Lie algebra type noncommutative phase spaces are Hopf algebroids

Stjepan Meljanac meljanac@irb.hr
Theoretical Physics Division, Institute Rudjer Bošković
Bijenička cesta 54, P.O.Box 180, HR-10002 Zagreb, Croatia

Zoran Škoda zoran.skoda@uhk.cz
Faculty of Science, University of Hradec Králové
Rokitanského 62, Hradec Králové, Czech Republic

Martina Stojić stojic@math.hr
Department of Mathematics, University of Zagreb
Bijenička cesta 3, HR-10000 Zagreb, Croatia

Abstract. For a noncommutative configuration space whose coordinate algebra is the universal enveloping algebra of a finite dimensional Lie algebra, it is known how to introduce an extension playing the role of the corresponding noncommutative phase space, namely by adding the commuting deformed derivatives in a consistent and nontrivial way, therefore obtaining certain deformed Heisenberg algebra. This algebra has been studied in physical contexts, mainly in the case of the kappa-Minkowski space-time. Here we equip the entire phase space algebra with a coproduct, so that it becomes an instance of a completed variant of a Hopf algebroid over a noncommutative base, where the base is the enveloping algebra.

Keywords: universal enveloping algebra, noncommutative phase space, deformed derivative, Hopf algebroid, completed tensor product

AMS classification: 16S30,16S32, 16S35, 16Txx

1. Introduction

Recently, a number of physical models has been proposed [1, 12, 19], where the background geometry is described by a noncommutative *configuration* space of Lie algebra type. Descriptively, its coordinate algebra is the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ with basis $\hat{x}_1, \ldots, \hat{x}_n$ (noncommutative coordinates). So-called κ-Minkowski space is the most explored example [11, 19, 20, 21]. That space has been used to build a model featuring the double special relativity, a framework modifying special relativity, proposed to explain some phenomena observed in the high energy gamma ray bursts.

The noncommutative *phase* space of the Lie algebra $\mathfrak{g}$ is introduced by enlarging $U(\mathfrak{g})$ with additional associative algebra generators, the *deformed derivatives*, which act on $U(\mathfrak{g})$ via an action $\blacktriangleright$ satisfying deformed Leibniz rules [23, 27]. The subalgebra generated by the deformed derivatives is commutative. In fact, this commutative algebra is a topological Hopf algebra isomorphic to the full algebraic dual $U(\mathfrak{g})^*$ of the enveloping algebra. In this article, we extend the coproduct of the topological Hopf algebra $U(\mathfrak{g})^*$ of deformed derivatives to a coproduct $\Delta : H \to H\hat{\otimes}_{U(\mathfrak{g})}H$ on the whole phase space H (and its completion $\hat{H}$); this coproduct is moreover a part of a (formally completed) Hopf algebroid structure on H over the noncommutative base algebra $\mathcal{A} = U(\mathfrak{g})$. Descriptively, a Hopf algebroid is an associative bialgebroid (Definition 2 in Subsection 1.1) with an antipode map (the antipode is treated in Section 7).

The notion of a Hopf algebroid in this paper is slightly modified regarding that the tensor product $\hat{\otimes}_{U(\mathfrak{g})}$ in the definition of the coproduct is understood in a completed sense; a part of the definition still needs the tensor products without completions. Our bialgebroid structure is similar but a bit weaker than the bialgebroid *internal* [3] to the tensor category of complete cofiltered vector spaces; a true internal variant is possible in a more intricate monoidal category involving filtrations of cofiltrations and is treated along with generalizations in [25].

The noncommutative phase space of Lie type is nontrivially isomorphic to an infinite-dimensional version $U(\mathfrak{g})\sharp U(\mathfrak{g})^* \cong U(\mathfrak{g})\sharp\hat{S}(\mathfrak{g}^*)$ of the Heisenberg double of $U(\mathfrak{g})$ [27]. Heisenberg doubles of finite dimensional Hopf algebras are known to carry a Hopf algebroid structure [7, 18]. However, our starting Hopf algebra $U(\mathfrak{g})$ is infinite-dimensional, though filtered by finite-dimensional pieces. While the generalities on such filtered algebras can be used to obtain the Hopf algebroid structure [25], we here use the specific features of $U(\mathfrak{g})$ instead, and in particular the matrix $\mathcal{O}$ introduced in the Section 3 and used to define the crucial part of the bialgebroid structure, the *target* map $\beta : \hat{x}_\alpha \mapsto \sum_\beta \hat{x}_\beta \otimes (\mathcal{O}^{-1})^\beta_\alpha$.

From a dual geometric viewpoint, where $U(\mathfrak{g})$ is viewed as the algebra of left invariant differential operators on a Lie group, the matrix $\mathcal{O}$ is interpreted as a transition matrix between a basis of left invariant and a basis of right invariant vector fields. Then our phase space appears as the algebra of formal differential operators $\mathrm{Diff}^\omega(G, e)$ around the unit e of the Lie group G integrating $\mathfrak{g}$. Every formal differential operator is a finite sum of products of the form $f_s D_s$ where D_s is an invariant differential operator (belonging to $U(\mathfrak{g})$) and f_s is a formal function (this decomposition amounts to a Hopf algebraic smash product in the algebraic part of the paper). By L. Schwartz's theorem [9], the space $J^\infty(G, e)$ of formal functions at e and $U(\mathfrak{g})$ are dual by evaluating the differential operator on a function at e; the duality equips $J^\infty(G, e)$ with a topological Hopf algebra structure with coproduct Δ^J. Then define the coproduct of the $J^\infty(G, e)$-bialgebroid $\mathrm{Diff}^\omega(G, e)$ by the *scalar ex-*

tension formula $\Delta(f_s D_s) = (f_s \otimes 1)\Delta_J(D_s)$, where the tensor product is over $U(\mathfrak{g})$ (and needs some completion). Though the noncommutativity of the base and completions makes it far more complicated, this is similar to the classical example ([29]), where the algebra $\mathrm{Diff}(M)$ of smooth differential operators on a manifold M becomes a bialgebroid over the commutative base $C^\infty(M)$ via the coproduct which multiplicatively extends the rule $X \mapsto 1 \otimes X + X \otimes 1$ for vector fields X and $f \mapsto f \otimes 1$ for functions f; the canonical embedding $C^\infty(M) \hookrightarrow \mathrm{Diff}(M)$ serves both as the source and the target map.

While we justified the initial formulas in Section 2 by the formal geometry on a Lie group (a wider picture in formal geometry will be exhibited in [26]), much of the paper is continued in the dual algebraic language dictated by the physical motivation where the Lie algebra generators are interpreted as deformed coordinates, rather than invariant vector fields. A different variant of this Hopf algebroid structure has been outlined in [15, 16, 17], for the special case when the Lie algebra is the κ-Minkowski space, at a physical level of rigor.

1.1. Algebraic preliminaries

We assume familiarity with bimodules, coalgebras, comodules, bialgebras, Hopf algebras, Hopf pairings and the Sweedler notation for comultiplications (coproducts) $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$, and right coactions $\rho(v) = \sum v_{(0)} \otimes v_{(1)}$ (with or without the explicit summation sign). We do not assume previous familiarity with Hopf algebroids. In noncommutative geometry, one interprets Hopf algebroids [2, 4, 7, 18] as formal duals to quantum groupoids.

The generic symbols for the multiplication map, comultiplication, counit and antipode will be $m, \Delta, \epsilon, \mathcal{S}$, with various subscripts and superscripts. All algebras are over a fixed ground field $\boldsymbol{k}$ of characteristic zero (in physical applications $\mathbb{R}$ or $\mathbb{C}$). The Einstein summation convention on repeated indices is assumed throughout the article. The opposite algebra of an associative algebra A is denoted A^{op}, and the coopposite coalgebra to $C = (C, \Delta)$ is $C^{\mathrm{co}} = (C, \Delta^{\mathrm{op}})$. Given a vector space V, denote its algebraic dual by $V^* := \mathrm{Hom}(V, \boldsymbol{k})$, and the corresponding symmetric algebras $S(V)$ and $S(V^*)$. If an algebra A is graded, we label its graded (homogeneous) components by upper indices, $A = \oplus_{i=0}^\infty A^i$, $A^i \cdot A^j \subset A^{i+j}$, and, if B is filtered, we label its filtered components $B_0 \subset B_1 \subset B_2 \subset \ldots$ by lower indices, $B_i \cdot B_j \subset B_{i+j}$ and $B = \cup_{i=0}^\infty B_i$. When applied to spaces, we use the hat symbol $\hat{}$ for completions. We often use the completion of the symmetric algebra $\hat{S}(V^*) = \varprojlim_i S_i(V^*) \cong \prod_i S^i(V^*)$ of a Lie algebra V (our main example is when V is the underlying space of a Lie algebra $\mathfrak{g}$) which is the completion of $S(V^*)$ with respect to the degree of polynomial; it may be identified with the formal power series ring $\boldsymbol{k}[[\partial^1, \ldots, \partial^n]]$ in n variables. For our purposes, it is the same to regard this ring, as well as the algebraic duals $U(\mathfrak{g})^*$ and $S(\mathfrak{g})^*$ of the enveloping and symmetric algebras, either as topological or as cofiltered algebras (see Appendix A.2); the continuous linear maps then translate as linear maps distributive over formal sums.

The n-th **Weyl algebra** A_n is the associative algebra generated by $x_1, \ldots, x_n$, $\partial^1, \ldots, \partial^n$ subject to relations $[x_\alpha, x_\beta] = [\partial^\alpha, \partial^\beta] = 0$ and $[\partial^\alpha, x_\beta] = \delta^\alpha_\beta$. It has a vector space basis formed by all expressions of the form $x_{\alpha_1} \cdots x_{\alpha_k} \partial^{\beta^1} \cdots \partial^{\beta^l}$; if we define the degree of this element as $\beta^1 + \ldots + \beta^l$, then A_n becomes a filtered algebra; it has no zero divisors and the elements of the degree at least k form an ideal $(A_n)_{\mathrm{deg} \geq k}$. Thus we can form the (semi)completed Weyl algebra $\hat{A}_n = \lim_s A_n/(A_n)_{\mathrm{deg}\geq s}$ ("completed by the degree"). In the geometric part of the paper we shall also consider the n-th **covariant Weyl algebra** A_n^{cov} where the position of the upper versus lower indices in the notation will be interchanged; hence $[\partial_\alpha, x^\beta] = \delta^\beta_\alpha$. Here we shall similarly dually complete by the dual degree which is $\alpha_1 + \ldots + \alpha_k$ on the basis elements $x^{\alpha_1} \cdots x^{\alpha_k} \partial_{\beta_1} \cdots \partial_{\beta_l}$ to obtain the completion $\hat{A}_n^{\mathrm{cov}}$. The correspondence $x^\alpha \mapsto \partial^\alpha$ and $\partial_\beta \mapsto x_\beta$ extends to the canonical antiisomorphism $\hat{A}_n^{\mathrm{cov}} \to \hat{A}_n$. The **Fock space** is the faithful representation of A_n on the polynomial algebra in $x_1, \ldots, x_n$ where each x_μ acts as the multiplication operator and ∂^μ as the partial derivative; this action extends continuously to a unique action of $\hat{A}_n$ also called Fock.

DEFINITION 1. *Let A be an algebra and B a bialgebra.*

A left action $\triangleright : B \otimes A \to A$ (right action $\triangleleft : A \otimes B \to A$), is a left (right) **Hopf action** *if $b \triangleright (aa') = \sum (b_{(1)} \triangleright a)(b_{(2)} \triangleright a')$ and $b \triangleright 1 = \epsilon(b)1$ (or, respectively, $(aa') \triangleleft b = \sum (a \triangleleft b_{(1)})(a \triangleleft b_{(2)})$ and $1 \triangleleft b = \epsilon(b)1$), for all $a, a' \in A$ and $b, b' \in B$. We then also say that A is a left (right) B-module algebra. As usual, we freely exchange actions and representations; thus by abuse of language we say that a representation $\psi : B \to \mathrm{End}(A)$ is a left Hopf action (representation) if $b \otimes a \mapsto \psi(b)(a)$ is a left Hopf action. Given a left Hopf action, the* **smash product** *$A \sharp B$ (for a right Hopf action, the smash product $B \sharp A$) is an associative algebra which is a tensor product vector space $A \otimes B$ ($B \otimes A$) with the multiplication bilinearly extending the formulas*

$$(a\sharp b)(a'\sharp b') = \sum a(b_{(1)} \triangleright a')\sharp b_{(2)} b', \quad a, a' \in A, b, b' \in B,$$
$$(b\sharp a)(b'\sharp a') = \sum bb'_{(1)} \sharp (a \triangleleft b'_{(2)})a', \quad a, a' \in A, b, b' \in B,$$

where, for emphasis, one writes $a \sharp b := a \otimes b$.

Note that $A\sharp 1$ and $1\sharp B$ are subalgebras in $A\sharp B$, canonically isomorphic to A and B. If B is a Hopf algebra with an antipode $\mathcal{S}$, we may replace a left Hopf action $\psi : B \to \mathrm{End}\, A$ by a homomorphism $\psi \circ \mathcal{S} : B^{\mathrm{co}} \to \mathrm{End}^{\mathrm{op}}\, A$, yielding a *right* Hopf action $\triangleleft : A \otimes B^{\mathrm{co}} \to A$, $\triangleleft : a \otimes b \mapsto a \triangleleft b := \mathcal{S}(b) \triangleright a$, and enabling us to define the smash product $B^{\mathrm{co}}\sharp A$. If B is cocommutative (for instance, $B^{\mathrm{co}} = B = U(\mathfrak{g})$ below) then $\mathcal{S}^2 = \mathrm{id}$ and there is an isomorphism $A\sharp B \cong B\sharp A$ of algebras, $a\sharp b \mapsto \sum b_{(1)}\sharp(a\triangleleft b_{(2)})$, with the inverse $b\sharp a \mapsto \sum (b_{(1)} \triangleright a)\sharp b_{(2)}$.

Simple examples of smash products are the Weyl algebras A_n (and completions $\hat{A}_n$). Indeed the symmetric algebra $S(V)$ of a vector space is a Hopf algebra with $\Delta(x) = 1 \otimes x + x \otimes 1$ for generators $x \in V$; and if V is a vector space spanned by $x_1, \ldots, x_n$ then there are canonical isomorphisms $\hat{A}_n = S(V)\sharp S(V)^* \cong S(V)^*\sharp S(V)$ where the smash products are constructed using the (right and left) Hopf actions of $S(V)$ on $S(V)^*$ defined using duality. More generally, replacing $S(V)$ by its noncommutative generalization – the universal enveloping algebra $U(\mathfrak{g})$ – we explictly construct in Section 3 certain smash products $H^L = U(\mathfrak{g}^L)\sharp S(\mathfrak{g})^*$ and $H^R = S(\mathfrak{g})^*\sharp U(\mathfrak{g}^R)$, both isomorphic as algebras to $\hat{A}_n$; their special smash product structures however give rise to a left and a right $U(\mathfrak{g})$-bialgebroid structures (in a completed sense).

DEFINITION 2. *[2, 7] A* **left bialgebroid** $(H, m, \alpha, \beta, \Delta, \epsilon)$ *over the* **base algebra** $\mathcal{A}$ *(shortly, left $\mathcal{A}$-bialgebroid) consists of*

- (**total algebra**) *an associative algebra H with multiplication m*
- *($\mathcal{A}$-bimodule structure on H) morphisms of algebras* **source** $\alpha : \mathcal{A} \to H$ *and* **target** $\beta : \mathcal{A}^{\mathrm{op}} \to H$ *satisfying $[\alpha(a), \beta(b)] = 0$ for all $a, b \in A$, hence equipping H with the structure of an $\mathcal{A}$-bimodule via the formula $a.h.b := \alpha(a)\beta(b)h$ for $a, b \in \mathcal{A}$ and $h \in H$;*
- *($\mathcal{A}$-coring structure on H, see Definition 5) $\mathcal{A}$-bimodule maps coproduct $\Delta : H \to H \otimes_{\mathcal{A}} H$ and the corresponding counit $\epsilon : H \to \mathcal{A}$ making (H, Δ, ϵ) into a comonoid (coalgebra) in the category of $\mathcal{A}$-bimodules equipped with the tensor product $\otimes_{\mathcal{A}}$ of $\mathcal{A}$-bimodules.*

In addition, Δ and ϵ need to be compatible with the multiplication m, but in more subtle way than in the bialgebra case. Namely, if the base $\mathcal{A}$ is noncommutative, $H \otimes_{\mathcal{A}} H$ does not inherit a well defined multiplication from the usual tensor product $H \otimes H$ over the ground field. Instead, one demands that the image of Δ is within the subspace $H \times_{\mathcal{A}} H$ consisting of all $\sum_i b_i \otimes b'_i$ in $H \otimes_{\mathcal{A}} H$ such that $\sum_i b_i \otimes b'_i \alpha(a) = \sum_i b_i \beta(a) \otimes b'_i$ for all $a \in \mathcal{A}$; it appears that $H \times_{\mathcal{A}} H$ automatically inherits the well-defined algebra structure from $H \otimes H$. We demand that after corestricting Δ to *this smaller codomain* $H \times_{\mathcal{A}} H$*,* Δ *becomes an algebra map. Similarly, the counit* ϵ *is not required to be an algebra map, but a weaker condition is assumed: the formula* $h \otimes a \mapsto \epsilon(h\alpha(a))$ *needs to define an action* $H \otimes A \to A$ *restricting to the multiplication* $A \otimes A \to A$*.*

1.2. Preliminaries on formal differential operators

If $\boldsymbol{k}$ is $\mathbb{R}$ or $\mathbb{C}$ then for any smooth manifold M of dimension n we denote by $C^\infty(M)$ the algebra of smooth functions. Given a point $e \in M$ and a natural number s, recall that an s-jet of functions around e is a class of equivalence of smooth functions defined locally around e, where two functions are equivalent if they are defined in some neighborhood of e and their Taylor series up to order s agree at e. All s-jets around e form a vector space $J^s(M,e)$ with canonical projections $J^{s+1}(M,e) \to J^s(M,e)$ and the inverse limit $J^\infty(M,e) = \varprojlim_s J^s(M,e)$ is by definition the space of formal functions around e; the spaces $J^s(M,e)$ with their canonical projections then form a cofiltration of $J^\infty(M,e)$. In any chosen coordinate chart around e the formal functions are represented by formal power series in $n = \dim M$ indeterminates. Similarly, one can consider s-jets of maps to other manifolds (including the coordinate charts viewed as maps to $\boldsymbol{k}^n$) and, in the limit, formal maps and formal charts. Since, by a theorem of É. Borel [5], each formal power series over $\mathbb{R}$ is a Taylor series of a non-unique smooth function, a formal function may be viewed as an ∞-jet of an actual but non-unique smooth function. Thus those quantities in differential geometry which depend only on their Taylor series have formal analogues, namely the ∞-jets of actual locally defined smooth quantities.

