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TIMOSHENKO BEAM THEORY 93 YEARS LATER – OVER BRIDGES TO NANOTUBES AND ULTRA LARGE SHIPS

Ivo Senjanović, Nikola Vladimir, Marko Tomić, Neven Hadžić

Abstract

An outline of the Timoshenko beam theory, which deals with deflection and cross-section rotation as the basic variables, is presented. It is modified by decomposing total deflection into pure bending deflection and shear deflection, and total rotation into bending rotation and axial shear angle. The governing equations are condensed into two independent equations of motion, one for flexural and another for axial shear vibrations. The solution is given for natural vibrations. Nonlocal stress parameter is taken into account for vibration analysis of nanotube embedded in an elastic medium. The 4th order partial differential equation for flexural vibration is extended to the 6th order one. Nanotube response to moving nanoparticle gravity load is analyzed by employing modal superposition method, separation of variables, Galerkin method and harmonic balance method. In addition coupled flexural and torsional vibrations of a thin-walled girder are considered. The modified Timoshenko beam theory is applied for flexural vibrations while the complex torsional beam theory with warping cross-section is worked out in an analogical way. For vibration analysis of nonprismatic ship hull an advanced beam finite element is created. Application of the developed theory is illustrated in the case of nanotube and ultra large container ship vibrations.

Keywords: Timoshenko beam theory; nanotube; moving load; thin-walled girder; flexural vibrations; torsional vibrations; container ship

Preface
Timoshenko's way from St. Petersburg to Stanford – via Zagreb written by Stjepan Jecić

In engineering mechanical, especially supporting structures, the essential elements are the so-called girders. They are of different designs, and their basic geometrical shapes are bars (beams), plates, walls, and shells, and variants thereof. In the first semesters of engineering studies, such as civil engineering, mechanical engineering and naval architecture, the strength of such elements is studied according to approximate theories adapted to the most common cases of actual structural designs. Thus, for instance, the calculation of beam elements is done according to the Euler-Bernoulli hypothesis which is valid for beams whose span l between the supports exceeds five heights h of the cross-section. Special cases are the beams on numerous mutually close supports with elastic characteristics and extreme cases of beams that rely on elastic bases. There are a lot of such cases known in engineering practice, such as railway tracks all the way to vessels – ships.

More accurate calculations of such special problems were the subject of research of scientists both of theoretical and applied mechanics at the end of the 19th century and further on intensively in all the years of the past twentieth century (e.g. E. Winkler, H. Zimmermann, W. Prager and others). Among these scientists special focus should be on a scientist theoretician, and above all pioneer of applied (engineering, technical) mechanics, Stephen Prokopovych Timoshenko. Born Ukrainian (in Shpotovka on 22 December 1878), he studied in St. Petersburg where he graduated in 1901 and acquired the university degree of an engineer of ways of communication. By starting to work at the Polytechnic Institute in St. Petersburg he acquired additional knowledge in theoretical mechanics under the supervision of very distinguished professor I.V. Mešćerski, and additional knowledge in mechanics of deformable bodies he acquired from the books and works of A. Föppel, whom he greatly admired. As a professor of mechanics he worked in Kiev (1906 – 1911) at the Faculty of Civil Engineering where he was also the Dean. From 1913, as professor of mechanics at the Polytechnics in St. Petersburg he held lectures and published a textbook on the Theory of Elasticity (1914). Scientifically he was dealing with problems of structural elements bending. He extended the Prandtl theory of boundary layer analogy to problems of torsion, and for the needs of shipbuilding of the Baltic fleet he developed procedures of calculating stability of stiffened panels. It is interesting to mention that at that time Timoshenko defined and determined the so-called shear centre for beams in which the resulting load does not pass through the centre of gravity of the girder cross-section. At the beginning of the First World War he published his original idea, observing the railway tracks as girders on an elastic bedding. He published a paper dealing with this topic in the Works of the St. Petersburg Ways of Communication Institute.

In the whirlwind of war and political events the life and work in St. Petersburg were becoming increasingly difficult. Timoshenko took refuge with his family in a relatively peaceful Kiev. However, war and revolutionary developments forced Stephen Timoshenko to move with quite a large group of refugees towards Sevastopol where, together with his former student J.M. Hlitčijev, he decided to move to Serbia. Passing across Turkey, after several weeks of toilsome travelling they arrived in Belgrade. The city was full of refugees and it was impossible to find any accommodation. After a short time, the Belgrade professor of mechanics Arnovljević found him accommodation at his sister's place in Zemun. Realizing that in 1919 the Technical Higher School was to be founded in Zagreb, he sent his job application and was admitted as the professor of Science of Strength of Materials starting in April 1920. He stayed in Zagreb until the end of the academic year 1922 where he established a Department for technical mechanics. However, his junior colleague Hlitčijev, a shipbuilding engineer, remained all his life in Belgrade as a distinguished university professor and member of the Serbian Academy of Sciences.

Although Stephen Timoshenko describes Zagreb as a city of pro-European appearance and culture, his life in this city was not comfortable. Zagreb could not offer any appropriate accommodation, so by the approval of the Rector he was placed with his family in the premises of the future laboratory for material testing. On improvised furniture the whole family lived for two years modestly and in a cramped space, which resulted in Timoshenko's decision in 1922 to move to the United States. Yet, even in such circumstances he was active scientifically, and retreated from undesirable visitors to the university library into peace and quiet.

Still while he was staying in Zemun, he used the time to develop arithmetic procedures of solving various problems important in shipbuilding. The continuation of intensive work in Zagreb brought to two major publications in the field of transversal vibrations of beams abandoning the Euler-Bernoulli hypothesis. It is, namely, in bending of short beams that the cross-sections deflect more from their original position and do not stay perpendicular to the neutral axis of the girder, and therefore the application of a simple theory of bending will lead to the wrong result greater than 1%. With the paper *On the correction for shear of the differential equation for transverse vibration of prismatic bars* and the paper *On the transverse vibrations of bars of uniform cross section* Timoshenko solved the problem in a way applicable to the engineering practice, although solving finite equations satisfying the given boundary conditions was a demanding job. Both papers were published in 1921 and 1922 in the Journal *Philosophical Magazine*, and in this Timoshenko received great help from the English mathematician E.H. Love, one of the founders of mathematical theory of elasticity. Today's powerful computers provide almost unlimited possibilities for the application of direct methods of searching for solutions numerically (e.g. finite difference method). However, new numerical methods, developed from the middle of the past century provide a better and more convenient tool. Among them, the method of finite elements is definitely unmatched. Today in literature the usual term for such problems related to the beams is *Timoshenko beam theory*.

In Zagreb Stephen Timoshenko has been remembered as an excellent scientist, and foremost as an excellent lecturer, delivering lectures initially in Russian, and later, with the help of an assistant in the Croatian language. He left Zagreb in the summer of 1922 moving with his family first to Philadelphia and then to Pittsburgh. In 1927 he became Professor at the University of Michigan. From 1936 until his retirement he was Professor of mechanics at the Stanford University. On several occasions he visited Europe. On two occasions, in 1958 and 1967, Timoshenko visited Kiev, which was described in more detail by the Kiev academician G.S. Pisarenko who was his host and wrote the afterword to the second edition of Timoshenko's book *Vospominanija*. In 1956 Timoshenko received a honorary doctorate of science of the University of Zagreb. On that occasion he visited the Faculty of Engineering, writing about it with words of praise in his *Memoirs* describing the huge progress and development from the time of his work in Zagreb. In Switzerland in 1964 he had the bad luck of breaking his leg and that same autumn his elder daughter Ana escorted him in a wheelchair to the opening of the International Congress of Mechanics in Munich. Timoshenko was greeted with a round of applause receiving thus tribute from more than 1,000 attendees as one of the greatest mechanical engineers of that time. Unable to return to the USA he remained until the end of his life (29 May 1972) with his daughter Ana in Wuppertal. His urn was laid next to his wife in Palo Alto in California.

1. INTRODUCTION

Beam is used as a structural element in many engineering structures like frame and grillage ones, [1,2,3]. Also, the whole complex structures like bridges, ship hulls, floating airports, etc. can be modeled as a beam. Hence, instead of 3D FEM model, beam model is used with cross-section properties determined as equivalent quantities of 2D sectional structure. In case of structure with large aspect ratio of height and length the Timoshenko beam theory is used, instead of the Euler-Bernoulli theory, since it takes both shear and rotary inertia into account. Their influence is especially pronounced in higher natural modes.

The Timoshenko beam theory was published 93 years ago, and during that long time it has been successfully used for static and dynamic analysis of any type of beam-like structures, [4,5]. That theory deals with two differential equations of motion with deflection and cross-section rotation as the basic variables. The system is reduced into a single the 4th order partial differential equation by Timoshenko [6], where only an approximate solution is given as commented in [7,8]. Almost in all papers the first approach with two differential equations is used in order to ensure control of exact and complete beam behavior, [8,9].

The Timoshenko beam theory is applied also for more complex problems as for instance beam vibrations on elastic foundation, [10], beam vibrations and buckling on elastic foundation, [11], vibrations of double-beam system with transverse and axial load, [12], vibration and stability of multiple beam systems, [13], beam response due to moving gravity and inertia load related to railway and highway bridges, [14-17], etc. Recent decades the Timoshenko beam theory is used in nanotechnology for vibration analysis of nanotubes exposed to moving nanoparticle load, [18-21].

Timoshenko's idea of shear and rotary inertia influence on deflection is not only limited to beams. These effects are also incorporated into the Mindlin thick plate theory as well as into its modification, as 2D problem, [22,23]. Timoshenko beam static deflection functions are often used as coordinate functions for thick plate vibration analysis by the Rayleigh-Ritz method, [24]. Furthermore, differential equation of beam torsion with shear influence is based on analogy with that for beam bending, [25]. Hence, in case of coupled flexural and torsional vibrations of a girder with open cross-section the same mathematical model is used for analysis of both responses.

The Timoshenko beam theory plays an important role in development of sophisticated beam finite element. Various finite elements have been worked out in the last decades. They are distinguished into the choice of interpolation functions for mathematical description of beam deflection and cross-section rotation. Application of the same order polynomials for both displacements leads to the so-called shear locking problem, since bending strain energy for a slender beam vanishes before shear strain energy. If static solution of Timoshenko beam is used for deflection and rotation functions the problem is overcome, [26].

In spite of the fact that enormous number of papers has been published by employing Timoshenko beam theory from 1922, it seems that all phenomena hidden in that theory are not yet investigated. For instance problem of beam response due to moving gravity and inertia force related to bridges is analyzed in literature in different ways. Presently, a systematic investigation of that problem is undertaken in [27] in order to establish the simplest mathematical formulation. Hence, original and modified Timoshenko beam theory is used in combination with Galerkin method and energy balance. The problem is solved completely in analytical way employing the perturbation method.

Motivated by the above described state-of-the art in this article modified Timoshenko beam theory, with bending deflection as the single based variable, and its advantages are presented, [28]. It is applied for nanotube vibrations due to moving nanoparticle load, [29], by employing results from [26]. Also, an advanced beam finite element for coupled flexural and torsional vibration analysis of thin-walled girder is worked out. Its application is illustrated in case of an ultra large container ship, [30].

2. OUTLINE OF THE TIMOSHENKO BEAM THEORY

Timoshenko beam theory deals with beam deflection and angle of rotation of cross-section, w and ψ, respectively, [4,5]. The sectional forces, i.e. bending moment and shear force read

$$M = D\frac{\partial \psi}{\partial x}, \qquad Q = S\left(\frac{\partial w}{\partial x} + \psi\right), \tag{1}$$

where $D=EI$ is flexural rigidity and $S = k_S GA$ is shear rigidity, A is cross-section area and I is its moment of inertia, k_S is shear coefficient, and E and $G = E/\left(2\left(1+\nu\right)\right)$ is Young's modulus and shear modulus, respectively. Value of shear coefficient depends on beam cross-section profile, [31].

Beam is loaded with transverse inertia load per unit length, and distributed bending moment

$$q_x = -m\frac{\partial^2 w}{\partial t^2}, \qquad m_x = -J\frac{\partial^2 \psi}{\partial t^2}, \tag{2}$$

where $m = \rho A$ is specific mass and $J = \rho I$ is mass moment of inertia, both per unit length.

Equilibrium of moments and forces

$$\frac{\partial M}{\partial x} - Q = -m_x, \qquad \frac{\partial Q}{\partial x} = -q_x \tag{3}$$

leads to two coupled differential equations

$$D\frac{\partial^2 \psi}{\partial x^2} - S\left(\frac{\partial w}{\partial x} + \psi\right) - J\frac{\partial^2 \psi}{\partial t^2} = 0 \tag{4}$$

$$S\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial \psi}{\partial x}\right) - m\frac{\partial^2 w}{\partial t^2} = 0\,. \tag{5}$$

From (5) yields

$$\frac{\partial \psi}{\partial x} = -\frac{\partial^2 w}{\partial x^2} + \frac{m}{S}\frac{\partial^2 w}{\partial t^2} \tag{6}$$

and by substituting (6) into (4) derived per x, one arrives at the single beam differential equation of motion

$$\frac{\partial^4 w}{\partial x^4}-\left(\frac{J}{D}+\frac{m}{S}\right)\frac{\partial^4 w}{\partial x^2 \partial t^2}+\frac{m}{D}\frac{\partial^2}{\partial t^2}\left(w+\frac{J}{S}\frac{\partial^2 w}{\partial t^2}\right)=0\,. \tag{7}$$

Once (7) is solved angle of rotation is obtained from (6) as

$$\psi=-\frac{\partial w}{\partial x}+\frac{m}{S}\int\frac{\partial^2 w}{\partial t^2}\mathrm{d}x+f\left(t\right), \tag{8}$$

where $f\left(t\right)$ is rigid body motion.

