Fractal analysis of unbounded sets in Euclidean spaces: complex dimensions and Lapidus zeta functions

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Relative fractal drum (A,Ω)

- $\emptyset \neq A \subset \mathbb{R}^N$
- δ-neighbourhood of A:

$$A_\delta = \{x \in \mathbb{R}^N : d(x,A) < \delta\}$$

- $\Omega \subset \mathbb{R}^N$, $|\Omega| < \infty$, $\exists \delta > 0$, such that $\Omega \subseteq A_\delta$, $r \in \mathbb{R}$
- **lower** r**-dimensional Minkowski content of** (A,Ω):

$$\underline{\mathcal{M}}^r(A,\Omega) := \liminf_{\delta \to 0^+} \frac{|A_\delta \cap \Omega|}{\delta^{N-r}}$$

- **upper** r**-dimensional Minkowski content of** (A,Ω):

$$\overline{\mathcal{M}}^r(A,\Omega) := \limsup_{\delta \to 0^+} \frac{|A_\delta \cap \Omega|}{\delta^{N-r}}$$

Relative box dimension

- **lower and upper box dimension of** (A,Ω):

  $\underline{\dim}_B(A,\Omega) = \inf\{r \in \mathbb{R} \ : \ \underline{\mathcal{M}}^r(A,\Omega) = 0\}$

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- $\underline{\dim}_B(A,\Omega) = \overline{\dim}_B(A,\Omega)$ $\Rightarrow$ $\exists \dim_B(A,\Omega)$

- if $\exists D \in \mathbb{R}$ such that

$$0 < \underline{\mathcal{M}}^D(A,\Omega) = \overline{\mathcal{M}}^D(A,\Omega) < \infty,$$

define (A,Ω) **Minkowski measurable** $\Rightarrow$ $D = \dim_B(A,\Omega)$

The relative distance zeta function [LapRaŽu]

- generalization of Professor Lapidus' definition of a zeta function associated to bounded (fractal) sets (Catania 2009)
- (A,Ω) RFD in $\mathbb{R}^N$, $|\Omega| < \infty$, $s \in \mathbb{C}$ and fix $\delta > 0$
- the **distance zeta function of** (A,Ω):

$$\zeta_A(s,\Omega;\delta) := \int_{A_\delta \cap \Omega} d(x,A)^{s-N}\, dx$$

Holomorphicity theorem for the relative distance zeta function

Theorem (Cited from [LapRaŽu])

(A,Ω) RFD in $\mathbb{R}^N$, then

(a) $\zeta_A(s,\Omega)$ is holomorphic on $\{\mathrm{Re}\, s > \overline{\dim}_B(A,\Omega)\}$, and

$$\zeta_A'(s,\Omega) = \int_{A_\delta\cap\Omega} d(x,A)^{s-N}\log d(x,A)\,dx$$

(b) $\mathbb{R}\ni s<\overline{\dim}_B(A,\Omega) \Rightarrow$ the integral defining $\zeta_A(s,\Omega)$ diverges

(c) $(\exists D=\dim_B(A,\Omega)<N)(\underline{\mathcal{M}}^D(A,\Omega)>0) \Rightarrow$
$\zeta_A(s,\Omega)\to+\infty$ when $\mathbb{R}\ni s\to D^+$

The relative tube zeta function [LapRaŽu]

(A,Ω) an RFD in $\mathbb{R}^N$ and fix $\delta > 0$

- the **tube zeta function of** (A,Ω):

$$\widetilde{\zeta}_A(s,\Omega;\delta) := \int_0^\delta t^{s-N-1}|A_t \cap \Omega|\,\mathrm{d}t$$

- the analog of the the holomorphicity theorem holds for $\widetilde{\zeta}_A(s,\Omega;\delta)$

- a functional equation connecting the two zeta functions:

$$\zeta_A(s,\Omega;\delta) = \delta^{s-N}|A_\delta \cap \Omega| + (N-s)\widetilde{\zeta}_A(s,\Omega;\delta)$$

Fractal tube formulas for relative fractal drums

- **The problem**: Derive an asymptotic formula for the relative tube function $t \mapsto |A_t \cap \Omega|$ as $t \to 0^+$ from the distance zeta function $\zeta_A(\,\cdot\,,\Omega)$ of (A,Ω).
- More precisely, express $|A_t \cap \Omega|$ as a sum of residues over the **complex dimensions** of (A,Ω).
- Apply this to derive a **Minkowski measurability criterion** for a large class of RFDs.

The idea of solving the problem

$$\widetilde{\zeta}_A(s,\Omega;\delta)=\int_0^{+\infty}t^{s-1}\left(\chi_{(0,\delta)}(t)t^{-N}|A_t\cap\Omega|\right)\,\mathrm{d}t$$

- **Mellin inversion theorem** $\Rightarrow$

Theorem (The integral tube formula [Ra])

(A,Ω) an RFD in $\mathbb{R}^N$ and fix $\delta>0$.
Then, for every $t\in(0,\delta)$ and $c>\overline{\dim}_B(A,\Omega)$, we have

$$|A_t\cap\Omega|=\frac{1}{2\pi\mathbb{i}}\int_{c-\mathbb{i}\infty}^{c+\mathbb{i}\infty}t^{N-s}\widetilde{\zeta}_A(s,\Omega;\delta)\,\mathrm{d}s. \tag{1}$$

- express (1) as a sum over the residues of $\widetilde{\zeta}_A(\,\cdot\,,\Omega)$

