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A Simple and Robust Method for Estimating Afterpulsing in Single Photon Detectors

Gerhard Humer, Momtchil Peev, Christoph Schaeff, Sven Ramelow, Mario Stipčević, and Rupert Ursin

***Abstract*—Single photon detectors are important for a wide range of applications each with their own specific requirements, which makes necessary the precise characterization of detectors. Here, we present a simple and cost-effective methodology of estimating the dark count rate, detection efficiency, and afterpulsing in single photon detectors purely based on their counting statistics. This methodology extends previous work [*IEEE J. Quantum Electron*., vol. 47, no. 9, pp. 1251–1256, Sep. 2011], [*Electron. Lett.*, vol. 38, no. 23, pp. 1468–1469, Nov. 2002]: 1) giving upper and lower bounds of afterpulsing probability, 2) demonstrating that the simple linear approximation, put forward for the first time in [*Electron. Lett.*, vol. 38, no. 23, pp. 1468–1469, Nov. 2002], yields an estimate strictly exceeding the upper bound of this probability, and 3) assessing the error when using this estimate. We further discuss the requirements on photon counting statistics for applying the linear approximation to different classes of single photon detectors.**

***Index Terms*—Afterpulsing, photodetectors, photodiodes.**

I. Introduction

SINGLE photon detection at telecom wavelengths has attracted significant research efforts due to its numerous applications in metrology and telecommunications as well as in quantum optics where it is particularly relevant for quantum key distribution (QKD).

Characterization of single photon detectors has become an important task in order to compare and select the right parameters for a specific application. Here we discuss and develop further a method for afterpulsing estimation, which uses a discrete, binned probability density function of the timing distances between the measured events. Based on the theoretical probability density function of time measurement events, as recorded by a perfect detector, which detects photons, generated by a light source at random times and independently one from the other, this method allows separating the imperfection in a very simple way. It even lets detector assessment using only the intrinsic dark counts. This method is a generalization of a procedure proposed in [1], [2], which is specifically designed for characterizing detectors operating in gated mode with the objective to obtain a robust estimate of the various performance parameters, especially the afterpulsing probability. The advancement presented in this paper extends the applicability to the free-running detection mode and allows using any light generation process if it can be approximated by a Poisson one. Importantly, this includes the intrinsic dark counts of the detector. Our method only requires the time-binned statistical measurement of detection events and is easily realizable in hardware allowing for a quick assessment of single photon counting detectors. Fundamentally, similar to [2] it is based on a linear regression fit of the detection events' histogram in contrast to an approximation (second order Taylor series expansion) of the afterpulsing waiting probability suggested in [1]. Simultaneously in contrast to [2] a precise mathematical derivation of the waiting probability of detection events is put forward and the waiting probabilities characterizing the different classes of events (source photons, dark counts, afterpulsing) are systematically studied. Moreover we derive bounds for the cumulative afterpulsing probability and use these for estimating the error introduced by the linear regression approach. In any case it should be underlined that unless the exact functional dependence of the afterpulsing probability as a function of time is known, our method (exactly as the approaches of [1], [2]) can only serve to find an upper bound of afterpulsing and thus verify that the detector performs better, i.e., has a lower afterpulsing probability than is determined by the linear regression. The well-known standard method [3], in contrast can exactly determine the afterpulsing probability as a function of time. The difference of the two approaches lies in the fact that while the standard method requires relatively advanced instrumentation including pulsed sources, the method discussed here does not even require a light source. So it can be used as quick approach to determine an upper bound of afterpulsing.

We have tested our results for different detector classes using simulation tools, and have also done an experimental proof of principle validation using a self-designed and implemented single photon detector (custom-made electronics with a commercial Indium Gallium Arsenide/Indium Phosphide single photon avalanche diode, PGA-400 by Princeton Lightwave, Inc.) that we had at our disposal.

Our paper is structured as follows: We first present the principal experimental setup and the theoretical background of our method, followed by an illustration based on measurement results and discussion. Our analysis includes the afterpulsing probability as well as the dark count rate. The theoretical analysis is

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G. Humer and M. Peev are with the Austrian Institute of Technology, Vienna 1220, Austria (e-mail: Gerhard.Humer@ait.ac.at; momtchil.peev@ait.ac.at).

