepsilon^2 - \frac{1}{2}\varepsilon + 1$.

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The case $\delta = 2$ and $\varepsilon \equiv 0 \pmod 4$

Pellian equation (2) becomes

$$X^2 - 2d(2d-1)Y^2 = \left(\frac{1}{2}(\varepsilon^2 - 2\varepsilon + 4)\right)^2. \tag{3}$$

If we set

$$X = \frac{1}{2}(\varepsilon^2 - 2\varepsilon + 4)U, \quad Y = \frac{1}{2}(\varepsilon^2 - 2\varepsilon + 4)V, \tag{4}$$

the equation (3) becomes

$$U^2 - 2d(2d-1)V^2 = 1. \tag{5}$$

The case $\delta = 2$ and $\varepsilon \equiv 0 \pmod 4$

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The case $\delta = 2$ and $\varepsilon \equiv 0 \pmod 4$

Equation (5) is a Pell equation which has infinitely many positive integer solutions (U, V), and consequently, there exist infinitely many positive integer solutions (X, Y) of (3) of the form (4).
We can easily get

$$\sqrt{2d(2d-1)} = [2d-1; \overline{2, 4d-2}].$$

The case $\delta = 2$ and $\varepsilon \equiv 0 \pmod 4$

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The case $\delta=2$ and $\varepsilon\equiv 0 \pmod 4$

Generally, nonnegative integer solutions of (5) are generated by recursive sequences

$$U_0=1,\ \ U_1=4d-1,\ \ U_{m+2}=2(4d-1)U_{m+1}-U_m,$$

$$V_0=0,\ \ V_1=2,\ \ V_{m+2}=2(4d-1)V_{m+1}-V_m,\ \ m\in\mathbb{N}_0. \quad (6)$$

By induction on m, one gets that

$$U_m\equiv 1 \pmod{(4d-2)},\ m\geq 0.$$

The case $\delta = 2$ and $\varepsilon \equiv 0 \pmod 4$

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$$U_0 = 1,\ \ U_1 = 4d-1,\ \ U_{m+2} = 2(4d-1)U_{m+1} - U_m,$$

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By induction on m, one gets that

$$U_m \equiv 1 \pmod{(4d-2)},\, m \geq 0.$$

The case $\delta = 2$ and $\varepsilon \equiv 0 \pmod 4$

It remains to compute the corresponding values of n which arise from

$$X = (4d-2)n + 2d\varepsilon,\ \ X = \frac{1}{2}(\varepsilon^2 - 2\varepsilon + 4)U.$$

We obtain

$$n = \frac{\frac{1}{2}(\varepsilon^2 - 2\varepsilon + 4)U - 2d\varepsilon}{4d-2}.$$

We want the above number n to be a positive integer.

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The case $\delta = 2$ and $\varepsilon \equiv 0 \pmod 4$

Congruences

$$\frac{1}{2}(\varepsilon^2-2\varepsilon+4)U-2d\varepsilon \equiv -(2d-1)\varepsilon \equiv 0 \pmod{(4d-2)},$$

show that all numbers n generated in the specified way are integers.
From the first recursive sequence in (7) we know that $U_m, m \geq 0$, are odd integers so we may conclude

$$\frac{1}{2}(\varepsilon^2-2\varepsilon+4)U-2d\varepsilon \equiv 2U \equiv 2 \pmod 4 \text{ and } 4d-2 \equiv 2 \pmod 4,$$

which implies that all such integers n are odd.

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□

The case $\delta = 2$ and $\varepsilon \equiv 0 \pmod 4$

The first few values of number n, which we get from U_1, U_2, U_3, are

$$\begin{cases} n = \frac{1}{2}(\varepsilon^2 - 3\varepsilon + 6), \\ d_1 = 1, \\ d_2 = \varepsilon^2 - 2\varepsilon + 5. \end{cases}$$

$$\begin{cases} n = \frac{1}{2}(\varepsilon^4 - 6\varepsilon^3 + 20\varepsilon^2 - 33\varepsilon + 34), \\ d_1 = \varepsilon^2 - 2\varepsilon + 5, \\ d_2 = \varepsilon^4 - 6\varepsilon^3 + 19\varepsilon^2 - 30\varepsilon + 29. \end{cases}$$

$$\begin{cases} n = \frac{1}{2}(\varepsilon^6 - 10\varepsilon^5 + 50\varepsilon^4 - 148\varepsilon^3 + 281\varepsilon^2 - 323\varepsilon + 198), \\ d_1 = \varepsilon^4 - 6\varepsilon^3 + 19\varepsilon^2 - 30\varepsilon + 29, \\ d_2 = \varepsilon^6 - 10\varepsilon^5 + 49\varepsilon^4 - 142\varepsilon^3 + 262\varepsilon^2 - 292\varepsilon + 169. \end{cases}$$

The case $\delta = 4$ and $\varepsilon \equiv 2 \pmod 4$

Theorem

If $\varepsilon \equiv 2 \pmod 4$, then there exist infinitely many positive odd integers n with the property that there exists a pair of positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = 4n + \varepsilon$.