Figure: The screen and the window

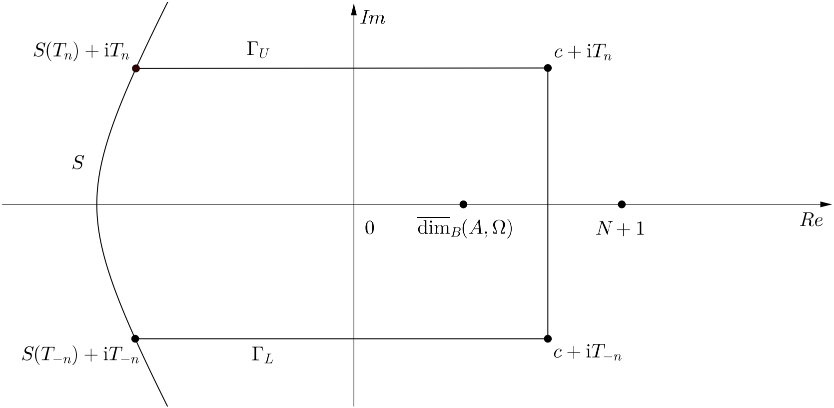


Figure: Using the residue theorem to express $|A_t \cap \Omega|$ as a sum over the complex dimensions of (A, Ω).

The screen and the window, admissibility

Definition (Adapted from [Lap–vFr])

$$\text{the \textbf{screen}:}\quad \boldsymbol{S} := \{S(\tau) + \mathbb{i}\tau \,:\, \tau \in \mathbb{R}\}$$

$S(\tau)$ bounded, real-valued, Lipschitz continuous:

$$|S(x) - S(y)| \leq \|S\|_{\mathrm{Lip}}|x - y|, \qquad \text{for all } x, y, \in \mathbb{R}$$

$$\inf S := \inf_{\tau\in\mathbb{R}} S(\tau) \quad \text{and} \quad \sup S := \sup_{\tau\in\mathbb{R}} S(\tau)$$

$$\text{the \textbf{window}:}\quad \boldsymbol{W} := \{s \in \mathbb{C} : \operatorname{Re} s \geq S(\operatorname{Im} s)\}$$

(A, Ω) is **admissible** if its relative tube (or distance) zeta function can be meromorphically extended to an open connected neighborhood of some window $\boldsymbol{W}$.

Languidity

Definition (Adapted from [Lap–vFr])

An admissible (A,Ω) is **languid** if for some $\delta>0$, $\widetilde{\zeta}_A(\,\cdot\,,\Omega;\delta)$ satisfies: $(\exists\kappa\in\mathbb{R})$, $(\exists C>0)$, $\exists(T_n)_{n\in\mathbb{Z}}$ such that $T_{-n}<0<T_n$ for $n\geq 1$ and $\lim_{n\to\pm\infty}|T_n|=+\infty$ satisfying

L1 For all $n\in\mathbb{Z}$ and all $\sigma\in(S(T_n),c)$,

$$|\widetilde{\zeta}_A(\sigma+\mathbb{i}\,T_n,\Omega;\delta)|\leq C(|T_n|+1)^{\kappa},$$

where $c>\overline{\dim}_B(A,\Omega)$ is some constant.

L2 For all $\tau\in\mathbb{R}$, $|\tau|\geq 1$,

$$|\widetilde{\zeta}_A(S(\tau)+\mathbb{i}\tau,\Omega;\delta)|\leq C|\tau|^{\kappa}.$$

Figure: Languidity

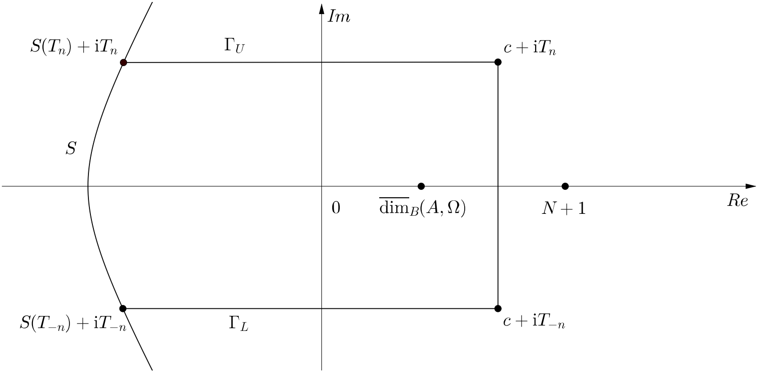


Figure: **Languidity** of an RFD roughly equals to **at most polynomial growth** of its tube zeta function along a suitable double sequence of segments and along the vertical direction of the screen.

Strong languidity

Definition (Adapted from [Lap–vFr])

(A, Ω) is **strongly languid** if

L1' For all $n \in \mathbb{Z}$ and all $\sigma \in (-\infty, c)$,

$$|\widetilde{\zeta}_A(\sigma + \mathbb{i}\, T_n, \Omega; \delta)| \leq C(|T_n| + 1)^{\kappa},$$

where $c > \overline{\dim}_B(A, \Omega)$ is some constant.
Additionally, $\exists (S_m)_{m \geq 1}$ such that $\sup S_m \to -\infty$ and $\sup_{m \geq 1} \|S_m\|_{\mathrm{Lip}} < \infty$, such that

L2' there exist $B, C > 0$ such that for all $\tau \in \mathbb{R}$ and $m \geq 1$,

$$|\widetilde{\zeta}_A(S_m(\tau) + \mathbb{i}\tau, \Omega; \delta)| \leq C B^{|S_m(\tau)|}(|\tau| + 1)^{\kappa}.$$

Complex dimensions of an RFD

Definition ([LapRaŽu])

Assume that (A,Ω) is admissible for some window W.
Visible complex dimensions of (A,Ω) (**with respect to** W):

$$\mathcal{P}(\zeta_A(\,\cdot\,,\Omega;\delta),W):=\{\omega\in W:\omega \text{ is a pole of } \zeta_A(\,\cdot\,,\Omega;\delta)\}.$$

$(W=\mathbb{C})\Rightarrow$ the set of **complex dimensions** of (A,Ω).