C. Schaeff, S. Ramelow, and R. Ursin are with the Institute for Quantum Optics and Quantum Information, Austrian Academy of Sciences Vienna A-1090, Austria (e-mail: Christoph.Schaeff@univie.ac.at; sven.ramelow@univie.ac.at; rupert.ursin@univie.ac.at).

M. Stipčević is with the Ruđer Bošković Institute, Center of Excellence for Advanced Materials and Sensors and Division of Experimental Physics, Zagreb 10000, Croatia (e-mail: stipcevi@gmail.com).

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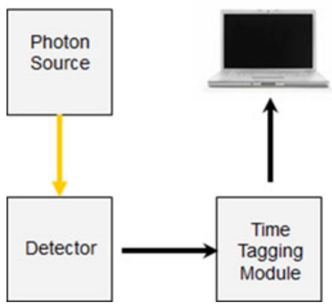

Fig. 1. Sketch of the setup used in the measurements. The photon source can be switched OFF and ON to illuminate the detector with light. All arrival times are recorded by the time-tagging module and stored on a computer.

similar to the approach of [1]. We however, take care to establish a precise formal setting (also augmented by specific dead-time related counting conventions, discussed in Appendix I). The main novel theoretical derivations on afterpulsing probability bounds are presented in Appendix III and not given in the main text of the paper to allow separation of methodological approach and application relevant material.

Finally, it should be stressed that the paper presents a probability framework that can be applied to estimate model parameters. To make the paper logically closed we have intentionally left detailed statistical considerations on sample sizes and respective confidence intervals of the model estimates. Clearly the latter are indispensable in any practical application of the suggested framework.

II. Experimental Setup

The general scheme for characterizing single photon detectors is shown in Fig. 1. One can typically use a tunable CW laser source augmented with attenuators and power splitters to reach low enough light levels. The source is to be connected to the detector, e.g., via a single mode fiber. The setup can however be also operated by just shutting the input port of the detector and utilizing the dark counts alone, provided their rate is sufficient (see, e.g., the discussion at the end of Appendix III).

The output pulses from the photon detector are precisely measured with a time-to-digital converter. We make use of the AIT development AIT TTM8000, a time-tagging-module (TTM) that provides eight independent input channels for continuous time of arrival measurements. In the basic mode, sufficient for our measurements, the timing resolution is 82 ps simultaneously on eight channels. The time stamps with a minimum recovery time interval of 6 ns between two subsequent ones are stored in a local temporary buffer and can be transferred to a computer via Gigabit Ethernet (max 25 MEvents/s). In high-resolution modes a resolution of less than 10 ps can be achieved simultaneously on two channels and down to 1 ps if one channel is used exclusively for Start and the other exclusively for Stop signals.

III. Method for Statistical Performance Assessment of Photon Detectors

Originally, in [1] and [2] time discretization has been considered, whereby the equidistant "time bins" have been de-

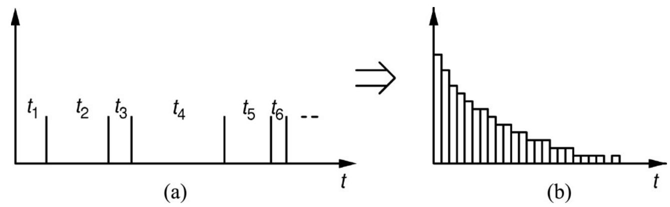

Fig. 2. Principle of the acquisition of time intervals. A sequence of time intervals t_i as measured by the time tagging unit (a) is graphically illustrated as a histogram with finite bin width (b).

fined as multiples of the gating period of the detector. Our first observation is that the concept of a time bin is well defined, whenever the number of time intervals elapsing after some event before the occurrence of a second one can be counted with a sufficient precision. This is also the case for a free running detector, if the elapsed time between a detection event and a subsequent one is measured using a time-tagging device, as shown in Fig. 2(a).

By means of the time-tagging unit the statistical distribution of waiting-times between two consecutive detection events can be precisely recorded. The recorded times can be graphically illustrated in a histogram as shown in Fig. 2(b). This histogram represents a discrete approximation of the waiting-time probability distribution for registering a first event after a trigger one. The bin width of the histogram can, in principle, be chosen arbitrarily, but there is a tradeoff between measurement time and approximation accuracy. A more detailed discussion on the relation of bin size and measurement time is given Appendix III.