3. MODIFIED TIMOSHENKO BEAM THEORY

3.1 Differential equations of motion

In order to make the beam theory more physically transparent, beam deflection w and angle of rotation ψ are split into their constitutive parts, [28], Fig. 1, i.e.

$$w=w_b+w_s, \qquad \psi=\varphi+\vartheta, \qquad \varphi=-\frac{\partial w_b}{\partial x}, \tag{9}$$

where w_b and w_s is beam deflection due to pure bending and transverse shear, respectively, and φ is angle of cross-section rotation due to bending, while ϑ is cross-section slope due to axial shear. Newly introduced phenomenology as well as additional explanations are thoroughly presented in [28]. Equilibrium equations (4) and (5) can be presented in the form with the separated variables w_b and w_s, and ϑ

$$D\frac{\partial^3 w_b}{\partial x^3}-J\frac{\partial^2}{\partial t^2}\left(\frac{\partial w_b}{\partial x}\right)+S\frac{\partial w_s}{\partial x}=D\frac{\partial^2 \vartheta}{\partial x^2}-S\vartheta-J\frac{\partial^2 \vartheta}{\partial t^2} \tag{10}$$

$$S\frac{\partial^2 w_s}{\partial x^2}-m\frac{\partial^2}{\partial t^2}\left(w_b+w_s\right)=-S\frac{\partial \vartheta}{\partial x}\,. \tag{11}$$

Since only two equations are available for three variables one can assume that flexural shear, w_s, and slope due to axial shear, ϑ, are not coupled. In that case, by setting both left and right hand side of (10) zero, yields from the former

$$w_s=-\frac{D}{S}\frac{\partial^2 w_b}{\partial x^2}+\frac{J}{S}\frac{\partial^2 w_b}{\partial t^2}\,. \tag{12}$$

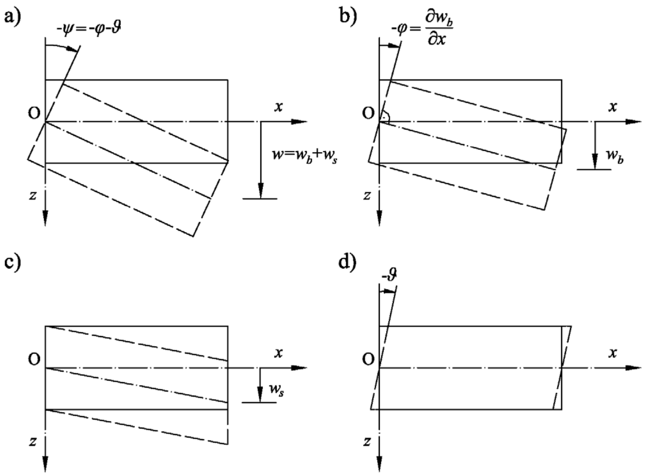

Fig. 1. Thick beam displacements; a – total deflection and rotation w, ψ , b – pure bending deflection and rotation w_b, φ, c – transverse shear deflection w_s, d – axial shear angle ϑ

Sl. 1. Pomaci debele grede: a – ukupni progib i kut zakreta w, ψ, b – progib čistog savijanja i kut zakreta w_b, φ, c – progib poprečnog smicanja w_s , d – kut uzdužnog smicanja ϑ

By substituting (12) into (11) differential equation for flexural vibrations is obtained, which is expressed with pure bending deflection

$$\frac{\partial^4 w_b}{\partial x^4} - \left(\frac{J}{D} + \frac{m}{S}\right)\frac{\partial^4 w_b}{\partial x^2 \partial t^2} + \frac{m}{D}\frac{\partial^2}{\partial t^2}\left(w_b + \frac{J}{S}\frac{\partial^2 w_b}{\partial t^2}\right) = \frac{S}{D}\frac{\partial \vartheta}{\partial x} \tag{13}$$

Disturbing function on the right hand side in (13) can be ignored due to assumed uncoupling of flexural and axial shear vibrations. Once w_b is determined, the total beam deflection, according to (9), reads

$$w = w_b - \frac{D}{S}\frac{\partial^2 w_b}{\partial x^2} + \frac{J}{S}\frac{\partial^2 w_b}{\partial t^2}. \tag{14}$$

The right hand side of (10) represents differential equation of axial shear vibrations

$$\frac{\partial^2 \vartheta}{\partial x^2} - \frac{S}{D}\vartheta - \frac{J}{D}\frac{\partial^2 \vartheta}{\partial t^2} = 0, \tag{15}$$

as given in [28].

3.2 General solution of flexural vibrations

Natural vibrations are harmonic, i.e. $w_b = W_b \sin \omega t$ and $\vartheta = \Theta \sin \omega t$, so that equations of motion (13) and (15) are related to the vibration amplitudes

$$\frac{\mathrm{d}^4 W_b}{\mathrm{d}x^4} + \omega^2 \left(\frac{J}{D} + \frac{m}{S} \right) \frac{\mathrm{d}^2 W_b}{\mathrm{d}x^2} + \omega^2 \frac{m}{D} \left(\omega^2 \frac{J}{S} - 1 \right) W_b = 0 \tag{16}$$

$$\frac{\mathrm{d}^2 \Theta}{\mathrm{d}x^2} + \frac{S}{D} \left(\omega^2 \frac{J}{S} - 1 \right) \Theta = 0 \, . \tag{17}$$

Amplitude of total deflection, according to (14), reads

$$W = \left(1 - \omega^2 \frac{J}{S} \right) W_b - \frac{D}{S} \frac{\mathrm{d}^2 W_b}{\mathrm{d}x^2} \, . \tag{18}$$

Eq. (16) is known in literature as a reliable alternative of Timoshenko differential equations, [32,33].

Solution of (16) can be assumed in the form $W_b = A e^{\gamma x}$ that leads to biquadratic equation

$$\gamma^4 + a\gamma^2 + b = 0 \, , \tag{19}$$

where

$$a = \omega^2 \left(\frac{J}{D} + \frac{m}{S} \right), \qquad b = \omega^2 \frac{m}{D} \left(\omega^2 \frac{J}{S} - 1 \right). \tag{20}$$

Roots of (19) read

$$\gamma = \alpha, -\alpha, i\beta, -i\beta \, , \tag{21}$$

where $i = \sqrt{-1}$ and

$$\begin{aligned} \alpha &= \frac{\omega}{\sqrt{2}} \sqrt{\sqrt{\left(\frac{m}{S} - \frac{J}{D} \right)^2 + \frac{4m}{D\omega^2}} - \left(\frac{m}{S} + \frac{J}{D} \right)}, \\ \beta &= \frac{\omega}{\sqrt{2}} \sqrt{\sqrt{\left(\frac{m}{S} - \frac{J}{D} \right)^2 + \frac{4m}{D\omega^2}} + \left(\frac{m}{S} + \frac{J}{D} \right)}. \end{aligned} \tag{22}$$

Hence, solution of Eq. (16) is obtained in the form

$$W_b = A_1 \sinh \alpha x + A_2 \cosh \alpha x + A_3 \sin \beta x + A_4 \cos \beta x \, . \tag{23}$$

By employing expressions for displacements and forces one arrives at

$$W = A_1\left(1-\omega^2\frac{J}{S}-\alpha^2\frac{D}{S}\right)\mathrm{sh}\,\alpha x + A_2\left(1-\omega^2\frac{J}{S}-\alpha^2\frac{D}{S}\right)\mathrm{ch}\,\alpha x$$
$$+A_3\left(1-\omega^2\frac{J}{S}+\beta^2\frac{D}{S}\right)\sin\beta x + A_4\left(1-\omega^2\frac{J}{S}+\beta^2\frac{D}{S}\right)\cos\beta x$$
$$\Phi = -\frac{\mathrm{d}W_b}{\mathrm{d}x} = -\left(A_1\alpha\,\mathrm{ch}\,\alpha x + A_2\alpha\,\mathrm{sh}\,\alpha x + A_3\beta\cos\beta x - A_4\beta\sin\beta x\right) \tag{24}$$
$$M = -D\frac{\mathrm{d}^2W_b}{\mathrm{d}x^2} = -D\left(A_1\alpha^2\mathrm{sh}\,\alpha x + A_2\alpha^2\mathrm{ch}\,\alpha x - A_3\beta^2\sin\beta x - A_4\beta^2\cos\beta x\right)$$
$$Q = -D\frac{\mathrm{d}^3W_b}{\mathrm{d}x^3} - \omega^2 J\frac{\mathrm{d}W_b}{\mathrm{d}x} = -D\left[A_1\alpha\left(\alpha^2+\omega^2\frac{J}{D}\right)\mathrm{ch}\,\alpha x + A_2\alpha\left(\alpha^2+\omega^2\frac{J}{D}\right)\mathrm{sh}\,\alpha x -\right.$$
$$\left.-A_3\beta\left(\beta^2-\omega^2\frac{J}{D}\right)\cos\beta x + A_4\beta\left(\beta^2-\omega^2\frac{J}{D}\right)\sin\beta x\right].$$

3.3 General solution of axial shear vibrations

Differential equations (17) for natural axial shear vibrations of beam reads

$$\frac{\mathrm{d}^2\Theta}{\mathrm{d}x^2} + \left(\omega^2\frac{J}{D} - \frac{S}{D}\right)\Theta = 0\,. \tag{25}$$

It is similar to the equation for rod stretching vibrations

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \omega_R^2\frac{m}{EA}u = 0\,. \tag{26}$$

The difference is the additional moment $S\Theta$, which is associated to inertia moment $\omega^2 J\Theta$, and represents reaction of an imagined rotational elastic foundation with stiffness equal to the shear stiffness *S*, as shown in Fig. 2.

Solution of (26) and corresponding axial force $N = EA\dfrac{\mathrm{d}u}{\mathrm{d}x}$ read

$$u = C_1\sin\chi x + C_2\cos\chi x,$$
$$N = EA\left(C_1\chi\cos\chi x - C_2\chi\sin\chi x\right), \tag{27}$$

where $\chi = \omega_R\sqrt{m/(EA)}$. Based on analogy between (25) and (26) one can write for shear slope angle and moment

$$\Theta = C_1 \sin\eta x + C_2 \cos\eta x,$$
$$M = D\left(C_1\eta\cos\eta x - C_2\eta\sin\eta x\right) \tag{28}$$

where

$$\eta = \sqrt{\omega^2 \frac{J}{D} - \frac{S}{D}}\,. \tag{29}$$

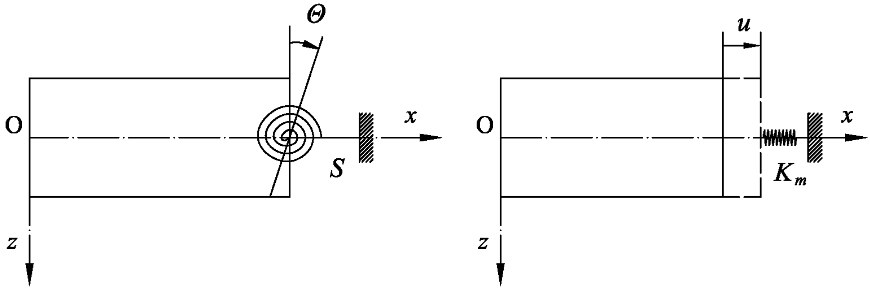

Fig. 2. Analogy between axial shear model and stretching model

Sl. 2. Analogija između modela uzdužnog smicanja i rastezanja

4. NONLOCAL VIBRATIONS OF A CARBON NANOTUBE DUE TO MOVING NANOPARTICLE GRAVITY FORCE

4.1 Differential equation of nonlocal vibrations

In the modified Timoshenko beam theory the basic variable is bending deflection w_b, so the total deflection reads $w = w_b + w_s$, where w_s is shear deflection (12). Nonlocal effect is included in the definition of beam bending moment and shear force of nanotube according to [18]

$$M - \mu\frac{\partial^2 M}{\partial x^2} = -D\frac{\partial^2 w_b}{\partial x^2}, \tag{30}$$

$$Q - \mu\frac{\partial^2 Q}{\partial x^2} = -D\frac{\partial^3 w_b}{\partial x^3} + J\frac{\partial^3 w_b}{\partial x \partial t^2}, \tag{31}$$

where $\mu = \left(e_0 a\right)^2$ is nonlocal parameter related to the second stress gradient, e_0 is a constant and a is an internal characteristic length, [18]. Dynamic equilibrium of bending moments and transverse forces acting on beam differential element reads, Fig. 3,

$$\frac{\partial M}{\partial x} - Q = -J\frac{\partial^3 w_b}{\partial x \partial t^2}, \tag{32}$$

$$\frac{\partial Q}{\partial x} = k_w w + m\frac{\partial^2 w}{\partial t^2} - q, \tag{33}$$

where k_w is stiffness of elastic support, $m = \rho A$ is mass per unit length, A is cross-section area, and q is distributed external excitation. By eliminating Q from (32) by (33) yields

$$\frac{\partial^2 M}{\partial x^2} = k_w w + m\frac{\partial^2 w}{\partial t^2} - J\frac{\partial^4 w_b}{\partial x^2 \partial t^2} - q, \tag{34}$$

while (33) can be written in the form

$$\frac{\partial^2 Q}{\partial x^2} = k_w\frac{\partial w}{\partial x} + m\frac{\partial^3 w}{\partial x \partial t^2} - \frac{\partial q}{\partial x}. \tag{35}$$

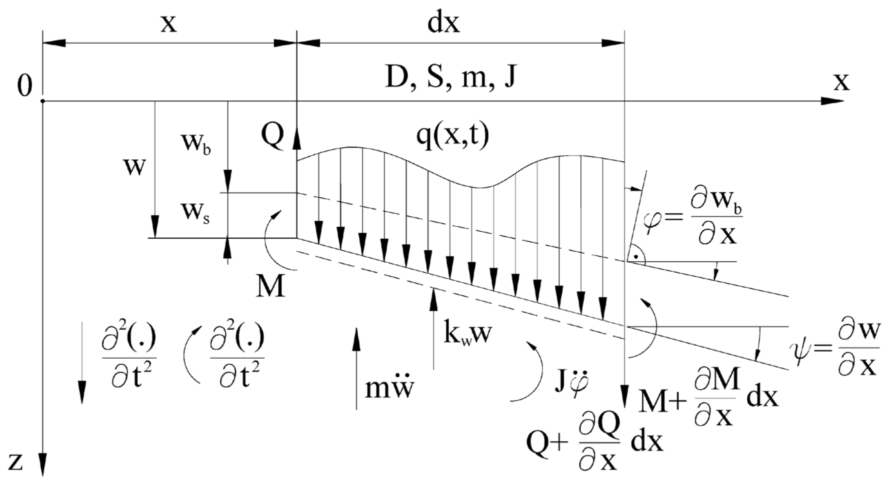

Fig. 3. Displacements and forces on differential element of a beam on elastic foundation

Sl. 3. Pomaci i sile na diferencijalnom elementu grede na elastičnoj podlozi

Substituting (34) and (35) into (30) and (31) respectively, one arrives at

$$M = -D\frac{\partial^2 w_b}{\partial x^2} + \mu\left(k_w w + m\frac{\partial^2 w}{\partial t^2} - J\frac{\partial^4 w_b}{\partial x^2 \partial t^2} - q \right), \tag{36}$$

$$Q = -D\frac{\partial^3 w_b}{\partial x^3} + J\frac{\partial^3 w_b}{\partial x \partial t^2} + \mu\left(k_w \frac{\partial w}{\partial x} + m\frac{\partial^3 w}{\partial x \partial t^2} - \frac{\partial q}{\partial x} \right). \tag{37}$$

