A new finite element formulation for vibration analysis of thick plates

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ABSTRACT: A new procedure for determining properties of thick plate finite elements, based on the modified Mindlin theory for moderately thick plate, is presented. Bending deflection is used as a potential function for the definition of total (bending and shear) deflection and angles of cross-section rotations. As a result of the introduced interdependence among displacements, the shear locking problem, present and solved in known finite element formulations, is avoided. Natural vibration analysis of rectangular plate, utilizing the proposed four-node quadrilateral finite element, shows higher accuracy than the sophisticated finite elements incorporated in some commercial software. In addition, the relation between thick and thin finite element properties is established, and compared with those in relevant literature.

KEY WORDS: Mindlin plate theory; Finite element formulation; Thick-thin plate relation; Vibration analysis; Shear locking.

INTRODUCTION

In plate theory, two mathematical models are distinguished, the well-known Kirchhoff thin plate and the Mindlin thick plate theory. In the former, shear influence on deflection is small and is therefore ignored. This theory is very well developed and the achievements are presented in Szilard's fundamental book (Szilard, 2004). The dynamic behaviour of thick plate is quite a complex problem, due to shear influence and rotary inertia, and is still an interesting subject of investigation. The first works are those of Reissner and Mindlin (Reissner, 1945; Mindlin, 1951), in which it is assumed that the plate cross-section remains a plane but not normal to the plate middle surface. As a result, the Mindlin theory deals with a system of three differential equations of motion with plate deflection and two angles of rotation of cross-section as unknown variables. This system is the starting point for all later developed variants. In the meantime, a large number of articles have been published and a comprehensive survey of literature up to 1994 can be found in (Liew et al., 1995).

Generally, there are two main approaches for solving the problem of thick plate natural vibrations, i.e. analytical methods for the solution of differential equations of motion, and numerical procedures based on the Rayleigh-Ritz energy method as well as the Finite Element Method (FEM). Different analytical methods are known depending on which functions are kept as fundamental ones in the reduction of the system of differential equations of motion. Some methods operate with three, two or even one function, for instance Wang (1994), Shimpi and Patel (2006) Endo and Kimura (2007) and Xing and Liu (2009), respectively. The application of analytical methods is relatively simple for simply supported plates and plates with simply supported two opposite edges. A sophisticated closed-form solution is derived in Xing and Liu (2009) for plate vibration analysis with any
combination of simply supported and clamped edges. The Rayleigh-Ritz method is widely used for arbitrary boundary conditions as well as for elastically supported edges. Accuracy depends on the chosen set of orthogonal functions for the assumed natural modes. For that purpose two dimensional polynomials can be used Liew et al. (1993), or functions of static Timoshenko beam deflection Dawe and Roufaeil (1980), Cheung and Zhou (2000). Efficient solution is also achieved applying the assumed mode method with static Timoshenko beam functions to thick bare plate Kim et al. (2012) as well as for some more complex problems (Cho et al., 2013; 2014).

The finite element method is a very powerful tool for the analysis of any problem (linear, nonlinear, static and dynamic) of engineering structures with a complicated configuration. Several finite elements for Mindlin plate have been developed and incorporated in commercial FEM software. They deal with three displacement fields, i.e. deflection and two rotations. Due to the impossibility to prescribe correct interdependence among deflection and rotations, the same order polynomials for the interpolation of all displacements are used. Consequently, so-called shear locking phenomenon arises since in transition from thick to thin plate, it is not possible to capture pure bending modes and zero shear strain constraints. There are a few procedures for solving shear locking problem in the FEM analysis, which are referred to in Falsone and Settineri (2012): reduced integration for shear terms (Zienkiewicz et al., 1971; Hughes et al., 1977), which is commonly used in commercial software; mixed formulation of hybrid finite elements (Lee and Wong, 1982; Auricchio and Taylor, 1995; Lovadina, 1998); Assumed Natural Strain (Hughes and Tezduyar, 1981; Bathe, 1996; Zienkiewicz and Taylor, 2000); and Discrete Shear Gap (DSG) (Bletzinger et al., 2000), and its combination with the mesh-free procedure (Liu et al., 2009; Nguyen-Xuan et al., 2010). Recently, a new shear locking free finite element formulation for static analysis of thick plate has been proposed in (Falsone and Settineri, 2012) based on an extension of the well-known Kirchhoff thin plate theory. The so-called fictitious deflection is used as a basic function, by which the other kinematic and static quantities are determined. It is also necessary to mention the worthwhile formulation of the mixed FEM and the Differential Quadrature Method (DQM) for longitudinal and transverse plate direction, respectively (Eftekhari and Jafari, 2013).