For uncorrelated events, i.e., if the probabilities for detecting events in different time slots are independent each from the other in time, the probability for the first detection event to occur n time slots (time bins) after a triggering detection event can be expressed using products of probabilities for such events to occur in single time slots. This assumption holds for an APD photon detector connected to a Poisson photon generation process(es) via a memoryless channel between them, as is the case in our setup.

Furthermore we explicitly assume that the probability for detecting an event after the triggering one is independent of the pre-history, i.e., that the state of the detector after recovering from registering a pulse (quenching the avalanche) is the same on the average[1], [4].

[1]Clearly the strength of each avalanche statistically fluctuates (see also [4]) and correspondingly the density of the trapped carriers varies. In this sense after the avalanche reset, the state of the detector is always different. However, it is safe to assume that this state is the same on the average. The statistical fluctuations can be integrated in the afterpulsing probability introduced in the text. However, this is an assumption that is not universally true. Consider the cases of very high rates, for which the average delay in time between detections starts to approach the delay between the triggering of the avalanche and the reset, or those, for which the dead time is too low – shorter than the time between trigger and reset. In such cases the assumption of memoryless channel is no longer fulfilled and it is probable that the analysis presented in this paper is no longer valid. (The authors thank the anonymous referee for providing important insights for this discussion.)

The principle of our approach (see also [1]) can be illustrated as follows: The probability that a detection event in a given time slot is followed by detection in some subsequent time slot, e.g., in the third time slot, after the initial one (we use the convention that the event initiating the counting procedure, corresponds to time slot 0) can be expressed as:

$$P_H(3) = P(3)[1 - P(1)][1 - P(2)]. \tag{1}$$

Here the probability of measuring the first subsequent event in the third time slot, $P_H(3)$, is a product of the probabilities of no detection event in the first and second time slots and that of a detection event in the third time slot. Generally, the probability that the first subsequent event is measured in time slot number *n*, is given by

$$P_H(n) = [1 - P_{ne}(n)] \prod_{i=1}^{n-1} P_{ne}(i), \tag{2}$$

where the following notation has been used:
$P_H(n)$ probability of an event to occur *n* time slots after a triggering one, with no detection events in between,
n, i time slot indices,
$P_{ne}(n)$ probability of no detection event in the *n*th time slot,
$P(n)$ probability of a detection event in the *n*th time slot.
Note: $P(n) = 1 - P_{ne}(n)$.

With detection events due to source photons, dark counts and afterpulsing we get

$$P_H(n) = [1 - (1 - P_S)(1 - P_d)(1 - P_a(n)] \times \prod_{i=1}^{n-1} [(1 - P_S)(1 - P_d)(1 - P_a(i))], \tag{3}$$

where
P_S probability to detect a source photon in one time slot,
P_d probability to detect a dark count in one time slot,
$P_a(i)$ probability to detect an afterpulse count in the ith time slot.

We note that $P_H(n)$ is a mass function of a discrete probability distribution defined over the positive integers $= 1, 2, 3, \ldots$. This statement is almost trivial from an intuitive point of view.

All values $P_H(n)$ are positive numbers that are smaller or equal to one as it follows from (2) or (3). Moreover the sum of $P_H(n)$ over n is equal to 1 as each term in the sequence of partial sums is equal to $\sum_{n=1}^{N} P_H(n) = 1 - \prod_{i=1}^{N} P_{ne}(i)$. The last quantity on the right hands side obviously tends to 0 with increasing value of N. In short the probability that some event will be detected at some time after the trigger is 1. The probability of no such detection tends to 0.

We assume further that source-photon detection events and dark count events can be described by a Poisson process with events occurring continuously and independently at a constant average rate:

$$P_H(n) = \left[1 - e^{-\mu_S \Delta t} e^{-\mu_d \Delta t}(1 - P_a(n))\right] \times \prod_{i=1}^{n-1} \left[e^{-\mu_S \Delta t} e^{-\mu_d \Delta t}(1 - P_a(i))\right], \tag{4}$$

or

$$P_H(n) = \left[1 - e^{-(\mu_S + \mu_d)\Delta t}(1 - P_a(n))\right] \times e^{-(\mu_S + \mu_d)(n-1)\Delta t} \prod_{i=1}^{n-1} [(1 - P_a(i))], \tag{5}$$

where

$\mu_S = \eta \lambda_{S_0}$ is the average number of detected source photons per unit time, i.e., the rate of detected source photons, λ_{S_0} is the rate of the source photons, and η is the detection efficiency, including any further attenuation;
$\mu_d = \lambda_d$ the average number of dark counts per unit time, i.e., the dark count rate;
Δt is the duration of the time slot.