Furthermore, inserting (37) into (33), equilibrium equation with nonlocal effect in terms of w and w_b is obtained

$$D\frac{\partial^4 w_b}{\partial x^4} - J\frac{\partial^4 w_b}{\partial x^2 \partial t^2} + m\frac{\partial^2 w}{\partial t^2} + k_w w - \mu\left(m\frac{\partial^4 w}{\partial x^2 \partial t^2} + k_w\frac{\partial^2 w}{\partial x^2} \right) = q - \mu\frac{\partial^2 q}{\partial x^2}. \tag{38}$$

Referring to (9) and (12), total deflection reads

$$w = w_b - \frac{D}{S}\frac{\partial^2 w_b}{\partial x^2} + \frac{J}{S}\frac{\partial^2 w_b}{\partial t^2}, \tag{39}$$

and substituting (39) into (38) partial differential equation of vibrations in terms of single variable w_b is obtained

$$\begin{aligned} &\mu\frac{m}{S}\frac{\partial^6 w_b}{\partial x^4 \partial t^2} - \mu\frac{mJ}{DS}\frac{\partial^6 w_b}{\partial x^2 \partial t^4} + \left(1 + \mu\frac{k_w}{S}\right)\frac{\partial^4 w_b}{\partial x^4} \\ &\quad - \frac{m}{S}\left[1 + \frac{SJ}{Dm} + \mu\frac{S}{D}\left(1 + \frac{k_w J}{Sm}\right)\right]\frac{\partial^4 w_b}{\partial x^2 \partial t^2} + \frac{mJ}{DS}\frac{\partial^4 w_b}{\partial t^4} \\ &\quad - \frac{k_w}{S}\left(1 + \mu\frac{S}{D}\right)\frac{\partial^2 w_b}{\partial x^2} + \frac{m}{D}\left(1 + \frac{k_w J}{Sm}\right)\frac{\partial^2 w_b}{\partial t^2} + \frac{k_w}{D}w_b = \frac{q}{D} - \frac{\mu}{D}\frac{\partial^2 q}{\partial x^2}. \end{aligned} \tag{40}$$

The equation (40) is of the sixth order, i.e. the fourth order per x and t. If $\mu = 0$ and $k_w = 0$ Eq. (40) is reduced to the modified Timoshenko equation, [28].

4.2 Natural vibrations

Natural vibrations are harmonic functions $w_b(x,t) = W_b(x)\sin\omega t$, where w_b is bending natural mode and ω is natural frequency. Now Eq. (40) is reduced to w_b, which depends on ω, and can be presented in the form

$$a\frac{\mathrm{d}^4 W_b}{\mathrm{d}x^4} + b\frac{\mathrm{d}^2 W_b}{\mathrm{d}x^2} + cW_b = 0, \tag{41}$$

where

$$\begin{aligned} a &= 1 + \mu\frac{k_w}{S} - \omega^2\mu\frac{m}{S}, \\ b &= -\frac{k_w}{S}\left(1 + \mu\frac{S}{D}\right) + \omega^2\frac{m}{S}\left[1 + \frac{SJ}{Dm} + \mu\frac{S}{D}\left(1 + \frac{k_w J}{Sm}\right)\right] - \omega^4\mu\frac{mJ}{DS}, \\ c &= \frac{k_w}{D} - \omega^2\frac{m}{D}\left(1 + \frac{k_w J}{Sm}\right) + \omega^4\frac{mJ}{DS}. \end{aligned} \tag{42}$$

Solution of (41) is assumed in exponential form $W_b = Ae^{\gamma x}$ that leads to biquadratic characteristic equation

$$a\gamma^4 + b\gamma^2 + c = 0. \tag{43}$$

with roots

$$\gamma_{1,2} = \pm\alpha,\ \gamma_{3,4} = \pm i\beta, \tag{44}$$

where

$$\alpha, \beta = \frac{1}{\sqrt{2a}}\sqrt{\sqrt{b^2 - 4ac} \mp b}. \tag{45}$$

Bending deflection is obtained in the form (23) and displacements and sectional forces, according to (18), (36) and (37) respectively, read

$$\begin{aligned} W &= \left(1 - \omega^2\frac{J}{S}\right)W_b - \frac{D}{S}\frac{\mathrm{d}^2 W_b}{\mathrm{d}x^2} \\ &= A_1 a_1 \sinh\alpha x + A_2 a_1 \cosh\alpha x + A_3 a_2 \sin\beta x + A_4 a_2 \cos\beta x, \\ \Phi &= \frac{\mathrm{d}W_b}{\mathrm{d}x} = A_1\alpha\cosh\alpha x + A_2\alpha\sinh\alpha x + A_3\beta\cos\beta x - A_4\beta\sin\beta x, \end{aligned}$$

$$\begin{aligned} M_x &= -b_1\frac{\mathrm{d}^2 W_b}{\mathrm{d}x^2} + b_2 W_b \\ &= A_1\left(-b_1\alpha^2 + b_2\right)\sinh\alpha x + A_2\left(-b_1\alpha^2 + b_2\right)\cosh\alpha x \\ &\quad + A_3\left(b_1\beta^2 + b_2\right)\sin\beta x + A_4\left(b_1\beta^2 + b_2\right)\cos\beta x, \end{aligned} \tag{46}$$

$$Q_x = -c_1 \frac{\mathrm{d}^3 W_b}{\mathrm{d}x^3} + c_2 \frac{\mathrm{d}W_b}{\mathrm{d}x}$$
$$= A_1\left(-c_1\alpha^2 + c_2\right)\alpha\cosh\alpha x + A_2\left(-c_1\alpha^2 + c_2\right)\alpha\sinh\alpha x$$
$$+ A_3\left(c_1\beta^2 + c_2\right)\beta\cos\beta x - A_4\left(c_1\beta^2 + c_2\right)\beta\sin\beta x,$$

where

$$a_1 = 1 - \omega^2\frac{J}{S} - \alpha^2\frac{D}{S},\ a_2 = 1 - \omega^2\frac{J}{S} + \beta^2\frac{D}{S},$$
$$b_1 = D + \mu\left[k_w\frac{D}{S} - \omega^2 J\left(1 + \frac{Dm}{SJ}\right)\right],\ b_2 = \mu\left(k_w - \omega^2 m\right)\left(1 - \omega^2\frac{J}{S}\right), \tag{47}$$
$$c_1 = D + \mu\left(k_w - \omega^2 m\right)\frac{D}{S},\ c_2 = -\omega^2 J + \mu\left(k_w - \omega^2 m\right)\left(1 - \omega^2\frac{J}{S}\right).$$

Integration constants A_k, $\mathrm{k} = 1,2,3,4$, are determined by satisfying boundary conditions. Simply supported beam is a special case since boundary conditions $W(0) = W(l) = 0$ and $M_x(0) = M_x(l) = 0$ are satisfied by trigonometric function $W_b = \sin\frac{i\pi x}{l}$. Substituting W_b into differential equation (41), and grouping terms of the same power of ω, the frequency equation is obtained

$$\tilde{a}\omega^4 - \tilde{b}\omega^2 + \tilde{c} = 0, \tag{48}$$

where

$$\tilde{a} = \frac{mJ}{DS}\left[1 + \mu\left(\frac{i\pi}{l}\right)^2\right],$$
$$\tilde{b} = \frac{m}{D}\left(1 + \frac{k_w J}{Sm}\right) + \frac{m}{S}\left[1 + \frac{SJ}{Dm} + \mu\frac{S}{D}\left(1 + \frac{k_w J}{Sm}\right)\right]\left(\frac{i\pi}{l}\right)^2 + \mu\frac{m}{S}\left(\frac{i\pi}{l}\right)^4, \tag{49}$$
$$\tilde{c} = \frac{k_w}{D} + \frac{k_w}{S}\left(1 + \mu\frac{S}{D}\right)\left(\frac{i\pi}{l}\right)^2 + \left(1 + \mu\frac{k_w}{S}\right)\left(\frac{i\pi}{l}\right)^4.$$

Solutions of Eq. (48) read

$$\omega_{1,2} = \frac{1}{\sqrt{2\tilde{a}}}\sqrt{\tilde{b} \pm \sqrt{\tilde{b}^2 - 4\tilde{a}\tilde{c}}}. \tag{50}$$

It is obvious that two frequency spectra are obtained as a characteristic of simply supported beam. They are shifted for

$$\omega_1^2-\omega_2^2=\frac{1}{\tilde{a}}\sqrt{\tilde{b}^2-4\tilde{a}\tilde{c}}. \tag{51}$$

4.3 Forced vibrations due to moving gravity force

Vibrations of a simply supported single-walled carbon nanotube (SWCNT), embedded in elastic medium and excited by weight of moving nanoparticle, are analyzed, Fig. 4. Modal superposition method and separation of variables approach are used for solving governing partial differential equation (40). Accordingly, bending deflection is assumed in the form

$$w_b\left(x,t\right)=\sum_{j=1}^{\infty}W_{bj}\left(x\right)T_j\left(t\right), \tag{52}$$

where $W_{bj}\left(x\right)$ is natural mode and $T_j\left(t\right)$ is unknown time dependent function. Eq. (52) is substituted into (40), and applying the Galerkin method, it is further multiplied with mode function $W_{bi}\left(x\right)$ and integrated over the beam length. As a result, the following modal equation in terms of generalized displacements is obtained

$$\begin{aligned}&\sum_{j=1}^{\infty}\frac{mJ}{DS}\left(I_{ij}^{(0)}-\mu I_{ij}^{(2)}\right)\ddddot{T}_j\\&+\sum_{j=1}^{\infty}\left\{\mu\frac{m}{S}I_{ij}^{(4)}-\frac{m}{S}\left[1+\frac{SJ}{Dm}+\mu\frac{S}{D}\left(1+\frac{k_wJ}{Sm}\right)\right]I_{ij}^{(2)}+\frac{m}{D}\left(1+\frac{k_wJ}{Sm}\right)I_{ij}^{(0)}\right\}\ddot{T}_j\\&+\sum_{j=1}^{\infty}\left[\left(1+\mu\frac{k_w}{S}\right)I_{ij}^{(4)}-\frac{k_w}{S}\left(1+\mu\frac{S}{D}\right)I_{ij}^{(2)}+\frac{k_w}{D}I_{ij}^{(0)}\right]T_j=\frac{1}{D}\int_0^l\left(q-\mu\frac{\partial^2q}{\partial x^2}\right)W_{bi}\mathrm{d}x,\end{aligned} \tag{53}$$

where $i=1,2,\dots$ and

$$I_{ij}^{(0)}=\int_0^l W_{bi}W_{bj}\mathrm{d}x,\ I_{ij}^{(2)}=\int_0^l W_{bi}W_{bj}''\mathrm{d}x,\ I_{ij}^{(4)}=\int_0^l W_{bi}W_{bj}''''\mathrm{d}x. \tag{54}$$

The above system of ordinary linear differential equations of the fourth order can be written in matrix notation

$$\left[N_{ij}\right]\left\{\ddddot{T}_j\right\}+\left[M_{ij}\right]\left\{\ddot{T}_j\right\}+\left[K_{ij}\right]\left\{T_j\right\}=\left\{F_i\right\}, \tag{55}$$

where

$$F_i = \frac{1}{D}\int_0^l \left(q - \mu \frac{\partial^2 q}{\partial x^2} \right) W_{bi} \mathrm{d}x \tag{56}$$

is the generalized excitation force. In the considered case one is faced with constant lumped nanoparticle gravity force, P, Fig. 4. Using partial integration

$\int_0^l \frac{\partial^2 q}{\partial x^2} \mathrm{W}_{bi} \mathrm{d}x = \int_0^l q \frac{\partial^2 \mathrm{W}_{bi}}{\partial x^2} \mathrm{d}x$ and $q = P/\mathrm{d}x$ one obtains

$$F_i = \frac{P}{D}\left[W_{bi}\left(x_P\right) - \mu W_{bi}''\left(x_P\right) \right], \tag{57}$$

where the force position is linearly changed due to constant velocity, $x_P = v_P t$. Hence, disturbing function F_i becomes time dependent. The same expression (57) is obtained in [18] by employing the Dirac δ function for transformation of distributed load, q, to concentrated one, P.

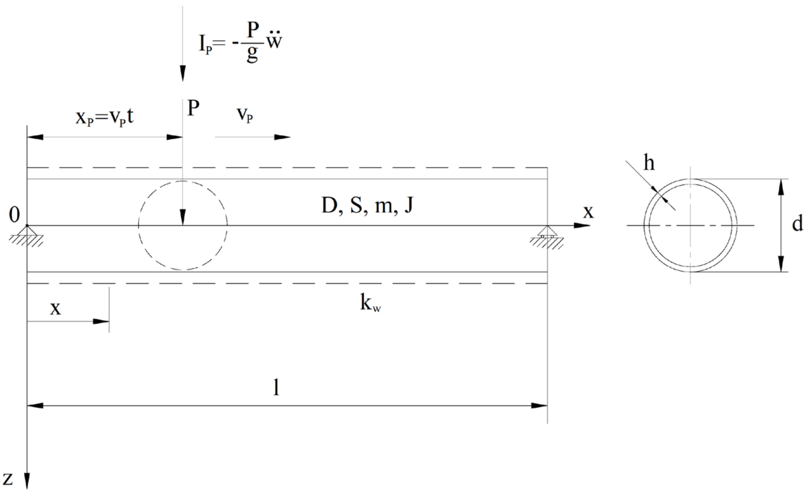

Fig. 4. Carbon nanotube embedded in elastic medium exposed to gravity and inertia force of moving nanoparticle

Sl. 4. Ugljična nanocijev uronjena u elastičnom mediju i izvrgnuta djelovanju gravitacijske i inercijske sile gibajuće nanočestice

Diagonal terms in (55) are dominant due to almost orthogonal modal functions as well as their derivatives in integrals (54). As a result, coupling of equations (55) is weak, and therefore an iteration procedure can be used for their solution. A typical equation can be written in the form

$$N_{ii}\left(\ddddot{T}_i\right)^{(k+1)}+M_{ii}\left(\ddot{T}_i\right)^{(k+1)}+K_{ii}T_i^{(k+1)}$$
$$=F_i-\sum_{j=1}^{\infty}\left(1-\delta_{ij}\right)\left(N_{ij}\left(\ddddot{T}_j\right)^k+M_{ij}\left(\ddot{T}_j\right)^k+K_{ij}T_j^k\right), \tag{58}$$

where δ_{ij} is the Kronecker delta and k is iteration step.

