Finite elements on non-linear manifolds of rotations or complete motion – relationships between objectivity, helicoidal interpolation and fixed-pole approach

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Summary. Finite-element implementation of mechanical problems defined on non-linear manifolds requires particular attention, since some of the important physical properties of equilibrium or motion are not necessarily automatically inherited from the underlying continuous governing equations. Here we review some of the features of the so-called objective, helicoidal and fixed-pole approaches and show that there exist interesting similarities between them even though their development has been motivated by clearly different demands. More detail is presented for the configuration space formed by a product of a three-dimensional vector space and a three-parametric orthogonal differential manifold as well as the non-linear non-orthogonal six-parametric differential manifold of complete motion.

Key words: objectivity, configuration-dependent interpolation, helicoidal interpolation, linked interpolation, fixed-pole approach, 3D rotation group, 6D group of complete motion

Introduction

The geometrically exact 3D beam theory developed by Simo in 1985 [1] along with its original finite-element implementation in 1986 [2] makes perhaps one of the most well-known excursions into applications of numerical techniques, the finite-element method in particular, to mechanical problems defined on non-linear manifolds. The non-linearity of the problem domain here shows up as a consequence of non-linearity of the manifold of 3D rotations – the special orthogonal Lie group. Other examples naturally include higher-dimensional cases of the Cosserat (or micropolar) continuum theory, such as shells with drilling rotations [3] or a 3D micro-polar continuum [4], but also more sophisticated implementations of the 1D (beam) theory defined on a complete six-parametric Lie group of complete motion [5, 6].

Special care need be exercised in the process of interpolation of the degrees of freedom that belong to the non-linear part of the configuration space lest one may end up in a numerical procedure that fails to inherit the important physical property of objectivity of the solution with respect to the frame of reference of the observer. A possible solution has been presented in [7] involving a configuration-dependent interpolation for the rotational degrees of freedom, but a number of alternative procedures have been suggested also, e.g. [8, 9].

A perhaps not so well-known relationship exists, however, between the interpolation for the rotational degrees of freedom presented in [7] and the so-called helicoidal interpolation given in [10], which was originally devised with a completely different objective in mind - to provide a solution which is independent of the choice of the beam reference axis. It turns out, interestingly, that so long as we limit our attention to two-node beam elements, the interpolation for the rotational degrees of freedom in these two sources turns out to be the same. On the other hand, the helicoidal interpolation appears to be more sophisticated in that it also produces a more elaborate, and in many senses beneficial, configuration-dependent interpolation for the
displacement field which, incidentally, turns out to be identical to that utilised for interpolation of the rotational field.

An attempt to generalise the helicoidal interpolation to beam elements with arbitrary number of nodes has been made in [11], but there clearly appears to remain present both a need and a potential for further development of the procedure proposed. The idea to apply the interpolation of the rotational field from [7] (which has been developed for a general n-node element) to the displacement field comes as very natural, but a closer inspection reveals that this is not necessarily so. The principal obstacle appears to lie in a conflict between the demands to secure the exact solution of a linear problem [12] and the solution independent of the position of the reference axis [10]. Intriguingly, this conflict disappears precisely and exclusively for the two-node elements, thus effectively asserting that the helicoidal interpolation is a genuinely two-noded interpolational concept.

The helicoidal interpolation also has links with the so-called fixed-pole approach [5]. Nonetheless, there is no limitation to two-node interpolation in the fixed-pole concept while, in contrast to the helicoidal interpolation, objectivity of the formulation is not provided unless specifically taken care of [13, 14]. The fixed-pole concept also provides a very natural setting for development of conservative and group-motion preserving energy-dissipative time-stepping schemes in implicit non-linear dynamics, where simultaneous conservation of both global momenta and energy is, if needed, much more easily attained than in the standard 'moving-pole' approaches.

Also, the fixed-pole concept effectively re-expresses governing equations of a problem in a fully non-linear manifold inhabited by the complete motion, where translations and rotations are only the specifically defined parts of a new six-dimensional motion parameter. This poses certain practical problems, as we end up in having to work with non-standard problem unknowns, which are not straightforward to relate to the standard displacements and rotations present in the existing finite elements. Also, they are awkward to utilise to define even relatively simple support conditions and an attempt to exploit its benefits, while by-passing its short-comings has been made in [13], where the standard degrees of freedom have been re-introduced at the nodal level. Again, objectivity of a finite-element implementation is not automatically guaranteed in the fixed-pole approach and an algorithm which enforces it may be devised in analogy with the procedure given in [7]. This follows as a consequence of the fact that the six-parametric group of complete motion is also a Lie group, and the governing equations of the problem defined on this group take a strikingly parallel form to those defining the kinematic and constitutive equations as well as the equations of motion for the rotational part in the 'moving-pole' approaches [14].