The case $\delta = 4$ and $\varepsilon \equiv 2 \pmod 4$

Let n be an odd integer, $\varepsilon \equiv 2 \pmod 4$ and d_1, d_2 divisors of $(n^2+1)/2$ where

$$d_1 + d_2 = 4n + \varepsilon.$$

Let $g = \gcd(d_1, d_2)$ and d is a positive integer which satisfies the equation

$$d_1 d_2 = \frac{g(n^2+1)}{2d}.$$

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The case $\delta = 4$ and $\varepsilon \equiv 2 \pmod 4$

From the identity

$$(d_2 - d_1)^2 = (d_1 + d_2)^2 - 4d_1 d_2,$$

we obtain the equation

$$X^2 - 2d(8d-g)Y^2 = 32dg + 2\varepsilon^2 dg - 4g^2, \tag{7}$$

where X, Y are $X = (16d - 2g)n + 4d\varepsilon$ and $Y = d_2 - d_1$.
For $g = 1$ (7) becomes

$$X^2 - 2d(8d-1)Y^2 = 2d(16+\varepsilon^2) - 4. \tag{8}$$

The right-hand side of (8) is always greater that zero.

The case $\delta = 4$ and $\varepsilon \equiv 2 \pmod 4$

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The right-hand side of (8) is always greater that zero.

The case $\delta = 4$ and $\varepsilon \equiv 2 \pmod 4$

If we take

$$d = \frac{1}{32}\varepsilon^2 - \frac{1}{8}\varepsilon + \frac{5}{8},$$

the right-hand side of (8) is a perfect square and Pellian equation (8) becomes

$$X^2 - 2d(8d-1)Y^2 = \left(\frac{1}{4}(\varepsilon^2 - 2\varepsilon + 16)\right)^2. \qquad (9)$$

We must notice that d is an integer if $\varepsilon \equiv 6 \pmod 8$, and it is not an integer if $\varepsilon \equiv 2 \pmod 8$. So, we split the proof in two subcases.

The case $\delta = 4$ and $\varepsilon \equiv 2 \pmod 4$

Let $\varepsilon \equiv 6 \pmod 8$. We set

$$X = \frac{1}{4}(\varepsilon^2 - 2\varepsilon + 16)W, \quad Y = \frac{1}{4}(\varepsilon^2 - 2\varepsilon + 16)Z. \tag{10}$$

and the equation (9) becomes

$$W^2 - 2d(8d-1)Z^2 = 1. \tag{11}$$

The equation (11) is a Pell equation which has infinitely many positive integer solutions (W, Z), and consequently, there exist infinitely many positive integer solutions (X, Y) of (9) of the form (10).

The case $\delta = 4$ and $\varepsilon \equiv 2 \pmod 4$

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The case $\delta = 4$ and $\varepsilon \equiv 2 \pmod 4$

All positive solutions of (11) are given by (W_m, Z_m) for some $m \geq 0$. Generally, nonnegative integer solutions of (11) are generated by recursive sequences

$$W_0 = 1, \;\; W_1 = 16d - 1, \;\; W_{m+2} = 2(16d-1)W_{m+1} - W_m, \quad (12)$$

$$Z_0 = 0, \;\; Z_1 = 4, \;\; Z_{m+2} = 2(16d-1)Z_{m+1} - Z_m, \;\; m \in \mathbb{N}_0.$$

The case $\delta = 4$ and $\varepsilon \equiv 2 \pmod 4$

By induction on m, one gets that

$$W_m \equiv 1 \pmod{(16d-2)}, \quad m \geq 0.$$

It remains to compute the corresponding values of n which arise from

$$X = (16d-2)n + 4d\varepsilon, \;\; X = \frac{1}{4}(\varepsilon^2 - 2\varepsilon + 16)W.$$

We obtain

$$n = \frac{\frac{1}{4}(\varepsilon^2 - 2\varepsilon + 16)W - 4d\varepsilon}{16d-2}.$$

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We obtain

$$n = \frac{\frac{1}{4}(\varepsilon^2 - 2\varepsilon + 16)W - 4d\varepsilon}{16d-2}.$$

The case $\delta = 4$ and $\varepsilon \equiv 2 \pmod 4$

The congruences

$$\frac{1}{4}(\varepsilon^2 - 2\varepsilon + 16)W - 4d\varepsilon \equiv (8d-1)(1-\frac{\varepsilon}{2}) \equiv 0 \pmod{(16d-2)}$$

show us that all numbers n generated in the specified way are integers.

