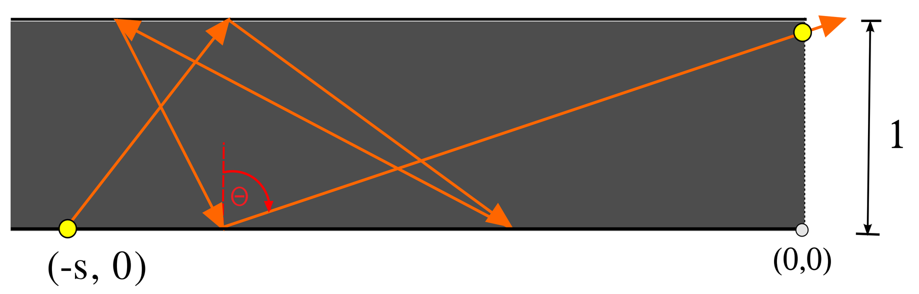
Random Reflections and Stopped Random Walks

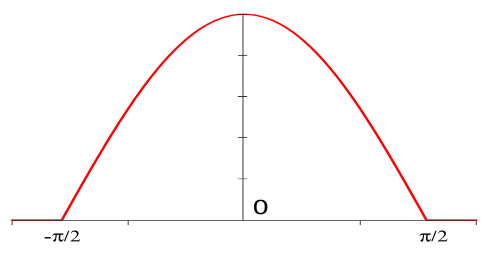
Tvrtko Tadić (joint work with Krzysztof Burdzy)
University of Washington, Seattle

UNIVERSITY of WASHINGTON

Light reflection in a semi-infinite strip

$$D = \{(x,y) \in \mathbb{R}^2 : x \le 0,\ 0 \le y \le 1\}$$





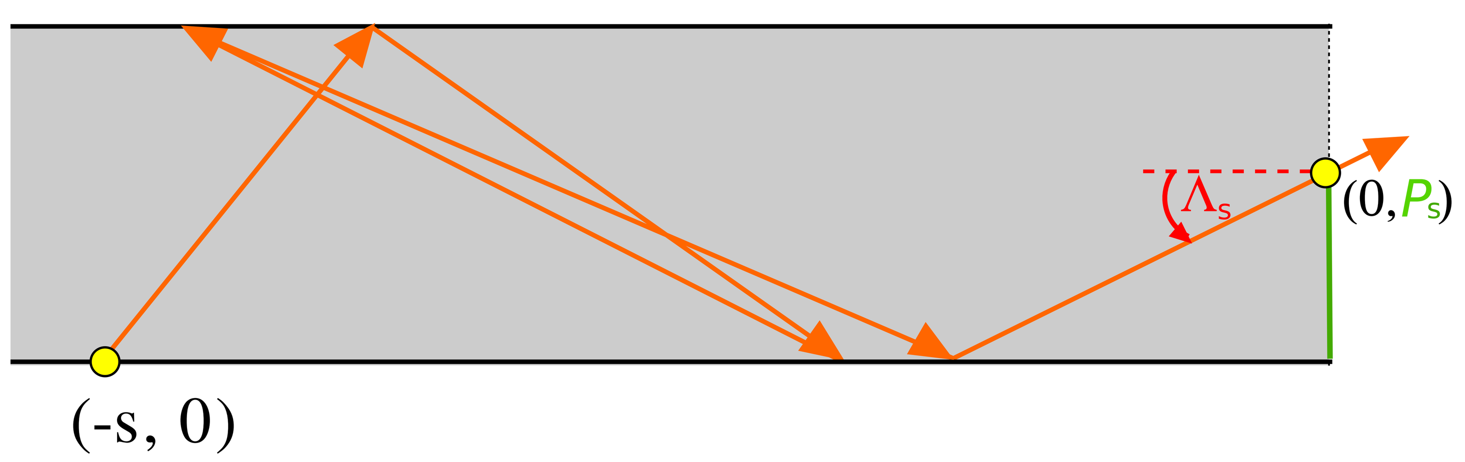
- We start at $(-s, 0)$ $(s > 0)$.
- At each point of the boundary ∂D,, the ray reflects at a random angle Θ with the normal pointing towards the interior of D. The density of Θ is given by

$$f(\theta) = \begin{cases} \frac{1}{2}\cos\theta, & \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \\ 0, & \text{otherwise.} \end{cases}$$