A regular differential operator $Q \in \mathrm{Diff}_s(M)$ of degree up to s is in every smooth chart a sum of the form $\sum_{|J|\leq s} q^J \partial_J$ where the sum is over multiindices $J = (j_1, \ldots, j_n) \in \mathbb{N}_0^n$ with $|J| = j_1 + \ldots + j_n \leq s$, and q^J is a smooth function defined over the chart. At the jet level, the ring of regular differential operators $\mathrm{Diff}(M) = \cup_{s\in\mathbb{N}} \mathrm{Diff}_s(M) \subset \mathrm{End}_{\mathbb{R}}(C^\infty(M))$ gives rise to the ring $\mathrm{Diff}^\omega(M,e) \subset \mathrm{End}_{\mathbb{R}}(J^\infty(M,e))$ of formal differential operators at e, namely the ∞-jets of regular differential operators around e. A formal differential operator at e is a sum $\sum_{|J|\leq s} q^J \partial_J$ where $q^J = q^J(x^1, \ldots, x^n)$ is a formal function at e; these sums can be viewed as elements of the semicompleted Weyl algebra $\hat{A}_n^{\mathrm{cov}}$. The evaluation of a differential operator at a function at e is a rule for a degenerate pairing between $\mathrm{Diff}^\omega(M,e)$ and $J^\infty(M,e)$. If $M = G$ is a Lie group and $e \in G$ the unit element, then it restricts to a nondegenerate pairing between the subspace $\mathrm{Diff}^{\omega,R}(G,e) \subset \mathrm{Diff}^\omega(G,e)$ of right invariant formal differential operators and $J^\infty(G,e)$.

For the later transition to the noncommutative point of view, it is useful to consider also the algebra of differential operators acting to the *left*, which is simply the opposite algebra $\mathrm{Diff}^{\mathrm{op}}(M) \subset \mathrm{End}^{\mathrm{op}}_{\mathbb{R}}(C^\infty(M))$ and its formal version $\mathrm{Diff}^{\omega,\mathrm{op}}(M,e)$. In order to stick to the Weyl algebra notation and commutation relations, after changing the order of operators we denote $x_\mu = (\partial_\mu)^{\mathrm{op}}$ and $\partial^\nu = (x^\nu)^{\mathrm{op}}$. The canonical antiisomorphism $\mathrm{Diff}^\omega(M,e) \to \mathrm{Diff}^{\omega,\mathrm{op}}(M,e)$ hence sends $\sum_{|J|\leq s} q^J \partial_J$ to $\sum_{|J|\leq s} x_J p^J(\partial^1,\ldots,\partial^n)$ where p^J is $(q^J)^{\mathrm{op}}$ (written as a formal function of ∂^μ). The latter sum can be viewed as belonging to the semicompleted Weyl algebra $\hat{A}_n$ with contravariant notation (as in 1.1).

2. Left versus right invariant differential operators

In Section 3 we shall introduce a noncommutative phase space H^L of Lie type and important matrix $\mathcal{O}$ which plays the central role in defining our Hopf algebroid structure. Geometrical origin of this matrix and related issues are clarified in this section using calculations relating left and right invariant vector fields.

Throughout the article, $\mathfrak{g}$ is a fixed Lie algebra over $\boldsymbol{k}$ of some finite dimension n. In a basis $\hat{x}_1,\ldots,\hat{x}_n$ of $\mathfrak{g}$, we define the structure constants $C^\lambda_{\mu\nu}$ by

$$[\hat{x}_\mu,\hat{x}_\nu] = C^\lambda_{\mu\nu}\hat{x}_\lambda. \tag{1}$$

Introduce the opposite Lie algebra $\mathfrak{g}^R$ generated by $\hat{y}_\mu$, where

$$[\hat{y}_\mu,\hat{y}_\nu] = -C^\lambda_{\mu\nu}\hat{y}_\lambda. \tag{2}$$

The Lie algebra $\mathfrak{g}^R$ is antiisomorphic to $\mathfrak{g}^L := \mathfrak{g}$ via $\hat{y}_\mu \mapsto \hat{x}_\mu$, inducing an isomorphism $U(\mathfrak{g}^L)^{\mathrm{op}} \cong U(\mathfrak{g}^R)$.

If $\boldsymbol{k}$ is $\mathbb{R}$ or $\mathbb{C}$ we also fix a Lie group G with unit e such that $\mathfrak{g}$ is its Lie algebra, realized as the algebra $\mathrm{Vect}^L(G)$ of left invariant vector fields on G, then $\mathfrak{g}^R \cong \mathrm{Vect}^R(G)$. The universal enveloping algebra $\mathfrak{g}^R \hookrightarrow U(\mathfrak{g}^R)$ can be realized as the algebra of right invariant differential operators on G, i.e. by embedding $\mathfrak{g}^R \cong \mathrm{Vect}^R(G) \hookrightarrow \mathrm{Diff}^R(G)$. If $R_g : G \to G$ is the right multiplication by $g \in G$ then a differential operator $D \in \mathrm{Diff}(G)$ is right invariant if $(R_{g*})_h D_h = D_{hg}$. Therefore $D_g = (R_{g*})_e D_e$ and every right invariant formal differential operator $D \in \mathrm{Diff}^{\omega,R}(G,e)$ at the unit e (cf. 1.2) extends to a unique right invariant analytic differential operator on G. Thus $U(\mathfrak{g}^R) \cong \mathrm{Diff}^{\omega,R}(G,e) \hookrightarrow \mathrm{Diff}^\omega(G,e)$ and the evaluation of differential operator at ∞-jets of smooth functions gives a pairing of $U(\mathfrak{g})$ and $J^\infty(G,e)$ which is nondegenerate by the L. Schwartz's theorem ([9]).

The generators of the universal enveloping algebra $U(\mathfrak{g}^L)$ and $U(\mathfrak{g}^R)$ are also denoted by $\hat{x}_\mu$ and $\hat{y}_\mu$, unlike the generators of the symmetric algebra $S(\mathfrak{g})$ which are denoted by $x_1, \ldots, x_n$ (without hat symbol) instead. Each element $Y \in \mathfrak{g}^R$ can be written as $Y = Y^\mu \hat{y}_\mu$, thus $Y^\mu : \mathfrak{g} \to \boldsymbol{k}$ may be taken as global coordinates on $\mathfrak{g}^R$ as a manifold. The exponential map $\exp : T_e G \to G$ restricts to a diffeomorphism from some star-shaped open neighborhood U of $0 \in \mathfrak{g}^R$ to some open neighborhood $V = \exp(U)$ of $e \in G$. Thus $w^\mu, \partial^w_\nu : TV \to \mathbb{R}$ given by

$$w^\mu(\exp(Y^\gamma \hat{y}_\gamma)) = Y^\mu,\ \partial^w_\nu = (d\exp)(\partial/\partial Y^\nu), \quad \mu,\nu = 1,\ldots,n, \quad (3)$$

form a system of coordinates on the tangent manifold TV. The corresponding multiplication by a coordinate and the derivative elements in $\mathrm{Diff}(V)$ satisfy the usual commutation relations $[\partial^w_\mu, w^\nu] = \delta^\nu_\mu$ generating a copy ${}^w A^{\mathrm{cov}}_n$ of the Weyl algebra A^{cov}_n (see 1.1). There is a well known formula (see [13], Chapter II or [24], Lecture 4 Cor. 1) for the differential $d\exp : TU \to TG$ of the exponential map $\exp|_U : U \to G$,

$$(d\exp)_Y = (R_{\exp Y*})_e \circ \frac{1 - e^{-\mathrm{ad}\, Y}}{\mathrm{ad}\, Y} \ \text{ for all }\ Y = Y^\gamma \hat{y}_\gamma \in \mathfrak{g}^R, \quad (4)$$

where the action of $\mathrm{ad}\, Y = \mathrm{ad}^R Y$ is understood in the sense of the identification $T_Y U \cong \mathfrak{g}^R$ of the tangent space at Y with $\mathfrak{g}^R$ (hence it is $-\mathrm{ad}\, Y$ in the sense of $\mathfrak{g}^L$-bracket). Let ${}^w\mathcal{C}$ be the matrix of functions $({}^w\mathcal{C})^\mu_\nu = C^\mu_{\nu\gamma} w^\gamma = -C^\mu_{\gamma\nu} w^\gamma : V \to \mathbb{R}$. For fixed $Y = Y^\mu \hat{y}_\mu \in \mathfrak{g}^R$, the calculation $(\mathrm{ad}\, Y)(\hat{y}_\nu) = [Y^\mu \hat{y}_\mu, \hat{y}_\nu]_{\mathfrak{g}^R} = -Y^\mu C^\gamma_{\mu\nu} \hat{y}_\gamma$ implies

$$(\mathrm{ad}\, Y)^N(\hat{y}_\beta) = ({}^w\mathcal{C}^N(\exp Y))^\gamma_\beta \hat{y}_\gamma, \quad w^\mu(\exp Y) = Y^\mu,\ \ N = 0,1,2,\ldots$$

By (4) we have $(R_{\exp Y*})_e \hat{y}_\alpha = (d\exp)_Y \circ \frac{\mathrm{ad}\, Y}{1-\exp(-\mathrm{ad}\, Y)} \hat{y}_\alpha$ which equals $(d\exp)_Y \left(\frac{-{}^w\mathcal{C}}{e^{-{}^w\mathcal{C}}-1}\right)^\beta_\alpha \hat{y}_\beta = \left(\frac{-{}^w\mathcal{C}}{e^{-{}^w\mathcal{C}}-1}\right)^\beta_\alpha (d\exp)_Y \hat{y}_\beta$; hence the basis $\hat{y}_\alpha : w \mapsto (R_{\exp Y*})_e(\hat{y}_\alpha)$ of the space of right invariant vector fields $\mathrm{Vect}^R(G)|_V$ is in the coordinates $w^1, \ldots, w^n, \partial^w_1, \ldots, \partial^w_n$ given by

$$\hat{y}^{\exp}_\alpha = (R_{\exp Y*})_e(\hat{y}_\alpha) = \left(\frac{-{}^w\mathcal{C}}{e^{-{}^w\mathcal{C}} - 1}\right)^\beta_\alpha \partial^w_\beta. \quad (5)$$

Notice that $\frac{-{}^w\mathcal{C}}{e^{-{}^w\mathcal{C}}-1}$ is a matrix of power series in $w^1, \ldots, w^n$, hence analytic in V. The map $()^{\exp} : \hat{y}_\mu \mapsto \hat{y}^{\exp}_\mu$ is an embedding of $U(\mathfrak{g})$ into the algebra of formal differential operators $\mathrm{Diff}^\omega(U,e)$ with the distinguished Weyl subalgebra ${}^w A^{\mathrm{cov}}_n$ in the coordinate chart $w^1, \ldots, w^n$. The same geometric embedding is obtained by Durov in a more general setting of formal geometry over general ring $\boldsymbol{k} \supset \mathbb{Q}$ in [10], formula (36), where ${}^w\mathcal{C}$ is denoted by M. Notice that $L_{\exp Y} = L_{\exp Y} R_{\exp(-Y)} R_{\exp Y} = R_{\exp Y} L_{\exp Y} R^{-1}_{\exp Y}$, hence using $\mathrm{Ad}_{\exp Y} = e^{\mathrm{ad}\, Y}$ we obtain

$$(L_{\exp Y*})_e = {}^w\mathcal{O}(\exp Y) \circ (R_{\exp Y*})_e = (R_{\exp Y*})_e \circ \mathrm{Ad}_{\exp Y},$$

$$(L_{\exp Y*})_e(\hat{y}_\alpha) = ((R_{\exp Y*})_e \circ e^{\mathrm{ad}\, Y})(\hat{y}_\alpha),$$

where

$$({}^w\mathcal{O})(\exp Y) = (L_{\exp Y*})_e \circ (R_{\exp(-Y)*})_{\exp Y} : T_{\exp Y}G \to T_{\exp Y}G, \quad (6)$$

$$\mathrm{Ad}_g := (L_{g*})_{g^{-1}} \circ (R^{-1}_{g*})_e : \mathfrak{g}^R \to \mathfrak{g}^R, \quad g \in G, \quad (7)$$

$$e^{{}^w\mathcal{C}}(\exp Y) = e^{\mathrm{ad}\, Y} = \mathrm{Ad}_{\exp Y}, \quad (8)$$

$${}^w\mathcal{O}^\beta_\alpha := (e^{{}^w\mathcal{C}})^\beta_\alpha, \quad (9)$$

hence the bases of $\mathrm{Vect}^R(G)|_V$ and $\mathrm{Vect}^L(G)|_V$ are related via ${}^w\mathcal{O}^\beta_\alpha$,

$$\hat{x}^{\exp}_\alpha = {}^w\mathcal{O}^\beta_\alpha \hat{y}^{\exp}_\alpha \quad (10)$$

$$\hat{x}^{\exp}_\alpha := (L_{\exp Y*})_e(\hat{y}_\alpha) = \left(\frac{{}^w\mathcal{C}}{e^{{}^w\mathcal{C}} - 1}\right)^\beta_\alpha \partial^w_\beta. \quad (11)$$

3. From differential operators to the deformed phase space

If $C^\lambda_{\mu\nu} = 0$ then $\hat{x}^{\exp}_\mu = \hat{y}^{\exp}_\mu = \partial^w_\mu$, which is not in the spirit of the interpretation in physics where $\hat{x}_\mu$ are often viewed as the analogue or deformation of commutative coordinates x_μ, cf. [1, 11, 12, 15, 16]. For that purpose most of the paper is written in somewhat dual language obtained as follows. Introduce the *antiisomorphism* $\mathrm{Diff}^\omega(G,e) \to \hat{A}_n$ (restricting to ${}^wA^{\mathrm{cov}}_n \to A_n$) mapping $w^\mu \mapsto \partial^\mu$ and $\partial^w_\nu \mapsto x_\nu$ and consequently ${}^w\mathcal{C} \mapsto \mathcal{C}$, $\frac{-{}^w\mathcal{C}}{e^{-{}^w\mathcal{C}}-1} \mapsto \phi$, $\hat{y}^{\exp}_\alpha \mapsto \hat{y}^\phi_\alpha := x_\beta \phi^\beta_\alpha$, $\frac{{}^w\mathcal{C}}{e^{{}^w\mathcal{C}}-1} \mapsto \tilde{\phi}$, $\hat{x}^{\exp} \mapsto \hat{x}^\phi := x_\beta \tilde{\phi}^\beta_\alpha$, ${}^w\mathcal{O} \mapsto \mathcal{O}$, where $\mathcal{C}$, $\mathcal{O}$, ϕ and $\tilde{\phi}$ are $n \times n$ matrices

$$\mathcal{C}^\alpha_\beta := C^\alpha_{\beta\gamma}\partial^\gamma, \quad \alpha,\beta = 1,\ldots,n, \quad \mathcal{O} := e^{\mathcal{C}}, \quad (12)$$

$$\phi := \frac{-\mathcal{C}}{e^{-\mathcal{C}} - 1} = \sum_{N=0}^\infty \frac{(-1)^N B_N}{N!}\mathcal{C}^N, \qquad \tilde{\phi} = \frac{\mathcal{C}}{e^{\mathcal{C}} - 1}. \quad (13)$$

The constants B_N are the Bernoulli numbers and the matrix entries $\phi^\beta_\alpha, \tilde{\phi}^\beta_\alpha, \mathcal{O}^\mu_\nu \in \hat{S}(\mathfrak{g}^*)$ are formal power series in the elements $\partial^1,\ldots,\partial^n$ of $S(\mathfrak{g})^*$ which correspond to the basis of $\mathfrak{g}^*$ dual to $\hat{x}_1,\ldots,\hat{x}_n$ of $\mathfrak{g}^L$. This is in agreement with the notation in the Weyl algebra A_n and in $\hat{A}_n \cong S(V)\sharp S(V)^*$ for $V = \mathfrak{g}$. The formula $\boldsymbol{\phi}_+(\hat{x}_\alpha)(\partial^\beta) := \phi^\beta_\alpha$ determines a linear map $\boldsymbol{\phi}_+(\hat{x}_\alpha) : \mathfrak{g}^* \to \hat{S}(\mathfrak{g}^*)$, which by the Leibniz rule and continuity extends to a unique continuous derivation $\boldsymbol{\phi}_+(\hat{x}_\alpha) \in \mathrm{Der}(\hat{S}(\mathfrak{g}^*))$. It is crucial that $\hat{x}_\alpha \mapsto \boldsymbol{\phi}_+(\hat{x}_\alpha)$ defines a Lie algebra homomorphism $\boldsymbol{\phi}_+ : \mathfrak{g}^L \to \mathrm{Der}(\hat{S}(\mathfrak{g}^*))$. Equivalently, $\boldsymbol{\phi}_+$ extends to a unique right *Hopf* action also denoted

$$\boldsymbol{\phi}_+ : U(\mathfrak{g}^L) \to \mathrm{End}^{\mathrm{op}}(\hat{S}(\mathfrak{g}^*)). \tag{14}$$

This induces the smash product $H^L := U(\mathfrak{g}^L)\sharp_{\boldsymbol{\phi}_+}\hat{S}(\mathfrak{g}^*)$ interpreted as the 'noncommutative phase space of Lie type'. (Warning: in [27] we used the notation $\boldsymbol{\phi}$ for the *left* Hopf action $\boldsymbol{\phi}_- = \boldsymbol{\phi}_+ \circ \mathcal{S}_{U(\mathfrak{g}^L)}$, where $\mathcal{S}_{U(\mathfrak{g}^L)} = \mathcal{S}^{-1}_{U(\mathfrak{g}^L)}$ is the antipode for $U(\mathfrak{g}^L)$, satisfying $\mathfrak{g}^L \ni h \mapsto -h$).