Motivated by the state of the art described above, in the present paper a new finite element formulation is proposed for thick plate vibration analysis. It is based on a new moderately thick plate theory presented in (Senjanović et al., 2013a; 2013b), where a system of governing differential equations of motion is reduced to one equation with bending deflection as an unknown function. It is actually a potential function by which total deflection and angles of rotations are determined, as well as bending and shear strains and sectional and inertia forces. This plate theory actually represents an extension of the modified Timoshenko beam theory (Senjanović and Fan, 1989; Senjanović and Grubišić, 1991; Senjanović et al., 2009; Senjanović and Vladimir, 2013). Due to strong interdependence among deflection and rotations, the shear locking phenomenon does not occur. Finite element stiffness and mass matrices are determined by employing an ordinary variational formulation (Zienkiewicz and Taylor, 2000).

OUTLINE OF NEW PLATE THEORY

Deformation of a thick rectangular plate is considered in the Cartesian coordinate system in Fig. 1. By following the idea from the modified Timoshenko beam theory, total deflection is decomposed into bending deflection and shear deflection (Senjanović et al., 2013a; 2013b)
\[ w(x, y, t) = w_0(x, y, t) + w_1(x, y, t). \]  

It is assumed that the angles of rotation of the plate cross-section are caused only by bending that is acceptable for the vibration of moderately thick plates in a lower frequency domain

\[ \psi_x = -\frac{\partial w_0}{\partial x}, \quad \psi_y = -\frac{\partial w_0}{\partial y}. \]  

Bending moments and twist moments are a function of plate curvatures and warping, respectively, i.e.

\[ M_x = -D \left( \frac{\partial^2 w_0}{\partial x^2} + \nu \frac{\partial^2 w_0}{\partial y^2} \right), \]  

\[ M_y = -D \left( \frac{\partial^2 w_0}{\partial y^2} + \nu \frac{\partial^2 w_0}{\partial x^2} \right), \]  

\[ M_{xx} = M_{yy} = -(1 - \nu)D \frac{\partial^2 w_0}{\partial x \partial y}, \]  

where

\[ D = \frac{Eh^3}{12(1 - \nu^2)} \]  

is plate flexural rigidity, and \( h, E, \) and \( \nu \) is plate thickness, Young's modulus of elasticity and Poisson's ratio, respectively.

Shear strain is the summation of the angle of rotation of the plate generatrix and its cross-section, i.e.

\[ \gamma_x = \frac{\partial w}{\partial x} + \psi_x = \frac{\partial w}{\partial x}, \]  

\[ \gamma_y = \frac{\partial w}{\partial y} + \psi_y = \frac{\partial w}{\partial y}. \]  

Thus, the shear forces read

\[ Q_x = S \frac{\partial w}{\partial x}, \quad Q_y = S \frac{\partial w}{\partial y}, \]  

where \( S = kGh \) is shear rigidity and \( k \) is shear coefficient.

Natural vibrations of plate is caused by inertia force and moments.
\[ q = \frac{m}{2} \frac{\partial^2 w}{\partial t^2}, \quad m_x = -J \frac{\partial^3 w_x}{\partial x \partial t^2}, \quad m_y = -J \frac{\partial^3 w_y}{\partial y \partial t^2}, \tag{10} \]

where \( m = \rho h \) is the plate mass per unit area, and \( J = \rho I = \rho h^3 / 12 \) is the mass moment of inertia of the cross-section per unit breadth.

By considering the equilibrium of vertical forces and moments around the \( x \) and \( y \) axis, and applying the above relations, one obtains a single differential equation of motion as shown in (Senjanović et al., 2013a)

\[
D \Delta w_b - J \left( \frac{1 + mD}{JS} \right) \frac{\partial^2}{\partial t^2} \Delta w_b + m \frac{\partial^2}{\partial t^2} \left( w_x + \frac{J}{mS} \frac{\partial^2 w_x}{\partial t^2} \right) = q(x, y, t), \tag{11} \]

where \( A(\cdot) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \) is the Laplace differential operator and \( q \) is the distributed excitation. Bending deflection \( w_b \) is actually a potential function since the remaining displacements \( w_x, \psi_x \) and \( \psi_y \) are defined by its derivatives. Hence, the total deflection (1), according to (Senjanović et al., 2013a) reads

\[ w = w_b + \frac{J}{S} \frac{\partial^2 w_b}{\partial t^2} - \frac{D}{S} \Delta w_b. \tag{12} \]

### FORMULATION OF FINITE ELEMENT PROPERTIES

A general finite element with \( n \) nodes and three d.o.f. per node, i.e. deflection and rotations around the \( x \) and \( y \) axis, is considered. The ordinary procedure for determining stiffness and the mass matrix in the case of a thin plate is used (Szilard, 2004). Bending deflection as a potential function is assumed in a polynomial form with a number of unknown coefficients which corresponds to the total number of d.o.f. \( N = 3n \)

\[ w_b = \{ a \} \{ P \}_b, \tag{13} \]

where \( \{ a \} \) is a row vector with terms \( a_i, i=0,1,...,N-1 \), and

\[ \{ P \}^T_b = \{ P \}_b = \{ 1, x, y, x^2, xy, y^2, \ldots \}. \tag{14} \]