**Brief outline of 3D geometrically exact beam theory and its original FE implementation**

The geometrically exact 3D beam theory provided by Simo in 1985 [1] makes one of the milestones in the development of the non-linear finite-element method by introducing a non-linear manifold, composed of a three-dimensional vector space of displacements and a three-parametric Lie group of the rotation tensors as the configuration space. To present it briefly, for a beam of length \( L \) in free flight the weak form of the power-balance law reads

\[
\int_0^L \left[ (v' + \hat{\mathbf{r}} w') \cdot \mathbf{n} + w' \cdot \mathbf{m} \right] dx + \int_0^L \left( \mathbf{v} \cdot \mathbf{k} + \mathbf{w} \cdot \mathbf{\pi} \right) dx = 0,
\]

where \( \mathbf{n} \) and \( \mathbf{m} \) are vectors of spatial stress and stress-couple resultants, \( \mathbf{k} = A_\rho \mathbf{v} \) and \( \mathbf{\pi} = \Lambda^t \mathbf{\rho} \mathbf{A}^t \mathbf{w} \) are the vectors of specific momentum and angular momentum with respect to the beam reference axis at a cross-section, \( \mathbf{r} \) and \( \Lambda \) are the position vector of the reference line and the orientation tensor of the principal axes of the cross-section with respect to their position in the reference state, which belong to a non-linear configuration space composed of a three-dimensional vector space \( \mathcal{R}^3 \) and the three-parametric special orthogonal group \( SO(3) \). A dot and a dash in the power-balance equation indicate differentiation with respect to time \( t \) and the beam-length parameter \( x \), a superimposed hat indicates a cross-product operator, \( \mathbf{v} = \dot{\mathbf{r}} \) and \( \mathbf{w} \) (for which \( \dot{\mathbf{w}} = \Lambda^t \dot{\Lambda} \)) are the velocity and the angular velocity vectors, \( A \) and \( \rho \).
are the cross-sectional area and density of the material, and $J_\rho$ is the tensor of cross-sectional moments of inertia.

Originally [2], only the velocity fields in the power-balance equation have been interpolated using Lagrangian polynomials $\dot{v}(x)$ via $\mathbf{v}(x) = \sum_{i=1}^{N} \dot{v}(x) \mathbf{v}_i$ and $\mathbf{w}(x) = \sum_{i=1}^{N} \dot{v}(x) \mathbf{w}_i$. For arbitrary nodal velocities, this has resulted in the nodal balance $\mathbf{g}^l = \mathbf{q}^l + \mathbf{q}^m = 0$ at any node $i = 1, \ldots, N$ with the nodal internal and inertial force vectors $\mathbf{q}^l_i$ and $\mathbf{q}^m_i$ as

$$\mathbf{q}^l_i = \int_0^L \left[ \begin{array}{cc} \dot{\mathbf{I}}' & 0 \\ -\mathbf{I}' & \dot{\mathbf{I}}' \end{array} \right] \left[ \begin{array}{c} \mathbf{n} \\ \mathbf{m} \end{array} \right] dx$$

and

$$\mathbf{q}^m_i = \int_0^L \dot{\mathbf{m}} \left( \mathbf{K} \right) dx.$$  \hspace{1cm} (1)

The system of non-linear equations $\mathbf{g}^l = \mathbf{q}^l + \mathbf{q}^m = 0$ (for $i = 1, \ldots, N$) may now be solved for the kinematic unknowns $\mathbf{r}(x)$ and $\mathbf{A}(x)$ using the Newton–Raphson solution procedure in which the linear part of the changes in these unknowns $\Delta \mathbf{r}$ and $\Delta \mathbf{\theta}$ (emerging from $\Delta \mathbf{A} = \Delta \mathbf{\theta} \mathbf{A}$) may be interpolated in the same manner as $\mathbf{v}(x)$ and $\mathbf{w}(x)$ [2].