The above formulation is general for a beam with arbitrary boundary conditions, where mode function W_{bi} is expressed in terms of trigonometric and hyperbolic functions. A simply supported beam is a special case since natural modes are single trigonometric functions, i.e. $W_{bi}=\sin\dfrac{i\pi x}{l}$, and moreover they are orthogonal as well as any combination of their even derivatives. As a result, all integrals (54) are zero for $j\neq i$, and for $j=i$ they take the following values

$$I_{ii}^{(0)}=\frac{l}{2},\ I_{ii}^{(2)}=-\left(\frac{i\pi}{l}\right)^2\frac{l}{2},\ I_{ii}^{(4)}=\left(\frac{i\pi}{l}\right)^4\frac{l}{2}. \tag{59}$$

Coupling terms on the left hand side in (57) disappear and the remaining diagonal elements in (53) read

$$N_{ii}=\frac{l}{2}a_i,\ M_{ii}=\frac{l}{2}b_i,\ K_{ii}=\frac{l}{2}c_i, \tag{60}$$

where coefficients a_i, b_i and c_i are identical to $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$ determined within natural vibration analysis, Eqs. (49), respectively. In this case, modal differential equation (58) takes the following form

$$N_{ii}\ddddot{T}_i+M_{ii}\ddot{T}_i+K_{ii}T_i=F_i^0\sin\Omega_i t, \tag{61}$$

where

$$F_i^0=\frac{P}{D}\left[1+\mu\left(\frac{i\pi}{l}\right)^2\right],\ \Omega_i=\frac{i\pi v_P}{l}, \tag{62}$$

is modal excitation amplitude and forcing frequency, respectively. Assuming homogenous solution of Eq. (61) in harmonic form, $T_i^h=\sin\omega_i t$, frequency equation (48) is obtained as in the case of differential equation of motion (41). Particular integral of (61) is assumed in the same form as disturbing function

$$T_i=C_i\sin\Omega_i t, \tag{63}$$

and substituting it into (61) yields

$$C_i = \frac{F_i^0}{\Omega_i^4 N_{ii} - \Omega_i^2 M_{ii} + K_{ii}}. \tag{64}$$

Denominator in (64), according to (48), is identical to the modal frequency equation if ω_i is written instead of Ω_i. Hence, one can write

$$K_{ii} = -\omega_i^4 N_{ii} + \omega_i^2 M_{ii}, \tag{65}$$

and substituting (65) into (64) yields

$$C_i = \frac{F_i^0}{\left(\Omega_i^2 - \omega_i^2\right)\left[\left(\Omega_i^2 + \omega_i^2\right) N_{ii} - M_{ii}\right]}. \tag{66}$$

Two singular values of excitation frequency are possible, i.e. $\Omega_i = \omega_i$ and $\Omega_i = \sqrt{M_{ii}/N_{ii} - \omega_i^2}$. The former is due to translatory and the latter due to rotary inertia. Since the natural frequency spectra, Eq. (50), are quite dense, it is obvious that a particular mode can easily fall into resonance, depending on nanoparticle velocity, v_p .

In order to reduce infinite resonant response to a finite value it is necessary to include damping into differential equation of vibrations. Nanotube is supported by elastic medium and can slide along it if axial force overcomes friction force, which is equal to the product of normal force and friction coefficient. Friction force is independent on the velocity, but causes reduction of vibration amplitude due to dissipation of the kinetic energy, [34]. Therefore, damping can be modeled as a viscous one, with intensity based on equivalence of dissipated energy of friction force and the assumed viscous force, [35].

Prescribing linear viscous damping force Eq. (61) reads

$$N_{ii}\ddddot{T}_i + M_{ii}\ddot{T}_i + V_{ii}\dot{T}_i + K_{ii}T_i = F_i^0 \sin\Omega_i t, \tag{67}$$

where V_{ii} is damping coefficient. Solution of (67) is assumed in the form

$$T_i = A_i \cos\Omega_i t + B_i \sin\Omega_i t. \tag{68}$$

Substituting (68) into (67) and equalizing coefficients of sine and cosine functions, system of two algebraic equations is obtained,

$$\begin{bmatrix} N_{ii}\Omega_i^4 - M_{ii}\Omega_i^2 + K_{ii} & V_{ii}\Omega_i \\ -V_{ii}\Omega_i & N_{ii}\Omega_i^4 - M_{ii}\Omega_i^2 + K_{ii} \end{bmatrix} \begin{Bmatrix} A_i \\ B_i \end{Bmatrix} = \begin{Bmatrix} 0 \\ F_i^0 \end{Bmatrix}. \tag{69}$$

Its solution, determined by the Cramer rule, reads

$$A_i = \frac{D_A^i}{D_0^i},\ B_i = \frac{D_B^i}{D_0^i}, \tag{70}$$

where

$$\begin{aligned} D_A^i &= -V_{ii}\Omega_i F_i^0, \\ D_B^i &= \left(N_{ii}\Omega_i^4 - M_{ii}\Omega_i^2 + K_{ii}\right) F_i^0, \\ D_0^i &= \left(N_{ii}\Omega_i^4 - M_{ii}\Omega_i^2 + K_{ii}\right)^2 + V_{ii}^2\Omega_i^2, \end{aligned} \tag{71}$$

are determinants of the system. Taking $C_i = \sqrt{A_i^2 + B_i^2}$, $A_i = C_i \sin\varepsilon_i$ and $B_i = C_i \cos\varepsilon_i$, Eq. (68) is transformed into the form

$$T_i = C_i \sin\left(\Omega_i t + \varepsilon_i\right), \tag{72}$$

where

$$C_i = \frac{F_i^0}{\sqrt{D_0^i}},\ \varepsilon_i = \arctan\frac{D_B^i}{D_A^i}, \tag{73}$$

is mode amplitude and phase angle, respectively. The first quantity can be presented in the well-known form for a single degree of freedom system

$$\begin{aligned} C_i &= C_{st}^i \mu_F^i,\ C_{st}^i = \frac{F_i^0}{K_{ii}}, \\ \mu_F^i &= \frac{1}{\sqrt{\left(1 - \frac{M_{ii}}{K_{ii}}\Omega_i^2 + \frac{N_{ii}}{K_{ii}}\Omega_i^4\right)^2 + \left(\frac{V_{ii}}{K_{ii}}\right)^2 \Omega_i^2}}, \end{aligned} \tag{74}$$

where C_{st}^i is modal static coefficient and μ_F^i is dynamic amplification factor. In resonances $C_i = F_i^0 / \omega_i V_{ii}$ and $\varepsilon_i = \pi/2$.

The above particular solution of Eq. (67) is steady state response and does not satisfy initial condition $T_i(0) = 0$ and $\dot{T}_i(0) = 0$. That causes transient free vibrations, which are represented with homogenous solution of (67). That solution can not be obtained analytically in closed form and therefore an approximate solution is used

$$T_i^h = A_i^h \,\mathrm{e}^{-\gamma_i \omega_i t} \cos \omega_i t + B_i^h \,\mathrm{e}^{-\gamma_i \omega_i t} \sin \omega_i t, \tag{75}$$

where ω_i is natural frequency of undamped dynamic system derived in Section 4.3, and $\gamma_i = V_{ii} / \left(2\omega_i M_{ii}\right)$ is nondimensional damping coefficient.

Particular solution of (67) is presented with (68), and total time function is $T_i^* = T_i^h + T_i$. Constants A_i^h and B_i^h are determined by satisfying initial conditions $T_i^*\left(0\right) = 0$ and $\dot{T}_i^*\left(0\right) = 0$,

$$A_i^h = -A_i, \; B_i^h = -\frac{\Omega_i}{\omega_i} B_i - \gamma_i A_i. \tag{76}$$

Hence, the complete time function yields

$$\begin{aligned} T_i^* &= A_i \left[\cos \Omega_i t - e^{-\gamma_i \omega_i t} \left(\cos \omega_i t + \gamma_i \sin \omega_i t \right) \right] \\ &\quad + B_i \left(\sin \Omega_i t - \frac{\Omega_i}{\omega_i} e^{-\gamma_i \omega_i t} \sin \omega_i t \right). \end{aligned} \tag{77}$$

In resonance $\Omega_i = \omega_i$ and according to Eqs. (70) and (71) $A_i = -F_i^0 / \left(2\gamma_i \omega_i^2\right)$ and $B_i = 0$, and Eq. (77) is reduced to

$$T_{i\mathrm{res}}^* = \frac{F_i^0}{2\omega_i^2 M_{ii}} \left[\frac{1}{\gamma_i} \left(e^{-\gamma_i \omega_i t} - 1 \right) \cos \omega_i t + e^{-\gamma_i \omega_i t} \sin \omega_i t \right]. \tag{78}$$

In case of conservative dynamic system, $\gamma_i = 0$, the second term in (78) is determined, while the first one takes undetermined form, i.e. $T_{i\mathrm{res}}^{*(1)} = \mathrm{Nom}/\mathrm{Denom} = 0/0$. That is only apparently, and the problem can be overcome if the exponential function is expanded into the power series $e^{-\gamma_i \omega_i t} = 1 - \gamma_i \omega_i t + \frac{1}{2}\left(\gamma_i \omega_i t\right)^2 \mp \dots$. Hence, one finds

$$T_{i\mathrm{res}}^* = \frac{F_i^0}{2\omega_i^2 M_{ii}} \left[e^{-\gamma_i \omega_i t} \sin \omega_i t - \omega_i t \left(1 - \frac{1}{2} \gamma_i \omega_i t \right) \cos \omega_i t \right]. \tag{79}$$

If $\gamma_i = 0$, Eq. (79) takes a simpler form

$$T_{i\mathrm{res}}^* = \frac{F_i^0}{2\omega_i^2 M_{ii}} \left(\sin \omega_i t - \omega_i t \cos \omega_i t \right). \tag{80}$$

While damping plays decisive role in resonant steady state response, which attains infinite value if damping is not present, Eq. (74), the resonant response starting from the rest is a smooth function. That is a reason why numerical time integration of differential equation of motion without damping force can be performed in resonance without difficulties.

It is obvious from (80) that envelope of amplitude is increased linearly by time to infinity. Since passing time of nanoparticle through nanotube is known, $t_P = l/v_P$, and $\Omega_i = i\pi v_P/l = \omega_i$, yields $\omega_i t_P = i\pi$. It means that number of time function half waves is equal to the mode number i. The time function achieves its maximum value at t_P, i.e. $T_{i\mathrm{res}}^{*(\max)} = i\pi F_i^0 / \left(2\omega_i^2 M_{ii}\right)$.

4.4 Numerical example

4.4.1 Basic data

Application of the presented theory is illustrated in the case of simply supported embedded SWCNT exposed to the influence of moving nanoparticle gravity load. Values of the basic parameters are chosen the same as in [18] in order to enable comparison of some results: Young's modulus $E = 1\ \mathrm{TPa} = 10^3 \mathrm{nN}/\left(\mathrm{nm}\right)^2$, mass density $\rho = 2300\ \mathrm{kg/m^3} = 2.3\cdot 10^{-15}\ \mathrm{nkg}/\left(\mathrm{nm}\right)^3$, outer diameter $d = 1\ \mathrm{nm}$, wall thickness $h = 0.34\ \mathrm{nm}$ and Poisson's ratio $\nu = 0.2$. The derived data are the following: cross-section area $A = d^2\pi/4 = 0.70497\ \left(\mathrm{nm}\right)^2$, moment of inertia of cross-section $I = \left[d^4 - \left(d-2h\right)^4\right]\pi/64 = 0.048573\ \left(\mathrm{nm}\right)^4$. The shear correction factor for hollow circle is determined according to [31], $k_s = 0.71376$. Values of nanotube length, l, stiffness of elastic medium, k_w, and nonlocal parameter, μ, are varied.

4.4.2 Natural vibrations

Natural frequencies for simply supported SWCNT are calculated according to Eq. (50), for different values of slenderness ratio l/d and nonlocal parameter μ. They are normalized by the first natural frequency of the simply supported Euler-Bernoulli beam, $\omega_{EB}^1 = \left(\pi/l\right)^2 \sqrt{EI/\rho A}$, so that frequency parameter reads $\lambda_i = \pi^2 \omega_i/\omega_{EB}^1 = \omega_i l^2 \sqrt{\rho A/EI}$. The obtained results for the first natural mode are listed in Table 1, [18] and compared with those from Refs. [18] and [21]. Values from Ref. [21] agree excellently with analytical ones, while those from Ref. [18] are slightly different.

Influence of elastic medium stiffness, k_w, on nanotube natural frequencies can be seen in Table 2. Values of frequency parameters are somewhat increased for higher stiffness as expected.

Table 1 - The first frequency parameter $\lambda_1 = \omega_1 l^2 \sqrt{\rho A/EI}$ for simply supported nanotube, $k_w = 0$, $k_s = 5/6$

l/d	μ (nm)	PS	Ref. [21]	Ref. [18]
10	0	9.7425	9.7443	9.7074
	1	9.2974	9.2931	9.2612
	2	8.9081	8.8994	8.8713
	3	8.5640	8.5517	8.5268
	4	8.2569	8.2419	8.2196
20	0	9.8373	9.8381	9.8281
	1	9.7183	9.7187	9.7090
	2	9.6035	9.6036	9.5942
	3	9.4928	9.4924	9.4834
	4	9.3857	9.3850	9.3763
50	0	9.8644	9.8645	9.8629
	1	9.8450	9.8451	9.8435
	2	9.8257	9.8258	9.8242
	3	9.8065	9.8066	9.8050
	4	9.7875	9.7875	9.7859

Table 2 - Frequency parameter $\lambda_i = \omega_i l^2 \sqrt{\rho A/EI}$ for simply supported nanotube, $l/d = 10$, $k_s = 0.71376$, $\mu = 0$

Mode no.	$k_w = 0$	$k_w = 10^{-5} E$	$k_w = 10^{-4} E$
1	9.7271	9.8317	10.7271
2	37.3631	37.3900	37.6315
3	79.2282	79.2406	79.3527
4	131.2700	131.2770	131.3430
5	190.1420	190.1470	190.1920
6	253.4390	253.4430	253.4760
7	319.5350	319.5380	319.5640
8	387.3590	387.3610	387.3830
9	456.2120	456.2140	456.2330
10	525.6390	525.6410	525.6580

4.4.3 Forced vibrations

Forced vibrations of SWCNT are performed by employing procedure presented in Section 4.3. Some vibration parameters are normalized. Modal damping coefficient is specified as $V_{ii} = 2\gamma_i \omega_i M_{ii}$, where γ_i is nondimensional parameter. Velocity parameter is defined as ratio of nanoparticle velocity and velocity which corresponds to the first natural mode of nanotube, i.e. $\alpha = v_P / v_1 = \Omega_1 / \omega_1 = \pi v_P / (\omega_1 l)$. Nondimensional time is $\tau = x_P / l = v_P t / l$. Vibration deflection is normalized by the static value due to lumped force P acting at the midsection, i.e. $\delta(x,t) = w(x,t) / w_P$, where $w_P = Pl^3 / 48EI$.