Regarding that $\mathfrak{g}^R$ is a Lie algebra with known structure constants, $-C^\alpha_{\beta\gamma}$, the formula (13) can be applied to it. This also gives the right Hopf action $\tilde{\boldsymbol{\phi}}_+ : U(\mathfrak{g}^R) \to \mathrm{End}^{\mathrm{op}}(\hat{S}(\mathfrak{g}^*))$, $\tilde{\boldsymbol{\phi}}_+(\hat{y}_\nu)(\partial^\mu) = \tilde{\phi}^\mu_\nu$; the right bialgebroid structure constructed below will however be based on the *left* Hopf action $\tilde{\boldsymbol{\phi}}_- = \tilde{\boldsymbol{\phi}}_+ \circ \mathcal{S}_{U(\mathfrak{g}^R)} : U(\mathfrak{g}^R) \to \mathrm{End}(\hat{S}(\mathfrak{g}^*))$, $\tilde{\boldsymbol{\phi}}_-(-\hat{y}_\nu)(\partial^\mu) = \tilde{\phi}^\mu_\nu$. Thus we can define the smash product $H^R := \hat{S}(\mathfrak{g}^*)\sharp_{\tilde{\boldsymbol{\phi}}_-}U(\mathfrak{g}^R)$. Its generators are $\hat{y}_\mu, \partial^\mu$, $\mu = 1, \ldots, n$, completing in ∂^μ-s. In addition to the relations in $U(\mathfrak{g}^R)$ and $\hat{S}(\mathfrak{g}^*)$, we also have

$$[\partial^\mu, \hat{y}_\nu] = \left(\frac{\mathcal{C}}{e^{\mathcal{C}} - 1}\right)^\mu_\nu.$$

Precomposing $()^{\exp} : U(\mathfrak{g}^R) \to \mathrm{Diff}^\omega(G, e)$ by the antiisomorphism $U(\mathfrak{g}^L) \to U(\mathfrak{g}^R)$, $\hat{x}_\mu \mapsto \hat{y}_\mu$ and postcomposing by the above antiisomorphism $\mathrm{Diff}^\omega(G, e) \to \hat{A}_n$ we obtain the monomorphism $U(\mathfrak{g}^L) \to U(\mathfrak{g}^R) \to \mathrm{Diff}^{\omega,R}(G, e) \to \hat{A}_n$ denoted $()^\phi : U(\mathfrak{g}^L) \to \hat{A}_n$, used in the rest of the article and called the ϕ-**realization** of $U(\mathfrak{g}^L)$ (by dually-formal differential operators). When complemented by the rule $\partial^\mu \mapsto \partial^\mu$, the ϕ-realization extends to a unique continuous isomorphism of algebras $U(\mathfrak{g}^L)\sharp_{\boldsymbol{\phi}_+}\hat{S}(\mathfrak{g}^*) \cong \hat{A}_n$, the ϕ-realization of H^L. Notice that $(\hat{x}_\nu)^\phi = \hat{x}^\phi_\nu = x_\rho\phi^\rho_\nu$. We commonly identify $\hat{S}(\mathfrak{g}^*)$ with the subalgebra $1\sharp\hat{S}(\mathfrak{g}^*)$ and $U(\mathfrak{g}^L)$ with $U(\mathfrak{g}^L)\sharp 1$. It follows that in H^L

$$[\partial^\mu, \hat{x}_\nu] = \left(\frac{-\mathcal{C}}{e^{-\mathcal{C}} - 1}\right)^\mu_\nu. \tag{15}$$

This identity justifies the interpretation of ∂^μ within H^L as deformed partial derivatives. The universal formula (13) for ϕ is, in this context, derived in [10] and H^L is studied in [23].

The map $J^\infty(G, e) \to \hat{S}(\mathfrak{g}^*)$, $w^\nu \mapsto \partial^\nu$ is an antiisomorphism of algebras and it can be combined with the realization of $U(\mathfrak{g}^L)^{\mathrm{op}}$ via

$\mathrm{Vect}^R(G,e)$ to compare with the opposite smash product algebra,

$$^w\hat{A}^{\mathrm{cov}}_n \cong \mathrm{Diff}^\omega(G,e) \cong J^\infty(G,e)\sharp\,\mathrm{Vect}^R(G,e) \cong (U(\mathfrak{g}^L)\sharp_{\phi_+}\hat{S}(\mathfrak{g}^*))^{\mathrm{op}}.$$

The smash product $J^\infty(G,e)\sharp\,\mathrm{Vect}^R(G,e)$ could be also directly observed using the duality between $J^\infty(G,e)$ and $\mathrm{Vect}^R(G,e)$.

Similarly to the ϕ-realization of $U(\mathfrak{g}^L)$, there is a $\tilde{\phi}$-realization of $U(\mathfrak{g}^R)$ exteding to an isomorphism $H^R \cong \hat{A}_n$ given by $\hat{y}_\nu \mapsto x_\rho\tilde{\phi}^\rho_\nu$, $\hat{S}(\mathfrak{g}^{R*}) \ni \partial^\nu \mapsto \partial^\nu \in \hat{A}_n$.

THEOREM 1. *There is a unique algebra isomorphism from $H^L = U(\mathfrak{g}^L)\sharp\hat{S}(\mathfrak{g}^*)$ to $H^R = \hat{S}(\mathfrak{g}^*)\sharp U(\mathfrak{g}^R)$ which fixes the commutative subalgebra $\hat{S}(\mathfrak{g}^*)$ (i.e. identifies $1\sharp\hat{S}(\mathfrak{g}^{L*})$ with $\hat{S}(\mathfrak{g}^{R*})\sharp 1$, $1\sharp\partial^\mu \mapsto \partial^\mu\sharp 1$), and which maps $\hat{x}_\nu \mapsto \hat{y}_\sigma\mathcal{O}^\sigma_\nu$, where $\mathcal{O} = e^{\mathcal{C}}$ is an invertible $n\times n$-matrix with entries $\mathcal{O}^\mu_\nu \in \hat{S}(\mathfrak{g}^*)$ and inverse $\mathcal{O}^{-1} = e^{-\mathcal{C}}$. After the identification, $[\hat{x}_\mu,\hat{y}_\nu] = 0$. Consequently, the images of $U(\mathfrak{g}^L) \hookrightarrow H^L$ and $U(\mathfrak{g}^R) \hookrightarrow H^R$ mutually commute. The following identities hold*

$$[\mathcal{O}^\lambda_\mu,\hat{y}_\nu] = C^\lambda_{\rho\nu}\mathcal{O}^\rho_\mu \tag{16}$$

$$[\mathcal{O}^\lambda_\mu,\hat{x}_\nu] = C^\rho_{\mu\nu}\mathcal{O}^\lambda_\rho \tag{17}$$

$$[(\mathcal{O}^{-1})^\lambda_\mu,\hat{x}_\nu] = -C^\lambda_{\rho\nu}(\mathcal{O}^{-1})^\rho_\mu \tag{18}$$

$$[(\mathcal{O}^{-1})^\lambda_\mu,\hat{y}_\nu] = -C^\rho_{\mu\nu}(\mathcal{O}^{-1})^\lambda_\rho \tag{19}$$

$$C^\tau_{\mu\nu}\mathcal{O}^\lambda_\tau = C^\lambda_{\rho\sigma}\mathcal{O}^\rho_\mu\mathcal{O}^\sigma_\nu, \quad C^\tau_{\mu\nu}(\mathcal{O}^{-1})^\lambda_\tau = C^\lambda_{\rho\sigma}(\mathcal{O}^{-1})^\rho_\mu(\mathcal{O}^{-1})^\sigma_\nu. \tag{20}$$

Proof. The isomorphism $H^L \cong H^R$ is the composition of the two isomorphisms, supplied by ϕ- and $\tilde{\phi}$-realizations $H^L \cong \hat{A}_n \cong H^R$. If we express $\hat{x}_\mu$ and $\hat{y}_\nu$ within $\hat{A}_n$ as $x_\rho\phi^\rho_\mu$ and $x_\sigma\tilde{\phi}^\sigma_\nu$ respectively, the commutation relation $[\hat{x}_\mu,\hat{y}_\nu] = 0$ becomes $[x_\rho\phi^\rho_\mu, x_\sigma\tilde{\phi}^\sigma_\nu] = 0$, which is the Proposition 5 (Appendix A.1). If $\boldsymbol{k} = \mathbb{R}$ this also easily follows using the antiisomorphism with the geometric picture in Section 2 where $[\hat{y}^{\mathrm{exp}}_\mu, \hat{x}^{\mathrm{exp}}_\nu] = 0$ because the left and right invariant vector fields commute. Comparing ϕ and $\tilde{\phi}$ (or using (10)), note that

$$\tilde{\phi} = \phi\, e^{-\mathcal{C}}, \qquad \hat{x}_\nu = \hat{y}_\mu(e^{\mathcal{C}})^\mu_\nu = \hat{y}_\mu\mathcal{O}^\mu_\nu. \tag{21}$$

Rewrite $[\hat{x}_\mu,\hat{y}_\nu]$ now as

$$[\hat{y}_\rho\mathcal{O}^\rho_\mu,\hat{y}_\nu] = [\hat{y}_\rho,\hat{y}_\nu]\mathcal{O}^\rho_\mu + \hat{y}_\lambda[\mathcal{O}^\lambda_\mu,\hat{y}_\nu] = \hat{y}_\lambda(-C^\lambda_{\rho\nu}\mathcal{O}^\rho_\mu + [\mathcal{O}^\lambda_\mu,\hat{y}_\nu]).$$

Starting with the evident fact $[\partial^\gamma,\hat{y}_\nu] \in \hat{S}(\mathfrak{g}^*)$, and using the induction, one shows $[\hat{S}(\mathfrak{g}^*),\hat{y}_\nu] \subset \hat{S}(\mathfrak{g}^*)$. Thus, $(-C^\lambda_{\rho\nu}\mathcal{O}^\rho_\mu + [\mathcal{O}^\lambda_\mu,\hat{y}_\nu]) \in \hat{S}(\mathfrak{g}^*)$. Elements $\hat{y}_\lambda$ are independent in H^R, which is here considered a right

$\hat{S}(\mathfrak{g}^*)$-module, hence $0 = \hat{y}_\lambda(-C^\lambda_{\rho\nu}\mathcal{O}^\rho_\mu + [\mathcal{O}^\lambda_\mu, \hat{y}_\nu])$ implies (16). Similarly, in $[\hat{x}_\mu, \hat{y}_\nu] = 0$ replace $\hat{y}_\nu$ with $\hat{x}_\lambda(\mathcal{O}^{-1})^\lambda_\nu$ to prove (18). To show (17), calculate $C^\lambda_{\mu\nu}\hat{y}_\rho\mathcal{O}^\rho_\lambda = C^\lambda_{\mu\nu}\hat{x}_\lambda = [\hat{x}_\mu, \hat{x}_\nu] = [\hat{y}_\rho\mathcal{O}^\rho_\mu, \hat{x}_\nu] = \hat{y}_\rho[\mathcal{O}^\rho_\mu, \hat{x}_\nu]$, hence $\hat{y}_\rho(C^\lambda_{\mu\nu}\mathcal{O}^\rho_\lambda - [\mathcal{O}^\rho_\mu, \hat{x}_\nu]) = 0$. For (19) we reason analogously with $[\hat{x}_\rho(\mathcal{O}^{-1})^\rho_\mu, \hat{y}_\nu]$. If in (16) and (18) we replace $\hat{y}_\nu$ (resp. $\hat{x}_\nu$) on the left by $\hat{y}_\rho(\mathcal{O}^{-1})^\rho_\nu$ (resp. $\hat{x}_\rho\mathcal{O}^\rho_\nu$), we get a quadratic (in $\mathcal{O}$ or $\mathcal{O}^{-1}$) expression on the right, which is then compared with (17) and (19) to obtain (20).

4. Actions ▶ and ◀ and some identities for them

There is a map $\epsilon_S : \hat{S}(\mathfrak{g}^*) \to \boldsymbol{k}$, taking a formal power series to its constant term ('evaluation at 0'). We introduce the 'black action' ▶ of H^L on $U(\mathfrak{g}^L)$ as the composition

$$H^L \otimes U(\mathfrak{g}^L) \hookrightarrow H^L \otimes H^L \xrightarrow{m} H^L \cong U(\mathfrak{g}^L)\sharp_{\phi_+}\hat{S}(\mathfrak{g}^*) \xrightarrow{\mathrm{id}\sharp\epsilon_S} U(\mathfrak{g}^L), \quad (22)$$

where m is the multiplication map. ▶ is the unique action for which $\partial^\mu \blacktriangleright 1 = 0$ for all μ and $\hat{f} \blacktriangleright 1 = \hat{f}$ for all $\hat{f} \in U(\mathfrak{g}^L)$. It follows that $\mathcal{O}^\mu_\nu \blacktriangleright 1 = \delta^\mu_\nu 1 = (\mathcal{O}^{-1})^\mu_\nu \blacktriangleright 1$ and $\hat{y}_\nu \blacktriangleright 1 = \hat{x}_\mu(\mathcal{O}^{-1})^\mu_\nu \blacktriangleright 1 = \delta^\mu_\nu\hat{x}_\mu = \hat{x}_\nu$. Similarly, the right black action ◀ of H^R on $U(\mathfrak{g}^R)$ is the composition

$$U(\mathfrak{g}^R) \otimes H^R \hookrightarrow H^R \otimes H^R \xrightarrow{m} H^R \cong \hat{S}(\mathfrak{g}^*)\sharp U(\mathfrak{g}^R) \xrightarrow{\epsilon_S\sharp\mathrm{id}} U(\mathfrak{g}^R),$$

characterized by $1 \blacktriangleleft \partial^\mu = 0$, and $1 \blacktriangleleft \hat{u} = \hat{u}$, for all $\hat{u} \in U(\mathfrak{g}^R)$. The actions ▶, ◀ and the smash products H^L, H^R can be described abstractly in terms of the pairings between $\hat{S}(\mathfrak{g}^*)$ and $U(\mathfrak{g}^L)$ or $U(\mathfrak{g}^R)$, or equivalently in the geometric picture, between $J^\infty(G, e)$ and $\mathrm{Vect}^L(G)$ or $\mathrm{Vect}^R(G)$ ([25]), but we stay here within a more explicit approach.

THEOREM 2. *For any $\hat{f}, \hat{g} \in U(\mathfrak{g}^L)$ the following identities hold*

$$\hat{x}_\alpha\hat{f} = (\mathcal{O}^\beta_\alpha \blacktriangleright \hat{f})\hat{x}_\beta \quad (23)$$

$$\mathcal{O}^\gamma_\alpha \blacktriangleright (\hat{g}\hat{f}) = (\mathcal{O}^\beta_\alpha \blacktriangleright \hat{g})(\mathcal{O}^\gamma_\beta \blacktriangleright \hat{f}) \quad (24)$$

$$(\mathcal{O}^{-1})^\gamma_\alpha \blacktriangleright (\hat{g}\hat{f}) = ((\mathcal{O}^{-1})^\gamma_\beta \blacktriangleright \hat{g})((\mathcal{O}^{-1})^\beta_\alpha \blacktriangleright \hat{f}) \quad (25)$$

$$\hat{y}_\alpha \blacktriangleright \hat{f} = \hat{f}\hat{x}_\alpha \quad (26)$$

$$(\hat{x}_\alpha \blacktriangleright \hat{f})\hat{g} = (\mathcal{O}^\beta_\alpha \blacktriangleright \hat{f})(\hat{x}_\beta \blacktriangleright \hat{g}) \quad (27)$$

Proof. We show (23) for monomials $\hat{f}$ by induction on the degree of monomial; by linearity this is sufficient. For the base of induction, it is sufficient to note $\mathcal{O}_\alpha^\beta \blacktriangleright 1 = \delta_\alpha^\beta$. For the step of induction, calculate for arbitrary $\hat{f}$ of degree k

$$\begin{aligned}\mathcal{O}_\alpha^\gamma \blacktriangleright (\hat{x}_\nu \hat{f}) &= [\mathcal{O}_\alpha^\gamma, \hat{x}_\nu \hat{f}] \blacktriangleright 1 + \hat{x}_\nu \hat{f} \mathcal{O}_\alpha^\gamma \blacktriangleright 1 \\ &= [\mathcal{O}_\alpha^\gamma, \hat{x}_\nu] \blacktriangleright \hat{f} + \hat{x}_\nu [\mathcal{O}_\alpha^\gamma, \hat{f}] \blacktriangleright 1 + \hat{x}_\nu \hat{f} \delta_\alpha^\gamma \\ &= C_{\alpha\nu}^\beta \mathcal{O}_\beta^\gamma \blacktriangleright \hat{f} + \hat{x}_\nu (\mathcal{O}_\alpha^\gamma \blacktriangleright \hat{f}) \\ &= (C_{\alpha\nu}^\beta + \delta_\alpha^\beta \hat{x}_\nu)(\mathcal{O}_\beta^\gamma \blacktriangleright \hat{f}) \\ &= (\mathcal{O}_\alpha^\beta \blacktriangleright \hat{x}_\nu)(\mathcal{O}_\beta^\gamma \blacktriangleright \hat{f}),\end{aligned}$$

and use this result in the following:

$$\begin{aligned}\hat{x}_\alpha \hat{x}_\nu \hat{f} &= (\mathcal{O}_\alpha^\beta \blacktriangleright \hat{x}_\nu) \hat{x}_\beta \hat{f} \\ &= (\mathcal{O}_\alpha^\beta \blacktriangleright \hat{x}_\nu)(\mathcal{O}_\beta^\gamma \blacktriangleright \hat{f}) \hat{x}_\gamma \\ &= (\mathcal{O}_\alpha^\gamma \blacktriangleright (\hat{x}_\nu \hat{f})) \hat{x}_\gamma.\end{aligned}$$

Thus (23) holds for $\hat{f}$-s of degree $k+1$, hence, by induction, for all. Along the way, we have also shown (24) for $\hat{g}$ of degree 1 and $\hat{f}$ arbitrary. Now we do induction on the degree of $\hat{g}$: replace $\hat{g}$ with $\hat{x}_\mu \hat{g}$ and calculate

$$\begin{aligned}\mathcal{O}_\alpha^\gamma \blacktriangleright ((\hat{x}_\mu \hat{g}) \hat{f}) &= (\mathcal{O}_\alpha^\beta \blacktriangleright \hat{x}_\mu)(\mathcal{O}_\beta^\gamma \blacktriangleright (\hat{g} \hat{f})) \\ &= (\mathcal{O}_\alpha^\beta \blacktriangleright \hat{x}_\mu)(\mathcal{O}_\beta^\sigma \blacktriangleright \hat{g})(\mathcal{O}_\sigma^\gamma \blacktriangleright \hat{f}) \\ &= (\mathcal{O}_\alpha^\sigma \blacktriangleright (\hat{x}_\mu \hat{g}))(\mathcal{O}_\sigma^\gamma \blacktriangleright \hat{f}).\end{aligned}$$

The proof of (25) is similar to (24) and left to the reader. To show (26), we use (23) and the equality $\hat{y}_\alpha = \hat{x}_\beta (\mathcal{O}^{-1})_\alpha^\beta$ in H^L:

$$\begin{aligned}\hat{x}_\beta (\mathcal{O}^{-1})_\alpha^\beta \blacktriangleright \hat{f} &= \hat{x}_\beta \blacktriangleright ((\mathcal{O}^{-1})_\alpha^\beta \blacktriangleright \hat{f}) = (\mathcal{O}_\beta^\gamma \blacktriangleright ((\mathcal{O}^{-1})_\alpha^\beta \blacktriangleright \hat{f})) \hat{x}_\gamma \\ &= ((\mathcal{O}_\beta^\gamma (\mathcal{O}^{-1})_\alpha^\beta) \blacktriangleright \hat{f}) \hat{x}_\gamma = \delta_\alpha^\gamma \hat{f} \hat{x}_\gamma = \hat{f} \hat{x}_\alpha\end{aligned}$$

Finally, (27) follows from (23) by multiplying from the right with $\hat{g}$, and using $\hat{x}_\beta \blacktriangleright \hat{g} = \hat{x}_\beta \hat{g}$ and $\hat{x}_\alpha \blacktriangleright \hat{f} = \hat{x}_\alpha \hat{f}$.