The set of **principal complex dimensions of** (A,Ω):

$$\dim_{PC}(A,\Omega):=\{\omega\in\mathcal{P}(\zeta_A(\,\cdot\,,\Omega;\delta),W):\operatorname{Re}\omega=\overline{\dim}_B(A,\Omega)\}.$$

Tube formula via the distance zeta function

- $V^{[k]}_{(A,\Omega)}(t)$ the k-th primitve function of $|A_t \cap \Omega|$
- $k \in \mathbb{N}$: $(s)_0 := 1$ $(s)_k := s(s+1)\cdots(s+k-1)$
- $k \in \mathbb{Z}$: $(s)_k := \frac{\Gamma(s+k)}{\Gamma(s)}$

Theorem (Pointwise formula with error term [Ra])

- (A,Ω) **d-languid** for some κ_d and $\overline{\dim}_B(A,\Omega) < N$
- $k > \kappa_d$ a nonnegative integer

Then, for every $t \in (0,\delta)$ we have

$$V^{[k]}_{(A,\Omega)}(t) = \sum_{\omega \in \mathcal{P}(\zeta_A(\,\cdot\,,\Omega),W)} \operatorname{res}\left(\frac{t^{N-s+k}}{(N-s)_{k+1}}\zeta_A(s,\Omega;\delta),\omega\right) + R^{[k]}(t).$$

Theorem (...continued)

The error term $R^{[k]}$ is given by the absolutely convergent integral

$$R^{[k]}(t) = \frac{1}{2\pi\mathbb{i}} \int_S \frac{t^{N-s+k}}{(N-s)_{k+1}} \zeta_A(s,\Omega;\delta)\,\mathrm{d}s.$$

We have the following pointwise error estimate:

$$R^{[k]}(t) = O(t^{N-\sup S+k}) \quad \text{as} \quad t \to 0^+.$$

Moreover, $(\forall\tau \in \mathbb{R})(S(\tau) < \sup S) \Rightarrow$

$$R^{[k]}(t) = o(t^{N-\sup S+k}) \quad \text{as} \quad t \to 0^+.$$

Exact tube formula in case of strong languidity

Theorem (Exact pointwise tube formula [Ra])

- (A, Ω) **strongly d-languid** for some $\delta > 0$, $\kappa_d \in \mathbb{R}$
- $k > \kappa_d - 1$ a nonnegative integer and $\overline{\dim}_B(A, \Omega) < N$

Then, for every $t \in (0, \min\{1, \delta, B^{-1}\})$ we have

$$V^{[k]}_{(A,\Omega)}(t) = \sum_{\omega \in \mathcal{P}(\zeta_A(\,\cdot\,,\Omega),\mathbb{C})} \operatorname{res}\left(\frac{t^{N-s+k}}{(N-s)_{k+1}} \zeta_A(s, \Omega), \omega\right).$$

Here, B is the constant appearing in **L2'**.

When can we apply the tube formula at level $k = 0$?

- tube formula with error term: **if** $\kappa_d < 0$
- exact tube formula: **if** $\kappa_d < 1$

Distributional fractal tube formulas

$$|A_t \cap \Omega| = \sum_{\omega \in \mathcal{P}(\zeta_A(\,\cdot\,,\Omega),W)} \operatorname{res}\left(\frac{t^{N-s}}{N-s}\zeta_A(s,\Omega),\omega\right) + R^{[0]}(t)$$

- removing the restriction on κ_d we derive a tube formula only in the sense of **Schwartz's distributions**
- exact analogs of the the tube formula with and without the error term hold distributionally for **any** exponent $\kappa_d \in \mathbb{R}$ and **any** $k \in \mathbb{Z}$

The Minkowski measurability criterion

Theorem (Minkowski measurability criterion [Ra])

- (A,Ω) such that $\exists D := \dim_B(A,\Omega)$ and $D < N$
- (A,Ω) *d*-languid for a screen passing between the critical line $\{\operatorname{Re} s = D\}$ and all the complex dimensions of (A,Ω) with real part strictly less than D

Then, the following is equivalent:

(a) (A,Ω) is Minkowski measurable.

(b) D is the only pole of $\zeta_A(\,\cdot\,,\Omega)$ located on the critical line $\{\operatorname{Re} s = D\}$ and it is simple.

$$\mathcal{M}^D(A,\Omega) = \frac{\operatorname{res}(\zeta_A(\,\cdot\,,\Omega), D)}{N-D}$$

- $(a) \Rightarrow (b)$: from the distributional tube formula and the **Uniqueness theorem for almost periodic distributions** due to **Schwartz**

- $(b) \Rightarrow (a)$: a consequence of a **Tauberian theorem** due to **Wiener** and **Pitt** (conditions can be considerably weakened)

- the assumption $D < N$ can be removed by appropriately embedding the RFD in $\mathbb{R}^{N+1}$

Theorem (Bound for the upper Minkowski content [Ra])

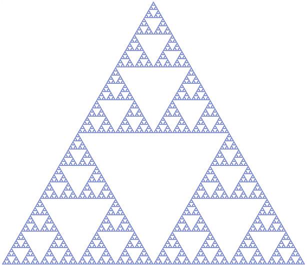
- (A,Ω) such that $\overline{D} := \overline{\dim}_B(A,\Omega) < N$
- $\zeta_A(\,\cdot\,,\Omega)$ mero. extendable to a neighborhood of $\{\operatorname{Re} s = \overline{D}\}$
- $\overline{D}$ is its simple pole
- $\{\operatorname{Re} s = \overline{D}\}$ contains another pole of $\zeta_A(\,\cdot\,,\Omega)$ different from $\overline{D}$
- let

$$\lambda_{(A,\Omega)} := \inf\left\{|\overline{D}-\omega| \,:\, \omega \in \dim_{PC}(A,\Omega) \setminus \{\overline{D}\}\right\}$$