All numerical calculations for forced vibrations are performed by employing 30 natural modes, that was necessary to stabilize the first 6 digits of response. Fig. 5 shows time history of maximum relative deflection of nanotube during nanoparticle motion from the rest, $t = 0$, up to passing time, t_P. Relief presentation points out moving of maximum deflection simultaneously with nanoparticle. Maximum relative deflection as function of velocity parameter, determined by step $\Delta\alpha = 0.01$, is shown in Fig. 6. Response is rapidly increased for higher values of $\sqrt{\mu}$.

Influence of damping response is analyzed in case of resonance, $\alpha = 1$. Time history of relative midsection deflection within time period of double value of passing time of nanoparticle is shown in Fig. 7. Response is dominated by the first natural mode, especially for higher damping value. Reason is that damping force is proportional to natural frequency, $V_{ii} = 2\gamma_i \omega_i$. Contribution of the other natural modes is more pronounced when there is no damping.

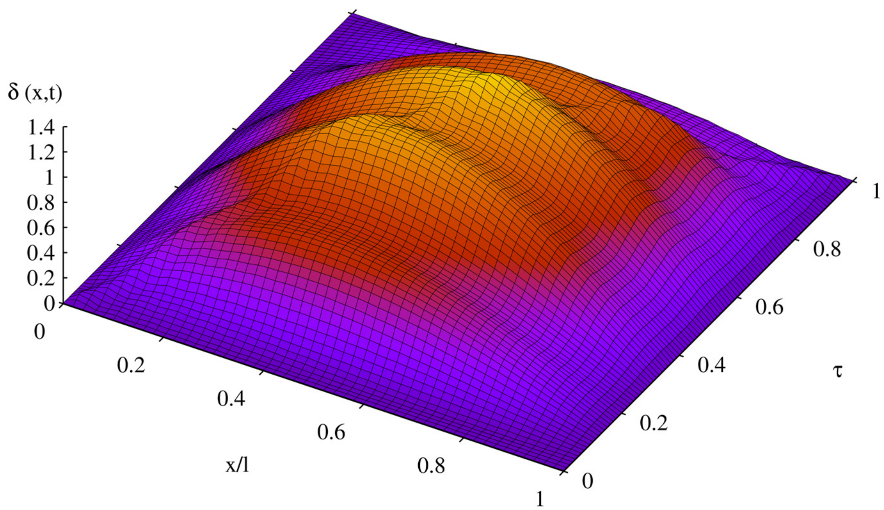

Fig. 5. Time history of relative deflection of nanotube, $l/d = 10$, $k_w = 0$, $P/g = 0$, $\sqrt{\mu}$ = 1 nm, $\alpha = 0.1$, $\gamma_i = 0.0$

Sl. 5. Vremenska promjena relativnog gibanja nanocijevi, $l/d = 10$, $k_w = 0$, $P/g = 0$, $\sqrt{\mu}$ = 1 nm, $\alpha = 0.1$, $\gamma_i = 0.0$

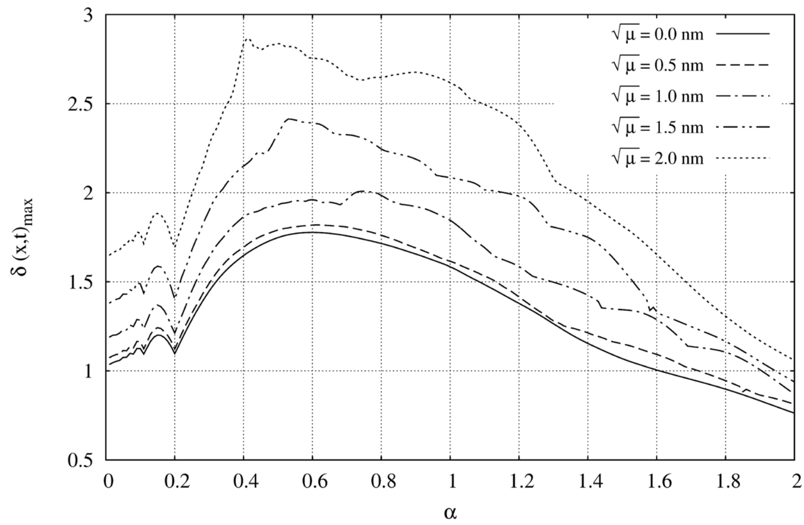

Fig. 6. Maximum relative deflection versus velocity parameter, $l/d = 10$, $k_w = 0$, $P/g = 0$, $\gamma_i = 0.0$

Sl. 6. Maksimalni relativni progib u ovisnosti o parametru brzine, $l/d = 10$, $k_w = 0$, $P/g = 0$, $\gamma_i = 0.0$

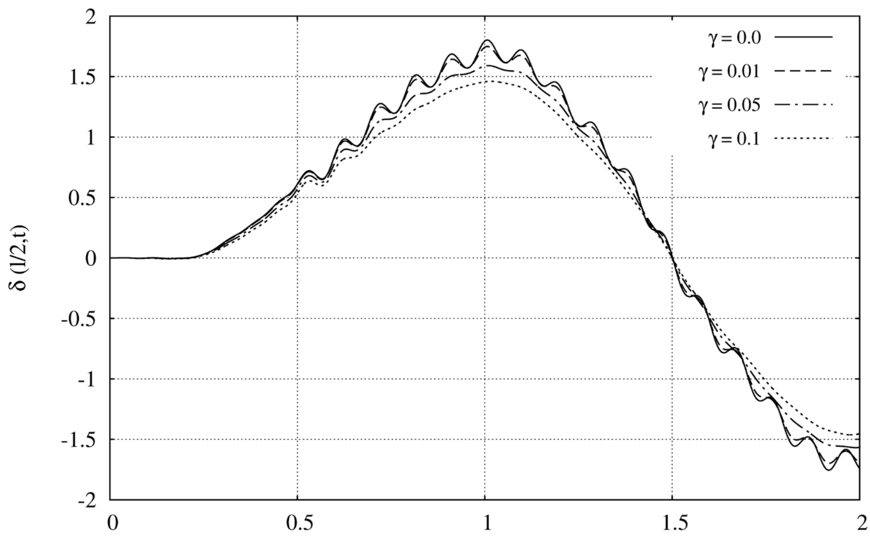

Fig. 7. Time history of relative midsection deflection in resonance, $l/d = 10$, $\alpha = 1$, $k_w = 10^{-5}E$, $\sqrt{\mu}$ = 1 nm, $P/g = 0$

Sl. 7. Vremenska promjena relativnog gibanja po sredini nanocijevi u rezonanciji

5. COUPLED HORIZONTAL AND TORSIONAL VIBRATIONS OF ULTRA LARGE SHIPS

5.1 General

Increased sea transport requires building of ultra large container ships which are quite flexible. Therefore, their strength has to be checked by hydroelastic analysis [36]. The methodology of hydroelastic analysis is described in [37]. It includes the definition of the structural model, ship and cargo mass distributions, and geometrical model of ship wetted surface. Hydroelastic analysis is based on the modal superposition method. First, dry natural vibrations of ship hull are calculated. Then, modal hydrostatic stiffness, modal added mass, modal damping and modal wave load are determined. Finally, the calculation of wet natural vibrations is performed and transfer functions for determining ship structural response to wave excitation are obtained [38].

The intention of this Section is to present an advanced numerical procedure based on the beam and thin-walled girder theories for reliable calculation of dry natural vibrations of container ships, as an important step in their hydroelastic analysis. A ship hull, as an elastic nonprismatic thin-walled girder, performs longitudinal, vertical, horizontal and torsional vibrations. Since the cross-sectional centre of gravity and centroid, as well as the shear centre positions are not identical, coupled longitudinal and vertical, and horizontal and torsional vibrations occur, respectively.

The distance between the centre of gravity and centroid for longitudinal and vertical vibrations, as well as distance between the former and shear centre for horizontal and torsional vibrations are negligible for conventional ships. Therefore, in the above cases ship hull vibrations are usually analysed separately. However, the shear centre in ships with large hatch openings is located outside the cross-section, i.e. below the keel, and therefore the coupling of horizontal and torsional vibrations is extremely high.

The above problem is rather complicated due to geometrical discontinuity of the hull cross-section. The accuracy of the solution depends on the reliability of stiffness parameters determination, i.e. of bending, shear, torsional and warping moduli. The finite element method is a powerful tool to solve the above problem in a successful way. One of the first solutions for coupled horizontal and torsional hull vibrations, dealing with the finite element technique, is given in [39,40]. Generalised and improved solutions are presented in [41,42]. An advanced theory of thin-walled girder with application to ship vibrations is worked out in [30].

5.2 Differential equations of beam vibrations

Referring to the flexural beam theory [31], the total beam deflection, w, consists of the bending deflection, w_b, and the shear deflection, w_s, while the angle of cross-section rotation depends only on the former, Fig. 8

$$w = w_b + w_s, \quad \varphi = \frac{\partial w_b}{\partial x}. \tag{81}$$

The cross-sectional forces are the bending moment and the shear force

$$M = -EI_b \frac{\partial \varphi}{\partial x}, \; Q = GA_s \frac{\partial w_s}{\partial x}, \tag{82}$$

where E and G are the Young's and shear modulus, respectively, while I_b and A_s are the moment of inertia of cross-section and shear area, respectively.

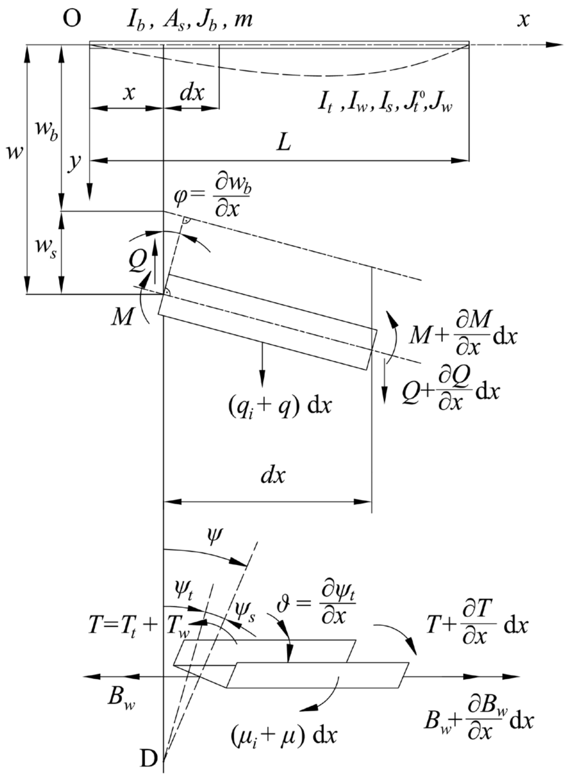

Fig. 8. Beam bending and torsion

Sl. 8. Savijanje i uvijanje grede

The inertia load consists of the distributed transverse load, q_i, and the bending moment, μ_i, and in the case of coupled horizontal and torsional vibration is specified as

$$q_i = -m\left(\frac{\partial^2 w}{\partial t^2} + c\frac{\partial^2 \psi}{\partial t^2}\right),\ \mu_i = -J_b \frac{\partial^2 \varphi}{\partial t^2}, \tag{83}$$

where m is the distributed ship and added mass, J_b is the moment of inertia of ship mass about z-axis, and c is the distance between the centre of gravity and the shear centre, $c = z_G - z_S$, Fig 9.

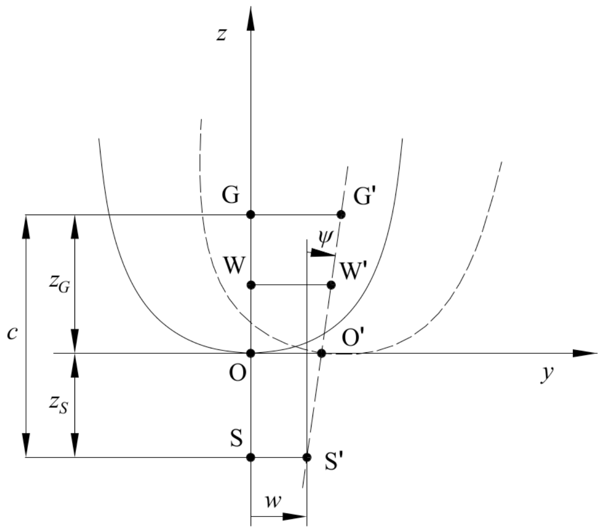

Fig. 9. Cross-section of a thin-walled girder

Sl. 9. Poprečni presjek tankostijenog nosača

Concerning torsion, the total twist angle, ψ (which should be distinguished from the cross-section rotation, ψ, within the Timoshenko beam theory, Sections 2 and 3), consists of the pure twist angle, ψ_t, and the shear contribution, ψ_s, while the second beam displacement, which causes warping (deplanation) of cross-section, is variation of the pure twist angle, i.e. Fig. 8 [30]

$$\psi = \psi_t + \psi_s,\ \vartheta = \frac{\partial \psi_t}{\partial x}. \tag{84}$$

The cross-sectional forces include the pure torsional torque, T_t, warping bimoment, B_w, and additional torque due to restrained warping, T_w, i.e.

$$T_t = GI_t\vartheta,\ B_w = -EI_w\frac{\partial\vartheta}{\partial x},\ T_w = GI_s\frac{\partial\psi_s}{\partial x},, \tag{85}$$

where I_t, I_w and I_s are the torsional modulus, warping modulus and shear inertia modulus, respectively.

The inertia load consists of the distributed torque, μ_{ti}, and the bimoment, b_i, presented in the following form:

$$\mu_{ti} = -J_t\frac{\partial^2\psi}{\partial t^2} - mc\frac{\partial^2 w}{\partial t^2},\ b_i = -J_w\frac{\partial^2\vartheta}{\partial t^2}, \tag{86}$$

where J_t is the polar moment of inertia of ship and added mass about the shear centre, and J_w is the bimoment of inertia of ship mass about the warping centre, Fig. 9.

