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Strain-invariant and path-independent modified fixed-pole non-linear finite element

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Abstract. The topic of this work is the family of recently developed novel beam finite elements. The elements are geometrically non-linear, while the constitutive law is taken to be linear. In order to form a finite element, the principle of virtual work is used and the iterative rotational parameters are interpolated. The finite elements are developed using a special case of the spatial description of kinematic quantities, the so-called fixed-pole description, but with a modification at the nodal level which enables using the standard degrees of freedom. The formulations in which the rotational update is interpolated using Lagrangian interpolation are bound to exhibit strain non-invariant and path-dependent behavior. In the presented family of finite elements, an invariant update of strain measures is implemented which solves the problem of strain non-invariance, while the use of the generalized shape functions produces path-independent results in two of the three proposed elements. These two interventions result in a family of geometrically exact strain-invariant and path-independent elements, which is demonstrated by a numerical example.

1 Introduction

In this paper we present a recently developed family of geometrically non-linear, spatial beam finite elements, based on the so-called fixed-pole approach. The fixed-pole approach was first introduced by Borri and Bottasso in 1994 [1] and thoroughly researched in a series of subsequent papers [2, 3, 4]. However, in all of the papers the fixed-pole concept is closely intertwined with the so-called helicoidal interpolation of the kinematic quantities. The helicoidal interpolation, which assumes that the reference axis of the beam element has a shape of a spatial helix and that both the translational and the rotational strain measures along it should be constant, also solved the problem of invariance of strain measures, but was developed only for a two-noded element. This idea was recently generalised to an element of an arbitrary order by Papa Dukić et. al. [5].

In our most recent paper [6] we implemented and analysed only the fixed-pole approach (i.e. the configuration-tensor approach), separated from the helicoidal interpolation. A formulation which uses standard degrees of freedom and is therefore compatible with standard, displacement-based finite element meshes, and is also strain-invariant and path-independent, was developed and will be briefly described here.

2 Modified fixed-pole approach

One of the key results of Bottasso and Borri is a unique update procedure for both the displacements and rotations. They have achieved this by first describing the kinematic quantities with respect to a fixed-pole (see also [1]) and then realising that the material description and the fixed-pole description are related via a *configuration* tensor. This approach was thoroughly explained in [4, 7]. However, the same results may be obtained alternatively, without any need to deal with Lie group theory associated with the configuration tensor. Imagine a beam cross-section with its spatial stress and stress-couple resultants $\mathbf{n}$ and $\mathbf{m}$ acting at the cross-section defined with a position vector $\mathbf{r}$. Now, let us reduce them to a fixed-pole (in this case, the origin of the spatial coordinate system) to obtain

$$\begin{Bmatrix} \overline{\mathbf{n}} \\ \overline{\mathbf{m}} \end{Bmatrix} = \begin{Bmatrix} \mathbf{n} \\ \mathbf{m} + \mathbf{r} \times \mathbf{n} \end{Bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \widehat{\mathbf{r}} & \mathbf{I} \end{bmatrix} \begin{Bmatrix} \mathbf{n} \\ \mathbf{m} \end{Bmatrix} , \tag{1}$$

where $\widehat{\mathbf{r}}$ is a skew-symmetric matrix satisfying $\widehat{\mathbf{r}}\mathbf{v} = \mathbf{r} \times \mathbf{v}$ for any 3D vector $\mathbf{v}$. We then use the standard Reisner-Simo kinematic equations [8], with translational and rotational spatial strain measures defined as