Related models

- Comets et al., *Billiards in a General Domain with Random Reflections* (2009)
- Comets et al., *Knudsen Gas in a Finite Random Tube : Transaport Diffusions and First Passage Properties* (2010)
- Angel et al., *Deterministic approximations of random reflectors* (2013)
- Evans, *Stochastic billiards on general tables* (2001)

Object of interest - distribution of the exiting angle and position



We are interested in joint distribution of:

1. the position P_s of the ray's exiting point on the exiting segment;
2. the angle Λ_s adjacent to the normal of the exiting segment;

as $s \to \infty$.

Random walk representation

The points of reflection are given by

$$\{(-s + S_k, (1 + (-1)^k)/2)\ :\ 0 \le k \le N_s - 1\}.$$

- $S_0 = 0$, $S_n = S_{n-1} + X_n$;
- we stop after S goes over s:
$$(S_n : n \le N(s)),$$
where
$$N_s = \inf\{n \ge 0 : S_n > s\}.$$

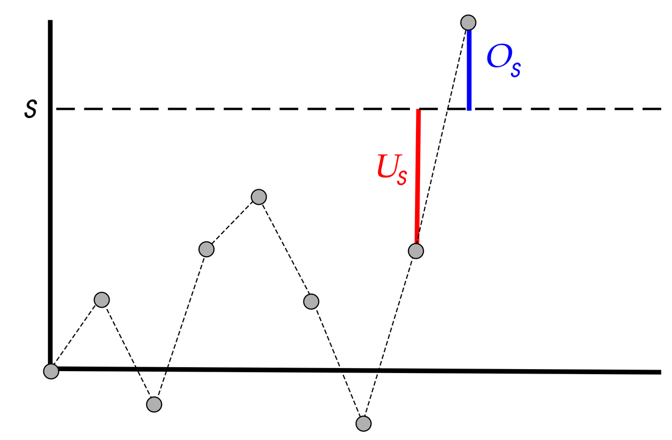
- $Y_1 \overset{d}{=} -Y_1$ (symmetric)
- $\mathbb{P}(Y_1 \le y) = \frac{1}{2} + \frac{y}{2\sqrt{1+y^2}}$;
- $\mathbb{P}(Y_1 > y) \sim \frac{1}{4y^2}$;
- $\mathbb{E}[|Y_1|] = 1$, $\mathbb{E}(Y_1) = 0$;
- $\mathbb{E}[Y_1^2] = +\infty$.

Overshoot and undershoot

Overshoot and undershoot:

$O_s = S_{N_s} - s$ and $U_s = s - S_{N_s - 1}$.

It is known that $O_s \overset{\mathbb{P}}{\to} \infty$ and $U_s \overset{\mathbb{P}}{\to} \infty$.



Proposition (Burdzy, Tadić, 2014)