From recursive sequence (12) and because of the following congruences

$$\frac{1}{4}(\varepsilon^2 - 2\varepsilon + 16)W - 4d\varepsilon \equiv 2W \equiv 2 \pmod 4 \text{ and } 16d-2 \equiv 2 \pmod 4$$

we can conclude that such integers n are odd.

The case $\delta = 4$ and $\varepsilon \equiv 2 \pmod 4$

The congruences

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The case $\delta = 4$ and $\varepsilon \equiv 2 \pmod 4$

The first few values of number n, which we get from W_1, W_2, W_3, are

$$\begin{cases} n = \frac{1}{4}(\varepsilon^2 - 3\varepsilon + 18), \\ d_1 = 1 \\ d_2 = \varepsilon^2 - 2\varepsilon + 17. \end{cases}$$

$$\begin{cases} n = \frac{1}{4}(\varepsilon^4 - 6\varepsilon^3 + 44\varepsilon^2 - 105\varepsilon + 322), \\ d_1 = \varepsilon^2 - 2\varepsilon + 17, \\ d_2 = \varepsilon^4 - 6\varepsilon^3 + 43\varepsilon^2 - 102\varepsilon + 305. \end{cases}$$

$$\begin{cases} n = \frac{1}{4}(\varepsilon^6 - 10\varepsilon^5 + 86\varepsilon^4 - 388\varepsilon^3 + 1529\varepsilon^2 - 3155\varepsilon + 5778), \\ d_1 = \varepsilon^4 - 6\varepsilon^3 + 43\varepsilon^2 - 102\varepsilon + 305, \\ d_2 = \varepsilon^6 - 10\varepsilon^5 + 85\varepsilon^4 - 382\varepsilon^3 + 1486\varepsilon^2 - 3052\varepsilon + 5473. \end{cases}$$

The case $\delta = 4$ and $\varepsilon \equiv 2 \pmod 4$

Now, we deal with the case when $\varepsilon \equiv 2 \pmod 8$, where $\varepsilon = 8k+2,\ \ k \in \mathbb{N}_0$.

For $g = d_1 = \frac{1}{4}\varepsilon^2 + 4$, the equation (7) becomes

$$X^2 - 2d(8d-g)Y^2 = \frac{2d-1}{4}\varepsilon^4 + 8\varepsilon^2(2d-1) + 64(2d-1).$$

The right-hand side of the equation will be a perfect square if $2d-1$ is a perfect square. Motivated by the experimental data, we take

$$d = \frac{1}{512}\varepsilon^4 - \frac{1}{64}\varepsilon^3 + \frac{7}{64}\varepsilon^2 - \frac{5}{16}\varepsilon + \frac{41}{32}.$$

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So, the equation (7) becomes

$$X^2 - 2d(8d-g)Y^2 = \left(\frac{1}{32}(\varepsilon^2+16)(\varepsilon^2-4\varepsilon+20)\right)^2. \qquad (13)$$

We consider the corresponding Pell equation

$$U^2 - 2d(8d-g)V^2 = 1. \qquad (14)$$

The case $\delta = 4$ and $\varepsilon \equiv 2 \pmod 4$

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The case $\delta = 4$ and $\varepsilon \equiv 2 \pmod 4$

Let (U_0, V_0) be the least positive integer solution of (14). That equation has infinitely many solutions. From (14) we get that

$$U^2 \equiv 1 \pmod{(16d-2g)}.$$

Motivated by experimental data, we can also set

$$d_2 = d_1^2 - 16kd_1, \quad k \in \mathbb{N}_0.$$

For $Y = d_2 - d_1$ we get

$$Y = \left(\frac{1}{4}\varepsilon^2+4\right)^2 - (2\varepsilon-3)\left(\frac{1}{4}\varepsilon^2+4\right) = \frac{\varepsilon^4}{16} - \frac{\varepsilon^3}{2} + \frac{11\varepsilon^2}{4} - 8\varepsilon + 28.$$

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$$d_2 = d_1^2 - 16kd_1, \;\; k \in \mathbb{N}_0.$$