$$\boldsymbol{\gamma} = \mathbf{r}' - \mathbf{t}_1 \tag{2}$$

$$\boldsymbol{\kappa} = \boldsymbol{\Lambda}'\boldsymbol{\Lambda}^T . \tag{3}$$

with $\mathbf{t}_1$ as the unit vector orthogonal to the cross-section in the deformed state in the spatial coordinate system and $\boldsymbol{\Lambda}$ as the rotation matrix of a cross-section with respect to the origin of the spatial orthogonal frame $\{\mathbf{e}_1\,, \mathbf{e}_2\,, \mathbf{e}_3\}$ ($\boldsymbol{\Lambda} \in SO(3)$, $\det \boldsymbol{\Lambda} = +1$, $\boldsymbol{\Lambda}^{-1} = \boldsymbol{\Lambda}^T$). Using (2) and (3) we can write the strain energy either using the spatial description or the fixed-pole description. Since the strain energy

$$\phi = \frac{1}{2} \int_0^L (\boldsymbol{\gamma} \cdot \mathbf{n} + \boldsymbol{\kappa} \cdot \mathbf{m}) \, \mathrm{d}x = \frac{1}{2} \int_0^L (\overline{\boldsymbol{\gamma}} \cdot \overline{\mathbf{n}} + \overline{\boldsymbol{\kappa}} \cdot \overline{\mathbf{m}}) \, \mathrm{d}x \tag{4}$$

remains invariant to the change of strain measures, it follows that

$$\overline{\boldsymbol{\kappa}} = \boldsymbol{\kappa} ,$$
$$\overline{\boldsymbol{\gamma}} = \boldsymbol{\gamma} + \mathbf{r} \times \boldsymbol{\kappa} ,$$

are the fixed-pole strain measures. Varying (4) results in the virtual work of internal forces, which, after introducing the fixed-pole virtual displacements (see [7, 6] for details)

$$\overline{\delta\mathbf{r}} = \delta\mathbf{r} + \mathbf{r} \times \delta\boldsymbol{\vartheta} , \tag{5}$$

takes the elegant form

$$V_i = \int_0^L (\overline{\delta\mathbf{r}}' \cdot \overline{\mathbf{n}} + \delta\boldsymbol{\vartheta}' \cdot \overline{\mathbf{m}}) \, \mathrm{d}x . \tag{6}$$

Note that $\overline{\delta \mathbf{r}}$ in (5) is a non-integrable quantity, i.e. no existence of any $\bar{\mathbf{r}}$ is implied. Since all the forces can be transported to the fixed-pole, the fixed-pole point loading may be defined in an analogous manner as (1)

$$\begin{Bmatrix} \overline{\mathbf{F}}_0 \\ \overline{\mathbf{M}}_0 \end{Bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \hat{\mathbf{r}} & \mathbf{I} \end{bmatrix} \begin{Bmatrix} \mathbf{F}_0 \\ \mathbf{M}_0 \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} \overline{\mathbf{F}}_L \\ \overline{\mathbf{M}}_L \end{Bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \hat{\mathbf{r}} & \mathbf{I} \end{bmatrix} \begin{Bmatrix} \mathbf{F}_L \\ \mathbf{M}_L \end{Bmatrix}, \tag{7}$$

which enables us to define the virtual work of external forces. Note that, as well as in the previous case, the conjugate virtual displacements to the fixed-pole forces are $\overline{\delta \mathbf{r}}$ and $\delta\boldsymbol{\vartheta}$. Deciding to interpolate $\overline{\delta \mathbf{r}}$ and $\delta\boldsymbol{\vartheta}$ using Lagrangian polynomials

$$\begin{Bmatrix} \overline{\delta \mathbf{r}} \\ \delta\boldsymbol{\vartheta} \end{Bmatrix} = \sum_{i=1}^{N} I^i \begin{Bmatrix} \overline{\delta \mathbf{r}}_i \\ \delta\boldsymbol{\vartheta}_i \end{Bmatrix}, \tag{8}$$

we obtain the fixed-pole virtual work equation

$$\overline{G}^h \equiv \sum_{i=1}^{N} \left\langle \overline{\delta \mathbf{r}}_i^T \quad \delta\boldsymbol{\vartheta}_i^T \right\rangle \overline{\mathbf{g}}^i = 0 \quad \Rightarrow \quad \overline{\mathbf{g}}^i = \mathbf{0}, \tag{9}$$