Considering the equilibrium of a differential element, one can write for flexural vibrations

$$\frac{\partial M}{\partial x} = Q + \mu_i,\ \frac{\partial Q}{\partial x} = -q_i - q\,, \tag{87}$$

and for torsional vibrations [29]

$$\frac{\partial B_w}{\partial x} = T_w + b_i,\ \frac{\partial T_t}{\partial x} + \frac{\partial T_w}{\partial x} = -\mu_{ti} - \mu. \tag{88}$$

The above equations can be reduced to two coupled partial differential equations as follows. Substituting Eqs. (82) into the first of Eqs. (87) yields

$$\frac{\partial w_s}{\partial x} = -\frac{EI_b}{GA_s}\frac{\partial^2\varphi}{\partial x^2} + \frac{J_b}{GA_s}\frac{\partial^2\varphi}{\partial t^2}\,. \tag{89}$$

By inserting the second of Eqs. (82) and the first of (83) into the second of (87) leads

$$EI_b\frac{\partial^4\varphi}{\partial x^4} + m\frac{\partial^2\varphi}{\partial t^2} - \left(J_b + m\frac{EI_b}{GA_s}\right)\frac{\partial^4\varphi}{\partial x^2\partial t^2} + \frac{mJ_b}{GA_s}\frac{\partial^4\varphi}{\partial t^4} + mc\frac{\partial^3\psi}{\partial x\partial t^2} = \frac{\partial q}{\partial x}\,. \tag{90}$$

In similar way, substituting the second and third of Eqs. (85) into the first of Eqs. (88) yields

$$\frac{\partial\psi_s}{\partial x} = -\frac{EI_w}{GI_s}\frac{\partial^2\vartheta}{\partial x^2} + \frac{J_w}{GI_s}\frac{\partial^2\vartheta}{\partial t^2}\,. \tag{91}$$

By inserting Eqs. (85) into the second of Eqs. (88) one finds

$$EI_w\frac{\partial^4\vartheta}{\partial x^4}-GI_t\frac{\partial^2\vartheta}{\partial x^2}+J_t\frac{\partial^2\vartheta}{\partial t^2}-\left(J_w+J_t\frac{EI_w}{GI_s}\right)\frac{\partial^4\vartheta}{\partial x^2\partial t^2}+\frac{J_w}{GI_s}\frac{\partial^4\vartheta}{\partial t^4}+mc\frac{\partial^3 w}{\partial x\partial t^2}=\frac{\partial\mu}{\partial x} \quad (92)$$

Furthermore, ψ in (90) can be split into $\psi_t+\psi_s$ and the later term can be expressed with (91). Similar substitution can be done for $w=w_b+w_s$ in (92), where w_s is given with (89). Thus, taking into account that $\varphi=\partial w_b/\partial x$ and $\vartheta=\partial\psi_t/\partial x$, Eqs (90) and (92) after integration per x read

$$\begin{aligned}&EI_b\frac{\partial^4 w_b}{\partial x^4}+m\frac{\partial^2 w_b}{\partial t^2}-\left(J_b+m\frac{EI_b}{GA_s}\right)\frac{\partial^4 w_b}{\partial x^2\partial t^2}+\frac{mJ_b}{GA_s}\frac{\partial^4 w_b}{\partial t^4}\\&+mc\left(\frac{\partial^2\psi_t}{\partial t^2}-\frac{EI_w}{GI_s}\frac{\partial^4\psi_t}{\partial x^2\partial t^2}+\frac{J_w}{GI_s}\frac{\partial^4\psi_t}{\partial t^4}\right)=q\end{aligned} \quad (93)$$

$$\begin{aligned}&EI_w\frac{\partial^4\psi_t}{\partial x^4}-GI_t\frac{\partial^2\psi_t}{\partial x^2}+J_t\frac{\partial^2\psi_t}{\partial t^2}-\left(J_w+J_t\frac{EI_w}{GI_s}\right)\frac{\partial^4\psi_t}{\partial x^2\partial t^2}+\frac{J_w}{GI_s}\frac{\partial^4\psi_t}{\partial t^4}\\&+mc\left(\frac{\partial^2 w_b}{\partial t^2}-\frac{EI_b}{GA_s}\frac{\partial^4 w_b}{\partial x^2\partial t^2}+\frac{J_b}{GA_s}\frac{\partial^4 w_b}{\partial t^4}\right)=\mu.\end{aligned} \quad (94)$$

After solving Eqs. (93) and (94) the total deflection and twist angle are obtained by employing (89) and (91)

$$w=w_b+w_s=w_b-\frac{EI_b}{GA_s}\frac{\partial^2 w_b}{\partial x^2}+\frac{J_b}{GA_s}\frac{\partial^2 w_b}{\partial t^2}+f(t) \quad (95)$$

$$\psi=\psi_t+\psi_s=\psi_t-\frac{EI_w}{GI_s}\frac{\partial^2\psi_t}{\partial x^2}+\frac{J_w}{GI_s}\frac{\partial^2\psi_t}{\partial t^2}+g(t), \quad (96)$$

where f(t) and g(t) are integration functions, which depend on initial conditions.

The main purpose of developing differential equations of vibrations (93) and (94) is to get insight into their constitution, position and role of the stiffness and mass parameters, and coupling, which is realized through the inertia terms. If the pure torque T_t is excluded from the above theoretical consideration, it is obvious that the complete analogy between bending and torsion exists, [43].

Application of Eqs. (93) and (94) is limited to prismatic girders. For more complex problems, like ship hull, the finite element method, as a powerful tool, is on disposal.

The shape functions of beam finite element for vibration analysis have to satisfy the following consistency relations for harmonic vibrations obtained from Eqs. (95) and (96), [44]

$$w = w_b + w_s = \left(1 - \omega^2 \frac{J_b}{GA_s}\right) w_b - \frac{EI_b}{GA_s} \frac{\mathrm{d}^2 w_b}{\mathrm{d}x^2} \tag{97}$$

$$\psi = \psi_t + \psi_s = \left(1 - \omega^2 \frac{J_w}{GI_s}\right) \psi_t - \frac{EI_w}{GI_s} \frac{\mathrm{d}^2 \psi_t}{\mathrm{d}x^2}. \tag{98}$$

5.3 Beam finite element

The properties of a finite element for the coupled flexural horizontal and torsional vibration analysis can be derived from the total element energy. The total energy consists of the strain energy, the kinetic energy, the work of the external lateral load, q, and the torque, μ, and the work of the boundary forces. Thus, according to [40, 44],

$$\begin{aligned} E_{tot} &= \frac{1}{2}\int_0^l \left[EI_b \left(\frac{\partial^2 w_b}{\partial x^2}\right)^2 + GA_s \left(\frac{\partial w_s}{\partial x}\right)^2 + EI_w \left(\frac{\partial^2 \psi_t}{\partial x^2}\right)^2 + GI_s \left(\frac{\partial \psi_s}{\partial x}\right)^2 + GI_t \left(\frac{\partial \psi_t}{\partial x}\right)^2 \right] \mathrm{d}x \\ &+ \frac{1}{2}\int_0^l \left[m\left(\frac{\partial w}{\partial t}\right)^2 + J_b \left(\frac{\partial^2 w_b}{\partial x \partial t}\right)^2 + 2mc \frac{\partial w}{\partial t}\frac{\partial \psi}{\partial t} + J_w \left(\frac{\partial^2 \psi_t}{\partial x \partial t}\right)^2 + J_t \left(\frac{\partial \psi}{\partial t}\right)^2 \right] \mathrm{d}x \\ &- \int_0^l (qw + \mu\psi)\mathrm{d}x + (Qw - M\varphi + T\psi + B_w \vartheta)_0^l, \end{aligned} \tag{99}$$

where l is the element length.

Since the beam has four displacements, $w, \varphi, \psi, \vartheta$, a two-node finite element has eight degrees of freedom, i.e. four nodal shear-bending and torsion-warping displacements respectively, Fig. 10,

$$\{U\} = \begin{Bmatrix} w(0) \\ \varphi(0) \\ w(l) \\ \varphi(l) \end{Bmatrix}, \quad \{V\} = \begin{Bmatrix} \psi(0) \\ \vartheta(0) \\ \psi(l) \\ \vartheta(l) \end{Bmatrix}. \tag{100}$$

Therefore, the basic beam displacements, w_b and ψ_t, can be presented as the third-order polynomials

$$\begin{aligned} &w_b = \langle a_k \rangle \{\xi^k\}, \ \psi_t = \langle d_k \rangle \{\xi^k\}, \ k = 0,1,2,3, \\ &\xi = \frac{x}{l}, \ \langle \dots \rangle = \{ \dots \}^T. \end{aligned} \tag{101}$$

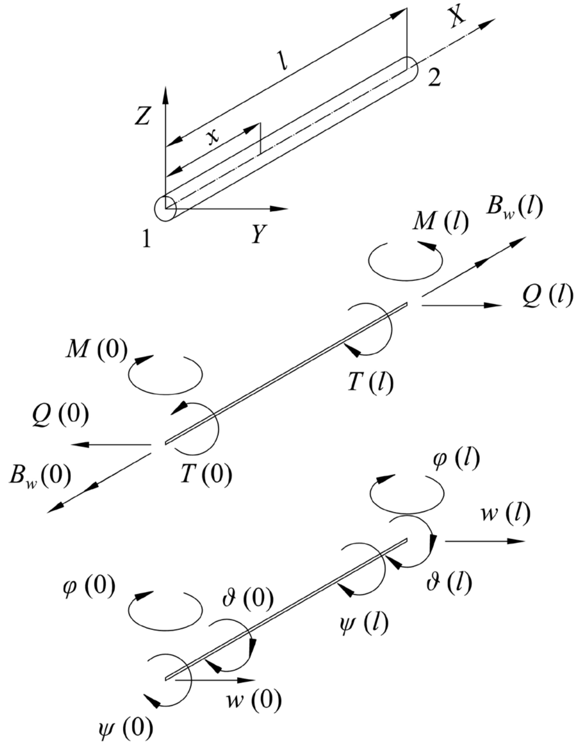

Fig. 10. Beam finite element

Sl. 10. Gredni konačni element

Furthermore, satisfying alternately the unit value for one of the nodal displacement $\{U\}$ and zero values for the remaining displacements, and doing the same for $\{V\}$, it follows that:

$$\begin{aligned} & w_b = \langle w_{bi} \rangle \{U\},\ w_s = \langle w_{si} \rangle \{U\},\ w = \langle w_i \rangle \{U\}, \\ & \psi_t = \langle \psi_{ti} \rangle \{V\},\ \psi_s = \langle \psi_{si} \rangle \{V\},\ \psi = \langle \psi_i \rangle \{V\},\ i = 1,2,3,4, \end{aligned} \tag{102}$$

where w_{bi}, w_{si}, w_i and ψ_{ti}, ψ_{si}, ψ_i are the shape functions specified below by employing relations (97) and (98)

$$\begin{aligned} & w_{bi} = \langle a_{ik} \rangle \{\xi^k\},\ w_{si} = \langle b_{ik} \rangle \{\xi^k\},\ w_i = \langle c_{ik} \rangle \{\xi^k\} \\ & \psi_{ti} = \langle d_{ik} \rangle \{\xi^k\},\ \psi_{si} = \langle e_{ik} \rangle \{\xi^k\},\ \psi_i = \langle f_{ik} \rangle \{\xi^k\} \end{aligned} \tag{103}$$

$$\left[a_{ik}\right]=\frac{1}{\alpha\left(\alpha+12\beta\right)}\begin{bmatrix}\alpha+6\beta & 0 & -3\alpha & 2\alpha\\ -4\beta\left(\alpha+3\beta\right)l & \alpha\left(\alpha+12\beta\right)l & -2\alpha\left(\alpha+3\beta\right)l & \alpha^2 l\\ 6\beta & 0 & 3\alpha & -2\alpha\\ -2\beta\left(\alpha-6\beta\right)l & 0 & -\alpha\left(\alpha-6\beta\right)l & \alpha^2 l\end{bmatrix} \tag{104}$$

$$\begin{aligned} b_{i0} &= -\left(1-\alpha\right)a_{i0}-2\beta a_{i2}\\ b_{i1} &= -\left(1-\alpha\right)a_{i1}-6\beta a_{i3}\\ b_{i2} &= -\left(1-\alpha\right)a_{i2}\\ b_{i3} &= -\left(1-\alpha\right)a_{i3}, \end{aligned} \tag{105}$$

$$\left[c_{ik}\right]=\left[a_{ik}\right]+\left[b_{ik}\right],\quad i=1,2,3,4,\quad k=0,1,2,3 \tag{106}$$

$$\alpha=1-\omega^2\frac{J_b}{GA_s},\ \beta=\frac{EI_b}{GA_s l^2} \tag{107}$$

Constitution of torsional matrices $\left[d_{ik}\right]$, $\left[e_{ik}\right]$ and $\left[f_{ik}\right]$ is the same as $\left[a_{ik}\right]$, $\left[b_{ik}\right]$ and $\left[c_{ik}\right]$, but parameters α and β have to be exchanged with

$$\eta=1-\omega^2\frac{J_w}{GI_s},\ \gamma=\frac{EI_w}{GI_s l^2} \tag{108}$$

according to (98).