Note that this interpolation (along with a suitable time-stepping procedure) is sufficient to completely define the dynamic problem, too, even though in [15] additional interpolations are also provided for the incremental rotations, angular velocities and accelerations, in conjunction with the Newmark time-integration scheme. In any case, however, this approach, as well as a variety of related approaches, turn out to be incapable of algorithmic preservation of the important mechanical property of objectivity of the solution with respect to the choice of the observer, also implying a solution which ceases to be strain-invariant with respect to a rigid-body motion.

**Objective finite-element interpolation of 3D rotations**

In the objective formulation for geometrically exact higher-order beam elements [7] the position vector of the beam reference axis is taken to coincide with the line of centroids and has been interpolated in a standard Lagrangian manner. The rotations, in contrast, have been interpolated very differently: the rotation matrix $\mathbf{A}(x)$ has been *multiplicatively* decomposed into a part constant for the whole beam and rigidly attached to a node $I$ ($\mathbf{A}_I$) and the part due to a *local* rotation $\mathbf{\Psi}^l$ with respect to that orientation as $\mathbf{A}(x) = \mathbf{A}_I \exp \mathbf{\Psi}^l(x)$, where $\exp \mathbf{\Psi}^l = \mathbf{I} + \alpha \mathbf{\Psi}^l + \beta (\mathbf{\Psi}^l)^2$, $\alpha = \sin \mathbf{\Psi}^l / \mathbf{\Psi}^l$ and $\beta = (1 - \cos \mathbf{\Psi}^l) / (\mathbf{\Psi}^l)^2$. The local rotation $\mathbf{\Psi}^l(x)$ is next interpolated in the standard Lagrangian way, where the local nodal rotations $\mathbf{\Psi}_I^l$ are extracted from $\exp \mathbf{\Psi}_I^l = \mathbf{A}_I^T \mathbf{A}_I$.

The Newton–Raphson increment $\Delta \mathbf{\theta}$ has been found in the form $\Delta \mathbf{\theta} = \sum_{i=1}^{N} \mathbf{N}^i(\mathbf{A}) \Delta \mathbf{\theta}_i$ with

$$\mathbf{N}^i = \sum_{j=1}^{N} \sum_{k=1}^{N} \Delta_{kj}^i \mathbf{A}_I \left\{ \delta_{lk} \left[ \mathbf{I} - \mathbf{H}(\mathbf{\Psi}_I^l) \sum_{m=1}^{N} \mathbf{I}_m \mathbf{H}^{-1}(\mathbf{\Psi}_m^l) \right] + \mathbf{H}(\mathbf{\Psi}_I^l) \mathbf{I} \mathbf{H}^{-1}(\mathbf{\Psi}_j^l) \right\} \mathbf{A}_I^T$$  \hspace{1cm} (2)

and $\mathbf{H}(\mathbf{\Psi}^l) = \mathbf{I} + \beta \mathbf{\Psi}^l + \gamma (\mathbf{\Psi}^l)^2$, $\gamma = \frac{\mathbf{\Psi}^l - \sin \mathbf{\Psi}^l / (\mathbf{\Psi}^l)^3}$, $\Delta_{kj}^i = 1$ for $i = j = k$ and $\Delta_{kj}^i = 0$ otherwise.

**Helicoidal interpolation**

The helicoidal interpolation [10] follows from a requirement that the finite-element solution should be invariant to the choice of the beam reference axis and consistent with the configuration space, in particular with $SO(3)$ and its core properties of orthogonality and unimodularity.

The FE solution will be invariant to the choice of the beam reference axis if the position vector and the rotation tensor are interpolated using the same interpolation functions [10, 11]. The simplest example of this, of course, is the standard Lagrangian interpolation for both fields.
Choosing to interpolate \( \omega \) the following corresponding six-dimensional nodal internal and inertial force vectors equation results in co-ordinate system is taken for the fixed pole. Substituting this into the earlier power-balance section. Further, they defined six-dimensional fixed-pole velocity, stress-resultant and specific the velocity vector as seen by an observer rigidly attached to the frame rotating with the cross-section. This result is interesting in its own right as it obviously by-passes the anomalous presence of shape-functions in the internal force vector, which are known to be responsible for shear locking. Additional consequence is that this set-up leads to a relatively simple satisfaction of the energy

\[
\begin{align*}
\mathbf{r}(x_1) &= \sum_{i=1}^{2} N_i \mathbf{r}_i \quad \text{and} \quad \mathbf{\Lambda}(x_1) = \sum_{i=1}^{2} N_i \mathbf{A}_i \Rightarrow \Delta \mathbf{\vartheta}(x_1) = \sum_{i=1}^{2} N_i \Delta \mathbf{\vartheta}_i, \quad (3)
\end{align*}
\]

with the generalised interpolation functions \( N_i \) identical to those given earlier. It is important to emphasise, however, that the proposed helicoidal interpolation makes sense only for two–noded elements. For the cases of (uncoupled) constant bending and shearing (as well as, of course, constant axial force and torsional moment) this interpolation provides exact solution irrespective of the amount of loading and deformation.