Then, we have the following upper bound:

$$\overline{\mathcal{M}}^{\overline{D}}(A,\Omega) \leq \frac{3\lambda_{(A,\Omega)}}{2\pi\left(1-\mathrm{e}^{-\frac{2\pi(N-\overline{D})}{\lambda_{(A,\Omega)}}}\right)}\operatorname{res}(\zeta_A(\,\cdot\,,\Omega),\overline{D}).$$

Figure: The Sierpiński gasket



- an example of a **self-similar fractal spray** with a generator G being an open equilateral triangle and with **scaling ratios** $r_1 = r_2 = r_3 = 1/2$
- $(A, \Omega) = (\partial G, G) \cup \bigcup_{j=1}^{3} (r_j A, r_j \Omega)$

Example (The Sierpiński gasket)

$$\zeta_A(s;\delta) = \frac{6(\sqrt{3})^{1-s}2^{-s}}{s(s-1)(2^s-3)} + 2\pi\frac{\delta^s}{s} + 3\frac{\delta^{s-1}}{s-1}$$

$$\mathcal{P}(\zeta_A) = \{0,1\} \cup \left(\log_2 3 + \frac{2\pi}{\log 2}\mathbb{i}\mathbb{Z}\right)$$

By letting $\omega_k := \log_2 3 + \mathbf{p}k\mathbb{i}$ and $\mathbf{p} := 2\pi/\log 2$ we have that

$$|A_t| = \sum_{\omega\in\mathcal{P}(\zeta_A)} \operatorname{res}\left(\frac{t^{2-s}}{2-s}\zeta_A(s;\delta),\omega\right)$$

$$= t^{2-\log_2 3}\frac{6\sqrt{3}}{\log 2}\sum_{k=-\infty}^{+\infty}\frac{(4\sqrt{3})^{-\omega_k}t^{-\mathbf{p}k\mathbb{i}}}{(2-\omega_k)(\omega_k-1)\omega_k} + \left(\frac{3\sqrt{3}}{2}+\pi\right)t^2,$$

valid pointwise for all $t \in (0, 1/2\sqrt{3})$.

Tube formula for self-similar fractal sprays

- in general, for a **self-similar fractal spray** we have a **generator** G and a **"ratio list"** $\{r_1, r_2, \dots, r_J\}$, $r_j > 0$ such that $\sum_{j=1}^{J} r_j^N < 1$
- λ_k are built as all possible words of multiples of the ratios r_j.
- $A := \partial(\sqcup \lambda_k G)$ $\quad$ $\Omega := \sqcup \lambda_k G$

Theorem ([Ra])

$\overline{\dim}_B(\partial G, G) < N$. Then, $(A, \Omega) = (\partial G, G) \sqcup \bigsqcup_{j=1}^{J} (r_j A, r_j \Omega)$ and

$$|A_t \cap \Omega| = \sum_{\omega \in (\mathfrak{D} \cap W) \cup \mathcal{P}(\zeta_{\partial G}(\,\cdot\,, G), W)} \operatorname{res}\left(\frac{t^{N-s}\zeta_{\partial G}(s, G)}{(N-s)\left(1 - \sum_{j=1}^{J} r_j^s\right)}, \omega\right) + \mathcal{R}(t),$$

where $\mathfrak{D}$ is the set of complex solutions of $\sum_{j=1}^{J} r_j^s = 1$.

Cantor sets of higher order

Example (The Cantor set of second order [Ra])

C the standard middle-third Cantor set in $[0,1]$, $\Omega := [0,1]$.
$G := \Omega \setminus C$; scaling ratios $r_1 = r_2 = 1/3$.

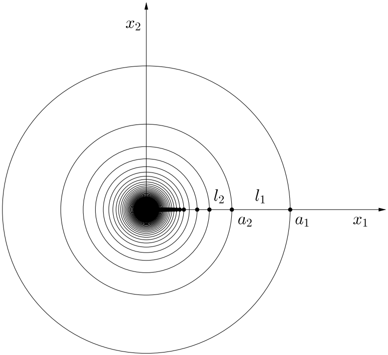
$$\zeta_{C_2}(s,\Omega_2) = \frac{3^s\,\zeta_C(s,\Omega)}{3^s-2} = \frac{3^s}{2^{s-1}s(3^s-2)^2}$$

$$\mathcal{P}(\zeta_{C_2}(\,\cdot\,\Omega_2)) = \{0\} \cup \left(\log_3 2 + \frac{2\pi}{\log 3}\mathbb{i}\mathbb{Z}\right)$$

$$|(C_2)_t \cap \Omega_2| = t^{1-\log_3 2}\Big(\log t^{-1}\,G(\log t^{-1}) + H(\log t^{-1})\Big) + 2t$$

$G, H\colon \mathbb{R} \to \mathbb{R}$ nonconstant, periodic with $T = \log 3$.