Now we state an analogue of the Theorem 2 for $\blacktriangleleft$.

THEOREM 3. *For any $\hat{f}, \hat{g} \in U(\mathfrak{g}^R)$ the following identities hold*

$$\hat{f} \hat{y}_\alpha = \hat{y}_\beta (\hat{f} \blacktriangleleft (\mathcal{O}^{-1})_\alpha^\beta), \tag{28}$$

$$(\hat{g} \hat{f}) \blacktriangleleft \mathcal{O}_\alpha^\gamma = (\hat{g} \blacktriangleleft \mathcal{O}_\alpha^\beta)(\hat{f} \blacktriangleleft \mathcal{O}_\beta^\gamma), \tag{29}$$

$$(\hat{g} \hat{f}) \blacktriangleleft (\mathcal{O}^{-1})_\alpha^\gamma = (\hat{g} \blacktriangleleft (\mathcal{O}^{-1})_\beta^\gamma)(\hat{f} \blacktriangleleft (\mathcal{O}^{-1})_\alpha^\beta) \tag{30}$$

$$\hat{f} \blacktriangleleft \hat{z}_\alpha = \hat{y}_\alpha \hat{f}, \tag{31}$$

$$\hat{g}(\hat{f} \blacktriangleleft \hat{y}_\alpha) = (\hat{g} \blacktriangleleft \hat{y}_\beta)(\hat{f} \blacktriangleleft (\mathcal{O}^{-1})^\beta_\alpha), \tag{32}$$

where

$$\hat{z}_\alpha := \mathcal{O}^\beta_\alpha \hat{y}_\beta = \mathcal{O}^\beta_\alpha \hat{x}_\rho (\mathcal{O}^{-1})^\rho_\beta \in H^L \cong H^R. \tag{33}$$

$$[\hat{z}_\alpha, \hat{z}_\beta] = C^\gamma_{\alpha\beta} \hat{z}_\gamma \tag{34}$$

5. Completed tensor product and bimodules

In this section, we discuss the completed tensor products needed for the coproducts ($\Delta_{S(\mathfrak{g}^*)}$ in this and Δ^L and Δ^R in the next section), introduce the maps $\alpha^L, \beta^L, \alpha^R, \beta^R$ and use them to define $U(\mathfrak{g}^L)$-bimodule structure on H^L and $U(\mathfrak{g}^R)$-bimodule structure on H^R.

Note that $S(\mathfrak{g}) = \oplus_{i=0}^\infty S^i(\mathfrak{g}) = \cup_{i=0}^\infty S_i(\mathfrak{g})$ carries a graded and $U(\mathfrak{g}) = \cup_i U_i(\mathfrak{g})$ a *filtered* Hopf algebra structure. Both structures are induced along quotient maps from the tensor bialgebra $T(\mathfrak{g})$. By the PBW theorem, the linear map

$$\xi : S(\mathfrak{g}) \to U(\mathfrak{g}), \qquad x_{i_1} \cdots x_{i_r} \mapsto \frac{1}{r!} \sum_{\sigma \in \Sigma(r)} \hat{x}_{i_{\sigma(1)}} \cdots \hat{x}_{i_{\sigma(r)}}, \tag{35}$$

is an isomorphism of filtered coalgebras whose inverse ξ^{-1} may be identified with the projection to the associated graded ring [6, 9, 10]. The isomorphism ξ is related to the ϕ-realization from Section 3 (hence to the exponential map in the geometric picture in Section 2) as follows. Consider the Fock action $\triangleright$ of $\hat{A}_n$ on $S(\mathfrak{g})$ and the ϕ-realization $()^\phi : U(\mathfrak{g}) \to \hat{A}_n$. For each $f, g \in S(\mathfrak{g})$, $\xi(f) \cdot_{U(\mathfrak{g})} \xi(g) = \xi(\xi(f)^\phi \triangleright g)$ and this property uniquely characterizes ξ.

For a multiindex $K = (k_1, \ldots, k_n) \in \mathbb{N}_0^n$, denote $|K| := k_1 + \ldots + k_n$, $x_K := x_1^{k_1} \cdots x_n^{k_n}$ and $\hat{x}_K := \hat{x}_1^{k_1} \cdots \hat{x}_n^{k_n}$. The multiindices add up componentwise. The partial order on $\mathbb{N}_0^n$ induced by the componentwise $<$ is also denoted $<$. If J, K are multiindices the rule $\langle x_k, \partial^J \rangle := J!\, \delta^J_K$ continuously in the first factor and linearly extends to a unique map $\langle , \rangle : S(\mathfrak{g}) \otimes \hat{S}(\mathfrak{g}^*) \to \boldsymbol{k}$ which is a nondegenerate pairing, hence it identifies $\hat{S}(\mathfrak{g}^*) \cong S(\mathfrak{g})^*$. This is the unique Hopf pairing extending the duality between $\mathfrak{g}$ and $\mathfrak{g}^*$ where $\hat{S}(\mathfrak{g}^*)$ is the topological Hopf algebra with elements in $\mathfrak{g}^*$ primitive. By duality, the linear map

$$\xi^T : U(\mathfrak{g})^* \longrightarrow S(\mathfrak{g})^* \cong \hat{S}(\mathfrak{g}^*) \tag{36}$$

transpose (dual) to ξ (see (35)) is an isomorphism of cofiltered algebras.

The inclusions of filtered components $U_k(\mathfrak{g}) \subset U_{k+1}(\mathfrak{g}) \subset U(\mathfrak{g})$ induce epimorphisms of dual vector spaces $U(\mathfrak{g})^* \to U_{k+1}(\mathfrak{g})^* \to U_k(\mathfrak{g})^*$, hence a complete *cofiltration* on $U(\mathfrak{g})^* = \varprojlim_k U_k(\mathfrak{g})^*$ (see Appendix A.2). For each finite level k, $U_k(\mathfrak{g})$ is finite dimensional, hence $(U_k(\mathfrak{g}) \otimes U_l(\mathfrak{g}))^* \cong U_k(\mathfrak{g})^* \otimes U_l(\mathfrak{g})^*$. Thus the multiplication $U_k(\mathfrak{g}) \otimes U_l(\mathfrak{g}) \to U_{k+l}(\mathfrak{g}) \subset U(\mathfrak{g})$ dualizes to $\Delta_{k,l} : U(\mathfrak{g})^* \to U_k(\mathfrak{g})^* \otimes U_l(\mathfrak{g})^*$. The inverse limits $\varprojlim_k \Delta_{k,k}$ and $\varprojlim_p \varprojlim_q \Delta_{p,q}$ agree and define the coproduct $\Delta_{U(\mathfrak{g})^*} := \varprojlim_k \Delta_{k,k} : U(\mathfrak{g})^* \to \varprojlim_k U_k(\mathfrak{g})^* \otimes U_k(\mathfrak{g})^* \cong \varprojlim_p \varprojlim_q U_p(\mathfrak{g})^* \otimes U_q(\mathfrak{g})^*$. The right-hand side is by definition the completed tensor product, $U(\mathfrak{g})^* \hat{\otimes} U(\mathfrak{g})^*$. (For completed tensoring of *elements* and *maps* we below often use simplified notation, $\otimes$.) Coproduct $\Delta_{U(\mathfrak{g})^*}$ transfers, along the isomorphism $\xi^T : U(\mathfrak{g})^* \xrightarrow{\cong} S(\mathfrak{g})^*$ of cofiltered algebras (see (36)), to the topological coproduct on the completed symmetric algebra $\hat{S}(\mathfrak{g}^*) \cong S(\mathfrak{g})^*$ (cf. [23]),

$$\Delta_{\hat{S}(\mathfrak{g}^*)} : \hat{S}(\mathfrak{g}^*) \to \hat{S}(\mathfrak{g}^*) \hat{\otimes} \hat{S}(\mathfrak{g}^*).$$

This construction can be performed both for $\mathfrak{g}^L$ and $\mathfrak{g}^R$. The canonical isomorphism of Hopf algebras $U(\mathfrak{g}^R) \cong U(\mathfrak{g}^L)^{\mathrm{op}}$ induces the isomorphism of dual cofiltered Hopf algebras $U(\mathfrak{g}^R)^* \cong (U(\mathfrak{g}^L)^*)^{\mathrm{co}}$, commuting with ξ^T, hence inducing an isomorphism of Hopf algebras $\hat{S}(\mathfrak{g}^{R*}) \cong \hat{S}(\mathfrak{g}^{L*})^{\mathrm{co}}$ fixing the underlying algebra $\hat{S}(\mathfrak{g}^*)$. Thus, the coproduct on $\hat{S}(\mathfrak{g}^{R*})$ is $\Delta^{\mathrm{op}}_{\hat{S}(\mathfrak{g}^{L*})}$, hence we just write $\hat{S}(\mathfrak{g}^*)$ and use the algebra identification, with the (co)opposite signs $\hat{S}(\mathfrak{g}^*)^{\mathrm{co}}$ or $\Delta^{\mathrm{op}}_{\hat{S}(\mathfrak{g}^*)}$ when needed.

As discussed in [23, 27], the coproduct is equivalently characterized by

$$P \blacktriangleright (\hat{f}\hat{g}) = m(\Delta_{\hat{S}(\mathfrak{g}^*)}(P)(\blacktriangleright \otimes \blacktriangleright)(\hat{f} \otimes \hat{g})), \tag{37}$$

for all $P \in \hat{S}(\mathfrak{g}^*)$ (for instance, $P = \partial^\mu$) and all $\hat{f}, \hat{g} \in U(\mathfrak{g})$. Using the action $\blacktriangleright$ we assumed that we embedded $\hat{S}(\mathfrak{g}^*) \hookrightarrow H^R \cong \hat{A}_n$. The right hand version of (37) is that for all $\hat{u}, \hat{v} \in U(\mathfrak{g}^R)$ and $Q \in \hat{S}(\mathfrak{g}^*)$,

$$(\hat{u}\hat{v}) \blacktriangleleft Q = m((\hat{u} \otimes \hat{v})(\blacktriangleleft \otimes \blacktriangleleft)\Delta_{\hat{S}(\mathfrak{g}^*)}(Q)). \tag{38}$$

DEFINITION 3. *The homomorphism $\alpha^L : U(\mathfrak{g}^L) \hookrightarrow H^L$ is the inclusion $U(\mathfrak{g}^L) \to U(\mathfrak{g}^L)\sharp 1 \hookrightarrow U(\mathfrak{g}^L)\sharp \hat{S}(\mathfrak{g}^*) = H^L$ and $\alpha^R : U(\mathfrak{g}^R) \to H^R$ is the inclusion $\alpha^R : U(\mathfrak{g}^R) \to 1\sharp U(\mathfrak{g}^R) \hookrightarrow \hat{S}(\mathfrak{g}^*)\sharp U(\mathfrak{g}^R) = H^R$. Thus, in our writing conventions, $\alpha^L(\hat{f}) = \hat{f}$ and $\alpha^R(\hat{u}) = \hat{u}$. Likewise, $\beta^L : U(\mathfrak{g}^L)^{\mathrm{op}} \to H^L$ and $\beta^R : U(\mathfrak{g}^R)^{\mathrm{op}} \to H^R$ are the unique antihomomorphisms of algebras extending the formulas (cf. (33))*

$$\begin{array}{l} \beta^L(\hat{x}_\mu) = \hat{x}_\rho(\mathcal{O}^{-1})^\rho_\mu = \hat{y}_\mu \in H^L. \\ \beta^R(\hat{y}_\alpha) := \mathcal{O}^\rho_\alpha \hat{y}_\rho = \mathcal{O}^\rho_\alpha \hat{x}_\sigma (\mathcal{O}^{-1})^\sigma_\rho = \hat{z}_\alpha \in H^R. \end{array} \tag{39}$$

The extension β^L exists, because the extension of the map $\hat{x}_\mu \mapsto \hat{y}_\mu$ on $\mathfrak{g}$ to the antihomomorphism $\beta^L_{T(\mathfrak{g})} : T(\mathfrak{g}) \to H^L$ maps $[\hat{x}_\alpha, \hat{x}_\beta] - C^\gamma_{\alpha\beta}\hat{x}_\gamma$ to $[\hat{y}_\beta, \hat{y}_\alpha] - C^\gamma_{\alpha\beta}\hat{y}_\gamma = 0$; similarly for β^R, using (34).

PROPOSITION 1. *(i) H^L is a $U(\mathfrak{g}^L)$-bimodule via the formula $a.h.b := \alpha^L(a)\beta^L(b)h$, for all $a, b \in U(\mathfrak{g}^L)$, $h \in H^L$. Likewise, H^R is a $U(\mathfrak{g}^R)$-bimodule via $a.h.b := h\beta^R(a)\alpha^R(b)$, for all $a, b \in U(\mathfrak{g}^R)$, $h \in H^R$.* From now on these bimodule structures are assumed.

(ii) For any $\hat{f}, \hat{g} \in U(\mathfrak{g}^L)$ and any $\hat{u}, \hat{v} \in U(\mathfrak{g}^R)$,

$$\beta^L(\hat{g}) \blacktriangleright \hat{f} = \hat{f}\hat{g}, \quad \hat{u} \blacktriangleleft \beta^R(\hat{v}) = \hat{v}\hat{u}. \tag{40}$$

Proof. (i) The bimodule property of commuting of the left and the right $U(\mathfrak{g}^L)$-action is ensured by $[\hat{x}_\mu, \hat{y}_\nu] = 0$. For the $U(\mathfrak{g}^R)$-actions it boils down to $[\hat{y}_\mu, \mathcal{O}^\rho_\nu \hat{x}_\sigma (\mathcal{O}^{-1})^\sigma_\rho] = 0$, which follows from the Theorem 1.

(ii) follows from (26) and (31), by induction on the filtered degree of $\hat{g}$ (respectively, of $\hat{v}$).

PROPOSITION 2. *Let $\hat{H}^L := U(\mathfrak{g}^L)\hat{\sharp}\hat{S}(\mathfrak{g}^*)$ and $\hat{H}^R := \hat{S}(\mathfrak{g}^*)\hat{\sharp}U(\mathfrak{g}^R)$ be the completed smash product algebras defined in Theorem 6. Then*

(i) the factorwise multiplication $(m \otimes m)(\mathrm{id} \otimes \tau \otimes \mathrm{id}) : (H^L \otimes H^L) \otimes (H^L \otimes H^L) \to (H^L \otimes H^L)$ (where τ switches the factors) extends to the unique map $(H^L \hat{\otimes} H^L) \otimes (H^L \hat{\otimes} H^L) \to (H^L \hat{\otimes} H^L)$ (note that the middle $\otimes$ is not completed!) distributive over formal sums in each of the two $H^L \hat{\otimes} H^L$-factors. Likewise for H^R in place of H^L.

(ii) The inclusions $H^L \hat{\otimes} H^L \to \hat{H}^L \hat{\otimes} \hat{H}^L$, $H^R \hat{\otimes} H^R \to \hat{H}^R \hat{\otimes} \hat{H}^R$, $H^L \hat{\otimes}_{U(\mathfrak{g}^L)} H^L \to \hat{H}^L \hat{\otimes}_{U(\mathfrak{g}^L)} \hat{H}^L$ and $H^R \hat{\otimes}_{U(\mathfrak{g}^R)} H^R \to \hat{H}^R \hat{\otimes}_{U(\mathfrak{g}^R)} \hat{H}^R$ are onto;

(iii) The actions $\blacktriangleright$ and $\blacktriangleleft$ extend to the actions of the completed algebra $\blacktriangleright : \hat{H}^L \otimes U(\mathfrak{g}^L) \to U(\mathfrak{g}^L)$ and $\blacktriangleleft : U(\mathfrak{g}^R) \otimes \hat{H}^R \to U(\mathfrak{g}^R)$.

Proof. (i) The proof is in the vein of the proof of Theorem 6.

(ii) The cofiltered components $(H^L)_r = (\hat{H}^L)_r$ agree, hence both sides of the tensor product inclusions have also equal cofiltered components. Therefore, the completions are the same.

For (iii) extend the recipe from (22) and notice that $\mathrm{id}\hat{\sharp}\epsilon_S$ kills also all elements in $U(\mathfrak{g}^L)\hat{\sharp}\hat{S}(\mathfrak{g}^*)$ not in $U(\mathfrak{g}^L)\sharp\hat{S}(\mathfrak{g}^*)$ with the result in $U(\mathfrak{g})$. On the other hand, *there are no completed actions $\hat{H}^L \hat{\otimes} U(\mathfrak{g}^L) \to U(\mathfrak{g}^L)$* and $U(\mathfrak{g}^R)\hat{\otimes}\hat{H}^R \to U(\mathfrak{g}^R)$ extending $\blacktriangleright$ and $\blacktriangleleft$.