Total static deflection, according to (12), reads

\[ w = \{ a \} \left( \frac{D}{S} \frac{\partial^2 \{ P \}_b}{\partial x^2} - \frac{D}{S} \frac{\partial^2 \{ P \}_b}{\partial y^2} \right). \tag{15} \]

Angles of rotation (2) yield

\[ \psi_x = -\{ a \} \frac{\partial \{ P \}_b}{\partial x}, \quad \psi_y = -\{ a \} \frac{\partial \{ P \}_b}{\partial y}. \tag{16} \]
By taking coordinate values $x_l$ and $y_l$ for each node, $l=1,2,...,n$, into account in Eqs. (15) and (16), the relation between the nodal displacements and the unknown coefficients $a_i$ is obtained

$$\{\delta\} = [C]\{a\}, \quad (17)$$

where $[C]$ includes $x_l$ and $y_l$ and

$$\{\delta\} = \begin{bmatrix} \{\delta\}_1 \\ \vdots \\ \{\delta\}_n \end{bmatrix}, \quad \{\delta\}_i = \begin{bmatrix} w_i \\ \varphi_i \\ y_i \end{bmatrix}. \quad (18)$$

Now, for the given nodal displacement vector $\{\delta\}$, the corresponding coefficient vector $\{a\}$ can be determined from (17)

$$\{a\} = [C]^{-1}\{\delta\}. \quad (19)$$

Substituting (19) into (13) yields

$$w_s = \{\phi\}_s \{\delta\}, \quad (20)$$

where

$$\{\phi\}_s = \{P\}_s [C]^{-1} \quad (21)$$

is the vector of the bending shape functions.

In a similar way, shear deflection can be presented in the form

$$w_s = \{\phi\}_s \{a\}, \quad (22)$$

where according to (15)

$$\{P\}_s = -\frac{D}{S} \frac{\partial^2 \{P\}_s}{\partial x^2} - \frac{D}{S} \frac{\partial^2 \{P\}_s}{\partial y^2}. \quad (23)$$

Substituting (19) into (22) yields

$$w_s = \{\phi\}_s \{\delta\}, \quad (24)$$

where

$$\{\phi\}_s = \{P\}_s [C]^{-1} \quad (25)$$
is the vector of the shear shape functions.

Total deflection according to (1) reads

\[ w = \{ \phi \} \{ \delta \}, \quad (26) \]

where

\[ \{ \phi \} = \{ \phi \}_b + \{ \phi \}_s, \quad (27) \]

is the vector of the total shape functions.

Columns of the inverted matrix \( [C] \) are vectors of coefficients \( a_i \) obtained for the unit value of particular nodal displacements

\[ [C]^{-1} = \{ A \}_1 \{ A \}_2 \ldots \{ A \}_N, \quad (28) \]

where

\[ \{ A \}_j = \{ A \}_j = \{ a_i \}' a_i \ldots a_i \}' \quad (29) \]

Bending curvatures and warping are presented in the form

\[ \{ \kappa \}_b = - \left[ L \right] \{ \delta \}, \quad (30) \]

Substituting (20) with (21) into (30) yields

\[ \{ \kappa \}_b = - \left[ L \right] \{ \delta \}, \quad (31) \]

where

\[ [L]_b = [H]_b [C]^{-1}, \quad (32) \]

\[ [H]_b = \begin{bmatrix} \frac{\partial^3 \{ P \}_b}{\partial x^3} \\ \frac{\partial^3 \{ P \}_b}{\partial y^3} \\ 2 \frac{\partial^3 \{ P \}_b}{\partial x \partial y} \end{bmatrix}, \quad (33) \]
Now it is possible to determine the bending stiffness matrix by employing a general formulation from the finite element method based on the variational principle as shown in (Zienkiewicz and Taylor, 2000)

\[ \mathbf{K}_b = \int \left[ \mathbf{L}_b \right]^T \left[ \mathbf{D}_b \right] \left[ \mathbf{L}_b \right] \, dA, \]

(34)

where

\[
\left[ \mathbf{D}_b \right] = D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}
\]

(35)

is the matrix of plate flexural rigidity. Furthermore, substituting (32) with (33) into (34) yields

\[ \mathbf{K}_b = [\mathbf{C}]^T \left[ \mathbf{B} \right] [\mathbf{C}]^{-1}, \]

(36)

where symbolically \([\mathbf{C}]^T = \left( [\mathbf{C}]^{-1} \right)^T\) and

\[ \left[ \mathbf{B} \right] = \int \left[ \mathbf{H} \right]^T \left[ \mathbf{D}_b \right] \left[ \mathbf{H} \right] dA. \]

(37)

By taking (33) and (35) into account, (37) can be presented in the form

\[ \left[ \mathbf{B} \right] = D \left[ \mathbf{I}_1 \right] + \nu \left( \left[ \mathbf{I}_2 \right] + \left[ \mathbf{I}_3 \right] \right) + \left[ \mathbf{I}_4 \right] + 2(1-\nu)[\mathbf{I}_5], \]