If we attempted to apply this result to a higher–order element by simply substituting \( N \) for 2 in (3)\(_1\), we would realise that the exact result, even in the limit case of the analysis becoming linear, cannot be achieved anymore. Still, such a solution is quite legitimate and in [11] it has been analysed numerically. To analyse the linear solution, it is instructive to isolate the linear part of the generalised interpolation \( \mathbf{N}^i = I_i(\mathbf{I} + \mathbf{\psi} - \mathbf{\psi}_0/2) \) [11] and compare it to the interpolation representing the exact linear solution \( \mathbf{N}^i = I_i(\mathbf{I} + \mathbf{\psi} - \mathbf{\psi}_0/N) \) [12].

The very close similarity between these results suggests that the original helicoidal interpolation may be generalised to higher order elements by applying a modification to the generalised interpolation \( N_i \) used for the interpolation of the positions. This avenue has been also pursued in [11]. It has to be noted that in this way the interpolation for the position vector again becomes different from the interpolation for the rotations thus spoiling the original requirement which the interpolation sought should provide, i.e. invariance of the solution with respect to the choice of the beam reference axis.

**Fixed-pole approach**

In [5] Bottasso and Borri thoroughly investigated the idea of replacing the stress-couple resultant \( \mathbf{m} \) and the specific angular momentum \( \mathbf{\pi} \) at the beam reference axis with another stress-couple resultant \( \mathbf{\tilde{m}} \) and specific angular momentum \( \mathbf{\tilde{\pi}} \), defined with respect to a unique point for the whole structure - the fixed pole, i.e. \( \mathbf{\tilde{m}} = \mathbf{r} \times \mathbf{n} + \mathbf{m} \) and \( \mathbf{\tilde{\pi}} = \mathbf{r} \times \mathbf{k} + \mathbf{\pi} \) if the origin of the spatial co-ordinate system is taken for the fixed pole. Substituting this into the earlier power-balance equation results in

\[
\int_0^L \left( \mathbf{\omega}' \cdot \mathbf{n} + \mathbf{w}' \cdot \mathbf{\tilde{m}} \right) dx + \int_0^L \left( \mathbf{\tilde{v}} \cdot \mathbf{k} + \mathbf{w} \cdot \mathbf{\tilde{\pi}} \right) dx = 0,
\]

where \( \mathbf{\tilde{v}} = \mathbf{v} + \mathbf{r} \times \mathbf{w} \) is the velocity vector as seen by an observer rigidly attached to the frame rotating with the cross-section. Further, they defined six-dimensional fixed-pole velocity, stress-resultant and specific momentum vectors as

\[
\omega = \left\{ \mathbf{\tilde{v}}, \mathbf{w} \right\}, \quad \mathbf{s} = \left\{ \mathbf{n}, \mathbf{\tilde{m}} \right\}, \quad \mathbf{p} = \left\{ \mathbf{k}, \mathbf{\tilde{\pi}} \right\}. \quad (4)
\]

Choosing to interpolate \( \omega(x) \), an alternative nodal balance \( \mathbf{\tilde{g}}_i \equiv \mathbf{\tilde{q}}_i^\text{f} + \mathbf{\tilde{q}}_i^\text{m} = \mathbf{0} \) is obtained with the following corresponding six-dimensional nodal internal and inertial force vectors

\[
\mathbf{\tilde{q}}_i^\text{f} = \int_0^L \mathbf{I}' \mathbf{s} dx \quad \text{and} \quad \mathbf{\tilde{q}}_i^\text{m} = \int_0^L \mathbf{I}' \mathbf{p} dx. \quad (5)
\]

This result is interesting in its own right as it obviously by-passes the anomalous presence of shear locking. Additional consequence is that this set-up leads to a relatively simple satisfaction of the energy.
and momentum conservation properties in discrete non-linear 3D beam dynamics without the need to re-parametrise the rotation field using tangent-scaled rotations as in [16].