Here, similar to [1], we have taken into account that the distribution of events generated by a Poissonian process in any time window of duration Δt is the Poisson distribution with mean equal to the average number of events in this window. The probability of detection no photons from one of these sources in a Δt time window is then equal to the 0th term of the respective Poisson distribution, i.e., $e^{-\mu_S \Delta t}$ or $e^{-\mu_d \Delta t}$. Taking the logarithm of (5), we get

$$\ln(P_H(n)) = \ln\left[1 - e^{-(\mu_S + \mu_d)\Delta t}(1 - P_a(n))\right] - (\mu_S + \mu_d)(n-1)\Delta t + \ln\left\{\prod_{i=1}^{n-1}[(1 - P_a(i))]\right\}. \tag{6}$$

To demonstrate the application of (6) we consider two specific cases: detection without and with afterpulsing.

A. Detection Without Afterpulsing ($P_a(n) = 0$)

Although this case is physically unrealistic it is instructive and will be used subsequently taking appropriate limits. For this case we get,

$$\ln(P_H(n)) = \ln\left(1 - e^{-(\mu_S + \mu_d)\Delta t}\right) - (\mu_S + \mu_d)(n-1)\Delta t, \tag{7}$$

or

$$\ln(P_H(n)) = -(\mu_S + \mu_d)n\Delta t + \ln\left(1 - e^{-(\mu_S + \mu_d)\Delta t}\right) + (\mu_S + \mu_d)\Delta t. \tag{8}$$

Clearly this is a linear function in n, $f(n) = -\mu n \Delta t + c$, where

$$\mu = \mu_S + \mu_d, \tag{9}$$

and

$$c = \ln\left(1 - e^{-(\mu_S + \mu_d)\Delta t}\right) + (\mu_S + \mu_d)\Delta t. \tag{10}$$

The measurement procedure for this case is then as follows.

1) Switch OFF the photon source and collect sufficient data (due to dark counts) to obtain a statistically significant

histogram. Then apply (8) to obtain μ using a linear regression. Since the source is switched off, $\mu_S = 0$ and one can easily obtain $\mu_d = \mu$.

2) Switch ON the Poisson photon source. Then apply (8)–(10) to determine $\mu = \mu_S + \mu_d$ using linear regression. Since μ_d has already been estimated in the previous step, we can then obtain $\mu_S = \mu - \mu_d$.

If λ_{S_0}, the rate of photons generated by the source, is independently measured, one can further obtain an estimate of the detection efficiency η as:

$$\eta = \frac{\mu_S}{\lambda_{S_0}}. \tag{11}$$

We stress again that this simple characterization procedure is valid under the assumption that there is no afterpulsing, which is unphysical, but it still yields good approximate values in case of small or negligible afterpulsing probability.

B. Detection With Afterpulsing ($P_a(n) > 0$)

1) $P_a(n)$ Modeled With an Exponential Decay: A simple and realistic model of after pulsing [1] represents the probability density function $P_a(t)$ in (6) as a decreasing exponential of the elapsed time. Then the probability for afterpulse in a time slot n is:

$$P_a(n\Delta t) = P_{a_0} e^{-\frac{n\Delta t}{\tau_0}} \Delta t. \tag{12}$$

Note that the probability is a dimensionless number and P_{a_0} has the dimension of rate $\left[\frac{1}{s}\right]$. More elaborate studies [5] have shown that the decay can be even more precisely described by means of a sum of exponentials or a power function with a rational negative exponent. In any case all descriptions rely on a function that quickly decays with elapsed time. Equation (12) in particular assumes an exponential decay for the trapped carriers with effective de-trapping lifetime τ_0 and associated amplitude P_{a_0}, which is related to the number of trapped carriers. Here, as above, Δt is the bin width of the histogram. We mention that