(38)

where

\[ \left[ \mathbf{I}_1 \right] = \int \frac{\partial^2 \left\{ P \right\}_x}{\partial x^2} \frac{\partial^2 \left\{ P \right\}_y}{\partial y^2} dA, \]

(39)

\[ \left[ \mathbf{I}_2 \right] = \int \frac{\partial^2 \left\{ P \right\}_x}{\partial x^2} \frac{\partial \left\{ P \right\}_y}{\partial y} dA = \left[ \mathbf{I}_3 \right], \]

(40)

\[ \left[ \mathbf{I}_3 \right] = \int \frac{\partial^2 \left\{ P \right\}_x}{\partial y^2} \frac{\partial \left\{ P \right\}_y}{\partial x} dA, \]

(41)

\[ \left[ \mathbf{I}_4 \right] = \int \frac{\partial \left\{ P \right\}_x}{\partial x} \frac{\partial \left\{ P \right\}_y}{\partial y} dA, \]

(42)
According to (9) the shear strain vector reads

\[
\{ \gamma \} = \begin{bmatrix} \frac{\partial w_x}{\partial x} \\ \frac{\partial w_y}{\partial x} \\ \frac{\partial w_y}{\partial y} \end{bmatrix},
\]

(43)

and by taking (24) with (25) into account, one obtains

\[
\{ \gamma \} = [L] \{ \delta \},
\]

(44)

where

\[
[L] = [H] [C]^{-1}.
\]

(45)

\[
[H] = \begin{bmatrix} \frac{\partial \{P\}_x}{\partial x} \\ \frac{\partial \{P\}_x}{\partial y} \\ \frac{\partial \{P\}_y}{\partial x} \\ \frac{\partial \{P\}_y}{\partial y} \end{bmatrix}.
\]

(46)

Analogously to (34), the shear stiffness matrix is presented in the form

\[
[K] = \int [L]^T [D] [L] \, dA,
\]

(47)

where

\[
[D] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\]

Substituting (45) with (46) into (47) yields

\[
[K] = [C]^{-T} [S] [C]^{-1},
\]

(48)

where

\[
[S] = \int [H]^T [H] \, dA.
\]

(49)

By taking (46) into account, (49) can be presented in the form

\[
[S] = S \left( [I]_x + [I]_y \right),
\]

(50)

where
The complete stiffness matrix is

\[
[K] = \begin{bmatrix} \partial\partial \int \frac{\partial}{\partial x} \frac{\partial}{\partial x} dA \\ \partial\partial \int \frac{\partial}{\partial y} \frac{\partial}{\partial y} dA \end{bmatrix} + \begin{bmatrix} \partial\partial \int \frac{\partial}{\partial x} \frac{\partial}{\partial y} dA \end{bmatrix} + \begin{bmatrix} \partial\partial \int \frac{\partial}{\partial y} \frac{\partial}{\partial x} dA \end{bmatrix}.
\]

According to the general formulation of the mass matrix in the finite element method based on the variational principle (Zienkiewicz and Taylor, 2000), one can write

\[
[M] = m \int \{\phi\} \{\phi\} dA,
\]

where \(\{\phi\}\) is the vector of the total shape functions (27). Taking (21) and (25) into account yields

\[
[M] = m[C]^{-T} [I]_b [C]^{-1},
\]

where

\[
[I]_b = \int (P) (P) dA.
\]

and \(\{P\} = \{P\}_s + \{P\}_s\), Eqs. (14) and (23).

RELATIONSHIP BETWEEN THICK AND THIN FINITE ELEMENT PROPERTIES

If the shape functions of thick and thin plate are determined by the same polynomial, it is interesting to find out the relation between their properties. For this purpose, let us decompose the coefficient matrix \([C]\) into the bending matrix and shear contribution

\[
[C] = [C]_b + [C]_s,
\]

where \([C]_b\) includes all terms without shear stiffness \(S\), while \([C]_s\) contains only terms with \(S\). Eq. (57) can also be presented in the form

\[
[C] = ([I] + [E])[C]_b,
\]

where \([I]\) is the identity matrix and
\[
[E] = [C], [C]^{-1}.
\]

Furthermore, substituting (58) into (36) and employing the inverse of the product, \(([A][B])^{-1} = [B]^{-1}[A]^{-1}\), and the transpose of the product, \(([A][B])^T = [B]^T[A]^T\), one arrives at

\[
[K]_b = ([I] + [E]^{-1})^{-1}([K]_b^T + [K]_s^T)([I] + [E])^{-1},
\]

where

\[
[K]_b^T = [C]_b^{-T}[B][C]_b^{-1}
\]

is the bending stiffness matrix of thin plate.