DEFINITION 4. *The right ideal $I \subset H^L \otimes H^L$ is generated by the set of all elements of the form $\beta^L(\hat{f}) \otimes 1 - 1 \otimes \alpha^L(\hat{f})$ where $\hat{f} \in H^L$. In other words, I is the kernel of the canonical map $H^L \otimes H^L \to H^L \otimes_{U(\mathfrak{g}^L)} H^L$.*

The right ideal $I' \subset H^L \otimes H^L$ *is the set of all* $\sum_i h_i \otimes h'_i \in H^L \otimes H^L$ *such that*

$$\sum_{i,j} (h_i \blacktriangleright \hat{f}_j)(h'_i \blacktriangleright \hat{g}_j) = 0, \quad \text{for all} \ \sum_j \hat{f}_j \otimes \hat{g}_j \in U(\mathfrak{g}^L) \otimes U(\mathfrak{g}^L).$$

Similarly, $\tilde{I} := \ker(H^R \otimes H^R \to H^R \otimes_{U(\mathfrak{g}^R)} H^R)$ *is the left ideal in* $H^R \otimes H^R$ *generated by all elements of the form* $\alpha^R(\hat{u}) \otimes 1 - 1 \otimes \beta^R(\hat{u})$, $\hat{u} \in U(\mathfrak{g}^R)$*, and* $\tilde{I}'$ *is the left ideal in* $H^R \otimes H^R$ *consisting of all* $\sum_i h_i \otimes h'_i$ *such that* $\sum_{i,j} (\hat{u}_j \blacktriangleleft h_i)(\hat{v}_j \blacktriangleleft h'_i) = 0$ *for all* $\sum_j \hat{u}_j \otimes \hat{v}_j \in U(\mathfrak{g}^R) \otimes U(\mathfrak{g}^R)$*. The completions (Appendix A.2) of the ideals* I, I' *and* $\tilde{I}, \tilde{I}'$ *are denoted* $\hat{I}, \hat{I}' \subset H^L \hat{\otimes} H^L \cong \hat{H}^L \hat{\otimes} \hat{H}^L$ *and* $\hat{\tilde{I}}, \hat{\tilde{I}}' \subset H^R \hat{\otimes} H^R \cong \hat{H}^R \hat{\otimes} \hat{H}^R$*, respectively.*

More generally, for $r \geq 2$, let $I^{(r)}$ be the kernel of the canonical projection $(H^L)^{\otimes r} := H^L \otimes H^L \otimes \ldots \otimes H^L$ (r factors) to the tensor product of $U(\mathfrak{g}^L)$-bimodules $H^L \otimes_{U(\mathfrak{g}^L)} H^L \otimes_{U(\mathfrak{g}^L)} \ldots \otimes_{U(\mathfrak{g}^L)} H^L$. $I^{(r)}$ coincides with the smallest right ideal in the tensor product algebra $(H^L)^{\otimes r}$ which contains $1^{\otimes k} \otimes I \otimes 1^{\otimes(r-k-2)}$ for $k = 0, \ldots, r-2$. Let $I'^{(r)}$ be the set of all elements $\sum_i h_{1i} \otimes h_{2i} \otimes \ldots h_{ri} \in (H^L)^{\hat{\otimes} r}$ such that for every $\sum_j u_{1j} \otimes u_{2j} \otimes \ldots \otimes u_{rj} \in U(\mathfrak{g}^L)^{\otimes r}$

$$\sum_{i,j} (h_{1i} \blacktriangleright u_{1j})(h_{2i} \blacktriangleright u_{2j}) \cdots (h_{ri} \blacktriangleright u_{rj}) = 0.$$

LEMMA 1. *(i) There is a nondegenerate Hopf pairing*

$$\langle , \rangle_\phi : U(\mathfrak{g}) \otimes \hat{S}(\mathfrak{g}^*) \to \boldsymbol{k}, \qquad \langle \hat{u}, P \rangle_\phi := \boldsymbol{\phi}_+(\hat{u})(P)(1),$$

where the action on 1 *is the Fock action (on* 1 *this amounts to evaluating* $\epsilon_{\hat{S}(\mathfrak{g}^*)}$*). It satisfies the* Heisenberg double identity

$$P \blacktriangleright \hat{u} = \sum \langle \hat{u}_{(2)}, P \rangle_\phi \hat{u}_{(1)} \quad \textit{for} \quad P \in \hat{S}(\mathfrak{g}^*) \quad \textit{and} \quad \hat{u} \in U(\mathfrak{g}).$$

(ii) For multiindices J_1, J_2, J, K *such that* $J_1 + J_2 = J$,

$$\boldsymbol{\phi}_+(\hat{x}_K)(\partial^J) = \sum_{K_1+K_2=K} \frac{K!}{K_1! K_2!} \boldsymbol{\phi}_+(\hat{x}_{K_2})(\partial^{J_1}) \boldsymbol{\phi}_+(\hat{x}_{K_1})(\partial^{J_2}).$$

(iii) $\boldsymbol{\phi}_+(\hat{x}_K)(\partial^J) \in \hat{S}(\mathfrak{g}^*)_{|J|-|K|}$ *if* $|K| < |J|$.

(iv) $\boldsymbol{\phi}_+(\hat{x}_K)(\partial^J) - K!\,\delta^K_J \in \hat{S}(\mathfrak{g}^*)_1$ *if* $|K| = |J|$.

(v) For multiindices K, J *and for the basis* $\{\partial^K \in S(\mathfrak{g}^*)\}_K$ *the identities* $\langle \hat{x}_J, \partial^K \rangle_\phi = K!\,\delta^K_J$ *hold if* $K \geq J$ *(in partial order for multiindices), but in general not otherwise.*

(vi) There is a unique family $\{\partial^{\{K\}} \in \hat{S}(\mathfrak{g}^*)\}_K$ *which for* *__all__* *multi-indices* K, J *satisfies* $\langle \hat{x}_J, \partial^{\{K\}}\rangle_\phi = K!\,\delta^K_J$.

(vii) Let $f \in \hat{S}(\mathfrak{g}^*)$. *Then* $\forall r \in \mathbb{N}_0$, $f_r = \sum_J \frac{1}{J!}\langle \hat{x}_J, f\rangle_\phi \partial_r^{\{J\}} \in S(\mathfrak{g})_r$, *where the sum is finite because* $\partial_r^{\{J\}} = 0$ *if* $r < |J|$. *Thus, there is a formal sum representation* $f = \varprojlim_r f_r = \sum_J \frac{1}{J!}\langle \hat{x}_J, f\rangle_\phi \partial^{\{J\}}$.

(viii) $\partial^J = \sum_{|K|\geq|J|} d_{K,J}\partial^{\{K\}}$ *for some* $d_{K,J} \in \boldsymbol{k}$.

Proof. (i) is a part of the content of Theorems 3.3 and 3.5 in [27].

(ii) $\boldsymbol{\phi}_+$ is a right Hopf action, hence the identity follows from the formula $\Delta(\hat{x}_K) = \sum_{K_1+K_2=K} \frac{K!}{K_1!K_2!}\hat{x}_{K_1}\otimes\hat{x}_{K_2}$ for the cocommutative coproduct in $U(\mathfrak{g})$.

(iii) This follows by a simple induction on $|J| - |K|$ using (ii) and $\boldsymbol{\phi}_+(1)(\partial^L) = \partial^L \in \hat{S}(\mathfrak{g}^*)$.

(iv) follows by induction on $|K|$ using (ii), (iii) and $\boldsymbol{\phi}_+(\hat{x}_\mu)(\partial^\mu) = \phi^\nu_\mu$, which by (13) equals δ^ν_μ up to a summand in $\hat{S}(\mathfrak{g}^*)_1$.

(v) This is an application of the formula for $\langle,\rangle_\phi$ in (i) to the results (iii) and (iv); indeed the elements in $\hat{S}(\mathfrak{g}^*)_1$ vanish when applied to 1.

(vi) Denote, as in Appendix A.2, by $\pi_r : \hat{S}(\mathfrak{g}^*) \to S(\mathfrak{g}^*)_r$ and $\pi_{r,r+s} : S(\mathfrak{g}^*)_{r+s} \to S(\mathfrak{g}^*)_r$ the canonical projections. By [27], 3.4, the isomorphism $\xi^T : U(\mathfrak{g})^* \to \hat{S}(\mathfrak{g}^*)$ (see (36)) of cofiltered algebras identifies the pairing $\langle,\rangle_\phi$ with the evaluation pairing $\langle,\rangle_U : U(\mathfrak{g}) \otimes U(\mathfrak{g})^* \to \boldsymbol{k}$. By the properties of $\langle,\rangle_U$, for each $r \in \mathbb{N}_0$, the induced pairing $\langle,\rangle_r : U(\mathfrak{g})_r \otimes \hat{S}(\mathfrak{g}^*)_r \to \boldsymbol{k}$ characterized by $\langle \hat{u}, \pi_r(P)\rangle_r = \langle \hat{u}, P\rangle_\phi$ for each $\hat{u} \in U(\mathfrak{g})$, $P \in \hat{S}(\mathfrak{g}^*)$ is *nondegenerate*. Thus there is a basis $\{\partial_r^{\{K\}}\}_{|K|\leq r}$ of the cofiltered component $\hat{S}(\mathfrak{g}^*)_r$ dual to the basis $\{\hat{x}_L\}_{|L|\leq r}$ of the filtered component $U(\mathfrak{g})_r$. Now $\ker\,\pi_{r,r+s} = \mathrm{Span}\,\{\partial^J,\, r < |J| \leq r+s\}$. By (v) $\langle U(\mathfrak{g})_r, \ker\,\pi_{r,r+s}\rangle_{r+s} = 0$. Therefore for all K, L if $|K| \leq r$, $|L| \leq r$ then $\delta^K_L = \langle \hat{x}_L, \partial^{\{K\}}_{r+s}\rangle_{r+s} = \langle \hat{x}_L, \pi_{r,r+s}(\partial^{\{K\}}_{r+s})\rangle_r = \langle \hat{x}_L, \partial^{\{K\}}_r\rangle_r$. By nondegeneracy, $\pi_{r,r+s}(\partial^{\{K\}}_{r+s}) = \partial^{\{K\}}_r$. Therefore $\exists!\,\partial^{\{K\}} \in \hat{S}(\mathfrak{g}^*)_{r+s}$ such that $\pi_r(\partial^{\{K\}}) = \partial^{\{K\}}_r$ for $r \geq |K|$ and $\pi_r(\partial^{\{K\}}) = 0$ for $r < |K|$; then the requirements of (vi) hold for $\{\partial^{\{K\}}\}_K$.

(vii) is now straightforward and (viii) follows from (v) and (vii).

THEOREM 4. *(i) The restriction of* $\blacktriangleright$: $H^L \otimes U(\mathfrak{g}^L) \to U(\mathfrak{g}^L)$ *to* $\hat{S}(\mathfrak{g}^*)\otimes U(\mathfrak{g}^L) \to U(\mathfrak{g}^L)$ *turns* $U(\mathfrak{g}^L)$ *into a* **faithful** *left* $\hat{S}(\mathfrak{g}^*)$*-module.*

(ii) The right ideals I, I' *agree and the left ideals* $\tilde{I}, \tilde{I}'$ *agree.*

(iii) More generally, $I^{(r)} = I'^{(r)}$, $\tilde{I}^{(r)} = \tilde{I}'^{(r)}$ *for* $r \geq 2$.

(iv) Statements (ii) and (iii) hold also for the completed ideals.

Proof. We show part (ii) for the right ideals, $I = I'$; the method of the proof easily extends to the left ideals, and to (i), (iii) and (iv).

Let $\sum_\sigma \hat{f}_\sigma \otimes \hat{g}_\sigma \in I$ and $v = \hat{x}_{\mu_1} \cdots \hat{x}_{\mu_k}$ a monomial in $U(\mathfrak{g}^L)$. Then

$$(\beta^L(v) \blacktriangleright \hat{f}_\sigma)\hat{g}_\sigma - \hat{f}_\sigma \alpha^L(v) \blacktriangleright \hat{g}_\sigma = (\hat{y}_{\mu_k} \cdots \hat{y}_{\mu_1} \blacktriangleright \hat{f}_\sigma)\hat{g}_\sigma - \hat{f}_\sigma \hat{x}_{\mu_1} \cdots \hat{x}_{\mu_k} \blacktriangleright \hat{g}_\sigma,$$

which is zero by Eq. (26) and induction on k. Thus, by linearity, $I \subset I'$.

It remains to show the converse inclusion, $I' \subset I$. Suppose on the contrary that there is an element $\sum_\lambda h_\lambda \otimes h'_\lambda$ in I', but not in I; then after adding any element in I the sum is still in I' and not in I. Observe that $\hat{x}_J \partial^K \otimes \hat{x}_{J'} \partial^{K'} = \hat{x}_J \partial^K \otimes \alpha^L(\hat{x}_{J'})\partial^{K'} = \beta^L(\hat{x}_{J'})\hat{x}_J \partial^K \otimes \partial^{K'} \bmod I$. The tensor factor $\beta(\hat{x}_{J'})\hat{x}_J \partial^K$ belongs to $H^L \subset \hat{H}^L$, hence it is also a formal linear combination of elements of the form $\hat{x}_{J''}\partial^{K''}$. Therefore, without loss of generality, we can assume

$$\sum_\lambda h_\lambda \otimes h'_\lambda = \sum_{J,K,L} a_{JKL} \hat{x}_J \partial^K \otimes \partial^L. \tag{41}$$

Using Lemma 1 (vi),(vii),(viii) we can in (41) uniquely express ∂^K as a formal sum in $\partial^{\{K\}}$ and ∂^L as a formal sum in $\partial^{\{L\}}$. Therefore, we can write $\sum_\lambda h_\lambda \otimes h'_\lambda$ as a formal sum

$$\sum_\lambda h_\lambda \otimes h'_\lambda = \sum_{J,K,L} b_{JKL} \hat{x}_J \partial^{\{K\}} \otimes \partial^{\{L\}},$$

for some coefficients $b_{JKL} \in \boldsymbol{k}$. The assumption $\sum_\lambda h_\lambda \otimes h'_\lambda \in I'$ implies

$$\sum_\lambda (h_\lambda \blacktriangleright \hat{x}_M)(h'_\lambda \blacktriangleright \hat{x}_N) = 0.$$

Choose multiindices M and N such that $(|M|, |N|)$ is a minimal bidegree for which b_{JMN} does not vanish for at least some J. By Lemma 1 (i), the formula $\Delta(\hat{x}_M) = \sum_{M_1+M_2=M} \frac{M!}{M_1!M_2!} \hat{x}_{M_1} \otimes \hat{x}_{M_2}$ for the coproduct in $U(\mathfrak{g})$, and Lemma 1 (vi)

$$\partial^{\{K\}} \blacktriangleright \hat{x}_M = \sum_{M_1+M_2=M} \binom{M}{M_2} \langle \hat{x}_{M_2}, \partial^{\{K\}} \rangle_\phi \, \hat{x}_{M_1} = \begin{cases} \binom{M}{K} \hat{x}_{M-K}, & M \geq K \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, using the minimality of $(|M|, |N|)$, only the summand with $M = K$ and $N = L$ contributes to the sum and

$$0 = \sum_\lambda (h_\lambda \blacktriangleright \hat{x}_M)(h'_\lambda \blacktriangleright \hat{x}_N) = \sum_J b_{JMN} \hat{x}_J,$$

hence by the linear independence of monomials $\hat{x}_J$, all $b_{JMN} = 0$, in contradiction to the existence of J with b_{JMN} different from 0.

6. Bialgebroid structures

Let us now use the shorter notation $\mathcal{A}^L := U(\mathfrak{g}^L)$, $\mathcal{A}^R := U(\mathfrak{g}^R)$. A suggestive symbol $\mathcal{A}$ denotes an abstract algebra in the axioms where either $\mathcal{A}^L$ or $\mathcal{A}^R$ (or both) may substitute in here intended examples. In this section, we equip the isomorphic associative algebras H^L and H^R with different structures: H^L is a left $\mathcal{A}^L$-bialgebroid and H^R is a right $\mathcal{A}^R$-bialgebroid. We start by exhibiting the coring structures of these bialgebroids; an $\mathcal{A}$-coring is an analogue of a coalgebra where the ground field is replaced by a noncommutative algebra $\mathcal{A}$.

DEFINITION 5. *[2, 8] Let $\mathcal{A}$ be a unital algebra and C an $\mathcal{A}$-bimodule with left action $(a,c)\mapsto a.c$ and right action $(c,a)\mapsto c.a$. A triple (C,Δ,ϵ) is an $\mathcal{A}$-**coring** if*

(i) $\Delta : C \to C\otimes_{\mathcal{A}} C$ and $\epsilon : C\to\mathcal{A}$ are $\mathcal{A}$-bimodule maps; they are called the coproduct (comultiplication) and the counit;

(ii) Δ is coassociative: $(\Delta\otimes_{\mathcal{A}}\mathrm{id})\circ\Delta = (\mathrm{id}\otimes_{\mathcal{A}}\Delta)\circ\Delta$, where in the codomain the associativity isomorphism $(C\otimes_{\mathcal{A}}C)\otimes_{\mathcal{A}}C\cong C\otimes_{\mathcal{A}}(C\otimes_{\mathcal{A}}C)$ for the $\mathcal{A}$-bimodule tensor product is understood;

(iii) The counit axioms $(\epsilon\otimes_{\mathcal{A}}\mathrm{id})\circ\Delta\cong\mathrm{id}\cong(\mathrm{id}\otimes_{\mathcal{A}}\epsilon)\circ\Delta$ hold, where the identifications of $\mathcal{A}$-bimodules $C\otimes_{\mathcal{A}}\mathcal{A}\cong C$, $c\otimes a\mapsto c.a$ and $\mathcal{A}\otimes_{\mathcal{A}}C\cong C$, $a\otimes d\mapsto a.d$ are understood.

PROPOSITION 3. *(i) $\exists!$ linear maps $\Delta^L : H^L\to H^L\hat{\otimes}_{\mathcal{A}^L}H^L$ and $\Delta^R : H^R\to H^R\hat{\otimes}_{\mathcal{A}^R}H^R$ such that Δ^L and Δ^R respectively satisfy*

$$P\blacktriangleright(\hat{f}\hat{g}) = m(\Delta^L(P)(\blacktriangleright\otimes\blacktriangleright)(\hat{f}\otimes\hat{g})),\quad \hat{f},\hat{g}\in\mathcal{A}^L,\ P\in H^L,\qquad(42)$$

$$(\hat{u}\hat{v})\blacktriangleleft Q = m((\hat{u}\otimes\hat{v})(\blacktriangleleft\otimes\blacktriangleleft)\Delta^R(Q)),\quad \hat{u},\hat{v}\in\mathcal{A}^R,\ Q\in H^R.\qquad(43)$$

(ii) Δ^L is the unique left $\mathcal{A}^L$-module map $H^L\to H^L\hat{\otimes}H^L$ extending $\Delta_{\hat{S}(\mathfrak{g}^*)} : \hat{S}(\mathfrak{g}^*)\to\hat{S}(\mathfrak{g}^*)\hat{\otimes}\hat{S}(\mathfrak{g}^*)\subset H^L\hat{\otimes}H^L$. Likewise, Δ^R is the unique right $\mathcal{A}^R$-module map extending $\Delta_{\hat{S}(\mathfrak{g}^*)}$ from $\hat{S}(\mathfrak{g}^*)$ to H^R. Equivalently,

$$\Delta^L(\hat{f}\sharp P) = \hat{f}\Delta_{\hat{S}(\mathfrak{g}^*)}(P),\qquad \Delta^R(Q\sharp\hat{v}) = \Delta_{\hat{S}(\mathfrak{g}^*)}(Q)\hat{v},\qquad(44)$$

for all $P,Q\in\hat{S}(\mathfrak{g}^*)$, $\hat{f}\in\mathcal{A}^L$ and $\hat{v}\in\mathcal{A}^R$.
In particular, $\Delta^L(\hat{x}_\mu) = \hat{x}_\mu\otimes_{\mathcal{A}^L}1$ and $\Delta^R(\hat{y}_\mu) = 1\otimes_{\mathcal{A}^R}\hat{y}_\mu$.