$$P_a = \sum_{i=1}^{\infty} P_{a_0} e^{-\frac{i\Delta t}{\tau_0}} \Delta t < 1, \tag{13}$$

is the total probability for an afterpulse after detecting an event. The complementary probability $P_{na} = 1 - P_a$ is the probability of no afterpulse after a detection. Detector design naturally aims at low total after pulse probability. One technical means to reach this goal is blocking the detector electrically after it fires when registering an event for a dead time $\tau_\delta = n_\delta \Delta t$, where we have assumed for convenience that the dead time is proportional to an integer number of time bins. Indeed, with dead time, we get

$$P_{a,\delta} = \sum_{i=n_\delta}^{\infty} P_{a_0} e^{-\frac{i\Delta t}{\tau_0}} \Delta t = e^{-\frac{n_\delta \Delta t}{\tau_0}} \sum_{i=1}^{\infty} P_{a_0} e^{-\frac{i\Delta t}{\tau_0}} \Delta t < P_a. \tag{14}$$

Note further that (12) is explicitly independent of the pre-history of the detector before firing the initiating event, an assumption that is fully compatible with the general pre-history independence assumption, discussed above. In any case this assumption is certainly satisfied for normal detector operation and particularly if we have chosen a sufficiently long τ_δ, as otherwise it might be the case that a detector firing, soon enough after the first trigger may lead to increasingly dense occupation of trapped carrier levels.

As shown in Appendix I when afterpulsing and dead time are considered, the discrete probability distribution $P_H(n)$ of registering a first event in the time slot n after an initialization at time slot 0 is to be replaced by a discrete probability $P_{H,\delta}(n), n = 1, 2, 3, \ldots$, which depends on the dead time and for which counting starts after the elapsing of the dead time. It can be written (see (A1.3), (A1.4)) as,

$$P_{H,\delta}(n) = \left[1 - e^{-(\mu_S+\mu_d)\Delta t}\left(1 - P_{a_0}(\tau_\delta) e^{-\frac{n\Delta t}{\tau_0}} \Delta t\right)\right] \times e^{-(\mu_S+\mu_d)(n-1)\Delta t} \prod_{i=1}^{n-1} \left[\left(1 - P_{a_0}(\tau_\delta) e^{-\frac{i\Delta t}{\tau_0}} \Delta t\right)\right]. \tag{15}$$

Here, with the increase of the dead time, the afterpulsing probabilities tend to 0 even for low values of n and the description correspondingly tends to that of the afterpulsing free case, as can be expected intuitively. Correspondingly, in logarithmic form (15) can be cast as

$$\begin{aligned}\ln(P_{H,\delta}(n)) = {} & \ln\Big[1 - e^{-(\mu_S+\mu_d)\Delta t} \\ & \times \left(1 - P_{a_0}(\tau_\delta) e^{-\frac{n\Delta t}{\tau_0}} \Delta t\right)\Big] \\ & - (\mu_S + \mu_d)(n-1)\Delta t + R_\delta(n),\end{aligned} \tag{16}$$

where

$$R_\delta(n) = \sum_{i=1}^{n-1} \ln\left(1 - P_{a_0}(\tau_d) e^{-\frac{i\Delta t}{\tau_0}} \Delta t\right). \tag{17}$$

With a direct numeric fit of two histograms (obtained with the source switched ON and OFF) one can in principle evaluate $\mu_S, \mu_d, P_{a_0}(\tau_\delta)$ and τ_0. In this process it will be a significant advantage if one is able to reduce the potential ambiguity in numeric fitting by giving an analytic expression of (16) and particularly a functional expression of the term $R_\delta(n)$. In Appendix II we derive such a an analytic expression for the continuous limit of $\frac{P_{H,\delta}(n)}{\Delta t}$, for $\Delta t \to 0, n \to \infty, n\Delta t \to t$, i.e., for the continuous probability density function $P_{H,\delta}(t)$, from which the discrete probability density $P_{H,\delta}(n)$ is obtained by partial integration in the intervals $n\Delta t$. The result is (see (A2.2))

$$\begin{aligned}\ln(P_{H,\delta}(t)) = {} & \ln\left[\mu_S + \mu_d + P_{a_0}(\tau_\delta) e^{-\frac{t}{\tau_0}}\right] - (\mu_S + \mu_d)t \\ & - P_{a_0}(\tau_\delta)\tau_0\left(1 - e^{-\frac{t}{\tau_0}}\right).\end{aligned} \tag{18}$$