In a similar way, the matrix of shear stiffness (48) can be presented in the form (60), and the complete stiffness matrix of thick plate reads

\[
[K] = ([I] + [E]^{-1})^{-1}([K]_b^T + [K]_s^T)([I] + [E])^{-1},
\]

where

\[
[K]_s^T = [C]_s^{-T}[S][C]_s^{-1}
\]

The mass matrix (55) can also be decomposed if the relation \(\{P\} = \{P\}_b + \{P\}_s\) is taken into account. Hence, one obtains

\[
[M] = ([I] + [E]^{-1})^{-1}([M]_b + [M]_s + [M]_s + [M]_b)([I] + [E])^{-1},
\]

where

\[
[M]_b = m[C]_b^{-T}\int\{P\}_b\{P\}_b^T dA[C]_b^{-1},
\]

\[
[M]_s = m[C]_s^{-T}\int\{P\}_s\{P\}_s^T dA[C]_s^{-1} = [M]_s^T,
\]

\[
[M]_s = m[C]_s^{-T}\int\{P\}_s\{P\}_s^T dA[C]_s^{-1}.
\]

The matrix \([M]_b\) is the bending mass matrix of thin plate, \([M]_s\) is the shear mass matrix of thick plate, and \([M]_{bs}\) and \([M]_{sb}\) are mass matrices of bending and shear coupling.
Distributed excitation is transmitted to the nodes by the shape functions

\[ \{ F \} = \int_a \{ \phi \} q dA. \]  

(68)

Taking (21) and (25) into account yields

\[ \{ F \} = [C]^T \int_a \{ P \} q dA. \]  

(69)

Furthermore, by employing (58) and \( \{ P \} = \{ P \}_b + \{ P \}_s \), one arrives at

\[ \{ F \} = (\{ I \} + [E]^T) \left( \{ F \}_b + \{ F \}_s \right), \]  

(70)

where

\[ \{ F \}_b = [C]^T \int_a \{ P \}_b q dA \]  

(71)

\[ \{ F \}_s = [C]^T \int_a \{ P \}_s q dA \]  

(72)

are the vector of the nodal forces due to bending and shear, respectively.

An observation on the relevant literature related to the above subject is given in Appendix.

### RECTANGULAR FINITE ELEMENT

The four-node rectangular finite element with nodal displacements is shown in Fig. 2. The dimensionless coordinates \( \xi = x / a \) and \( \eta = y / b \) are introduced. The ordinary polynomial for the thin plate finite element is applied for bending (Holand and Bell, 1970).

![Rectangular finite element](image_url)  

Fig. 2 Rectangular finite element.
\begin{equation}
\{P\}_b = \{1, \xi, \eta, \xi^2, \xi\eta, \eta^2, \xi^3, \xi^2\eta, \xi\eta^2, \eta^3, \xi^3\eta, \xi^2\eta^2\}. 
\end{equation}

The shear polynomial according to (23) reads

\begin{equation}
\{P\}_s = -\{0, 0, 0, 2\alpha, 0, 2\beta, 6\alpha\xi, 2\alpha\eta, 2\beta\xi, 6\beta\eta, 6\alpha\xi\eta, 6\beta\xi\eta\},
\end{equation}

where

\begin{equation}
\alpha = \frac{D}{S'a}, \quad \beta = \frac{D}{S'b}.
\end{equation}

\{P\}_s is for two order lower polynomial than \{P\}_b which is important for the possible avoidance of shear locking. The total deflection, Eq. (15), is

\begin{equation}
w = \{a\} \{P\}_b + \{P\}_s\},
\end{equation}

and the angles of rotation. Eqs. (16)

\begin{equation}
\psi_x = -\frac{1}{a} \{a\} \frac{\partial \{P\}_b}{\partial \xi}, \quad \psi_y = -\frac{1}{b} \{a\} \frac{\partial \{P\}_b}{\partial \eta}.
\end{equation}

By taking the coordinates \(\xi\) and \(\eta\) defined in Fig. 2 into account, matrix \([C]\) is obtained in the following form

\begin{equation}
[C] = \begin{bmatrix}
1 & 0 & 0 & -2\alpha & 0 & -2\beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{b} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{a} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1-2\alpha & 0 & -2\beta & 1-6\alpha & 0 & -2\beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{b} & 0 & \frac{1}{b} & 0 & 0 & \frac{1}{b} & 0 & 0 & \frac{1}{b} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{a} & 0 & -\frac{2}{a} & 0 & 0 & -\frac{3}{a} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1-2\alpha & 1 & 1-2\beta & 1-6\alpha & 1-2\alpha & 1-2\beta & 1-6\beta & 1-6\alpha & 1-6\beta & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{b} & 0 & \frac{1}{b} & \frac{2}{b} & 0 & \frac{1}{b} & \frac{2}{b} & \frac{3}{b} & \frac{1}{b} & \frac{3}{b} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{a} & 0 & -\frac{2}{a} & -\frac{1}{a} & 0 & -\frac{3}{a} & -\frac{2}{a} & -\frac{1}{a} & 0 & -\frac{3}{a} & -\frac{1}{a} & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & -2\alpha & 0 & 1-2\beta & 0 & -2\alpha & 0 & 1-6\beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{b} & 0 & 0 & \frac{2}{b} & 0 & 0 & 0 & \frac{3}{b} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{a} & 0 & 0 & -\frac{1}{a} & 0 & 0 & 0 & -\frac{1}{a} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}.
\end{equation}
The stiffness and mass matrix can be determined directly by employing Eqs. (53) and (55), respectively, or in an indirect way by Eqs. (62) and (64). The decomposition of matrix $[C]$ into $[C]_b$ and $[C]_s$ is very simple. The inversion of the matrices can be done analytically by a Computer Algebra System (CAS). For illustration, matrix $[E]$ is obtained in the following form:

$$[E] = \begin{bmatrix} 6\alpha + 6\beta & 4b\beta & -4\alpha a & -6\alpha & 0 & -2\alpha a & 0 & 0 & 0 & -6\beta & 2b\beta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6\alpha & 0 & 2\alpha a & 6\alpha + 6\beta & 4b\beta & 4\alpha a & -6\beta & 2b\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -6\beta & -2b\beta & 0 & 6\alpha + 6\beta & -4b\beta & 4\alpha a & -6\alpha & 0 & 2\alpha a \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6\beta & -2b\beta & 0 & 0 & 0 & 0 & -6\alpha & 0 & -2\alpha a & 6\alpha + 6\beta & -4b\beta & -4\alpha a \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (79)$$

**TRIANGULAR FINITE ELEMENT**

The three-node triangular finite element in the Cartesian coordinate system with nodal displacements is shown in Fig. 3. The ordinary polynomial for triangular finite element of thin plate is used for bending

$$\{P\}_b = \{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3\}, \quad (80)$$

The shear polynomial is obtained according to (23)

$$\{P\}_s = -\frac{D}{S}\{0, 0, 0, 2, 6x, 2(x + y), 6y\}. \quad (81)$$

The total deflection is

$$w = \{a\}\{\{P\}_b + \{P\}_s\}, \quad (82)$$
where \( \{ a \} = \{ a_1, \ldots, a_8 \} \). Angles of rotation are determined according to (16), i.e.

\[
\psi_0 = -\{ a \} \left\{ 0,1,0,2x,y,0,3x^2,2xy+y^2,0 \right\}
\]
\[
\psi_j = -\{ a \} \left\{ 0,0,1,0,x,2y,0,x^2+2xy,3y^2 \right\}.
\]  

Matrix \([C]\) in expression for nodal displacements, Eqs. (17) and (18), is obtained by taking nodal coordinates \( x_l \) and \( y_l \), \( l=1,2,3 \), into account

\[
\begin{bmatrix}
1 & x_1 & y_1^2 - 2 \frac{D}{S} x_1 y_1 & y_1^3 - 2 \frac{D}{S} y_1 \cr
0 & -1 & 0 & -2x_1 & -y_1 & 0 & -3x_1^2 & -\left( 2x_1 y_1 + y_1^2 \right) & 0 \\
0 & 0 & -1 & 0 & -x_1 & -2y_1 & 0 & -\left( x_1^2 + 2x_1 y_1 \right) & -3y_1^2 \\
1 & x_2 & y_2^2 - 2 \frac{D}{S} x_2 y_2 & y_2^3 - 2 \frac{D}{S} y_2 \\
0 & -1 & 0 & -2x_2 & -y_2 & 0 & -3x_2^2 & -\left( 2x_2 y_2 + y_2^2 \right) & 0 \\
0 & 0 & -1 & 0 & -x_2 & -2y_2 & 0 & -\left( x_2^2 + 2x_2 y_2 \right) & -3y_2^2 \\
1 & x_3 & y_3^2 - 2 \frac{D}{S} x_3 y_3 & y_3^3 - 2 \frac{D}{S} y_3 \\
0 & -1 & 0 & -2x_3 & -y_3 & 0 & -3x_3^2 & -\left( 2x_3 y_3 + y_3^2 \right) & 0 \\
0 & 0 & -1 & 0 & -x_3 & -2y_3 & 0 & -\left( x_3^2 + 2x_3 y_3 \right) & -3y_3^2
\end{bmatrix}.
\]  

In order to determine bending stiffness matrix it is necessary to define matrix \([H]_s\), Eq. (33). Hence, one finds

\[
[H]_s = \begin{bmatrix}
0 & 0 & 0 & 2 & 0 & 0 & 6x & 2y & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 2x & 6y & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 4(x+y) & 0 & 0
\end{bmatrix}.
\]  

Matrix \([B]\), Eq. (37), in the bending stiffness matrix, Eq. (36), is given in the Cartesian coordinate system. In order to make integration over the element area possible, triangular coordinates \( \zeta \) and \( \eta \) are introduced. According to Fig. 4 one can write