with $\overline{\mathbf{g}}^i$ as the fixed-pole nodal residual

$$\overline{\mathbf{g}}^i \equiv \overline{\mathbf{q}}_i^i - \overline{\mathbf{q}}_e^i = \mathbf{0}, \tag{10}$$

and

$$\overline{\mathbf{q}}_i^i = \int_0^L I^{i\prime} \begin{Bmatrix} \overline{\mathbf{n}} \\ \overline{\mathbf{m}} \end{Bmatrix} \mathrm{d}x, \tag{11}$$

$$\overline{\mathbf{q}}_e^i = \delta_1^i \begin{Bmatrix} \overline{\mathbf{F}}_0 \\ \overline{\mathbf{M}}_0 \end{Bmatrix} + \delta_N^i \begin{Bmatrix} \overline{\mathbf{F}}_L \\ \overline{\mathbf{M}}_L \end{Bmatrix}, \tag{12}$$

as the nodal vectors of internal and external forces, respectively. Unfortunately, the unknowns of a non-linear system of equations (10) are *non-standard*. Although the non-standard results can be easily transformed to the standard position vectors and orientations, there are a few implementation complications. The first one is that the definition of the boundary conditions is not straightforward. The other problem is that these unknowns make it unable to combine these elements with standard finite element meshes. In order to try to keep the spirit of the fixed-pole approach, but also to have the standard system unknowns, we use the relationship (5) at a nodal level

$$\overline{\delta \mathbf{r}}_i = \delta \mathbf{r}_i + \mathbf{r}_i \times \delta\boldsymbol{\vartheta}_i, \tag{13}$$

and simply substitute it in (9) so that the virtual work equation becomes

$$\overline{G}^h \equiv \sum_{i=1}^{N} \left\langle \delta \mathbf{r}_i^T \quad \delta\boldsymbol{\vartheta}_i^T \right\rangle \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\hat{\mathbf{r}}_i & \mathbf{I} \end{bmatrix} \overline{\mathbf{g}}^i = \sum_{i=1}^{N} \left\langle \delta \mathbf{r}_i^T \quad \delta\boldsymbol{\vartheta}_i^T \right\rangle \tilde{\mathbf{g}}^i = 0 \quad \Rightarrow \quad \tilde{\mathbf{g}}^i = \mathbf{0}, \tag{14}$$

	$\mathbf{r}$	$\Delta\mathbf{r}$
MFP1	$\sum_{k=1}^{N} I^k \mathbf{r}_k$	$\sum_{j=1}^{N} I^j [\Delta\mathbf{r}_j + (\mathbf{r}_j - \mathbf{r}) \times \Delta\boldsymbol{\vartheta}_j]$
MFP2	$\sum_{k=1}^{N} I^k \mathbf{r}_k$	$\sum_{j=1}^{N} I^j \Delta\mathbf{r}_j$
MFP3	$\mathbf{r}_{new}(x) = \mathbf{r}_{old}(x) + \Delta\mathbf{r}(x)$	$\sum_{j=1}^{N} I^j [\Delta\mathbf{r}_j + (\mathbf{r}_j - \mathbf{r}) \times \Delta\boldsymbol{\vartheta}_j]$

Table 1: Interpolation options for MFP elements

with

$$\tilde{\mathbf{g}}^i \equiv \tilde{\mathbf{q}}_i^i - \tilde{\mathbf{q}}_e^i = \mathbf{0}\,, \tag{15}$$

and

$$\tilde{\mathbf{q}}_i^i = \int_0^L I^{i'} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \widehat{\mathbf{r} - \mathbf{r}_i} & \mathbf{I} \end{bmatrix} \begin{Bmatrix} \mathbf{n} \\ \mathbf{m} \end{Bmatrix} \, \mathrm{d}x \tag{16}$$

$$\tilde{\mathbf{q}}_e^i = \delta_1^i \begin{Bmatrix} \mathbf{F}_0 \\ \mathbf{M}_0 \end{Bmatrix} + \delta_N^i \begin{Bmatrix} \mathbf{F}_L \\ \mathbf{M}_L \end{Bmatrix}\,. \tag{17}$$