In [1] an approximation of the term $R_\delta(n)$ to the second order has been obtained. This approximation is generally justified as a consequence of the quick, hyper-exponential decay of $e^{R_\delta(n)}$, which follows from (18). Here we apply instead a different approximation, proposed earlier in [2] that is both intuitive and simple to apply. The basis of this approximation (also pointed out in [2]) is the fact that for sufficiently large values of i, the corresponding terms in the sum in (17) quickly tend to zero and one can use a Cauchy convergence test to show that the sum itself approaches a constant:

$$\lim_{n\to\infty} R_\delta(n) = R_{0,\delta}. \tag{19}$$

Thus, for sufficiently large n, the following approximation (derived also in [2]) holds true:

$$\ln(P_{H,\delta}(n)) \approx -\mu n\Delta t + \ln\left(1 - e^{-\mu\Delta t}\right) + \mu\Delta t + R_{0,\delta} \tag{20}$$

where we have again denoted $\mu = \mu_S + \mu_d$ and taken into account that for the considered values of n, $1 - P_{a_0}(\tau_d)\, e^{-\frac{n\Delta t}{\tau_0}} \approx 1$. This is a linear function similar to that given in (8), whereby, importantly, the slope is given again by $\mu\Delta t$ and the additive constant is now:

$$c_\delta = \ln(1 - e^{-\mu\Delta t}) + \mu\Delta t + R_{0,\delta}. \tag{21}$$

Geometrically the graph of the function in (16) asymptotically tends to the linear function in (20). The important condition for the linearization to hold is that elapsed time $(n + n_\delta)\Delta t$ is sufficiently larger than the afterpulsing lifetime τ_0, namely that afterpulses have virtually all died-off by the nth time slot. For example, for InGaAs/InP operating at temperatures higher than –50 °C, one can safely assume that virtually all afterpulses die off after $\sim$5 μs [6–9]. In what follows we define this period to be a "maximum life time" τ, after which, e.g., there remains less than 5% of probability of afterpulsing events. This implies that $\tau \approx 3\tau_0$, which in turn gives $\tau_0 \approx 1.66\ \mu$s for the discussed case.

A procedure to determine an estimate for the parameters μ and c_δ under the assumptions given above can be then summarized as follows:

1) Collect sufficient data to obtain a statistically significant histogram by measuring "in the dark," or, if the dark count rate is insufficient, using a low-level light from a CW source to speed up data acquisition so that the mean detection frequency allows to generate a histogram in, e.g., 1–2 min. Generate a histogram $P_{H,\delta}(n)$ for time intervals between 0 and, e.g., 5 to 20 τ and collecting a sufficient amount of samples—e.g., 10^6. (A discussion on the choice of this rate, the histogram "step" width and the number of samples is given in Appendix III.)
2) Fit the linear approximation given in (20) for the linear part of the histogram $P_H(n)$,i.e., in the region essentially fee of afterpulses. This will yield estimates of the constant parameters μ and c_δ.
3) For the range of low to medium values of n (corresponding to time intervals between 0 and 2 τ) determine $P_{a_0}(\tau_\delta)$ and τ_0 by using the explicit expressions in (16)–(17). Alternatively one can perform direct numeric fitting of the full curve by using the analytic formula in (18). In any case, only two parameters ($P_{a_0}(\tau_\delta)$ and τ_0) remain to be determined instead of four, a fact which greatly simplifies the task.

2) Arbitrary or Unknown Model of the Afterpulsing Process: Generally, afterpulsing can be more complex than in the simplified exponential model given in (12) and [1]. For solid state avalanche photodiodes there is a convincing theoretical and experimental evidence that afterpulses are caused by one or more types of trapping centers each with its own trapping probability and lifetime [5]. In cases when one type of trap is predominant (as in [8]) the presented simple model may be sufficiently accurate.