$$\lim_{s\to\infty} \mathbb{P}\left(\frac{U_s}{O_s + U_s} \le t\right) = t^2.$$

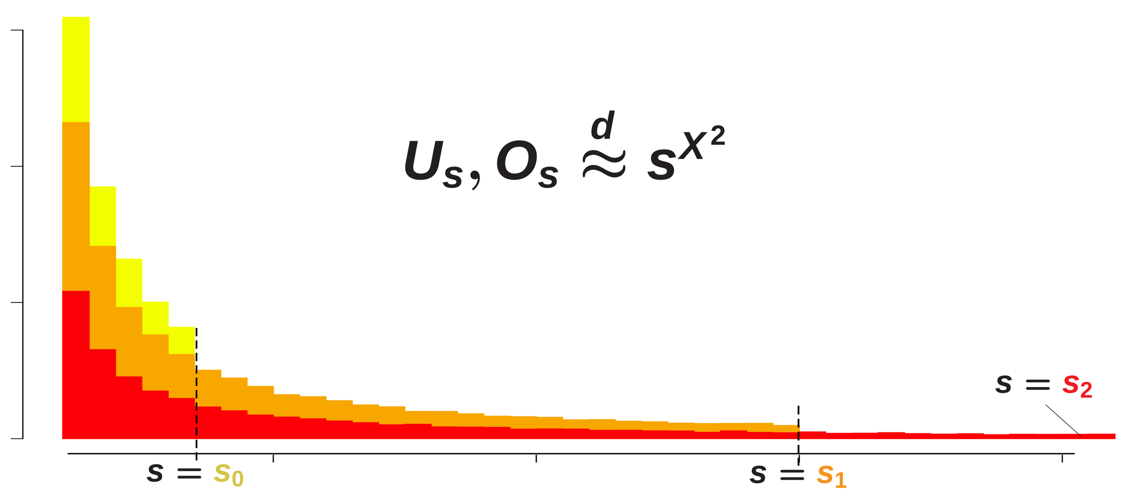
Domain of attraction of the normal law

Recall, $\mathbb{P}(Y_1 > y) \sim \frac{1}{2y^2}$. Y_1 is in the domain of attraction of the normal distribution.

Theorem (Burdzy, Tadić 2014, Erickson 1970)

$$\left(\sqrt{\frac{\log U_s}{\log s}}, \sqrt{\frac{\log O_s}{\log s}}\right) \overset{d}{\to} (X, X)$$

where $X \sim U[0,1]$.



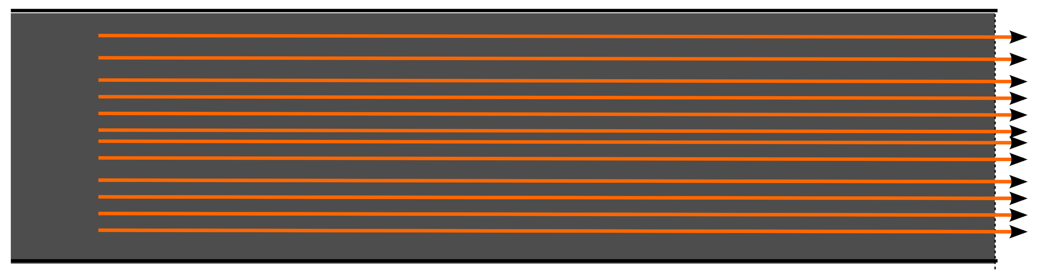
Uniform law

Λ_s angle, P_s position

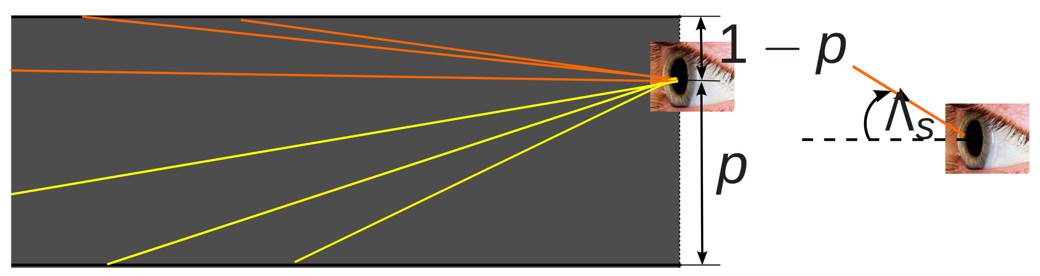
Theorem (Burdzy, Tadić, 2014)

As $s \to \infty$

$$(\Lambda_s, P_s) \overset{d}{\to} (0, U[0,1]).$$



What does the eye see?