(iii) $\Delta^L(\mathcal{O}^\mu_\nu) = \mathcal{O}^\gamma_\nu\otimes_{\mathcal{A}^L}\mathcal{O}^\mu_\gamma$, $\Delta^L(\mathcal{O}^{-1})^\mu_\nu = (\mathcal{O}^{-1})^\mu_\gamma\otimes_{\mathcal{A}^L}(\mathcal{O}^{-1})^\gamma_\nu$,
$\Delta^R(\mathcal{O}^\mu_\nu) = \mathcal{O}^\gamma_\nu\otimes_{\mathcal{A}^R}\mathcal{O}^\mu_\gamma$, $\Delta^R(\mathcal{O}^{-1})^\mu_\nu = (\mathcal{O}^{-1})^\mu_\gamma\otimes_{\mathcal{A}^R}(\mathcal{O}^{-1})^\gamma_\nu$,
$\Delta^L(\hat{y}_\nu) = \Delta^L(\hat{x}_\mu(\mathcal{O}^{-1})^\mu_\nu) = \hat{x}_\mu(\mathcal{O}^{-1})^\mu_\gamma\otimes_{\mathcal{A}^L}(\mathcal{O}^{-1})^\gamma_\nu = 1\otimes_{\mathcal{A}^L}\hat{y}_\nu$,
$\Delta^R(\hat{x}_\nu) = \Delta^R(\hat{y}_\mu\mathcal{O}^\mu_\nu) = (1\otimes\hat{y}_\mu)(\mathcal{O}^\beta_\nu\otimes\mathcal{O}^\mu_\beta) = \mathcal{O}^\beta_\nu\otimes_{\mathcal{A}^R}\hat{x}_\beta = \hat{x}_\nu\otimes_{\mathcal{A}^R}1$.

(iv) $(H^L,\Delta^L,\epsilon^L)$ and $(H^R,\Delta^R,\epsilon^R)$ satisfy the axioms for $\mathcal{A}^L$-coring and $\mathcal{A}^R$-coring respectively, provided we replace the tensor product of

bimodules by the completed tensor of (cofiltered) bimodules and the counit axioms modify to $(\epsilon\hat{\otimes}_{\mathcal{A}}\mathrm{id})\circ\Delta \cong j \cong (\mathrm{id}\hat{\otimes}_{\mathcal{A}}\epsilon)\circ\Delta$ *where, instead of the identity,* j *is the canonical map into the completion (say,* $j^L : H^L \hookrightarrow \hat{H}^L \cong H^L\hat{\otimes}\boldsymbol{k} \cong \boldsymbol{k}\hat{\otimes}H^L$*).*

Taking into account our bimodule structures, the counit axioms, Definition 5 (iii), read

$$\begin{array}{ll}\sum \alpha^L(\epsilon^L(h_{(1)}))h_{(2)} = h = \sum \beta^L(\epsilon^L(h_{(2)}))h_{(1)}, & h \in H^L \\ \sum h_{(2)}\beta^R(\epsilon^R(h_{(1)})) = h = \sum h_{(1)}\alpha^R(\epsilon^R(h_{(2)})), & h \in H^R.\end{array} \tag{45}$$

(v) The coring structures from (iv) canonically extend to an internal $\mathcal{A}^L$*-coring* $(\hat{H}^L,\hat{\Delta}^L,\hat{\epsilon}^L)$ *and an internal* $\mathcal{A}^R$*-coring* $(\hat{H}^R,\hat{\Delta}^R,\hat{\epsilon}^R)$ *(see [3]) in the category of complete cofiltered vector spaces with* $\hat{\otimes}$*-tensor product (see Proposition 2 and Appendix A.2). Bimodule structures on* $\hat{H}^L$*,* $\hat{H}^R$ *involve homomorphisms* $\hat{\alpha}^L := j^L\circ\alpha^L$*,* $\hat{\alpha}^R := j^R\circ\alpha^R$*, and antihomomorphisms* $\hat{\beta}^L := j^L\circ\beta^L$*,* $\hat{\beta}^R := j^R\circ\beta^R$*, where* $j^L : H^L \hookrightarrow \hat{H}^L$ *and* $j^R : H^R \hookrightarrow \hat{H}^R$ *are the canonical inclusions.*

Proof. The equivalence of the two statements in (ii) is evident. By Theorem 4 (ii), the formulas (42) and (43) determine $\Delta^L(P)$ and $\Delta^R(Q)$ uniquely. To show the existence, we set the values of Δ^L and Δ^R by (44) and check that (42) and (43) hold. By (37) and (38) we already know this for $P, Q \in \hat{S}(\mathfrak{g}^*)$. Using the action axiom for $\blacktriangleright$, observe that

$$\begin{array}{rcl}\hat{x}_\mu \blacktriangleright (P \blacktriangleright (\hat{f}\hat{g})) & = & \hat{x}_\mu\cdot m(\Delta^L(P)(\blacktriangleright\otimes\blacktriangleright)(\hat{f}\otimes\hat{g})) \\ & = & m(\hat{x}_\mu\Delta^L(P)(\blacktriangleright\otimes\blacktriangleright)(\hat{f}\otimes\hat{g})) \\ & \stackrel{(44)}{=} & m(\Delta^L(\hat{x}_\mu P)(\blacktriangleright\otimes\blacktriangleright)(\hat{f}\otimes\hat{g}))\end{array}$$

for all $\hat{f},\hat{g} \in \mathcal{A}^L$, hence (42) holds for all $P \in H^L$. Likewise check (43) for all $Q \in H^R$. Conclude (i). The statement in (ii) that Δ^L, Δ^R extend $\Delta_{\hat{S}(\mathfrak{g}^*)}$, $\Delta^{\mathrm{op}}_{\hat{S}(\mathfrak{g}^*)}$ is the statement that (42),(43) specialize to (37),(38) when $P, Q \in \hat{S}(\mathfrak{g}^*)$. The rest of (ii) follows from uniqueness in (i).

(iii) By Theorem 4 (ii), the first 4 formulas follow from (24),(25), (29),(30). The formulas for $\Delta^L(\hat{y}_\alpha)$ and $\Delta^R(\hat{x}_\alpha)$ are straightforward.

(iv) To show that Δ^L is an $\mathcal{A}^L$-bimodule map note that by (ii) Δ^L commutes with the left $\mathcal{A}^L$-action. It remains to show that Δ^L also commutes with the right $\mathcal{A}^L$-action. This is sufficient to check on the generators $\hat{x}_\mu$ of $\mathcal{A}^L$ and arbitrary $P \in \hat{S}(\mathfrak{g}^*)$:

$$\begin{array}{rcl}\Delta^L(P).\hat{x}_\mu & = & \sum P_{(1)}\otimes_{\mathcal{A}^L}\beta(\hat{x}_\mu)P_{(2)} \\ & = & \sum P_{(1)}\otimes_{\mathcal{A}^L}\alpha(\hat{x}_\nu)(\mathcal{O}^{-1})^\nu_\mu P_{(2)} \\ & = & \sum \beta(\hat{x}_\nu)P_{(1)}\otimes_{\mathcal{A}^L}(\mathcal{O}^{-1})^\nu_\mu P_{(2)} \\ & = & \sum \hat{x}_\gamma(\mathcal{O}^{-1})^\gamma_\nu P_{(1)}\otimes_{\mathcal{A}^L}(\mathcal{O}^{-1})^\nu_\mu P_{(2)} \\ & = & \Delta^L(\hat{x}_\nu(\mathcal{O}^{-1})^\nu_\mu P) \\ & = & \Delta^L(\beta(\hat{x}_\mu)(P))\end{array}$$

By Theorem 4 (iii) for $r = 3$, the action axiom for $\blacktriangleright$ and associativity in H^L implies the coassociativity of Δ^L.

We exhibit the counits ϵ^L and ϵ^R (and their completed versions $\hat\epsilon^L : \hat H^L \to \mathcal{A}^L$, $\hat\epsilon^R : \hat H^R \to \mathcal{A}^R$) by the corresponding actions on 1,

$$\epsilon^L(h) := h \blacktriangleright 1_{\mathcal{A}^L}, \quad \epsilon^R(h) := 1_{\mathcal{A}^R} \blacktriangleleft h. \tag{46}$$

The counit axioms (45) for ϵ^L are checked on the generators $\hat x_\mu$:

$$\sum \alpha(\epsilon^L(\hat x_{\mu(1)}))\hat x_{\mu(2)} = \alpha(\epsilon^L(\hat x_\mu))1 = \hat x_\mu,$$
$$\sum \beta(\epsilon^L(\hat x_{\mu(2)}))\hat x_{\mu(1)} = \beta(\epsilon^L(1))\hat x_\mu = \hat x_\mu.$$

Similarly, one checks the counit identities for ϵ^R.

Using formal expressions in the completions, (v) is straightforward.

DEFINITION 6. *(Modification of [2, 4, 7], cf. 2). Given an algebra $\mathcal{A}$, a* **formally completed left $\mathcal{A}$-bialgebroid** $(H, m, \alpha, \beta, \Delta, \epsilon)$ *consists of the following data. H is a cofiltered vector space and (H, m) an associative algebra with multiplication m distributive with respect to the formal sums in each argument (Appendix A.2) and the factorwise multiplication on $H \otimes H$ extends to a multiplication $(H\hat\otimes H) \otimes (H\hat\otimes H)$ distributive with respect to the formal sums in each argument; $\alpha : \mathcal{A} \to H$ and $\beta : \mathcal{A}^{\mathrm{op}} \to H$ are fixed algebra homomorphisms with commuting images; H is equipped with a structure of an $\mathcal{A}$-bimodule via the formula $a.h.a' := \alpha(a)\beta(a')h$; $\Delta : H \to H\hat\otimes_{\mathcal{A}} H$ is an $\mathcal{A}$-bimodule map, coassociative and with counit $\epsilon : H \to \mathcal{A}$ understood with respect to the completed tensor product $\hat\otimes$ and the counit axiom modifies to $(\epsilon\hat\otimes_{\mathcal{A}}\mathrm{id})\circ\Delta \cong j \cong (\mathrm{id}\hat\otimes_{\mathcal{A}}\epsilon)\circ\Delta$ where $j : H \to H\hat\otimes\boldsymbol{k} \cong \hat H \cong \boldsymbol{k}\hat\otimes H$ is the canonical map into the completion; both Δ and ϵ should be distributive with respect to formal sums. It is required that*

(i) ϵ is a **left character** *on the $\mathcal{A}$-ring (H, m, α) in the sense that the formula $h \otimes \hat f \mapsto \epsilon(h\alpha(\hat f))$ defines an action $H \otimes \mathcal{A} \to \mathcal{A}$ extending the left regular action $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$;*

(ii) the coproduct $\Delta : H \to H\hat\otimes_{\mathcal{A}} H$ corestricts to the **formal Takeuchi product**

$$H\hat\times_{\mathcal{A}} H \subset H\hat\otimes_{\mathcal{A}} H$$

which is by definition the $\mathcal{A}$-subbimodule, consisting of all formal sums $b = \sum_\lambda b_\lambda \otimes_{\mathcal{A}} b'_\lambda$, where also $\sum_\lambda b_\lambda \otimes_{\boldsymbol{k}} b'_\lambda \in H\hat\otimes_{\boldsymbol{k}} H$ is formal and such that

$$\sum_\lambda b_\lambda \otimes_{\mathcal{A}} b'_\lambda \alpha(a) = \sum_\lambda b_\lambda \beta(a) \otimes_{\mathcal{A}} b'_\lambda, \ \forall a \in \mathcal{A}.$$

(iii) The corestriction $\Delta| : H \to H\hat\times_{\mathcal{A}} H$ is an algebra map.

Notice that $H\hat{\otimes}_{\mathcal{A}}H$ does not carry a well-defined multiplication induced from $H\hat{\otimes}H$, unlike $H\hat{\times}_{\mathcal{A}}H$ which does. This explains the need for (ii). Indeed, (ii) implies that for any $b\in H\hat{\times}H$ and for any formal sum $c=\sum_\mu c_\mu\otimes_{\mathcal{A}} c'_\mu\in H\hat{\otimes}_{\mathcal{A}}H$ the product $b\cdot c$ obtained by lifting b and c to $H\hat{\otimes}_{\boldsymbol{k}}H$ is a well defined element of $H\hat{\otimes}_{\mathcal{A}}H$ (does not depend on the lifting as a formal sum in $H\hat{\otimes}_{\boldsymbol{k}}H$); if moreover $c\in H\hat{\times}_{\mathcal{A}}H$ then $b\cdot c\in H\hat{\times}_{\mathcal{A}}H$. Thus $H\hat{\times}_{\mathcal{A}}H$ is an algebra and (iii) makes sense.

Interchanging the left and right sides in all modules and binary tensor products in the definition of a left $\mathcal{A}$-bialgebroid, we get a **right $\mathcal{A}$-bialgebroid** ([2]). The $\mathcal{A}$-bimodule structure on H is then given by $a.h.b := h\alpha(b)\beta(a)$. In short, $(H,m,\alpha,\beta,\Delta,\epsilon)$ is a right $\mathcal{A}$-bialgebroid iff $(H,m,\beta,\alpha,\Delta^{\mathrm{op}},\epsilon)$ is a left $\mathcal{A}^{\mathrm{op}}$-bialgebroid; analogously with the completed versions.

Let us return to our candidate examples H^L and H^R. Regarding that $H^L\hat{\otimes}H^L=\hat{H}^L\hat{\otimes}\hat{H}^L$ it may be convenient to have all modules completed to start with, hence considering the completed smash product algebras $\hat{H}^L,\hat{H}^R$ (Theorem 6 and Proposition 2). One of the advantages is that for $\hat{H}^L$ and $\hat{H}^R$ the internal coring axiom is not modified for $\hat{\Delta}^L$. Still, the expectation that all objects and morphisms are in the completed sense is *not* true, as the multiplication $m:H^L\otimes H^L\to H^L$ can be extended to $\hat{H}^L\otimes\hat{H}^L\to\hat{H}^L$ but can not be extended to a function on the completed tensor product $\hat{H}^L\hat{\otimes}\hat{H}^L\to\hat{H}^L$ distributive over formal sums. The thesis [25] alternatively introduces a canonical tensor product on a more complicated category of filtrations of cofiltrations; it involves less drastic completions in general and admits a truly internal bialgebroid structure on H^L.

PROPOSITION 4. $(H^L,m,\alpha^L,\beta^L,\Delta^L,\epsilon^L)$ *and* $(\hat{H}^L,\hat{m},\hat{\alpha}^L,\hat{\beta}^L,\hat{\Delta}^L,\hat{\epsilon}^L)$ *have a structure of formally completed left $\mathcal{A}^L$-bialgebroids. Likewise,* $(H^R,m,\alpha^R,\beta^R,\Delta^R,\epsilon^R)$ *and* $(\hat{H}^R,\hat{m},\hat{\alpha}^R,\hat{\beta}^R,\hat{\Delta}^R,\hat{\epsilon}^R)$ *are formally completed right $\mathcal{A}^R$-bialgebroids.*

Proof. The coring axioms are checked in Proposition 3.

To check that the rule $\sum_\lambda h_\lambda\otimes\hat{f}_\lambda\mapsto\sum_\lambda\epsilon^L(h_\lambda\alpha(\hat{f}_\lambda))$ (for finite sums) is an action and (i) holds for ϵ^L, observe from the definition (46) that $\epsilon^L(h\alpha(\hat{f}))=h\alpha(\hat{f})\blacktriangleright 1=h\blacktriangleright\hat{f}$, for all $\hat{f}\in\mathcal{A}^L$, $h\in H^L$. Analogously check (i) for $\epsilon^R,\hat{\epsilon}^L,\hat{\epsilon}^R$.

To show that $\hat{\Delta}^L$ corestricts to the formal Takeuchi product $\hat{H}^L\hat{\times}_{\mathcal{A}^L}\hat{H}^L$, calculate for $P\in\hat{H}^L$ and $\hat{f},\hat{g},\hat{h}\in\mathcal{A}^L$,

$$\begin{array}{rcl}((P_{(1)}\hat{\beta}^L(\hat{g})\blacktriangleright\hat{f})\cdot(P_{(2)}\blacktriangleright\hat{h}) & = & (P_{(1)}\blacktriangleright(\hat{\beta}^L(\hat{g})\blacktriangleright\hat{f}))\cdot(P_{(2)}\blacktriangleright\hat{h})\\ & \overset{(40)}{=} & (P_{(1)}\blacktriangleright(\hat{f}\hat{g}))\cdot(P_{(2)}\blacktriangleright\hat{h})\\ & = & P\blacktriangleright(\hat{f}\hat{g}\hat{h})\\ & = & (P_{(1)}\blacktriangleright\hat{f})\cdot((P_{(2)}\hat{\alpha}^L(\hat{g}))\blacktriangleright\hat{h}),\end{array}$$

thus, by Theorem 4 (ii), (iv), $P_{(1)}\hat{\beta}^L(\hat{g})\otimes_{\mathcal{A}^L} P_{(2)} = P_{(1)}\otimes_{\mathcal{A}^L} P_{(2)}\hat{\alpha}^L(\hat{g})$, hence $\hat{\Delta}^L(P)\in \hat{H}^L\hat{\times}_{\mathcal{A}^L}\hat{H}^L$.

We now check directly that the corestriction $\hat{\Delta}^L : \hat{H}^L \to \hat{H}^L\hat{\times}_{\mathcal{A}^L}\hat{H}^L$ is a homomorphism of algebras,

$$\hat{\Delta}^L(h_1h_2) = \hat{\Delta}^L(h_1)\hat{\Delta}^L(h_2) \quad \text{for all } h_1, h_2 \in H^L.$$

To this aim, recall that $\Delta_{\hat{S}(\mathfrak{g}^*)} : \hat{S}(\mathfrak{g}^*) \to \hat{S}(\mathfrak{g}^*)\hat{\otimes}\hat{S}(\mathfrak{g}^*)$ is a homomorphism, and that by Proposition 3 (ii), $\hat{\Delta}^L|_{1\sharp\hat{S}(\mathfrak{g}^*)}$ is the composition

$$1\sharp\hat{S}(\mathfrak{g}^*) \cong \hat{S}(\mathfrak{g}^*) \stackrel{\Delta_{\hat{S}(\mathfrak{g}^*)}}{\longrightarrow} \hat{S}(\mathfrak{g}^*)\hat{\otimes}\hat{S}(\mathfrak{g}^*) \hookrightarrow \hat{H}^L\hat{\times}_{\mathcal{A}^L}\hat{H}^L,$$

hence homomorphism as well (the inclusion is a homomorphism, because the product is factorwise). We use this when applying to the tensor factor $P_{(2)}Q$ in the calculation

$$\begin{array}{rcl}
\hat{\Delta}^L((u\sharp P)(v\sharp Q)) &=& \hat{\Delta}^L(u(P_{(1)}\blacktriangleright v)\sharp P_{(2)}Q)\\
&=& (u(P_{(1)}\blacktriangleright v)\sharp P_{(2)}Q_{(1)})\otimes(1\sharp P_{(3)}Q_{(2)}).\\
&=& [(u\sharp P_{(1)})(v\sharp Q_{(1)})]\otimes(1\sharp P_{(2)}Q_{(2)})\\
&=& (u(P_{(1)}\blacktriangleright v)\sharp P_{(2)}Q_{(1)})\otimes(1\sharp P_{(3)}Q_{(2)}).\\
&=& [(u\sharp P_{(1)})\otimes(1\sharp P_{(2)})][(v\sharp Q_{(1)})\otimes(1\sharp Q_{(2)})]\\
&=& \hat{\Delta}^L(u\sharp P)\hat{\Delta}^L(v\sharp Q).
\end{array}$$

7. The antipode and Hopf algebroid

A Hopf algebroid is roughly a bialgebroid with an antipode. In the literature, there are several nonequivalent versions. In the framework of G. Böhm [2], there are two variants which are equivalent if the antipode is bijective (as it is here the case): nonsymmetric and symmetric. The *nonsymmetric* involves one-sided bialgebroid with an antipode map satisfying axioms which involve both the antipode map and its inverse. The *symmetric* version involves two bialgebroids and axioms neither involve nor require the inverse of the antipode. We choose this version here, because we naturally constructed two actions, $\blacktriangleright$ and $\blacktriangleleft$, which lead to the two coproducts, Δ^L and Δ^R, as exhibited in Section 6.