The most striking of all, however, is the relationship between $\mathbf{w}$ and $\mathbf{w}$, which are not only the elements of a three-dimensional and a six-dimensional vector space, but are in fact elements of the vector spaces which are topologically equivalent to the three-dimensional Lie algebra of skew-symmetric tensors $so(3)$ and the six-dimensional Lie algebra of tensors of the form $\hat{\mathbf{w}} = \begin{bmatrix} \dot{\mathbf{w}} & \mathbf{v} \\ \mathbf{0} & \mathbf{w} \end{bmatrix}$, which in [5] Bottasso in Borri have named $sr(6)$. Exponentiation of the elements of these algebras, of course, results in the corresponding elements of Lie groups, the Lie group $SO(3)$ of rotation tensors $\mathbf{A}$ and $SR(6)$, the so-called Lie group of rigid motions $C = \begin{bmatrix} \Lambda & \mathbf{r} \Lambda \\ \mathbf{0} & \Lambda \end{bmatrix}$. The system of non-linear equations may now be solved for the complete motion $C$ using the Newton–Raphson solution procedure in which the linear part of the change in this unknown $\Delta \mathbf{C}$ (emerging from $\Delta \mathbf{C} = \Delta \mathbf{c} \mathbf{C}$) may be interpolated in the same manner as $\mathbf{w}$ [14]. The parallels between the elements of $SO(3)$ and the elements of $SR(6)$ were investigated thoroughly in [5] where it was shown that the complete problem of motion of a 3D beam may be made mathematically equivalent to the problem of rotational motion of the 3D beam.

These results have been re-derived in detail in [14], where special emphasis has been placed onto objectivity of the formulations defined on $SR(6)$ which, owing to the parallels with the non-linear manifold $SO(3)$, naturally must suffer from the same short-comings. A remarkably similar result is obtained for the generalised interpolation that should be applied to $\Delta \mathbf{C}$ to provide the objective finite-element solution. The configuration tensor $C(x)$ has been multiplicatively decomposed into a part constant for the whole beam and defined at a node $I$ ($C_I$) and the part due to a local configurational change $\Phi^I_i$ with respect to that configuration as $C(x) = C_I \exp \Phi^I_i(x)$ (see [5, 14] for details on exponentiation in $SR(6)$). The local configurational change $\Phi^I_i(x)$ (which consists of the upper part $P^I_i$ and the lower part $\Psi^I_i$) is next interpolated in the standard Lagrangian way, where the local nodal configurational changes $\Phi^I_i$ are extracted from $\exp \Phi^I_i = C_I^{-1}C_i$. It follows that $\Delta \mathbf{C} = \sum_{i=1}^{N} J^i(C) \Delta \mathbf{c}_i$ with

$$J^i = \sum_{j=1}^{N} \sum_{k=1}^{N} \Delta_{jk} C_I \left\{ I - X(\Phi^I_i) \sum_{m=1}^{N} I_m X^{-1}(\Phi^I_m) \right\} X(\Phi^I_i) I_k X^{-1}(\Phi^I_j) \right\} C_I^{-1}. \quad (6)$$

where $X(\Phi^I_i) = \begin{bmatrix} H(\Psi^I_i) & B(P^I_i, \Psi^I_i) \\ 0 & H(\Psi^I_i) \end{bmatrix}$, $B(P^I_i, \Psi^I_i) = \beta \mathbf{P}^I_i + \frac{\alpha - 2\beta}{\varepsilon^2} (P^I_i \cdot \Psi^I_i) \mathbf{P}^I_i + \frac{\beta - 3\gamma}{\varepsilon^2} (P^I_i \cdot \Psi^I_i) \mathbf{P}^I_i + \gamma \left( \Psi^I_i \mathbf{P}^I_i + \mathbf{P}^I_i \Psi^I_i \right)$.

**Conclusions**

The so-called objective, helicoidal and fixed-pole interpolations are reviewed and shown to possess many interesting similarities despite the fact that they have been developed with quite different demands in mind. The well-known equivalence between the rotational part of the mechanical problem of motion of a 3D flexible beam defined on $SO(3)$ and the complete motion defined on $SR(6)$ is shown to extend to the issue of objectivity of the finite-element solution with respect to the position of an observer allowing for a completely corresponding solution process.

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