For $Y = d_2 - d_1$ we get

$$Y = \left(\frac{1}{4}\varepsilon^2+4\right)^2 - (2\varepsilon-3)\left(\frac{1}{4}\varepsilon^2+4\right) = \frac{\varepsilon^4}{16} - \frac{\varepsilon^3}{2} + \frac{11\varepsilon^2}{4} - 8\varepsilon + 28.$$

The case $\delta = 4$ and $\varepsilon \equiv 2 \pmod 4$

Let (U_0, V_0) be the least positive integer solution of (14). That equation has infinitely many solutions. From (14) we get that

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From (13), we obtain:

$$X = \frac{(\varepsilon^2+16)(\varepsilon^6-16\varepsilon^5+140\varepsilon^4-768\varepsilon^3+3120\varepsilon^2-8704\varepsilon+14400)}{2048}.$$

We claim that X satisfies the congruence

$$X \equiv 4d\varepsilon \pmod{(16d-2g)}. \tag{15}$$

Indeed,

$$16d-2g = \frac{\varepsilon^4}{32}-\frac{\varepsilon^3}{4}+\frac{5\varepsilon^2}{4}-5\varepsilon+\frac{25}{2},$$

$$X-4d\varepsilon = \left(\frac{\varepsilon^4}{32}-\frac{\varepsilon^3}{4}+\frac{5\varepsilon^2}{4}-5\varepsilon+\frac{25}{2}\right)\left(\frac{\varepsilon^4}{64}-\frac{\varepsilon^3}{8}+\frac{13\varepsilon^2}{16}-\frac{9\varepsilon}{4}+9\right).$$

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From $n = \frac{X-4d\varepsilon}{16d-2g}$, we get

$$n = \frac{\varepsilon^4}{64} - \frac{\varepsilon^3}{8} + \frac{13\varepsilon^2}{16} - \frac{9\varepsilon}{4} + 9 = 64k^4 + 28k^2 + 7,$$

and we see that n is an odd integer.

The case $\delta = 4$ and $\varepsilon \equiv 2 \pmod 4$

We set $X = X_0$ and $Y = Y_0$. Because (X_0, Y_0) is a solution of (13), solutions of (13) are also

$$X_i+\sqrt{2d(8d-g)}Y_i = \left(X_0+\sqrt{2d(8d-g)}Y_0\right)\left(U_0+\sqrt{2d(8d-g)}V_0\right)^{2i}, \quad i=0,1,2,\ldots \tag{16}$$

From the equation (16), we get

$$X_i \equiv U_0^{2i}X_0 \equiv X_0 \equiv 4d\varepsilon \pmod{(16d-2g)}.$$

So, there are infinitely many solutions (X_i, Y_i) of (13) where X_i satisfies the congruence (15).

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The case $\delta = 4$ and $\varepsilon \equiv 2 \pmod 4$

Therefore, by

$$n_i = \frac{X_i - 4d\varepsilon}{16d - 2g}, \quad i = 0, 1, 2, \ldots$$

we get infinitely many integers n_i with the required properties. It is easy to see that number n_i defined in this way is odd. Indeed, we have $16d - 2g \equiv 2 \pmod 4$, $X_0 \equiv 2 \pmod 4$, and since (14) implies that U_0 is odd and V_0 is even, we get from (15) that

$$X_i - 4d\varepsilon \equiv X_i \equiv U_0^{2i} X_0 \equiv X_0 \equiv 2 \pmod 4,$$

so every n_i, $i = 0, 1, 2, \ldots$, is an odd integer.

□

The case $\varepsilon=0$ and $\delta=2$

Proposition

There exist infinitely many positive odd integers n with the property that there exists a pair of positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1+d_2=2n$. These solutions satisfy $\gcd(d_1,d_2)=1$ and $d_1 d_2=\frac{n^2+1}{2}$.

The case $\varepsilon = 0$ and $\delta = 2$

- Let n be an odd integer and let d_1, d_2 be positive divisors of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = 2n$.
- Let $g = \gcd(d_1, d_2)$. Then $g|(2n)$ and $g|(n^2+1)$ which implies that $g|((2n)^2+4)$ so we can conclude that $g|4$.
- Because g is the greatest common divisor of d_1, d_2 and d_1, d_2 are odd numbers, we can also conclude that g is an odd number.
- So, $g = 1$.
- Like we did in the proofs of the previous theorems, we define a positive integer d which satisfies the equation $d_1 d_2 = \frac{n^2+1}{2d}$.

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The case $\varepsilon=0$ and $\delta=2$

From the identity

$$(d_2-d_1)^2=(d_1+d_2)^2-4d_1d_2,$$

we can easily obtain

$$(2d-1)n^2-2dy^2=1, \tag{17}$$

where $d_2-d_1=2y$.