DEFINITION 7. *Given two algebras $\mathcal{A}^L$ and $\mathcal{A}^R$ with fixed isomorphism $(\mathcal{A}^L)^{\mathrm{op}} \cong \mathcal{A}^R$, a* **symmetric Hopf algebroid** *([2]) is a pair of a left $\mathcal{A}^L$-bialgebroid H^L and a right $\mathcal{A}^R$-bialgebroid H^R, isomorphic and identified as algebras $H \cong H^L \cong H^R$, such that the compatibilities*

$$\begin{array}{c}
\alpha^L\circ\epsilon^L\circ\beta^R = \beta^R,\ \ \beta^L\circ\epsilon^L\circ\alpha^R = \alpha^R,\\
\alpha^R\circ\epsilon^R\circ\beta^L = \beta^L,\ \ \beta^R\circ\epsilon^R\circ\alpha^L = \alpha^L,
\end{array} \tag{47}$$

hold between the source and target maps $\alpha^L, \alpha^R, \beta^L, \beta^R$*, and the counits* ϵ^L, ϵ^R*; the comultiplications* Δ^L *and* Δ^R *satisfy the compatibility relations*

$$(\Delta^R \otimes_{\mathcal{A}^L} \mathrm{id}) \circ \Delta^L = (\mathrm{id} \otimes_{\mathcal{A}^R} \Delta^L) \circ \Delta^R \tag{48}$$

$$(\Delta^L \otimes_{\mathcal{A}^R} \mathrm{id}) \circ \Delta^R = (\mathrm{id} \otimes_{\mathcal{A}^L} \Delta^R) \circ \Delta^L \tag{49}$$

and there is a map $\mathcal{S} : H \to H$*, called the* **antipode** *which is an antihomomorphism of algebras and satisfies*

$$\begin{array}{c} \mathcal{S} \circ \beta^L = \alpha^L, \qquad \mathcal{S} \circ \beta^R = \alpha^R \\ m \circ (\mathcal{S} \otimes \mathrm{id}) \circ \Delta^L = \alpha^R \circ \epsilon^R \\ m \circ (\mathrm{id} \otimes \mathcal{S}) \circ \Delta^R = \alpha^L \circ \epsilon^L \end{array} \tag{50}$$

A **formally completed symmetric Hopf algebroid** *is defined analogously as a pair of left and right formally completed bialgebroid with antipode* $\mathcal{S}$ *satisfying (50) and the compatibilities (47),(48),(49) satisfied with the tensor products replaced with the completed ones.*

THEOREM 5. *Data* $\mathcal{A}^L = U(\mathfrak{g}^L)$*,* $\mathcal{A}^R = U(\mathfrak{g}^R)$ *together with either*

(i) $\hat{H}^L := U(\mathfrak{g}^L)\hat{\sharp}\hat{S}(\mathfrak{g}^*)$*,* $\hat{H}^R := \hat{S}(\mathfrak{g}^*)\hat{\sharp}U(\mathfrak{g}^R)$*,* $\hat{\epsilon}^L, \hat{\epsilon}^R, \hat{\alpha}^L, \hat{\beta}^L, \hat{\alpha}^R, \hat{\beta}^R$ *from Section 5 and* $\hat{\Delta}^L$*,* $\hat{\Delta}^R$ *defined in Section 6,*

(ii) or $H^L = U(\mathfrak{g}^L)\sharp\hat{S}(\mathfrak{g}^*)$*,* $H^R := \hat{S}(\mathfrak{g}^*)\sharp U(\mathfrak{g}^R)$*,* $\epsilon^L, \epsilon^R, \alpha^L, \beta^L, \alpha^R, \beta^R$ *from Section 5 and* Δ^L*,* Δ^R *defined in Section 6,*

form a formally completed symmetric Hopf algebroid. The antipode map for (i) is the unique homomorphism of algebras $\mathcal{S} : \hat{H} \to \hat{H}$ *distributive over formal sums and such that*

$$\mathcal{S}(\partial^\nu) = -\partial^\nu,$$

and the antipode for (ii) is its restriction $\mathcal{S} = \mathcal{S}| : H \to H$*. The antipode* $\mathcal{S}$ *is bijective in both cases, and by distributivity over formal sums it follows that* $\mathcal{S}(\mathcal{O}) = \mathcal{S}(e^{\mathcal{C}}) = e^{-\mathcal{C}} = \mathcal{O}^{-1}$ *and*

$$\mathcal{S}(\hat{y}_\mu) = \hat{x}_\mu. \tag{51}$$

For a general Lie algebra $\mathfrak{g}$*,* $\mathcal{S}^2 \neq \mathrm{id}$*. More precisely,*

$$\mathcal{S}^2(\hat{y}_\mu) = \mathcal{S}(\hat{x}_\mu) = \hat{y}_\mu - C^\lambda_{\mu\lambda}, \quad \mathcal{S}^{-2}(\hat{x}_\mu) = \mathcal{S}^{-1}(\hat{y}_\mu) = \hat{x}_\mu - C^\lambda_{\mu\lambda} \tag{52}$$

$$\mathcal{S}^2(\hat{x}_\mu) = \hat{x}_\mu + C^\lambda_{\mu\lambda}, \quad \mathcal{S}^{-2}(\hat{y}_\mu) = \hat{y}_\mu + C^\lambda_{\mu\lambda}. \tag{53}$$

with the summation over λ *understood.*

Proof. In this proof, we simply write ϵ^L, Δ^L etc. without hat symbol, as it is not essential for the arguments below which work for both versions. We proved that the above data give bialgebroids (Proposition 4).

One checks the relations (47) on generators, for which $\alpha^R(\hat{y}_\mu) = \hat{y}_\mu$, $\beta^R(\hat{y}_\mu) = \mathcal{O}^\rho_\mu \hat{y}_\rho = \mathcal{O}^\rho_\mu \hat{y}_\sigma (\mathcal{O}^{-1})^\sigma_\rho$, $\alpha^L(\hat{x}_\mu) = \hat{x}_\mu$, $\beta^L(\hat{x}_\mu) = \hat{y}_\mu$.

Regarding that Δ^L and Δ^R restricted to $\hat{S}(\mathfrak{g}^*)$ coincide with $\Delta_{\hat{S}(\mathfrak{g}^*)}$, (48) and (49) restricted to $\hat{S}(\mathfrak{g}^*)$ reduce to the coassociativity. Algebra H^L is generated by $\hat{S}(\mathfrak{g}^*)$ and $\mathfrak{g}^L$, so it is enough to check (48),(49) also on $\hat{y}_\mu = \hat{x}_\nu (\mathcal{O}^{-1})^\nu_\mu$. This follows from the matrix identities

$$\Delta^L(\hat{x}\mathcal{O}^{-1}) = \hat{x}\mathcal{O}^{-1} \otimes_{\mathcal{A}^L} \mathcal{O}^{-1} = \mathcal{O}^{-1}\mathcal{O} \otimes_{\mathcal{A}^L} \hat{x}\mathcal{O}^{-1} = 1 \otimes_{\mathcal{A}^L} \hat{y},$$

$$\Delta^R(\hat{y}) = 1 \otimes_{\mathcal{A}^R} \hat{y} = \hat{y} \otimes_{\mathcal{A}^R} \mathcal{O}^{-1} = \hat{x}\mathcal{O}^{-1} \otimes_{\mathcal{A}^R} \mathcal{O}^{-1}.$$

Formula $\mathcal{S}(\partial^\mu) = -\partial^\mu$ clearly extends to a unique continuous antihomomorphism of algebras on the formal power series ring $\hat{S}(\mathfrak{g}^*)$. Similarly, by functoriality of $\mathfrak{g} \mapsto U(\mathfrak{g})$, the antihomomorphism of Lie algebras, $\mathcal{S} : \mathfrak{g}^R \to \mathfrak{g}^L$, $\hat{y}_\mu \mapsto \hat{x}_\mu$, extends to a unique antihomomorphism $U(\mathfrak{g}^R) \to U(\mathfrak{g}^L)$. Regarding that $U(\mathfrak{g}^R)$ and $\hat{S}(\mathfrak{g}^*)$ generate H^R, it is sufficient to check that $\mathcal{S}$ is compatible with the additional relations in the smash product, namely $[\partial^\mu, \hat{y}_\nu] = \left(\frac{\mathcal{C}}{e^{\mathcal{C}}-1}\right)^\mu_\nu$. Then $\mathcal{S}([\partial^\mu, \hat{x}_\nu]) = \mathcal{S}\left(\frac{-\mathcal{C}}{e^{-\mathcal{C}}-1}\right)^\mu_\nu = \left(\frac{\mathcal{C}}{e^{\mathcal{C}}-1}\right)^\mu_\nu = \left(e^{-\mathcal{C}}\frac{-\mathcal{C}}{e^{-\mathcal{C}}-1}\right)^\mu_\nu$, which equals $(e^{-\mathcal{C}})^\rho_\nu [\hat{x}_\rho, -\partial^\mu] = [\mathcal{S}(\hat{y}_\rho \mathcal{O}^\rho_\nu), -\partial^\mu] = [\mathcal{S}(\hat{x}_\nu), \mathcal{S}(\partial^\mu)]$.

To exhibit the inverse $\mathcal{S}^{-1}$, we similarly check that the obvious formulas $\mathcal{S}^{-1}(\hat{x}_\mu) = \hat{y}_\mu$, $\mathcal{S}^{-1}(\partial^\mu) = \partial^\mu$ define a unique continuous (that is, distributive over formal sums) antihomomorphism $\mathcal{S}^{-1} : H \to H$.

For (52) calculate $\mathcal{S}(\hat{x}_\mu) = \mathcal{S}(\hat{y}_\rho \mathcal{O}^\rho_\mu) = \mathcal{S}(\mathcal{O}^\rho_\mu)\mathcal{S}(\hat{y}_\rho) = (\mathcal{O}^{-1})^\rho_\mu \hat{x}_\rho = (\mathcal{O}^{-1})^\rho_\mu \hat{y}_\sigma \mathcal{O}^\sigma_\rho$ and use $[\mathcal{O}^\rho_\mu, \hat{y}_\sigma] = -C^\rho_{\tau\sigma}\mathcal{O}^\tau_\mu$ in the last step. Similarly, we get $\mathcal{S}^{-1}(\hat{y}_\mu) = \mathcal{O}^\rho_\mu \hat{x}_\sigma (\mathcal{O}^{-1})^\sigma_\rho$ and use $[\mathcal{O}^\rho_\mu, \hat{x}_\sigma] = -C^\rho_{\tau\sigma}\mathcal{O}^\tau_\mu$ for the second formula in (52). Notice that $\mathcal{S}^{-1}(\hat{y}_\mu) = \hat{z}_\mu$ from Theorem 3, formula (31). For (53) similarly use the matrix identities $\mathcal{S}^2(\hat{x}) = \mathcal{S}(\mathcal{O}^{-1}\hat{y}\mathcal{O}) = \mathcal{O}^{-1}\hat{x}\mathcal{O}$, $\mathcal{S}^{-2}(\hat{y}) = \mathcal{O}\hat{y}\mathcal{O}^{-1}$.

The formula $\mathcal{S}(\beta^L(\hat{x}_\mu)) = \mathcal{S}(\hat{y}_\mu) = \hat{x}_\mu = \alpha^L(\hat{x}_\mu)$ shows $\mathcal{S} \circ \beta^L = \alpha^L$ for the generators of $\mathcal{A}^L$. Likewise for the rest of the identities (50).

8. Conclusion and perspectives.

We have equipped the noncommutative phase spaces of Lie algebra type with the structure of a version of a Hopf algebroid over $U(\mathfrak{g})$. That roughly means that we have found a left $U(\mathfrak{g})$-bialgebroid H^L, and a right $U(\mathfrak{g})^{\mathrm{op}}$-bialgebroid H^R, which are canonically isomorphic as associative algebras $H^L \cong H^R$, and an antipode map $\mathcal{S}$ satisfying a number of axioms involving a completed tensor product $\hat{\otimes}$.

Hopf algebroids allow a version of Drinfeld's twisting cocycles studied earlier in the context of deformation quantization ([29]), and are a promising tool for extending many constructions to the noncommutative case, and a planned direction for our future work. One can find a cocycle which can be used to twist the Hopf algebroid corresponding to the abelian Lie algebra (i.e. the Hopf algebroid structure on the completion of the usual Weyl algebra) to recover the Hopf algebroid of the phase space for any other Lie algebra of the same dimension ([22]). More importantly for applications, along with the phase space one can systematically twist many geometric structures, including differential forms, from the undeformed to the deformed case. This has earlier been studied in the case of κ-spaces (e.g. in [15]), while the work for general finite-dimensional Lie algebras (and for some nonlinear star products) is in progress.

Acknowledgments. We thank L. El Kaoutit, V. Roubtsov, T. Brzeziński, T. Maszczyk and G. Böhm for reading fragments of this work and advice. We thank A. Borowiec, J. Lukierski and A. Pachoł for discussions. S.M. has been supported by Croatian Science Foundation, project IP-2014-09-9582. A part of the work has been done at IRB, Zagreb, and a substantial progress has also been made during the visit of Z.Š. to IHÉS in November 2012. Z.Š. thanks G. Garkusha for the invitation to present a related *Algebra and topology seminar* at Swansea, July 15, 2014.

Appendix

A.1 Commutation $[\hat{x}_\alpha, \hat{y}_\beta] = 0$

PROPOSITION 5. *The identity* $[\hat{x}_\mu, \hat{y}_\nu] = 0$ *holds in the realization* $\hat{x}_\mu = x_\sigma \phi^\sigma_\mu = x_\sigma \left(\frac{-\mathcal{C}}{e^{-\mathcal{C}}-1}\right)^\sigma_\mu$, $\hat{y}_\mu = x_\rho \tilde{\phi}^\rho_\mu = x_\rho \left(\frac{\mathcal{C}}{e^{\mathcal{C}}-1}\right)^\rho_\mu$, *where* $\mathcal{C}^\mu_\nu = C^\mu_{\nu\gamma}\partial^\gamma$ *(cf. the equations (15,12,13)).*

Proof. For any formal series $P = P(\partial)$ in ∂-s, $[P, \hat{x}_\mu] = \frac{\partial P}{\partial(\partial^\mu)} =: \delta_\mu P$. In particular (cf. [10]), from $[\hat{x}_\mu, \hat{x}_\nu] = C^\lambda_{\mu\nu}\hat{x}_\lambda$, one obtains a formal differential equation for ϕ^σ_μ,

$$(\delta_\rho \phi^\gamma_\mu)\phi^\rho_\nu - (\delta_\rho \phi^\gamma_\nu)\phi^\rho_\mu = C^\sigma_{\mu\nu}\phi^\gamma_\sigma. \tag{54}$$

By symmetry $C^i_{jk} \mapsto -C^i_{jk}$ the same equation holds with $(-\tilde{\phi}) = \frac{-\mathcal{C}}{e^{\mathcal{C}}-1}$ in the place of ϕ. Similarly, the equation $[\hat{x}_\mu, \hat{y}_\nu] = 0$, i.e. $[x_\gamma \phi^\gamma_\mu, x_\beta \tilde{\phi}^\beta_\nu] = 0$, is equivalent to

$$(\delta_\rho \phi^\gamma_\mu)\tilde{\phi}^\rho_\nu - (\delta_\rho \tilde{\phi}^\gamma_\nu)\phi^\rho_\mu = 0 \tag{55}$$

Recall that $\phi = \frac{-\mathcal{C}}{e^{-\mathcal{C}}-1} = \sum_{N=0}^\infty (-1)^N \frac{B_N}{N!}(\mathcal{C}^N)^i_j$, where B_N are the Bernoulli numbers, which are zero unless N is either even or $N = 1$.

Hence $\tilde{\phi} = \frac{\mathcal{C}}{e^{\mathcal{C}}-1} = \sum_{N=0}^{\infty} \frac{B_N}{N!}\mathcal{C}^N = \frac{B_1}{2}\mathcal{C} + \sum_{N\,\mathrm{even}}^{\infty} \frac{B_N}{N!}\mathcal{C}^N$ and $\phi - \tilde{\phi} = -2\frac{B_1}{2}\mathcal{C} = \mathcal{C}$. Notice that $\frac{\partial \mathcal{C}^\alpha_\beta}{\partial(\partial^\mu)} = C^\alpha_{\beta\mu}$. Therefore, subtracting (55) from (54) gives the condition

$$(\delta_\rho \phi^\gamma_\mu)\mathcal{C}^\rho_\nu - C^\gamma_{\nu\rho}\phi^\rho_\mu = C^\sigma_{\mu\nu}\phi^\gamma_\sigma.$$

$\mathcal{C}$ is homogeneous of degree 1 in ∂^μ-s, so we can split this condition into the parts of homogeneity degree N:

$$[\delta_\rho(\mathcal{C}^N)^\gamma_\mu]\mathcal{C}^\rho_\nu - (\delta_\rho \mathcal{C}^\gamma_\nu)(\mathcal{C}^N)^\rho_\mu = C^\sigma_{\mu\nu}(\mathcal{C}^N)^\gamma_\sigma, \tag{56}$$

where the overall factor of $(-1)^N B_N/N!$ has been taken out. Hence the proof is reduced to the following lemma:

LEMMA 2. *The identities (56) hold for $N = 0, 1, 2, \ldots$.*

Proof. For $N = 0$, (56) reads $C^\gamma_{\nu\mu} = C^\gamma_{\mu\nu}$, which is the antisymmetry of the bracket. For $N = 1$ it follows from the Jacobi identity:

$$(C^\gamma_{\mu\rho}C^\rho_{\nu\tau} - C^\gamma_{\nu\rho}C^\rho_{\mu\tau})\partial^\tau = C^\rho_{\mu\nu}C^\gamma_{\rho\tau}\partial^\tau.$$

Suppose now (56) holds for given $N = K \geq 1$. Then

$$C^\gamma_{\mu\nu}(\mathcal{C}^K)^\rho_\sigma \mathcal{C}^\gamma_\rho = [\delta_\rho(\mathcal{C}^K)^\rho_\mu]\mathcal{C}^\sigma_\nu \mathcal{C}^\gamma_\rho - C^\rho_{\nu\sigma}(\mathcal{C}^K)^\sigma_\mu \mathcal{C}^\gamma_\rho$$

By the usual Leibniz rule for δ_ρ, this yields

$$C^\gamma_{\mu\nu}(\mathcal{C}^K)^\rho_\sigma \mathcal{C}^\gamma_\rho = \delta_\rho(\mathcal{C}^{K+1})^\gamma_\rho \mathcal{C}^\sigma_\nu - (\mathcal{C}^K)^\rho_\mu C^\gamma_{\rho\sigma}\mathcal{C}^\sigma_\nu - C^\rho_{\nu\sigma}(\mathcal{C}^K)^\sigma_\mu \mathcal{C}^\gamma_\rho.$$

The identity (56) follows for $N = K+1$ if the second and third summand on the right hand side add up to $-C^\gamma_{\nu\sigma}(\mathcal{C}^{K+1})^\sigma_\mu$. After renaming the indices, one brings the sum of these two to the form

$$(\mathcal{C}^K)^\rho_\mu(-C^\sigma_{\nu\lambda}C^\gamma_{\rho\sigma} + C^\sigma_{\nu\rho}C^\gamma_{\lambda\sigma})\partial^\lambda = -(C^K)^\rho_\mu C^\sigma_{\rho\lambda}\partial^\lambda C^\gamma_{\nu\sigma} = -(C^{K+1})^\sigma_\mu C^\gamma_{\nu\sigma}$$

as required. The Jacobi identity is used for the equality on the left.