The non-linear equation (15) may be solved using the Newton-Raphson solution procedure. When linearising (15), account has to be taken of the fact that $\Delta\boldsymbol{\Lambda} = \widehat{\Delta\boldsymbol{\vartheta}}\boldsymbol{\Lambda}$ and $\Delta\boldsymbol{\gamma} = \Delta\mathbf{r}' - \Delta\boldsymbol{\vartheta} \times \mathbf{t}_1$ as well as $\Delta\boldsymbol{\kappa} = \mathbf{H}(\Delta\boldsymbol{\vartheta})\Delta\boldsymbol{\vartheta}'$ [8], where $\mathbf{H}(\Delta\boldsymbol{\vartheta}) = \mathbf{I} + (1 - \cos\Delta\vartheta)/\Delta\vartheta^2\,\widehat{\Delta\boldsymbol{\vartheta}} + (\Delta\vartheta - \sin\Delta\vartheta)/\Delta\vartheta^3\widehat{\Delta\boldsymbol{\vartheta}}^2$. In the present formulation, besides the orientation matrix $\boldsymbol{\Lambda}$ and the spatial strain measures $\boldsymbol{\gamma}$, $\boldsymbol{\kappa}$, the position vector $\mathbf{r}$ is also present in the integrals of the residual and the stiffness matrix, something not uniquely defined by the given interpolation (8) when applied to the Newton-Raphson iterative changes. To compute it, we propose three different interpolation options resulting in the family of modified fixed-pole finite elements (MFP) with different combinations of interpolations for $\mathbf{r}$ and $\Delta\mathbf{r}$ given in Table 1 (note that in MFP3 $\mathbf{r}$ is not interpolated, but computed directly at integration points). In all of the proposed elements, the iterative spins are interpolated using Lagrange polynomials

$$\Delta\boldsymbol{\vartheta} \doteq \sum_{j=1}^{N} I^j \Delta\boldsymbol{\vartheta}_j\,. \tag{18}$$

3 Family of strain-invariant and path-independent modified fixed-pole elements

The results in [6] show that the modified fixed-pole formulation presented in the previous section is not strain-invariant, and this is due to Lagrangian interpolation of the spins (18). Indeed, whenever rotational variables (iterative, incremental or total) are interpolated additively we are bound to encounter strain non-invariant results [9], and, with the exception of the total formulation, also path-dependent results. Although there are many successful solutions to the aforementioned problems (see eg. [10, 11, 12, 13, 14, 15]), in this work we want to solve the problem of strain-invariance and path-dependence with minimal interventions within the formulation itself. This is the reason we have chosen the generalised approach given by Jelenić and Crisfield [16] in which the total rotational matrix $\mathbf{\Lambda}(x)$ is decomposed using a reference orientation matrix $\mathbf{\Lambda}_I$ which is unique for the whole beam and rigidly attached to it, and an orientation matrix defining a local rotation $\mathbf{\Psi}^l(x)$ between the reference orientation matrix and the total orientation matrix, so that

$$\mathbf{\Lambda}(x) \doteq \mathbf{\Lambda}^h(x) = \mathbf{\Lambda}_I \exp \widehat{\mathbf{\Psi}}^{lh}(x) \,. \tag{19}$$