- a pole ω of order m generates factors of type $t^{N-\omega}(\log t^{-1})^{k-1}$ for $k = 1, \ldots, m$



Example (The fractal nest generated by the a-string)

$a > 0$, $a_j := j^{-a}$, $l_j := j^{-a} - (j+1)^{-a}$, $\Omega := B_{a_1}(0)$

$$\zeta_{A_a}(s;\Omega) = \frac{2^{2-s}\pi}{s-1} \sum_{j=1}^{\infty} l_j^{s-1}(a_j + a_{j+1})$$

Example (The fractal nest generated by the a-string)

$$\mathcal{P}(\zeta_{A_a}(\,\cdot\,,\Omega)) \subseteq \left\{1, \frac{2}{a+1}, \frac{1}{a+1}\right\} \cup \left\{-\frac{m}{a+1} : m \in \mathbb{N}\right\}$$

$a \neq 1,\ D := \frac{2}{1+a} \Rightarrow$

$$|(A_a)_t \cap \Omega| = \frac{2^{2-D} D\pi}{(2-D)(D-1)} a^{D-1} t^{2-D} + \left(4\pi\zeta(a) - 2\pi\right) t + \frac{\operatorname{res}\left(\zeta_{A_a}(\,\cdot\,,\Omega), \frac{1}{a+1}\right) t^{2-\frac{1}{a+1}}}{2-\frac{1}{a+1}} + O\left(t^2\right), \text{ as } t \to 0^+$$

$$|(A_1)_t \cap \Omega| = \operatorname{res}\left(\frac{t^{2-s}}{2-s}\zeta_{A_1}(s,\Omega), 1\right) + o(t) = 2\pi t \log t^{-1} + \text{const} \cdot t + o(t) \quad \text{as } t \to 0^+$$

Embeddings in higher dimensions

Proposition

- (A,Ω) with $\overline{\dim}_B(A,\Omega)=\overline{D}$ and fix $\delta\in(0,1)$

Then, the following functional equality holds:

$$\widetilde{\zeta}_{A\times\{0\}}(s,\Omega\times[-1,1];\delta)=2\int_0^{\pi/2}\frac{\widetilde{\zeta}_A(s,\Omega;\delta\sin\tau)}{\sin^{s-N-1}\tau}\,\mathrm{d}\tau \qquad (2)$$

for all $s\in\{\operatorname{Re} s>\overline{D}\}$.

Theorem ([Ra])

- (A,Ω) such that $\overline{D} := \overline{\dim}_B(A,\Omega) < N$ and fix $a > 0$

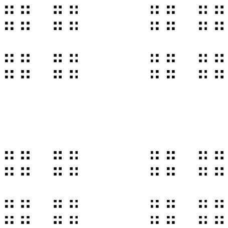
Then, the following functional equation is valid:

$$\zeta_{A\times\{0\}}(s,\Omega\times[-a,a]) = \frac{\sqrt{\pi}\Gamma\left(\frac{N-s}{2}\right)}{\Gamma\left(\frac{N+1-s}{2}\right)}\zeta_A(s,\Omega) + E(s;a). \qquad (3)$$

$E(s;a)$ is meromorphic on $\mathbb{C}$ with a set of simple poles contained in $\{N+2k : k\in\mathbb{N}_0\}$.

- complex dimensions of an RFD are independent of the ambient space
- determine complex dimensions of RFDs by decomposing them into relative fractal subdrums

Figure: The Cantor dust



- $A := C^{(1/3)} \times C^{(1/3)}$ $\Omega := (0,1)^2$
- (A,Ω) may be viewed as a **self-similar** RFD with scaling ratios $r_1 = r_2 = r_3 = r_4 = 1/3$ and the **base** RFD (A_0,Ω_0)
- Ω_0 is the 'middle open cross'
- A_0 is the union of Cantor sets contained in $\partial\Omega_1$

Complex dimensions of the Cantor dust

Example

Let $A := C^{(1/3)} \times C^{(1/3)}$ be the Cantor dust and $\Omega := [0,1]^2$. Then,

$$\zeta_A(s,\Omega) = \frac{8}{s(3^s-4)}\left(\frac{I(s)}{6^s} + \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{2-s}{2}\right)}\frac{\sqrt{\pi}}{6^s s(3^s-2)} + E(s;6^{-1})\right),$$

where $I(s) = 2^{-1}\mathrm{B}_{1/2}(1/2,(1-s)/2)$ is entire.

$$\mathcal{P}(\zeta_A(\,\cdot\,,\Omega)) \subseteq \left(\log_3 4 + \frac{2\pi}{\log 3}\mathbb{i}\mathbb{Z}\right) \cup \left(\log_3 2 + \frac{2\pi}{\log 3}\mathbb{i}\mathbb{Z}\right) \cup \{0\}.$$

- $\mathrm{B}_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}\,\mathrm{d}t$ is the incomplete beta function

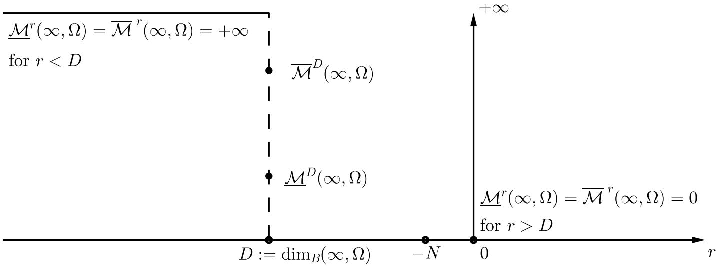
Minkowski content and dimension at infinity

- $\Omega \subseteq \mathbb{R}^N$, $|\Omega| < \infty$, $r \in \mathbb{R}$
- **upper r-dimensional Minkowski content** of (∞, Ω):

$$\overline{\mathcal{M}}^r(\infty, \Omega) := \limsup_{t \to +\infty} \frac{|B_t(0)^c \cap \Omega|}{t^{N+r}}$$

- **upper box dimension** of (∞, Ω):

$$\overline{\dim}_B(\infty, \Omega) := \sup\{r \in \mathbb{R} \, : \, \overline{\mathcal{M}}^r(\infty, \Omega) = +\infty\}$$