A.2 Cofiltered vector spaces and completions

We sketch the formalism treating the algebraic duals $U(\mathfrak{g})^*$ and $S(\mathfrak{g})^*$ of filtered algebras $U(\mathfrak{g}), S(\mathfrak{g})$ as cofiltered algebras. The reader can treat them alternatively as topological algebras: the basis of neighborhoods of 0 in the formal adic topology of $U(\mathfrak{g})^*$ and $S(\mathfrak{g})^*$ is given by the annihilator ideals Ann $U_i(\mathfrak{g})$ and Ann $S_i(\mathfrak{g})$, consisting of functionals vanishing on the i-th filtered component. A **cofiltration** on a vector space A is an inverse sequence of epimorphisms of its quotient spaces

$\ldots \to A_{i+1} \to A_i \to A_{i-1} \to \ldots \to A_0$; denoting the quotient maps $\pi_i : A \to A_i$ and $\pi_{i,i+k} : A_{i+k} \to A_i$, the identities $\pi_i = \pi_{i,i+k} \circ \pi_{i+k}$, $\pi_{i,i+k+l} = \pi_{i,i+k} \circ \pi_{i+k,i+k+l}$ are required to hold. The limit $\underleftarrow{\lim}_r A_r$ consists of *threads*, i.e. the sequences $(a_r)_{r\in\mathbb{N}_0} \in \prod_r A_r$ of compatible elements, $a_r = \pi_{r,r+k}(a_{r+k})$. The canonical map $A \to \hat{A}$ to the **completion** $\hat{A} := \underleftarrow{\lim}_r A_r$ is 1-1 if $\forall a \in A\ \exists r \in \mathbb{N}_0$ such that $\pi_r(a) \neq 0$. The cofiltration is **complete** if the canonical map $A \to \hat{A}$ is an isomorphism. **Strict morphisms** of cofiltered vector spaces $A \to B$ are the linear maps which induce the levelwise maps $A_r \to B_r$ on the quotients. (This makes the category of complete cofiltered vector spaces more rigid than the category of pro-vector spaces.) We say that $a = (a_r)_r$ has the **cofiltered degree** $\geq N$ if $a_r = 0$ for $r < N$. In our main example, $U_i(\mathfrak{g})^* := (U(\mathfrak{g})^*)_i := U(\mathfrak{g})^* / \mathrm{Ann}\, U_i(\mathfrak{g}) \cong (U_i(\mathfrak{g}))^*$ and similarly for $S(\mathfrak{g})^* \cong \hat{S}(\mathfrak{g}^*)$. We use lower indices both for filtrations and for cofiltrations (but upper for gradations!). Given a family of elements in A, $a : \Lambda \to A$, $\lambda \mapsto a_\lambda$, the expression ('abstract infinite sum') $\sum_{\lambda\in\Lambda} a_\lambda$ is called a **formal sum** if for each $r \geq 0$, there is only finitely many λ such that $\pi_r(a_\lambda) \neq 0$ hence $\pi_r(\sum_{\lambda\in\Lambda} a_\lambda) := \sum_{\lambda\in\Lambda} \pi_r(a_\lambda) \in A_r$ is well defined; and therefore there is well defined thread $(\pi_r(\sum_{\lambda\in\Lambda} a_\lambda))_r \in \hat{A}$, the value of the formal sum.

The usual tensor product $A \otimes B$ of cofiltered vector spaces is cofiltered with the r-th cofiltered component (see [25])

$$(A \otimes B)_r = \frac{A \otimes B}{\cap_{p+q=r} \ker\, \pi^A_p \otimes \ker\, \pi^B_q}. \tag{57}$$

$(A \otimes B)_r$ is an abelian group of finite sums of the form $\sum_\lambda a_\lambda \otimes b_\lambda \in A \otimes B$ modulo the additive relation of equivalence $\sim_r$ for which $\sum a_\mu \otimes b_\mu \sim_r 0$ iff $\pi_p(a_\mu) \otimes \pi_q(b_\mu) = 0$ in $A_p \otimes B_q$ for all p, q such that $p+q = r$. Define the **completed tensor product** $A \hat{\otimes} B = \underleftarrow{\lim}_r (A \otimes B)_r$, equipped with the same cofiltration, $(A\hat{\otimes}B)_r := (A \otimes B)_r$. An element in $A\hat{\otimes}B$ is thus the class of equivalence of a formal sum $\sum_\lambda a_\lambda \otimes b_\lambda$ such that for any p and q there are at most finitely many λ such that $\pi^A_p(a_\lambda) \otimes \pi^B_q(b_\lambda) \neq 0$. Alternatively, we can equip $A \otimes B$ with a bicofiltration ($\mathbb{N}_0 \times \mathbb{N}_0$-cofiltration), $(A \otimes B)_{r,s} = A_r \otimes B_s$. Observe the inclusions $\ker\, \pi_{r+s} \otimes \ker\, \pi_{r+s} \subset \cap_{p+q=r+s} \ker\, \pi_p \otimes \ker\, \pi_q \subset \ker\, \pi_r \otimes \ker\, \pi_s$, which induce projections $A_{r+s} \otimes B_{r+s} \twoheadrightarrow (A \otimes B)_{r+s} \twoheadrightarrow A_r \otimes B_s$ for all r, s; by passing to the limit we see that the completion with respect to the bicofiltration and with respect to the original cofiltration are equivalent (and alike statement for the convergence of infinite sums inside $A\hat{\otimes}B$). A linear map among cofiltered vector spaces is **distributive over formal sums** if it sends formal sums to formal sums summand by summand (formal version of σ-additivity). This property is weaker than being a strict morphism of complete cofiltered vector spaces. In fact ([25]), a linear map $f : C \to D$ is distributive over formal sums iff $\forall s\ \exists r$ and a linear map $f_{sr} : C_r \to D_s$ such that $\pi_s \circ f = f_{sr} \circ \pi_r$ (in the strict case we required $s = r$). If A and B are complete, we can also consider maps $A \otimes B \to C$ distributive over formal sums in each argument separately. Unlike the strict morphisms of cofiltered spaces, such a map does not need to extend to a map $A\hat{\otimes}B \to \hat{C}$ distributive over formal sums in $A\hat{\otimes}B$ (continuity in each argument separately does not imply the joint continuity).

A (strict) **cofiltered algebra** A (e.g. $\hat{S}(\mathfrak{g}^*)$) is a monoid internal to the $\boldsymbol{k}$-linear category of complete cofiltered vector spaces, strict morphisms and with the tensor product $\hat{\otimes}$ ([25]). The bilinear associative unital multiplication map $\hat{m} : A\hat{\otimes}A \to A$ is a strict morphism, hence inducing linear maps $m_r : (A \otimes A)_r \to A_r$ for all r. In other words, $A\hat{\otimes}A \ni \sum_\lambda a_\lambda \otimes b_\lambda \stackrel{\hat{m}}{\mapsto} \sum_\lambda a_\lambda \cdot b_\lambda \in A$, where $(\sum_\lambda a_\lambda \cdot b_\lambda)_r$ is an equivalence class in A_r of $(\pi_r \circ \hat{m})(\sum'_\lambda a_\lambda \otimes b_\lambda)$, where $\sum'$ denotes the *finite sum* over all λ such that $\exists(p, q)$ with $p+q = r$ and $\pi_p(a_\lambda)\otimes\pi_q(b_\lambda) \neq 0$.

Any vector subspace W of a cofiltered vector space V is cofiltered by $W_p := V_p \cap W$ with a canonical linear map $\varprojlim W_p \to \varprojlim V_p = \hat{V}$, whose image is a cofiltered subspace $\hat{W}_{\hat{V}} \subset \hat{V}$, the **completion** of W in $\hat{V}$. This is compatible with many additional structures, so defining the completions of sub(bi)modules and ideals (thus $\hat{I}$, $\hat{I}'$, $\hat{I}^{(r)}$, $\hat{I}'^{(r)}$, $\hat{\bar{I}}$, $\hat{\bar{I}}'$, $\hat{\bar{I}}^{(r)}$, $\hat{\bar{I}}'^{(r)}$ in Sections 5 and 6). If U is an associative algebra, A_U a right U-module and ${}_UB$ a left U-module, where both modules are cofiltered, then define $A\hat{\otimes}_U B$ as the quotient of $A\hat{\otimes}B$ by the completion of $\ker(A \otimes B \to A \otimes_U B)$ in $A\hat{\otimes}B$.

In this article, the completed tensor product $U(\mathfrak{g}^L)\hat{\otimes}\hat{S}(\mathfrak{g}^*)$ is defined by equipping the filtered ring $U(\mathfrak{g}^L)$ with the *trivial cofiltration* $U(\mathfrak{g}^L)$, in which every cofiltered component is the entire $U(\mathfrak{g})$ (and carries the discrete topology). The elements of $U(\mathfrak{g}^L)\hat{\otimes}\hat{S}(\mathfrak{g}^*)$ are given by the formal sums $\sum u_\lambda \otimes a_\lambda$ such that $\forall r$, $\pi_r(a_\lambda) = 0$ for all but finitely many λ. The basis of neighborhoods of 0 in $U(\mathfrak{g}^L)\hat{\otimes}\hat{S}(\mathfrak{g}^*)$ consists of the subspaces $\boldsymbol{k}\hat{f} \otimes \prod_{p>r} S^p(\mathfrak{g}^*)$ for all $\hat{f} \in U(\mathfrak{g})$ and $r \in \mathbb{N}$. The right Hopf action $a \otimes \hat{u} \mapsto \boldsymbol{\phi}_+(\hat{u})(a)$ *admits* a completed smash product:

THEOREM 6. *The multiplication in $H^L = U(\mathfrak{g}^L)\sharp_{\boldsymbol{\phi}_+}\hat{S}(\mathfrak{g}^*)$ extends to a unique multiplication $\hat{m}$ on $U(\mathfrak{g}^L)\hat{\otimes}\hat{S}(\mathfrak{g}^*)$ which distributes over formal sums in each argument, forming the* **completed smash product algebra** *$\hat{H}^L = U(\mathfrak{g}^L)\hat{\sharp}\hat{S}(\mathfrak{g}^*)$. Likewise, the multiplication on $H^R = \hat{S}(\mathfrak{g}^*)\sharp_{\tilde{\phi}_-}U(\mathfrak{g}^R)$ extends to $\hat{S}(\mathfrak{g}^*)\hat{\otimes}U(\mathfrak{g}^R)$ forming $\hat{H}^R = \hat{S}(\mathfrak{g}^*)\hat{\sharp}U(\mathfrak{g}^R)$.*

However, there are no cofiltered algebra structures on $\hat{H}^L$, because the multiplication does not distribute over formal sums in $H^L \hat{\otimes} H^L$.

Proof. The extended multiplication is well defined by a formal sum $\sum_{\lambda,\mu}(u_\lambda \sharp a_\lambda)(u'_\mu \sharp a'_\mu) = \sum_{\lambda,\mu} u_\lambda u'_{\mu(1)} \sharp \boldsymbol{\phi}_+(u'_{\mu(2)})(a_\lambda) a'_\mu$ if for all $r \in \mathbb{N}_0$ the number of pairs (μ, λ) such that $u_\lambda u'_{\mu(1)} \otimes \pi_r(\boldsymbol{\phi}_+(u'_{\mu(2)})(a_\lambda) \cdot a'_\mu) \neq 0$ (only Sweedler summation) is finite. There are only finitely many μ such that $\pi_r(a'_\mu) \neq 0$; only those contribute to the sum because $\pi_k(a)\pi_l(b) = 0$ implies $\pi_{k+l}(ab) = 0$ in any cofiltered ring. For each such μ fix a representation of $\Delta(u_\mu)$ as a finite sum $\sum_k u_{\mu(1)k} \otimes u_{\mu(2)k}$ and denote by $K(\mu)$ the maximal over k filtered degree of $u_{\mu(2)k}$ and by $L(\mu)$ the minimal cofiltered degree of a'_μ. By Lemma 1 (iii) and induction we see that if $a_\lambda \in \hat{S}(\mathfrak{g}^*)_s$ then $\boldsymbol{\phi}_+(u_{\mu(2)k})(a_\lambda) \in \hat{S}(\mathfrak{g}^*)_{s-K(\mu)}$. Hence for each μ there are only finitely many λ for which $s - K(\mu) + L(\mu) \leq r$. That is sufficient for the conclusion. More details will be exhibited elsewhere.

References

1. Amelino-Camelia G., Arzano M.: Coproduct and star product in field theories on Lie-algebra non-commutative space-times, Phys. Rev. D65:084044 (2002) arXiv:hep-th/0105120.
2. Böhm G.: Hopf algebroids, in Handbook of Algebra, Vol. 6, ed. by M. Hazewinkel, Elsevier 2009, 173–236; arXiv:0805.3806.
3. Böhm G.: Internal bialgebroids, entwining structures and corings, AMS Contemp. Math. 376 (2005) 207–226; arXiv:math.QA/0311244.
4. Böhm G, Szlachányi K.: Hopf algebroids with bijective antipodes: axioms, integrals and duals, Comm. Alg. 32 (11) (2004) 4433–4464; math.QA/0305136.
5. Borel É.: Sur quelques points de la théorie des fonctions, Annales Scientifiques de l'École Normale Supérieure, Sér. 3, 12 (1895) 9–55.
6. Bourbaki N.: Lie groups and algebras, Ch. I-III, Hermann, Paris, 1971 (Ch.I), 1972 (Ch.II-III) (French); Springer 1975, 1989 (Ch. I-III, English).
7. Brzeziński T., Militaru G.: Bialgebroids, $\times_A$-bialgebras and duality, J. Alg. 251 (2002) 279–294; arXiv:math.QA/0012164.
8. Brzeziński T., Wisbauer R.: Corings and comodules. London Math. Soc. Lect. Note Ser. 309, Cambridge Univ. Press 2003.
9. Cartier P.: A primer of Hopf algebras, in: Frontiers in Number Theory, Physics, and Geometry II, ed. P. Cartier et al. Springer 2007, 537–615.
10. Durov N., Meljanac S., Samsarov A., Škoda Z.: A universal formula for representing Lie algebra generators as formal power series with coefficients in the Weyl algebra, J. Alg. 309, n. 1, 318–359 (2007) arXiv:math.RT/0604096.
11. Govindarajan T. R., Gupta K. S., Harikumar E., Meljanac S., Meljanac D., Twisted statistics in κ-Minkowski spacetime, Phys. Rev. D 77 (2008), no. 10, 105010, 6 pp.
12. Halliday S., Szabo R. J.: Noncommutative field theory on homogeneous gravitational waves, J. Phys. A39 (2006) 5189–5226; arXiv:hep-th/0602036.
13. Helgason S.: Differential geometry, Lie groups and symmetric spaces, Acad. Press 1978; Amer. Math. Soc. 2001.

14. Kontsevich M.: Deformation quantization of Poisson manifolds, Lett. Math. Phys. 66 (2003), no. 3, 157–216; arXiv:q-alg/9709040.
15. Jurić T., Kovačević D., Meljanac S., κ-deformed phase space, Hopf algebroid and twisting, SIGMA 10 (2014) 106, 1–18; arXiv:1402.0397.
16. Jurić T., Meljanac S., Štrajn R.: κ-Poincaré-Hopf algebra and Hopf algebroid structure of phase space from twist,, Phys. Lett. A 377 (2013) 2472–2476; arXiv:1303.0994.
17. Jurić T., Meljanac S., Štrajn R.: Twists, realizations and Hopf algebroid structure of kappa-deformed phase space, Int. J. Mod. Phys. A 29 (2014) 1450022; arXiv:1305.3088.
18. Lu J-H.: Hopf algebroids and quantum groupoids, Int. J. Math. 7 (1996) 47–70; arXiv:q-alg/9505024.
19. Lukierski J., Nowicki A., Ruegg H., Tolstoy V. N.: q-deformation of Poincaré algebra, Phys. Lett. B 264 (1991) 331.
20. Lukierski J., Nowicki A.: Heisenberg double description of κ-Poincaré algebra and κ-deformed phase space, in Proc. XXI Int. Coll. Group Theor. Methods in Physics, ed. V.K. Dobrev, H.D. Doebner, Sofia (1997) 186–192, arXiv:q-alg/9702003.
21. Meljanac S., Samsarov A., Stojić Marko, Gupta K. S.: Kappa-Minkowski space-time and the star product realizations, Eur. Phys. J. C53:295–309 (2008) arXiv:0705.2471.
22. Meljanac S., Škoda Z.: Hopf algebroid twists for deformation quantization of linear Poisson structures, arXiv:1605.01376.
23. Meljanac S., Škoda Z.: Leibniz rules for enveloping algebras and a diagrammatic expansion, arXiv:0711.0149.
24. Postnikov M. M.: Lie groups and Lie algebras (Lectures in geometry: Semester 5), Mir Publishers, Moscow 1988.
25. Stojić Martina: Completed Hopf algebroids (in Croatian: Upotpunjeni Hopfovi algebroidi), Ph.D. thesis, University of Zagreb, in preparation.
26. Škoda Z., Stojić Martina: Formal charts, Lie algebra realizations and dualities for Hopf algebroids, in preparation.
27. Škoda Z.: Heisenberg double versus deformed derivatives, Int. J. Mod. Phys. A 26, Nos. 27 & 28 (2011) 4845–4854; arXiv:0806.0978.
28. Takeuchi, M.: Groups of algebras over $A \otimes \bar{A}$, J. Math. Soc. Japan 29:3 (1977), 459–492.
29. Xu, P.: Quantum groupoids, Commun. Math. Phys., 216:539–581 (2001) arXiv:q-alg/9905192.