As it is shown in [16] the approximated strain measures obtained when using the interpolation of local rotations are invariant to rigid body motion. Implementation of this interpolation into the modified fixed-pole formulation is straightforward with details given in [6, 7]. The above decomposition is equivalent to a generalised interpolation for the spins $\Delta\boldsymbol{\vartheta} \doteq \sum_{j=1}^{N} \tilde{\mathbf{I}}^j \Delta\boldsymbol{\vartheta}_j$ instead of that given in (18) with the generalised shape functions and their derivatives [16]

$$\tilde{\mathbf{I}}^j(x) = \begin{cases} \mathbf{I} - \mathbf{\Lambda}_I \mathbf{H}\left[\mathbf{\Psi}^{lh}(x)\right] \sum_{m=1}^{N}(1-\delta_I^m) I^m(x) \mathbf{H}^{-1}(\mathbf{\Psi}_m^l)\mathbf{\Lambda}_I^T \,, & \text{if } j = I \,, \\ \mathbf{\Lambda}_I \mathbf{H}\left[\mathbf{\Psi}^{lh}(x)\right] I^j(x) \mathbf{H}^{-1}(\mathbf{\Psi}_j^l)\mathbf{\Lambda}_I^T \,, & \text{if } j \neq I \,. \end{cases} \tag{20}$$

$$\tilde{\mathbf{I}}^{j\prime}(x) = \begin{cases} -\mathbf{\Lambda}_I \left\{ \mathbf{H}'\left[\mathbf{\Psi}^{lh}(x)\right] \sum_{m=1}^{N} (1-\delta_I^m) I^m(x) \mathbf{H}^{-1}(\mathbf{\Psi}_m^l) + \right. & \\ \left. +\mathbf{H}\left[\mathbf{\Psi}^{lh}(x)\right] \sum_{m=1}^{N} (1-\delta_I^m) I^{m\prime}(x) \mathbf{H}^{-1}(\mathbf{\Psi}_m^l) \right\} \mathbf{\Lambda}_I^T \,, & \text{if } j = I \,, \\ \mathbf{\Lambda}_I \left\{ \mathbf{H}'\left[\mathbf{\Psi}^{lh}(x)\right] I^j(x) + \mathbf{H}\left[\mathbf{\Psi}^{lh}(x)\right] I^{j\prime}(x) \right\} \mathbf{H}^{-1}(\mathbf{\Psi}_j^l)\mathbf{\Lambda}_I^T \,, & \text{if } j \neq I \,, \end{cases} \tag{21}$$

with $\mathbf{H}\left[\mathbf{\Psi}^{lh}(x)\right]$, $\mathbf{H}^{-1}\left[\mathbf{\Psi}^{lh}(x)\right]$ and $\mathbf{H}'\left[\mathbf{\Psi}^{lh}(x)\right]$ explicitly given in [16]. This results in a new family of elements – the generalised modified fixed-pole elements (GMFP). Expressions for the stiffness matrices can be found in [6] or [7].

4 Numerical example: $45°$ cantilever bend

We analyse a well-known spatial problem of a planar curved cantilever loaded with a vertical out-of-plane concentrated force of magnitude $F = 600$ at its tip, as shown in Figure 1. The geometric and material characteristics are given as follows: $A = A_2 = A_3 = 1$, $J_1 = 16.656 \times 10^{-2}$, $I_2 = I_3 = 8.3333 \times 10^{-2}$, $E = 10^7$ and $G = 0.5 \times 10^7$. The cantilever is in the horizontal plane and it represents one eighth of a circle of radius $R = 100$ and is modelled using eight equally long straight linear elements. $N - 1$ point Gaussian quadrature (reduced integration) is used for evaluating the internal forces vector and its stiffness matrix. The Newton-Raphson solution procedure is used for obtaining solutions inside each load increment, with two convergence criteria which must both be satisfied – the displacement norm $\delta_u = 10^{-7}$ and the residual norm $\delta_r = 10^{-7}$. The