Fig. 4 Triangular coordinates.
\[
x = x_i + x_{i1} \xi + x_{i2} \xi \eta,
\]
\[
y = y_i + y_{i1} \xi + y_{i2} \xi \eta,
\]
where \( x_{ij} = x_i - x_j \) and \( y_{ij} = y_i - y_j \). Furthermore, differential area of triangular element reads
\[
dA = J d\xi d\eta,
\]
where
\[
J = \begin{vmatrix}
\frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\
\frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta}
\end{vmatrix} = 2A_\xi.
\]
is Jacobian and \( A = \frac{1}{2} (x_{i1} y_{i2} - x_{i2} y_{i1}) \) is element area. Finally, one can write for matrix \([B]\), Eq. (37)
\[
[B] = 2A \int_0^1 \int_0^1 [H(\xi, \eta)] [D]_h [H(\xi, \eta)]_h d\xi d\eta.
\]
In a similar way matrix \([H]_s\), Eq. (46), in the shear stiffness matrix, Eqs. (48) and (49), takes the following constant form
\[
[H]_s = -\frac{D}{S} \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 6 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 6
\end{bmatrix}
\]
Hence, one finds for matrix \([S]\), Eq. (49), after integration
\[
[S] = AS [H]_s [H]_s^{-1} \begin{bmatrix}
0 \\
0 \\
\tilde{s}
\end{bmatrix}.
\]
where
\[
[s] = \begin{bmatrix}
36 & 12 & 0 \\
12 & 8 & 12 \\
0 & 12 & 36
\end{bmatrix}
\]
Mass matrix \([M]\) is defined by Eqs. (55) and (56), where by employing triangular coordinates
\[
[M]_0 = 2A \int_0^1 \int_0^1 \{P(\xi, \eta)\} \{P(\xi, \eta)\}_\xi d\xi d\eta.
\]
and \( \{P(\xi, \eta)\} = \{P(\xi, \eta)\}_\xi + \{P(\xi, \eta)\}_\eta \).

It is necessary to mention that application of the triangular coordinates in case that two element boundaries are parallel to the coordinate axes leads to singularity of element properties. Therefore, application of the area coordinates is preferable in general case (Zienkiewicz, 1971).
NUMERICAL EXAMPLES

Reliability of the developed finite element formulation and four noded rectangular element is analysed by three numerical examples of natural vibrations: simply supported square plate (SSSS), rectangular plate clamped on transverse edges and simply supported on longitudinal edges (CSCS), and rectangular plate for combined clamped, free, and simply supported boundary conditions (CFSS). All plates are modelled with $8 \times 8 = 64$ finite elements. Necessary data and nondimensional frequency parameters are listed in the title of Tables 1, 2 and 3. Thin and thick plates are considered and Present Solutions (PS) are compared with available ones determined analytically, Tables 1 and 2, and by Rayleigh-Ritz method, Table 3. Also, results obtained by NASTRAN (MSC, 2010) are included in the tables. In the most cases very high accuracy is achieved, somewhere better than that of NASTRAN results. PS and NASTRAN values for thick plates bound rigorous frequency parameters. The first four natural nodes for case CFSS generated by NASTRAN are shown in Fig. 5 for illustration.

Table 1 Frequency parameter $\mu = \omega a^2 \sqrt{\rho h / D}$ of square plate, case SSSS, $k = 0.86667$.

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*m,n* - mode identification number, *m* and *n* number of half waves in $x$ and $y$ direction.

Table 2 Frequency parameter $\lambda = \left(\frac{\omega b^2}{\pi^2}\right) \sqrt{\rho h / D}$ of rectangular plate, case CSCS, $a / b = 0.5 , k = 0.86667$.

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Table 3 Frequency parameter $\lambda = \left( \omega b^2 / \pi^2 \right) \sqrt{\rho h / D}$ of rectangular plate, case CFSS, $a / b = 0.4$, $k = 5/6$.

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</table>

In addition, the natural vibration of equilateral triangular plate (like tension leg platform), using the developed general triangular finite element is analysed. The plate is modelled with 25 finite elements and it is simply supported at corner nodes. Dimensionless frequency parameters obtained by the proposed procedure are given in Table 4, together with NAUTRAN solutions, where good agreement can be noticed. For illustration, the first four natural modes obtained by NAUTRAN are shown in Fig. 6.

Fig. 5 The first four natural modes of rectangular plate, case CFSS (NAUTRAN).

The convergence of the proposed finite element formulation is demonstrated in the case of a simply supported square plate. Natural frequencies are determined by the finite element model for three mesh densities, i.e. $6 \times 6 = 36$, $8 \times 8 = 64$ and $10 \times 10 = 100$ elements, and three values of a thickness ratio $h / a$: 0.001, 0.1 and 0.2. The obtained frequency parameters are listed in Table 5 and compared with the exact analytical solution as well as with the NAUTRAN and ABAQUS (Dassault Systèmes, 2008) results. In order to have better insight into the convergence, frequency parameter for the 1st, 4th and 7th modes are shown in Fig. 7. For thin plate ($h / a = 0.001$), the present solution converges to the exact value, faster than the NAUTRAN and ABAQUS results, which converge from the opposite sides. PS values for moderately thick plate ($h / a = 0.1$) are very close to the exact values for all three mesh densities, while the NAUTRAN and ABAQUS results converge to a lower value than the exact solution. The discrepancy is reduced for higher modes. In the case of thick plate ($h / a = 0.2$), the variation of the PS values, which are somewhat higher than the exact solution, is rather small. The NAUTRAN and ABAQUS results show the same tendency as in the previous case.
Table 4 Frequency parameter $\mu = \alpha a^2 \sqrt{\rho h / D}$ of equilateral triangular plate with simply supported corner nodes, $k=5/6$.