By substituting Eqs. (102) into (99) one obtains

$$E_{tot}=\frac{1}{2}\begin{Bmatrix}U\\V\end{Bmatrix}^T\begin{bmatrix}k_{bs} & 0\\ 0 & k_{ws}+k_t\end{bmatrix}\begin{Bmatrix}U\\V\end{Bmatrix}+\frac{1}{2}\begin{Bmatrix}\dot{U}\\\dot{V}\end{Bmatrix}^T\begin{bmatrix}m_{sb} & m_{st}\\ m_{ts} & m_{tw}\end{bmatrix}\begin{Bmatrix}\dot{U}\\\dot{V}\end{Bmatrix}-\begin{Bmatrix}q\\\mu\end{Bmatrix}^T\begin{Bmatrix}U\\V\end{Bmatrix}-\begin{Bmatrix}P\\R\end{Bmatrix}^T\begin{Bmatrix}U\\V\end{Bmatrix}, \tag{109}$$

where, assuming constant values of the element properties,

$$\left[k\right]_{bs}=\left[EI_b\int_0^l\frac{\mathrm{d}^2 w_{bi}}{\mathrm{d}x^2}\frac{\mathrm{d}^2 w_{bj}}{\mathrm{d}x^2}\mathrm{d}x+GA_s\int_0^l\frac{\mathrm{d}w_{si}}{\mathrm{d}x}\frac{\mathrm{d}w_{sj}}{\mathrm{d}x}\mathrm{d}x\right]$$ – bending-shear stiffness matrix,

$$\left[k\right]_{ws}=\left[EI_w\int_0^l\frac{\mathrm{d}^2\psi_{ti}}{\mathrm{d}x^2}\frac{\mathrm{d}^2\psi_{tj}}{\mathrm{d}x^2}\mathrm{d}x+GI_s\int_0^l\frac{\mathrm{d}\psi_{si}}{\mathrm{d}x}\frac{\mathrm{d}\psi_{sj}}{\mathrm{d}x}\mathrm{d}x\right]$$ – warping-shear stiffness matrix,

$$[k]_t = \left[GI_t \int_0^l \frac{\mathrm{d}\psi_{ti}}{\mathrm{d}x} \frac{\mathrm{d}\psi_{tj}}{\mathrm{d}x} \mathrm{d}x \right] \text{ – torsion stiffness matrix,}$$

$$[m]_{sb} = \left[m \int_0^l w_i w_j \mathrm{d}x + J_b \int_0^l \frac{\mathrm{d}w_{bi}}{\mathrm{d}x} \frac{\mathrm{d}w_{bj}}{\mathrm{d}x} \mathrm{d}x \right] \text{ – shear-bending mass matrix,}$$

$$[m]_{tw} = \left[J_t \int_0^l \psi_i \psi_j \mathrm{d}x + J_w \int_0^l \frac{\mathrm{d}\psi_{ti}}{\mathrm{d}x} \frac{\mathrm{d}\psi_{tj}}{\mathrm{d}x} \mathrm{d}x \right] \text{ – torsion-warping mass matrix,}$$

$$[m]_{st} = \left[mc \int_0^l w_i \psi_j \mathrm{d}x \right], \quad [m]_{ts} = [m]_{st}^T \text{ – shear-torsion mass matrix,}$$

$$\begin{aligned} \{q\} &= \left\{ \int_0^l q\, w_j \,\mathrm{d}x \right\} \text{ – shear load vector,} \\ \{\mu\} &= \left\{ \int_0^l \mu \psi_j \,\mathrm{d}x \right\} \text{ – torsion load vector,} \end{aligned} \tag{110}$$

The vectors $\{P\}$ and $\{R\}$ represent the shear-bending and torsion-warping nodal forces, respectively,

$$\{P\} = \begin{Bmatrix} -Q(0) \\ M(0) \\ Q(l) \\ -M(l) \end{Bmatrix}, \quad \{R\} = \begin{Bmatrix} -T(0) \\ B_w(0) \\ T(l) \\ -B_w(l) \end{Bmatrix}. \tag{111}$$

The above matrices are specified in [30], as well as the load vectors for linearly distributed loads along the element, i.e.

$$q = q_0 + q_1 \xi, \quad \mu = \mu_0 + \mu_1 \xi \,. \tag{112}$$

Shape functions of sectional forces are also given in [30].

The total element energy has to be at its minimum. Satisfying the relevant conditions

$$\frac{\partial E_{tot}}{\partial \{U\}} = \{0\}, \quad \frac{\partial E_{tot}}{\partial \{V\}} = \{0\} \tag{113}$$

and by employing the Lagrange equations of motion, the finite element equation yields

$$\{f\} = [k]\{\delta\} + [m]\{\ddot{\delta}\} - \{f\}_{q\mu}, \tag{114}$$

where

$$\begin{gathered}\{f\} = \begin{Bmatrix} P \\ R \end{Bmatrix}, \quad \{f\}_{q\mu} = \begin{Bmatrix} q \\ \mu \end{Bmatrix}, \quad \{\delta\} = \begin{Bmatrix} U \\ V \end{Bmatrix} \\ [k] = \begin{bmatrix} k_{bs} & 0 \\ 0 & k_{ws} + k_t \end{bmatrix}, \quad [m] = \begin{bmatrix} m_{sb} & m_{st} \\ m_{ts} & m_{tw} \end{bmatrix}. \end{gathered} \tag{115}$$

It is obvious that coupling between the bending and torsion occurs through the mass matrix only, i.e. by the coupling matrices $[m]_{st}$ and $[m]_{ts}$.

In the finite element equation (114), first the element properties related to bending and then those related to torsion appear. To make an ordinary finite element assembling possible, it is necessary to transform Eq. (114) in such a way that first all properties related to the first node are specified and then those belonging to the second one. Thus, the rearranged nodal force and displacement vectors read

$$\{\tilde{f}\} = \begin{Bmatrix} -Q(0) \\ M(0) \\ -T(0) \\ B_w(0) \\ Q(l) \\ -M(l) \\ T(l) \\ -B_w(l) \end{Bmatrix}, \quad \{\tilde{\delta}\} = \begin{Bmatrix} w(0) \\ \varphi(0) \\ \psi(0) \\ \varphi(0) \\ w(l) \\ \varphi(l) \\ \psi(l) \\ \varphi(l) \end{Bmatrix}. \tag{116}$$

The same transformation has to be done for the load vector $\{f\}_{q\mu}$ resulting in $\{\tilde{f}\}_{q\mu}$. The above vector transformation implies also the row and column exchange in the stiffness and mass matrices.

The element deflection refers to the shear centre as the origin of a local coordinate system. Since the vertical position of the shear centre varies along the ship's hull, it is necessary to prescribe the element deflection for a common line, in order to be able to assemble the elements. Thus, choosing the x-axis (base line) of the global coordinate system as the referent line, the following relation between the former and the latter nodal deflections exists:

$$w(0) = \bar{w}(0) + z_S \psi(0)$$
$$w(l) = \bar{w}(l) + z_S \psi(l), \tag{117}$$

where z_S is the coordinate of the shear centre, Fig. 9. Other displacements are the same in both coordinate systems. Twist angle ψ does not have influence on the cross-section rotation angle φ. The local displacement vector can be expressed as

$$\{\tilde{\delta}\} = [\tilde{T}]\{\bar{\tilde{\delta}}\}, \tag{118}$$

where $[\tilde{T}]$ is the transformation matrix

$$[\tilde{T}] = \begin{bmatrix} [T] & [0] \\ [0] & [T] \end{bmatrix}, \quad [T] = \begin{bmatrix} 1 & 0 & z_S & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{119}$$

Since the total element energy is not changed by the above transformations, a new element equation can be derived taking (118) into account. Thus, one obtains in the global coordinate system

$$\{\bar{\tilde{f}}\} = [\bar{\tilde{k}}]\{\bar{\tilde{\delta}}\} + [\bar{\tilde{m}}]\{\ddot{\bar{\tilde{\delta}}}\} - \{\bar{\tilde{f}}\}_{q\mu}, \tag{120}$$

where

$$\{\bar{\tilde{f}}\} = [\tilde{T}]^T \{\tilde{f}\}$$
$$[\bar{\tilde{k}}] = [\tilde{T}]^T [\tilde{k}][\tilde{T}]$$
$$[\bar{\tilde{m}}] = [\tilde{T}]^T [\tilde{m}][\tilde{T}] \tag{121}$$
$$\{\bar{\tilde{f}}\}_{q\mu} = [\tilde{T}]^T \{\tilde{f}\}_{q\mu}.$$

The first of the above expressions transforms the nodal torques into the form

$$-\bar{T}(0) = -T(0) - z_S Q(0)$$
$$\bar{T}(l) = T(l) + z_S Q(l). \tag{122}$$

5.4 Numerical procedure for vibration analysis

A ship's hull is modelled by a set of beam finite elements. Their assemblage in the global coordinate system, performed in the standard way, results in the matrix equation of motion, which may be extended by the damping forces

$$[K]\{\Delta\}+[C]\{\dot{\Delta}\}+[M]\{\ddot{\Delta}\}=\{F(t)\}, \tag{123}$$

where [K], [C] and [M] are the stiffness, damping and mass matrices, respectively; $\{\Delta\},\{\dot{\Delta}\}$ and $\{\ddot{\Delta}\}$ are the displacement, velocity and acceleration vectors, respectively; and $\{F(t)\}$ is the load vector.

In case of natural vibration $\{F(t)\} = \{0\}$ and the influence of damping is rather low for ship structures, so that the damping forces may be ignored. Assuming

$$\{\Delta\}=\{\phi\}\mathrm{e}^{i\omega t}, \tag{124}$$

where $\{\phi\}$ and ω is the mode vector and natural frequency respectively, Eq. (123) leads to the eigenvalue problem

$$\left([K]-\omega^2[M]\right)\{\phi\}=\{0\}, \tag{125}$$

which may be solved by employing different numerical methods[45]. The basic one is the determinant search method in which ω is found from the condition

$$\left|[K]-\omega^2[M]\right|=0 \tag{126}$$

by an iteration procedure. Afterwards, $\{\phi\}$ follows from (125) assuming unit value for one element in $\{\phi\}$.

The forced vibration analysis may be performed by direct integration of Eq. (123), as well as by the modal superposition method. In the latter case the displacement vector is presented in the form

$$\{\Delta\}=[\phi]\{X\}, \tag{127}$$

where $[\phi]=[\{\phi\}]$ is the undamped mode matrix and $\{X\}$ is the generalised displacement vector. Substituting (127) into (123), the modal equation yields

$$[k]\{X\}+[c]\{\dot{X}\}+[m]\{\ddot{X}\}=\{f(t)\}, \tag{128}$$

where

$$\begin{aligned}
&[k]=[\phi]^T[K][\phi]-\text{modal stiffness matrix}\\
&[c]=[\phi]^T[C][\phi]-\text{modal damping matrix}\\
&[m]=[\phi]^T[M][\phi]-\text{modal mass matrix}\\
&\{f(t)\}=[\phi]^T\{F(t)\}-\text{modal load vector.}
\end{aligned} \tag{129}$$

The matrices [k] and [m] are diagonal, while [c] becomes diagonal only in a special case, for instance if [C] = α_0 [M] + β_0 [K], where α_0 and β_0 are coefficients [44].

Solving (128) for undamped natural vibration, [k] = [$\omega^2 m$] is obtained, and by its backward substitution into (128) the final form of the modal equation yields

$$[\omega^2]\{X\}+2[\omega][\zeta]\{\dot{X}\}+\{\ddot{X}\}=\{\varphi(t)\}, \tag{130}$$

where

$$\begin{aligned}
&[\omega]=\left[\sqrt{\frac{k_{ii}}{m_{ii}}}\right]-\text{natural frequency matrix}\\
&[\zeta]=\left[\frac{c_{ij}}{2\sqrt{(k_{ii}m_{ii})}}\right]-\text{relative damping matrix}\\
&\{\varphi(t)\}=\left\{\frac{f_i(t)}{m_{ii}}\right\}-\text{relative load vector.}
\end{aligned} \tag{131}$$

If [ζ] is diagonal, the matrix Eq. (130) is split into a set of uncoupled modal equations.

The ship vibration is caused by the engine and propeller excitation forces, which are of periodical nature and therefore can be split into harmonics. Thus, the ship's hull response is obtained solving either (123) or (128). In both cases, the system of differential equations is transformed into a system of algebraic equations.

If hull vibration is induced by waves, the time integration of (123) or (128) has to be performed. Several numerical methods are available for this purpose, as for instance the Houbolt, the Newmark and the Wilson θ method, as well as the harmonic acceleration method [46,47].

It is important to point out that all stiffness and mass matrices of the beam finite element (and consequently those of the assembly) are frequency dependent quantities, due to coefficients α and η in the formulation of the shape functions, Eqs.

(107) and (108). That results in the physically consistent natural modes which are not orthogonal and therefore their application in the modal superposition method for forced vibration analysis is not practical, especially not in the case of time integration. Hence, it is preferable to use mathematical orthogonal modes for that purpose. They are created by the static displacement relations yielding from Eqs. (97) and (98) with $\omega = 0$, that leads to $\alpha = \eta = 1$. In that case all finite element matrices, defined with Eqs. (110), can be transformed into explicit form, as shown in [30].

5.5 Numerical example

The application of the improved theory and numerical procedure is illustrated in case of an 11400 TEU VLCS (Very Large Container Ship), Fig. 11. The main vessel particulars are the following:

Length overall	L_{oa} = 363.44 m
Length between perpendiculars	L_{pp} = 348 m
Breadth	B = 45.6 m
Depth	H = 29.74 m
Draught	T = 15.5 m
Displacement, full load	Δ_f = 171445 t
Displacement, ballast	Δ_b = 74977 t
Displacement, light weight	Δ_l = 37151 t
Engine power	P = 72240 kW
Ship speed	v = 24.7 kn

The midship section, which shows a double skin structure with the web frames and longitudinals, is presented in Figure 12. The ship hull stiffness properties are calculated by the program STIFF [48], based on the theory of thin-walled girders, [47]. Their distributions along the ship are shown in [30]. Influence of transverse bulkheads is taken into account by increasing value of torsional modulus $I_t^* = 1.9 I_t$ according to the theory presented in [49]. The lightweight loading condition, i.e. without containers, is considered. Ship mass distribution and its properties are also given in [30].

Dry natural vibrations, as prerogative for hydroelastic analysis, are calculated by the modified and improved program DYANA, [50]. The ship hull is divided into 50 beam finite elements. Ordinary finite elements for closed cross-sections are used for the ship bow, ship aft and in the engine room area.

Natural frequencies of vertical vibrations, and those of coupled horizontal and torsional vibrations are listed in Table 3 and 4 and are compared to the results obtained by 3D FEM analysis. Quite good agreement is achieved for the first few natural modes.