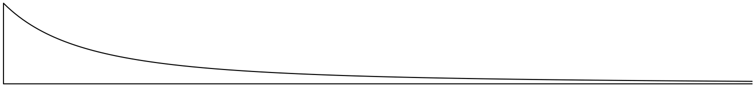
Example

$\alpha > 0$, $\beta > 1$, $a_j := j^{\alpha}$, $b_j := a_j + j^{-\beta}$

$$\Omega(\alpha,\beta) := \bigcup_{j=1}^{\infty} (a_j, b_j)$$

$$D := \dim_B(\infty, \Omega(\alpha,\beta)) = \frac{1-(\alpha+\beta)}{\alpha}, \quad \mathcal{M}^D(\infty, \Omega(\alpha,\beta)) = \frac{1}{\beta-1}$$

- we can obtain any value in $(-\infty, -1)$ for $\dim_B(\infty, \Omega(\alpha,\beta))$
- $\dim_B(\infty, \Omega(\alpha,\beta)) \to -\infty$ and $\mathcal{M}^D(\infty, \Omega(\alpha,\beta)) \to 0$ as $\beta \to +\infty$



Example

$\alpha > 1$, $\Omega := \{(x,y) \in \mathbb{R}^2 : x > 1,\ 0 < y < x^{-\alpha}\}$

$$D := \dim_B(\infty,\Omega) = -1-\alpha, \qquad \mathcal{M}^D(\infty,\Omega) = \frac{1}{\alpha-1}$$

- $\dim_B(\infty,\Omega) \to -\infty$ and $\mathcal{M}^D(\infty,\Omega) \to 0$ as $\alpha \to +\infty$

Example

$\Omega := \{(x,y) \in \mathbb{R}^2 : x > 1,\ 0 < y < \mathrm{e}^{-x}\} \Rightarrow \dim_B(\infty,\Omega) = -\infty$

$\Omega \subseteq \mathbb{R}^N$, $|\Omega| < \infty$, fix $T > 0$ **Lapidus zeta function** of (∞, Ω):

$$\zeta_\infty(s, \Omega) := \int_{B_T(0)^c \cap \Omega} |x|^{-s-N} \, \mathrm{d}x$$

Theorem (Holomorphicity theorem [Ra])

(*a*) $\zeta_\infty(\,\cdot\,, \Omega)$ is holomorphic on $\{\mathrm{Re}\, s > \overline{\dim}_B(\infty, \Omega)\}$.

(*b*) The half-plane from (*a*) is optimal.

(*c*) $(\exists D = \dim_B(\infty, \Omega))(\underline{\mathcal{M}}^D(\infty, \Omega) > 0) \Rightarrow$
$\zeta_\infty(s, \Omega) \to +\infty$ for $s \in \mathbb{R}$ and $s \to D^+$

Theorem (Zeta function via Hölder equivalent norms [Ra])

- $\|\cdot\|$ *another norm in* $\mathbb{R}^N$, $\alpha \in (-\infty, 1]$
- $\|x\| = |x| + O\left(|x|^\alpha\right), \quad \text{as} \quad |x| \to +\infty, \ x \in \Omega$

$\Rightarrow$ $\zeta_\infty(\,\cdot\,, \Omega) - \zeta_\infty(\,\cdot\,, \Omega; \|\cdot\|)$ *is holomorphic on* (*at least*)

$$\{\mathrm{Re}\, s > \overline{D} - (1 - \alpha)\}$$

The inverted relative fractal drum

- let $\Phi(x) := x/|x|^2$ be the **geometric inversion** on $\mathbb{R}^N$

Theorem (Inversive invariance of complex dimensions [Ra])

Ω be a Lebesgue measurable subset of $\mathbb{R}^N$, $\mathbf{O}$ the origin and fix $T > 0$. Then, we have

$$\zeta_\infty(s, \Omega; T) = \zeta_{\mathbf{O}}(s, \Phi(\Omega); 1/T).$$

- (∞, Ω) and $(\mathbf{O}, \Phi(\Omega))$ have identical complex dimensions
- $\overline{\dim}_B(\mathbf{O}, \Phi(\Omega)) = \overline{\dim}_B(\infty, \Omega)$ $\underline{\dim}_B(\mathbf{O}, \Phi(\Omega)) \leq \underline{\dim}_B(\infty, \Omega)$

Theorem (The residue connection [Ra])

- $\Omega \subseteq \mathbb{R}^N$, $|\Omega| < \infty$, such that $\dim_B(\infty, \Omega) = D < -N$
- $0 < \underline{\mathcal{M}}^D(\infty, \Omega) \leq \overline{\mathcal{M}}^D(\infty, \Omega) < \infty$
- $\zeta_\infty(\,\cdot\,, \Omega)$ mero. extendable to a neighborhood of $s = D$

Then, D is its simple pole and

$$\underline{\mathcal{M}}^D(\infty, \Omega) \leq \frac{\operatorname{res}(\zeta_\infty(\,\cdot\,, \Omega), D)}{-(D+N)} \leq \overline{\mathcal{M}}^D(\infty, \Omega).$$

Moreover, if Ω is Minkowski measurable at infinity, then

$$\operatorname{res}(\zeta_\infty(\,\cdot\,, \Omega), D) = -(D+N)\mathcal{M}^D(\infty, \Omega).$$

Corollary

If both, (∞, Ω) and $(\mathbf{O}, \Phi(\Omega))$ are Minkowski measurable, then

$$\mathcal{M}^D(\mathbf{O}, \Phi(\Omega)) = \frac{D+N}{D-N}\mathcal{M}^D(\infty, \Omega).$$

- Wiener–Pitt Tauberian theorem: sufficiency for Minkowski measurablity at infinity and an upper bound result

Example (The two parameter unbounded set $\Omega_\infty^{(a,b)}$ [Ra])