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Table 5 Convergence of frequency parameter $\mu = \alpha a^2 \sqrt{\rho h / D}$ of square plate, case SSSS, $k=5/6$.

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/*m,n*/ - mode identification number, $m$ and $n$ number of half waves in $x$ and $y$ direction.
CONCLUSION

Thick plate appears as a structural element in many engineering structures. In this paper, an outline of the modified Mindlin theory for moderately thick plate is presented. Instead of three variables, i.e. deflection and two rotations of cross-sections, the problem is reduced to only one, where bending deflection is used as a potential function for determining the remaining displacements, strains and sectional forces. A new formulation of thick plate finite elements based on this theory is proposed following the standard FEM procedure, thus ensuring variational consistency. The well-known shear locking problem, which accompanies the Mindlin theory, is overcome as a result of the strong interdependence among deflection and rotations which
reduce the order of the shear polynomial compared to the bending polynomial. The consistent relationship between thick and thin finite element properties is derived, so that the stiffness and mass matrix for the thick element can be determined directly or indirectly. Based on a systematic analysis of this subject, some shortcomings of a known similar relationship are noticed and discussed. The vibration analysis of a thick rectangular plate by a simple quadrilateral finite element and coarse mesh shows a higher level of accuracy than sophisticated finite elements incorporated in some commercial software packages. Developed triangular finite element applied for vibration analysis of the triangular plate, also gives good results. Avoidance of the shear locking problem, fast convergence and high accuracy are the advantages of the proposed finite element formulation for vibration analysis of moderately thick plates.

Shear locking is not a natural phenomenon. It is introduced by the inconsistent mathematical modelling of a physical task, resulting in a numerical problem. With finite elements based on the original Mindlin thick plate theory, it is not possible to ensure smooth transition from thick to thin plate, since the shear stiffness, which is dominant for higher modes, also remains dominant for lower modes, which is not realistic. Any attempt to eliminate shear locking results in complicated finite element properties and a decrease in accuracy. However, such elements can usually capture so-called missing modes in the higher frequency domain, which represent the coupling of ordinary bending-shear modes with in-plane shear modes (Lim et al., 2005). In contrast, the modified Mindlin theory is related to moderately thick plate and due to the strong interdependence among the deflection and rotations the shear locking problem in finite elements does not occur. Finite elements derived according to these two theories actually cover two regions of the problem, i.e. one from thin to moderately thick plates, and the other from moderately to fully thick plates, which partly overlap. Higher accuracy is obtained in the former types of finite elements than in the latter if moderately thick plate is an issue. Based on the above, it is not efficient to derive unique finite elements for solving shear locking and capturing high in-plane modes which are usually not of practical interest. Now, the proposed procedure for the formulation of shear locking-free finite elements may be used for the development of sophisticated thick plate elements with different shapes and a higher number of nodes.

ACKNOWLEDGMENT

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REFERENCES


APPENDIX – Consideration of the known relationship between thick and thin plate finite element properties

Stiffness matrix of thick plate finite element is derived in (Falsone and Settineri, 2012) by employing the simplified thick plate theory similar to that presented in (Senjanović et al., 2013a). So-called fictitious deflection as single variable is used, which corresponds to the bending deflection applied in the present paper. Stiffness matrix of thick plate finite element is obtained based on the known stiffness matrix of thin plate element. The following relation between those two matrices, written in the present notation, is found:

\[
[K] = [K]^0_t \left( [I] + [E] \right)^{-1}. \tag{A1}
\]

By comparing Eq. (A1) with (62) it is obvious that transformation matrix \([I] + [E]^0\) on the left side of (62) is not present in (A1). Reason is that shear influence on external nodal forces (70) is ignored in (Falsone and Settineri, 2012), and condition of equivalence of internal nodal forces for thick and thin finite element is used instead of balance of their strain energies. As a result stiffness matrix (A1) is not consistent and consequently it is not symmetric. Relation between equilibrium equations of thick and thin finite element should be of similar form like transformation of element equation from local to global coordinate system (Zienkiewicz and Taylor, 2000). Furthermore, explicitly specified matrix \([S]^0_t\) in (62) is missing in (A1), and the obtained stiffness matrix is not complete. Since this matrix is relevant for shear locking, it is not possible to consider that problem a priori.

Finite element approach to thick plate theory presented in (Falsone and Settineri, 2012) is actually an extension of the previously worked out simplification of the Timoshenko beam theory. Hence, the shortcomings of relation between thick and thin plate finite element properties are also related to the beam elements (Falsone and Settineri, 2011).