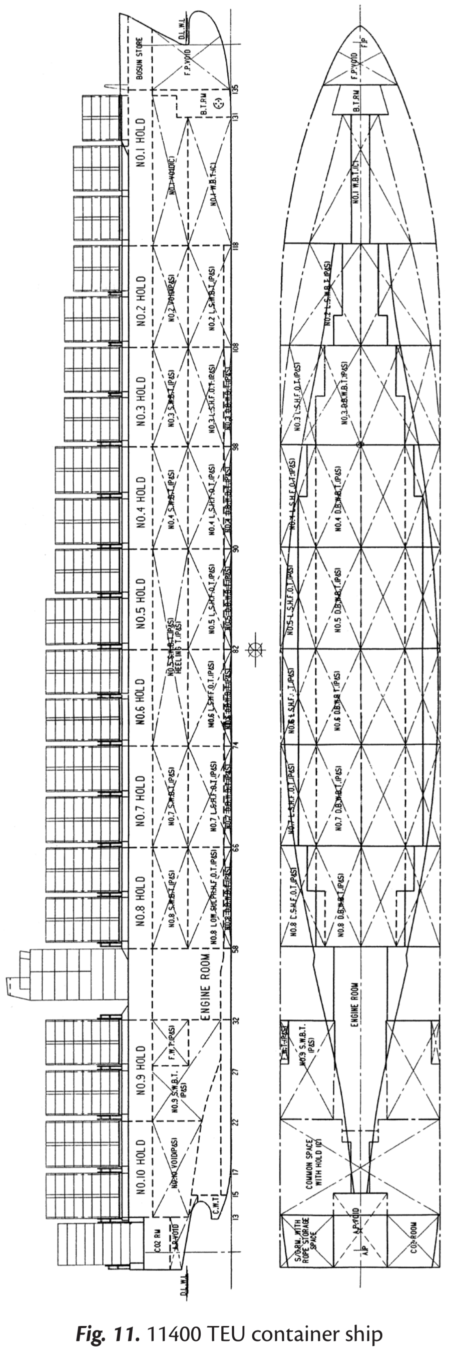

Fig. 11. 11400 TEU container ship

Sl. 11. Kontejnerski brod od 11400 TEU

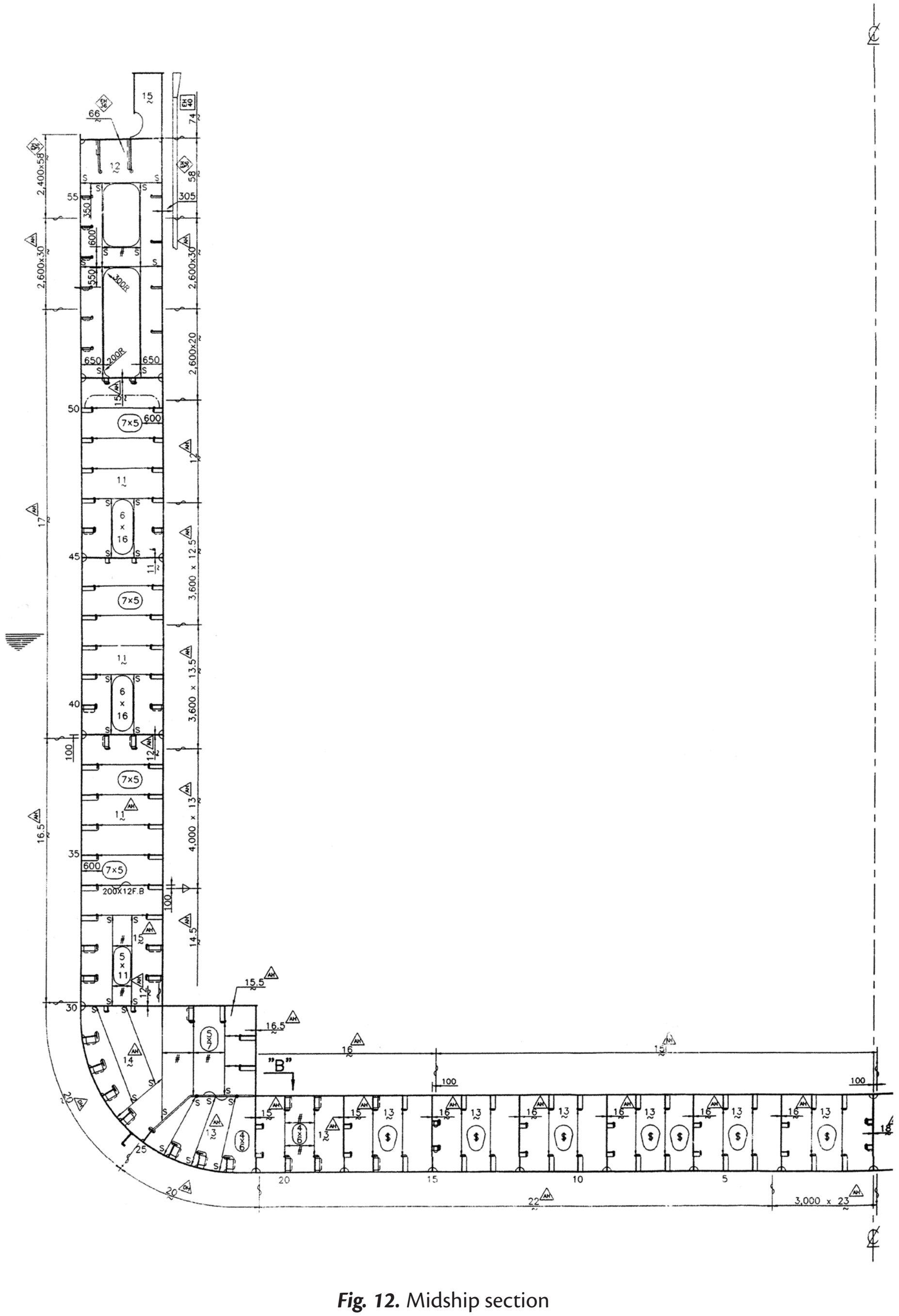

Fig. 12. Midship section

Sl. 12. Poprečni presjek kontejnerskog broda

Table 3 - Natural frequencies of vertical hull vibrations, ω_i (Hz)

Mode no.	1D FEM	3D FEM	Discrepancy %
1	1.149	1.159	-0.86
2	2.318	2.327	-0.39
3	3.695	3.654	1.12
4	5.457	5.409	0.89
5	6.913	6.605	4.66

Table 4 - Natural frequencies of coupled horizontal and torsional hull vibrations, ω_i (Hz)

Mode no.	Coupled modes	1D FEM	3D FEM	Discrepancy %
1	T1	0.639	0.638	0.16
2	T2+H1	1.056	1.076	-1.86
3	T3+H2	1.745	1.749	-0.23
4	T4+H3	2.233	2.429	-8.07
5	T2+H5	3.072	2.630	16.81
6	T5+H4	3.350	3.519	-4.80

Nodal displacements, i.e. translation and rotation of beam model, are transferred to the ship wetted surface. The first natural mode of vertical vibration as well as that of coupled horizontal and torsional vibrations are shown in Figs. 13 and 14, respectively. Also, the later mode determined by the 3D FEM analysis is shown in Fig. 15. 1D and 3D natural mode is of the same shape.

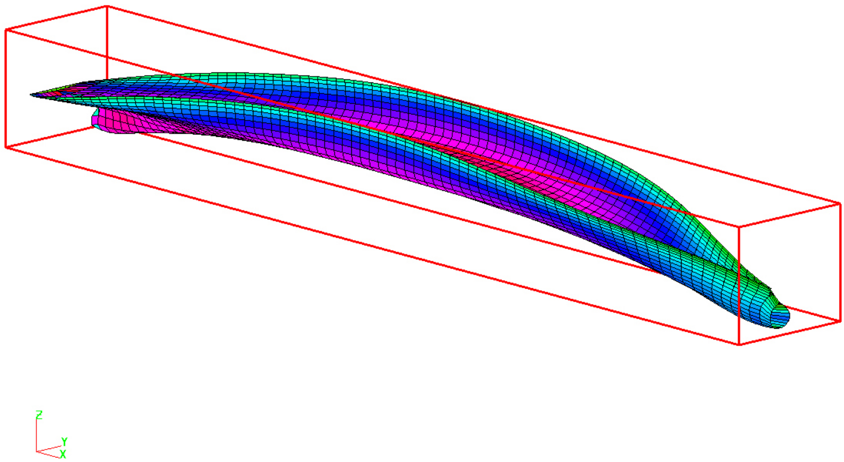

Fig. 13. The first natural mode of vertical vibrations, ω_1=1.149 Hz, 1D model

Sl. 13. Prvi prirodni oblik vertikalnih vibracija, ω_1=1.149 Hz, 1D model

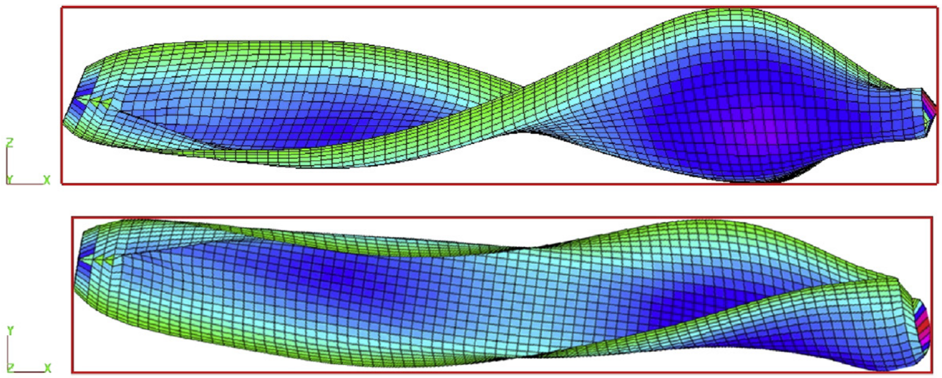

Fig. 14. The first natural mode of coupled horizontal and torsional vibrations, $\omega_1 = 0.639$ Hz, 1D model

Sl. 14. Prvi prirodni oblik spregnutih horizontalnih i torzijskih vibracija, $\omega_1 = 0.639$ Hz, 1D model

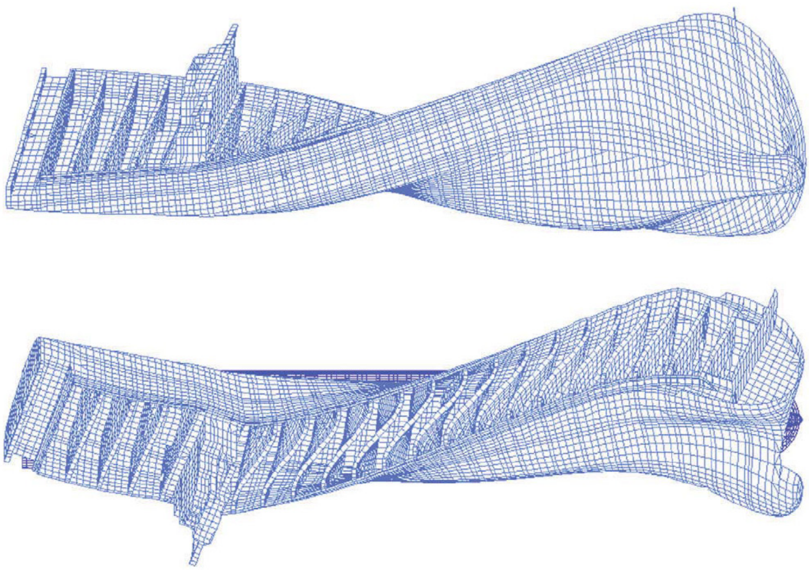

Fig. 15. The first natural mode of coupled horizontal and torsional vibrations, $\omega_1 = 0.638$ Hz, 3D FEM model

Sl. 15. Prvi prirodni oblik spregnutih horizontalnih i torzijskih vibracija, $\omega_1 = 0.638$ Hz, 3D FEM model

6. CONCLUSION

Writing of this paper was motivated by the fact that Timoshenko beam theory after 93 years is still actual. As pointed out in Introduction it has been used at the beginning for dynamic analysis of structural elements, and later on for structural analysis of bridges. Nowadays, its application is extended to large range of scale, from nanotubes to ultra large structures.

The paper is dealing with two presently interesting subjects, i.e. dynamic behavior of nanotube embedded in an elastic medium and exposed to moving nanoparticle gravity load, and vibrations of ultra large container ships. In both cases the modified Timoshenko beam theory is used, which results with simpler problem formulations. Nanotube response is determined semi-analytically and parametric analysis is performed emphasizing influence of damping on response.

Even complex structures like ultra large container ships can be modeled as a beam for global response analysis if flexural and torsional stiffness parameters of ship cross-section are determined in a sophisticated way by the advanced thin-walled girder theory and if influence of transverse bulkheads and relatively short engine room structure are taken into account in a proper way. Dry natural vibrations are prerogative for hydroelastic analysis of ship exposed to wave excitation by the modal superposition method. Such an analysis includes also determination of modal restoring stiffness, added mass and damping. Correlation of natural eigenpairs (frequency and modes) determined by the beam model with those obtained by 3D FEM model shows high reliability of the former.

Timoshenko beam theory is only one topic worked out by that genius scientist, but unique one due to long time and broad application. Probably, he could not imagine that application of his beam theory does not end with structural analysis of bridges.

Acknowledgement

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Timošenkova teorija grede nakon 93 godine – preko mostova do nanocijevi i ultra velikih brodova

Sažetak

Sažeto je prikazana Timošenkova teorija grede, koja operira s progibom i kutom zakreta poprečnog presjeka kao osnovnim varijablama. Teorija je nadalje modificirana rasprežući ukupni progib grede u progib čistog savijanja i progib uslijed smika, te ukupni kut zakreta u kut uslijed savijanja i kut uslijed uzdužnog smicanja. Osnovne jednadžbe su spregnute u dvije nezavisne jednadžbe gibanja, jedna za savojne, a druga za uzdužne smične vibracije. Jednadžbe su rješene za slučaj prirodnih vibracija. Razmatran je problem vibracija nanocijevi u elastičnom mediju uvođenjem parametara nelokalnog naprezanja. Parcijalna diferencijalna jednadžba 4-tog reda za savojne vibracije proširena je na jednadžbu 6-tog reda. Odziv nanocijevi na pomičnu gravitacijsku silu nanočestice analiziran je koristeći metodu superpozicije prirodnih oblika vibriranja, metodu separacije varijabli, Galerkinovu metodu i metodu ravnoteže harmonika. U nastavku razmotren je problem spregnutih savojnih i torzijskih vibracija tankostijenog nosača. Korištena je modificirana Timošenkova teorija grede za savojne vibracije. Složeni problem torzijskih vibracija uz vitoperenje poprečnog presjeka nosača formuiliran je na analogan način. Za potrebe analize vibracija neprizmatičnog brodskog trupa razvijen je sofisticirani gredni konačni element. Primjena razvijenih teorija ilustrirana je na primjerima analize vibracija nanocijevi i jednog ultra velikog kontejnerskog broda.

Ključne riječi: Timošenkova teorije grede; nanocijev; pomično opterećenje; tankostijeni nosač; savojne vibracije; torzijske vibracije; kontejnerski brod

Ivo Senjanović
University of Zagreb
Faculty of Mechanical Engineering
and Naval Architecture
Ivana Lučića 5, 10000 Zagreb
Croatia
e-mail: ivo.senjanovic@fsb.hr

Nikola Vladimir
University of Zagreb
Faculty of Mechanical Engineering
and Naval Architecture
Ivana Lučića 5, 10000 Zagreb
Croatia

Marko Tomić
University of Zagreb
Faculty of Mechanical Engineering
and Naval Architecture
Ivana Lučića 5, 10000 Zagreb
Croatia

Neven Hadžić
University of Zagreb
Faculty of Mechanical Engineering
and Naval Architecture
Ivana Lučića 5, 10000 Zagreb
Croatia

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