- $a \in (0, 1/2)$, $b \in (1 + \log_{1/a} 2, +\infty)$

$$\Omega_m^{(a,b)} := \{(x, y) \in \mathbb{R}^2 \, : \, x > a^{-m}, \; 0 < y < x^{-b}\}, \quad m \geq 1$$

$$\Omega_\infty^{(a,b)} := \bigsqcup_{m=1}^{\infty} \bigsqcup_{i=1}^{2^{m-1}} \left(\Omega_m^{(a,b)}\right)_j$$

- $\left(\Omega_m^{(a,b)}\right)_j$ are translated copies of $\Omega_m^{(a,b)}$

Proposition ([Ra])

$$\zeta_\infty(s,\Omega_\infty^{(a,b)};|\cdot|_\infty)=\frac{1}{s+b+1}\cdot\frac{1}{a^{-(s+b+1)}-2}$$

$$\mathcal{P}(\zeta_\infty(s,\Omega_\infty^{(a,b)}))=\{-(b+1)\}\cup\left(\log_{1/a}2-(b+1)+\frac{2\pi}{\log(1/a)}\mathbb{i}\mathbb{Z}\right)$$

$$\overline{\dim}_B(\infty,\Omega_\infty^{(a,b)})=\log_{1/a}2-(b+1)$$

- the **oscillatory period** of $\Omega_\infty^{(a,b)}$: $\mathbf{p}(a)=2\pi/\log(1/a)$
- $\mathbf{p}(a)\to 0$ as $a\to 0^+$

Proposition ([Ra])

$$\overline{\mathcal{M}}^D(\infty,\Omega_\infty^{(a,b)})=\frac{1}{b-1}\cdot\frac{a^{1-b}-1}{a^{1-b}-2},\quad \underline{\mathcal{M}}^D(\infty,\Omega_\infty^{(a,b)})>0$$

Definition (Quasiperiodicity at infinity)

$$|B_t(0)^c \cap \Omega| = t^{N+D}(G(\log t) + o(1)) \quad \text{as} \quad t \to +\infty$$

$G\colon \mathbb{R} \to [m, M]$, $m > 0$, $D \in (-\infty, -N]$ is a given constant

(a) G **transcendentally** n-quasiperiodic

(b) G **algebraically** n-quasiperiodic

- $D < -2$, $(a_n)_{n\geq 1}$ such that $0 < a_n < 1/2$ and $a_n \searrow 0^+$ as $n \to +\infty$
- $b_n := \log_{1/a_n} 2 - D - 1$ $\Rightarrow$ $\dim_B(\infty, \Omega_\infty^{(a_n,b_n)}) = D$
- for $n \in \mathbb{N}$:

$$\widetilde{\Omega}_n := \frac{1}{2^n}\Omega_\infty^{(a_n,b_n)}$$

- define Ω^∞ as the disjoint union of translates of $\widetilde{\Omega}_n$

Proposition (∞-quasiperiodic maximal hyperfractal [Ra])

$D \in (-3,-2) \Rightarrow \Omega^\infty$ is ∞-**quasiperiodic** at infinity with quasiperiods

$$T_n := \log(1/a_n), \quad n \in \mathbb{N}.$$

Ω^∞ is **Minkowski nondegenerate** at infinity and **maximally hyperfractal**; that is, the poles of the $\zeta_\infty(\infty, \Omega^\infty)$ are dense in $\{\operatorname{Re} s = D\}$, i.e., it is a natural boundary.

- $a_1 \in (0,1/2)$, $a_{n+1} := a_1^{\sqrt{p_n}}$, p_n the n-th prime number
  Besicovitch $\Rightarrow \Omega^\infty$ is **algebraically** ∞**-quasiperiodic**
- $a_n := 1/p_{n+1}$, p_n the n-th prime number
  Baker $\Rightarrow \Omega^\infty$ is **transcendentally** ∞**-quasiperiodic**
- truncating the union: $\Omega^m \Rightarrow m$-quasiperiodic sets

The ϕ-shell Minkowski content and dimension [Ra]

- $\Omega \subseteq \mathbb{R}$, Lebesgue measurable, $|\Omega| \in [0, \infty]$, $\phi > 1$, $r \in \mathbb{R}$
- **upper r-dimensional ϕ-shell Minkowski content** of (∞, Ω):

$$\overline{\mathcal{M}}_{\phi}^{r}(\infty, \Omega) := \limsup_{t \to +\infty} \frac{|B_{t,\phi t}(0) \cap \Omega|}{t^{N+r}}$$

- $B_{t,\phi t}(0) := B_t(0)^c \cap B_{\phi t}(0)$
- **ϕ-shell function** of (∞, Ω): $t \mapsto |B_{t,\phi t}(0) \cap \Omega|$
- **upper ϕ-shell box dimension** of (∞, Ω):

$$\overline{\dim}_B^{\phi}(\infty, \Omega) := \sup\{r \in \mathbb{R} : \overline{\mathcal{M}}_{\phi}^{r}(\infty, \Omega) = +\infty\}$$

- for standard RFDs: $|A_{t/\phi,t} \cap \Omega|$ $A_{t/\phi,t} = (A_{t/\phi})^c \cap A_t$

Proposition (Sets of finite measure [Ra])

$\Omega \subseteq \mathbb{R}^N$, $|\Omega| < \infty$. Then, for every $\phi > 1$ and $r < -N$ we have

$$\overline{\mathcal{M}}_\phi^r(\infty,\Omega) \le \overline{\mathcal{M}}^r(\infty,\Omega) \le \frac{1}{1-\phi^{N+r}}\,\overline{\mathcal{M}}_\phi^r(\infty,\Omega),$$

$$\frac{1}{1-\phi^{N+r}}\,\underline{\mathcal{M}}_\phi^r(\infty,\Omega) \le \underline{\mathcal{M}}^r(\infty,\Omega).$$