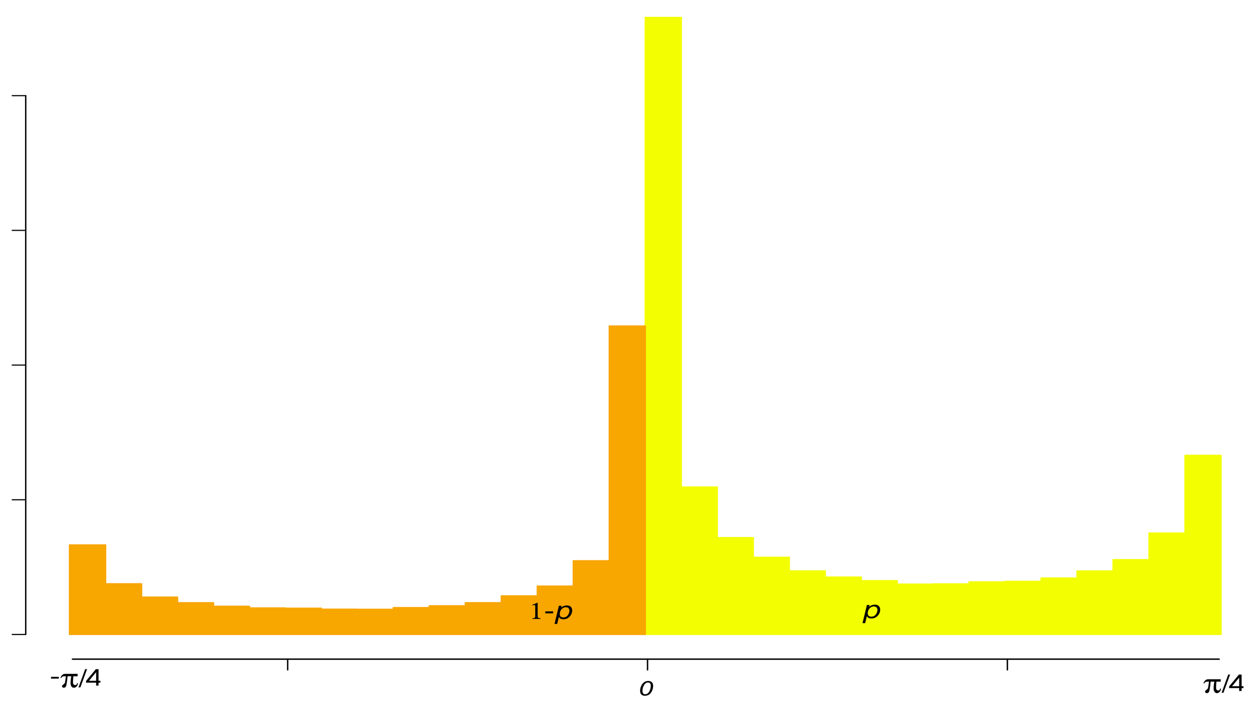


Theorem (Burdzy, Tadić, 2014)

$$\lim_{s\to\infty} \mathbb{P}\left(\sqrt{\frac{\log\cot|\Lambda_s|}{\log s}} \le l, u\Lambda_s < 0, P_s \le t\right) = \frac{l}{2}\cdot\begin{cases} 1 - (1-t)^2, & u = 1 \\ t^2, & u = -1. \end{cases}$$

Corollary (Burdzy, Tadić, 2014)

$$\lim_{s\to\infty} \mathbb{P}\left(\sqrt{\frac{\log\cot|\Lambda_s|}{\log s}} \le l, u\Lambda_s \le 0 \,\middle|\, P_s \in (p-\varepsilon, p]\right) = l\cdot\begin{cases} 1 - p + \varepsilon/2, & u = 1 \\ p - \varepsilon/2, & u = -1. \end{cases}$$



Department of Mathematics - University of Washington - Seattle, WA
Mail: tadic@uw.edu
WWW: http://www.math.washington.edu/~tadic