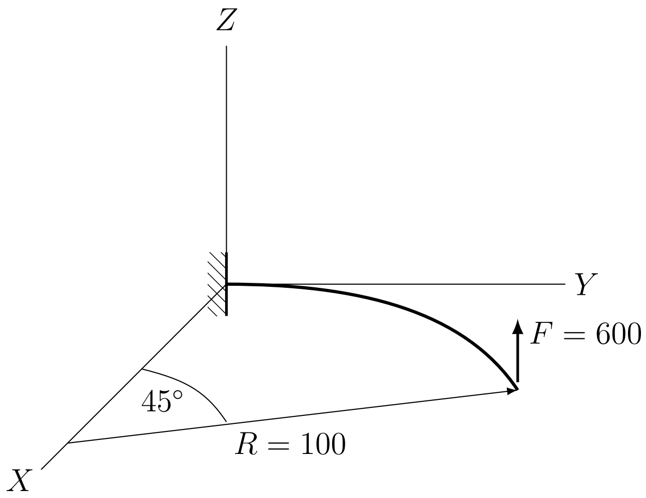

Figure 1: 45° cantilever bend

example was run using self-made programs coded in *Wolfram Mathematica*.

The load is divided into 3, 7, 10, 15 and 20 equal load increments and the problem is first run using MFP elements. The results in Table 2 show that the robustness of the proposed formulation is reduced in comparison to [17] or [16], regardless of the interpolation option applied – in all of the proposed interpolations the minimum number of equal load increments is 7. More importantly, all the elements exhibit path-dependent behaviour.

Formulation	Increments	u_1	u_2	u_3
MFP1	3	-	-	-
MFP1	7	13.48783	-23.47882	53.36984
MFP1	10	13.48789	-23.47877	53.36983
MFP1	15	13.48784	-23.47876	53.36980
MFP1	20	13.48782	-23.47876	53.36981
MFP2	3	-	-	-
MFP2	7	13.48784	-23.47883	53.36989
MFP2	10	13.48789	-23.47876	53.36983
MFP2	15	13.48785	-23.47877	53.36984
MFP2	20	13.48783	-23.47877	53.36985
MFP3	3	-	-	-
MFP3	7	13.48272	-23.53052	53.18321
MFP3	10	13.48194	-23.53197	53.15572
MFP3	15	13.48130	-23.53258	53.13641
MFP3	20	13.48092	-23.53273	53.12735

Table 2: Tip displacement components obtained using different load incrementation

Formulation	Increments	u_1	u_2	u_3
Invariant [16]	3	13.48286	-23.47949	53.37152
GMFP1	3	-	-	-
GMFP1	7	13.48286	-23.47949	53.37152
GMFP1	10	13.48286	-23.47949	53.37152
GMFP2	3	13.48286	-23.47949	53.37152
GMFP2	7	13.48286	-23.47949	53.37152
GMFP2	10	13.48286	-23.47949	53.37152
GMFP3	3	-	-	-
GMFP3	5	13.51873	-23.53052	53.33627
GMFP3	10	13.48871	-23.53052	53.15864
GMFP3	15	13.48871	-23.52958	53.15864
GMFP3	20	13.48737	-23.53049	53.13021

Table 3: Tip displacement components using different load incrementation – the generalised approach

Next, we run the example using GMFP elements. Results in Table 3 clearly show that the robustness is increased when generalised shape functions are implemented. Furthermore, GMFP1 and GMFP2 turn out to be path-independent while GMFP3 remains path-dependent.

5 Conclusions and future work

A family of novel spatial geometrically exact beam finite elements is presented which are conceptually founded in the fixed-pole approach proposed by Bottasso and Borri, of which we give our own interpretation called the modified fixed-pole formulation [6]. This is actually a family of elements because three interpolation options (each one formulating its own element) arise as a result of the fact that the position vector has to be approximated in some manner. As the modified fixed-pole formulation stems from the same framework as the standard, iterative elements given by Simo and Vu-Quoc, all of the proposed interpolation options exhibit path-dependence and non-invariance of strain measures. However, these interpolation options are enhanced by introducing the generalised shape functions given in [16], forming the generalised modified fixed-pole family of elements. Numerical example suggests that GMFP1 and GMFP2 are well behaved, strain-invariant and path-independent elements, while GMFP3 exhibits path-dependent behaviour. This is due to the fact that within this option the displacements are *not* interpolated, but directly updated at integration points on the basis of non-invariant interpolation of the iterative changes of the position vector and such an interpolation cannot be proven to be strain-invariant and path-independent. In future, we attempt to derive a strain-invariant and path-independent interpolation for the position vector for this interpolation option as well.