Corollary

$$\lim_{\phi\to+\infty}\overline{\mathcal{M}}_\phi^r(\infty,\Omega) = \overline{\mathcal{M}}^r(\infty,\Omega)$$

$$\overline{\dim}_B^\phi(\infty,\Omega) = \overline{\dim}_B(\infty,\Omega); \qquad \underline{\dim}_B^\phi(\infty,\Omega) \le \underline{\dim}_B(\infty,\Omega)$$

$$\exists D := \dim_B^\phi(\infty,\Omega) \Rightarrow \dim_B(\infty,\Omega) = D$$

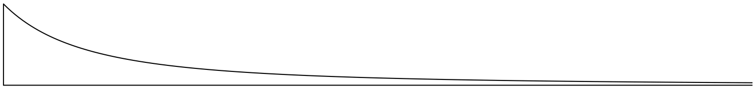
If Ω is ϕ-shell Minkowski measurable at infinity, then

$$\mathcal{M}^D(\infty,\Omega) = \frac{1}{1-\phi^{N+D}}\,\mathcal{M}_\phi^D(\infty,\Omega).$$

- $\dim_B^{\phi}(\infty,\mathbb{R}^N)=0$ $\quad$ $\mathcal{M}_{\phi}^{0}(\infty,\mathbb{R}^N)=\frac{\pi^{\frac{N}{2}}(\phi^N-1)}{\Gamma\left(\frac{N}{2}+1\right)}$
- $\underline{\dim}_B^{\phi}(\infty,\Omega)\le\overline{\dim}_B^{\phi}(\infty,\Omega)\le 0$
- $-N\le\overline{\dim}_B^{\phi}(\infty,\Omega)\le 0$

Example

$\Omega:=\{(x,y)\in\mathbb{R}^2\,:\,x>1,\ 0<y<x^{-1}\}\Rightarrow$
$\dim_B^{\phi}(\infty,\Omega)=-2$ and $\mathcal{M}_{\phi}^{-2}(\infty,\Omega)=\log\phi$



Example

$\Omega:=\left\{(x,y)\in\mathbb{R}^2\,:\,0<y<h\right\}\Rightarrow$
$\dim_B^{\phi}(\infty,\Omega)=-1$ and $\mathcal{M}_{\phi}^{-1}(\infty,\Omega)=2h(\phi-1)$

- $1 < \phi_1 < \phi_2$
- $\overline{\dim}_B^{\phi_1}(\infty, \Omega) = \overline{\dim}_B^{\phi_2}(\infty, \Omega)$ $\underline{\dim}_B^{\phi_1}(\infty, \Omega) \leq \underline{\dim}_B^{\phi_2}(\infty, \Omega)$

Theorem (Generalized Holomorphicity theorem [Ra])

Ω Lebesgue measurable subset of $\mathbb{R}^N$, $T > 0$ and $\phi > 1$ fixed. Then,

(a)

$$\zeta_\infty(s, \Omega) = \int_{{}_T\Omega} |x|^{-s-N}\,\mathrm{d}x$$

is holomorphic on the half-plane $\{\mathrm{Re}\, s > \overline{\dim}_B^{\phi}(\infty, \Omega)\}$.

(b) The half-plane from (a) is optimal.

(c) $(\exists D = \dim_B^{\phi}(\infty, \Omega))(\underline{\mathcal{M}}_{\phi}^{D}(\infty, \Omega) > 0) \Rightarrow$
$\zeta_\infty(s, \Omega) \to +\infty$ for $s \in \mathbb{R}$ and $s \to D^+$

Theorem (The generalized residue connection [Ra])

- $\phi > 1$ such that $D = \dim_B^{\phi}(\infty, \Omega)$ exists
- $0 < \underline{\mathcal{M}}_{\phi}^{D}(\infty, \Omega) \leq \overline{\mathcal{M}}_{\phi}^{D}(\infty, \Omega) < \infty$
- $\zeta_{\infty}(\,\cdot\,, \Omega)$ is mero. extendable to a neighborhood of $s = D$

Then, D is its simple pole.

$D \in [-N, 0] \Rightarrow$

$$\frac{1}{\phi^{N+D} \log \phi} \underline{\mathcal{M}}_{\phi}^{D}(\infty, \Omega) \leq \operatorname{res}(\zeta_{\infty}(\,\cdot\,, \Omega), D) \leq \frac{1}{\log \phi} \overline{\mathcal{M}}_{\phi}^{D}(\infty, \Omega)$$

$D \in (-\infty, -N) \Rightarrow$

$$\underline{\mathcal{M}}_{\phi}^{D}(\infty, \Omega) \leq -\frac{1 - \phi^{N+D}}{N + D} \operatorname{res}(\zeta_{\infty}(\,\cdot\,, \Omega), D) \leq \overline{\mathcal{M}}_{\phi}^{D}(\infty, \Omega)$$

Corollary

If Ω is ψ-shell Minkowski measurable at infinity for every $\psi \in (1, \phi)$, we have that

$$\operatorname{res}(\zeta_\infty(\,\cdot\,,\Omega), D) = \lim_{\psi \to 1^+} \frac{\mathcal{M}_\psi^D(\infty, \Omega)}{\log \psi}. \tag{4}$$

- $\Omega_\infty^{(a,b)}$ the two parameter set of **infinite** Lebesgue measure; that is, with $a \in (0, 1/2)$ and $b \in (\log_{1/a} 2, 1 + \log_{1/a} 2]$
- the limit (4) is also connected to the notion of **surface Minkowski content at infinity**
- **future work**: fractal tube formulas at infinity and a (ϕ-shell) Minkowski measurablity criterion at infinity
- **possible application**: PDEs on unbounded domains of finite or infinite volume, unbounded oscillations...

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