The case $\varepsilon = 0$ and $\delta = 2$

We will use the next lemma, which is Criterion 1 from [4] to check if there exists a solution for (17).

Lemma (Grelak, Grytczuk)

Let $a > 1$, b be positive integers such that $\gcd(a, b) = 1$ and $D = ab$ is not a perfect square. Moreover, let (u_0, v_0) denote the least positive integer solution of the Pell equation

$$u^2 - Dv^2 = 1.$$

Then equation $ax^2 - by^2 = 1$ has a solution in positive integers x, y if and only if

$$2a|(u_0+1) \text{ and } 2b|(u_0-1).$$

The case $\varepsilon = 0$ and $\delta = 2$

In order to find solutions of (17), for start we solve the Pell equation

$$U^2 - 2d(2d-1)V^2 = 1, \tag{18}$$

where $n = U, \; y = V$.

Its least positive integer solution is $(4d-1, 2)$.

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According to introduced Lemma, conditions that have to be satisfied are

$$2(2d-1)|4d \text{ and } 4d|(4d-2),$$

which is not true for $d\in\mathbb{N}$. So, for Pellian equation (17) there are no integer solutions (n,y) when $a=2d-1>1$.

Finally, we have to check the remaining case for $a=1$, which is the case that is not included in Lemma. The condition $a=1$ implies that $d=1$.

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The case $\varepsilon=0$ and $\delta=2$

From (17) and $d=1$, we get the Pell equation

$$n^2-2y^2=1, \tag{19}$$

which has infinitely many solutions $n=U_m,\ \ y=V_m,\ \ m\in\mathbb{N}_0$ where

$$U_0=1,\ \ U_1=3,\ \ U_{m+2}=6U_{m+1}-U_m,$$

$$V_0=0,\ \ V_1=2,\ \ V_{m+2}=6V_{m+1}-V_m,\ \ m\in\mathbb{N}_0.$$

The case $\varepsilon=0$ and $\delta=2$

From the first few values (U_i, V_i) which are

$$(U_0, V_0) = (1, 0),\ (U_1, V_1) = (3, 2),\ (U_2, V_2) = (17, 12), \ldots$$

we can easily generate (n, d_1, d_2)

$$(n, d_1, d_2) = (3, 1, 5),\ (17, 5, 29),\ (99, 29, 169), \ldots.$$

We have proved that in this case is $g=1$ and $d=1$, so we conclude that numbers d_1 and d_2 are coprime and that $d_1 d_2 = \frac{n^2+1}{2}$.

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□

The case $\varepsilon = 0$ and $\delta \equiv 2 \pmod 4$, $\delta \geq 6$

Theorem

Let $\delta \geq 6$ be a positive integer such that $\delta = 4k+2, k \in \mathbb{N}$. Then there does not exist a positive odd integer n with the property that there exists a pair of positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = \delta n$.

The case $\varepsilon = 0$ and $\delta \equiv 2 \pmod 4$, $\delta \geq 6$

- Suppose on the contrary that this is not so and let the number δ be the smallest positive integer $\delta = 4k+2$, $k \in \mathbb{N}$ for which there exists an odd integer n and a pair of positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = \delta n$.
- We assume that $g = \gcd(d_1, d_2) > 1$ which leads us to contradiction.
- So, in this case $g = 1$.

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From the identity

$$(d_2-d_1)^2=(d_1+d_2)^2-4d_1d_2,$$

and using $g=1$, we obtain

$$(\delta^2 d-2)n^2-d(d_2-d_1)^2=2.$$

We set $(d_2-d_1)=2y$ (number d_2-d_1 is an even number because d_1,d_2 are odd integers), and get

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If we divide both sides by 2 and define $\delta' = \frac{\delta}{2} = 2k+1$, the previous equation becomes

$$(2\delta'^2 d - 1)n^2 - 2dy^2 = 1. \tag{20}$$

We will use introduced Lemma from [4] to prove that the above Pell equation (20) has no solutions.
First, we find the least positive integer solution of the equation

$$u^2 - 2d(2\delta'^2 d - 1)v^2 = 1. \tag{21}$$

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The least positive integer solution of equation (21) is

$$(u_0, v_0) = (4\delta'^2 d - 1, 2\delta').$$

According to Lemma from [4] conditions that have to be satisfied are

$$(4\delta'^2 d - 2) | 4\delta'^2 d, \ \ 4d | (4\delta'^2 d - 2).$$

We can easily see that $4d|(4\delta'^2 d - 2)$ if and only if $4d|2$ which is not possible because d is an integer.

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