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References

[1] Borri, M., Bottasso, C. L. An intrinsic beam model based on a helicoidal approximation—Part I: Formulation. *Int. J. Numer. Methods Eng.*, **37** (13):2267–2289, 1994.

[2] Bottasso, C. L., Borri, M. Energy preserving/decaying schemes for non-linear beam dynamics using the helicoidal approximation. *Comput. Methods Appl. Mech. Eng.*, **143** (3-4):393–415, 1997.

[3] Bottasso, C. L., Borri, M. Integrating finite rotations. *Comput. Methods Appl. Mech. Eng.*, **164** (3-4):307–331, 1998.

[4] Borri, M., Trainelli, L., Bottasso, C. L. On Representations and Parameterizations of Motion. *Multibody Syst. Dyn.*, **4** (2-3):129–193, 2000.

[5] Papa Dukić, E., Jelenić, G., Gaćeša, M. Configuration-dependent interpolation in higher-order 2D beam finite elements. *Finite Elem. Anal. Des.*, **78**:47–61, 2014.

[6] Gaćeša, M., Jelenić, G. Modified fixed-pole approach in geometrically exact spatial beam finite elements. *Finite Elem. Anal. Des.*, **99**:39–48, 2015.

[7] Gaćeša, M. *Fixed-pole concept in 3D beam finite elements – relationship to standard approaches and analysis of different interpolations*. PhD thesis, University of Rijeka, 2015.

[8] Simo, J. C. A finite strain beam formulation. The three-dimensional dynamic problem. Part I. *Comput. Methods Appl. Mech. Eng.*, **49** (1):55–70, 1985.

[9] Crisfield, M. A., Jelenić, G. Objectivity of strain measures in the geometrically exact three-dimensional beam theory and its finite-element implementation. *Proc. R. Soc. A Math. Phys. Eng. Sci.*, **455** (1983):1125–1147, 1999.

[10] Ibrahimbegović, A., Taylor, R. On the role of frame-invariance in structural mechanics models at finite rotations. *Comput. Methods Appl. Mech. Eng.*, **191** (45):5159–5176, 2002.

[11] Ghosh, S., Roy, D. A frame-invariant scheme for the geometrically exact beam using rotation vector parametrization. *Comput. Mech.*, **44** (1):103–118, 2009.

[12] Zhong, H., Zhang, R., Xiao, N. A quaternion-based weak form quadrature element formulation for spatial geometrically exact beams. *Arch. Appl. Mech.*, **84** (12):1825–1840, 2014.

[13] Betsch, P., Steinmann, P. Frame-indifferent beam finite elements based upon the geometrically exact beam theory. *Int. J. Numer. Methods Eng.*, **54** (12):1775–1788, 2002.

[14] Romero, I., Armero, F. An objective finite element approximation of the kinematics of geometrically exact rods and its use in the formulation of an energy-momentum conserving scheme in dynamics. *Int. J. Numer. Methods Eng.*, **54** (12):1683–1716, 2002.

[15] Zupan, D., Saje, M. The three-dimensional beam theory: Finite element formulation based on curvature. *Comput. Struct.*, **81** (18-19):1875–1888, 2003.

[16] Jelenić, G., Crisfield, M. A. Geometrically exact 3D beam theory: implementation of a strain-invariant finite element for statics and dynamics. *Comput. Methods Appl. Mech. Eng.*, **171** (1-2):141–171, 1999.

[17] Simo, J. C., Vu-Quoc, L. A three-dimensional finite-strain rod model. Part II: Computational aspects. *Comput. Methods Appl. Mech. Eng.*, **58** (1):79–116, 1986.