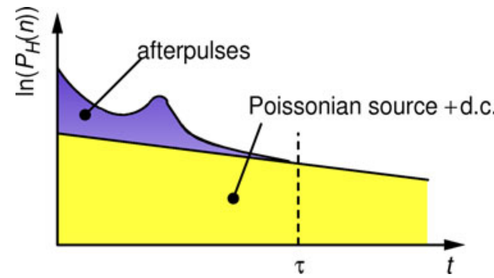

Fig. 3. Example of a histogram (drawn in log scale) representing an arbitrary afterpulsing model whose important property is that afterpulses eventually die off after a time τ. The histogram has a range due exclusively to dark counts and the Poissonian light source for $t > \tau$ and a range due to a combination of these and afterpulses for $t < \tau$.

In what follows we concentrate on the case of completely unknown afterpulsing model. Before turning to it, we would point out that the approach developed for the case of simple exponential decay can in principle be generalized to all cases, for which the afterpulsing model is known. Step 3 in the procedure, discussed above can always be carried out for known models, albeit with increasing uncertainty of the results if the number of model constants to be determined grows. It should be also be pointed out that in many physically relevant model cases an analogue of the analytic function of (18) can be derived (see Appendix II), which reduces the search space of a multidimensional fit and contributes to obtaining more robust estimates.

However, even if nothing on the afterpulsing model is known, a careful consideration of the method described above immediately reveals that the explicit functional time-slot dependence of $P_{H,\delta}(n)$ is not necessary in all steps of the parameter estimation procedure. The important point is that afterpulsing essentially fades out after a (relatively small) number of time slots and therefore the term $R_\delta(n)$ in (16) can be approximated by a constant in this regime, which in turn allows asymptotic linearization of the equation and getting (20)–(21). This holds true because from a physical point of view afterpulsing is caused by trapped carriers that are in metastable states and, irrespectively of the concrete mechanism, these inevitably decay after a while. For this reason it is evident that (20) is a universal asymptote and can be used to determine the constant parameters μ_S, μ_d and c_δ.

An important parameter which can be determined robustly in this case is the time τ, after which the experimental curve and the linear fit can no longer be differentiated, i.e., the time for which afterpulsing can be considered as already effectively "extinguished" (see Fig. 3). The corresponding time interval (or part of it) can then be used as, e.g., a dead-time for applications that are sensitive to effects of afterpulsing (for example, QKD).

Regardless of whether the afterpulsing model is known or not, and having estimated μ and c_δ by linearly fitting (20)–(21) in the afterpulsing free region $t > \tau$, one can directly get an upper bound of the total afterpulsing probability. Referring to

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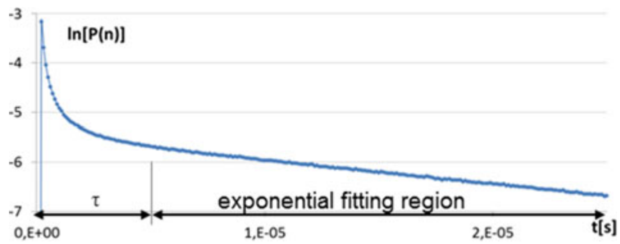

Fig. 4. Measured histogram using the detector's dark counts. Two regions are depicted: one containing virtually all afterpulses (from 0 to τ) and other containing virtually only either real photon detections or dark counts (from τ to 20 μs).

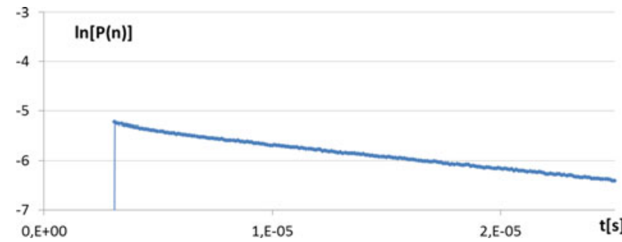

Fig. 5. The histogram of the waiting probability with a dead time of 3.0 μs.

the analysis of the total (cumulative) probability of afterpulsing in general, presented in Appendix III (cf. (A3.11)), it is then straightforward to see that

$$P_a < 1 - e^{R_{0,\delta}} = 1 - e^{\left(-\ln\left(1-e^{-\mu\Delta t}\right)-\mu\Delta t+c_\delta\right)}, \qquad (22)$$

where in the last step we have taken into account (21), which (as stated) holds independently of the afterpulsing mechanism, provided that the latter is compatible with the general assumptions discussed above.

Respectively, the number of afterpulses per Poissonian photon (source photon or dark count) can be bounded as follows (see Appendix III),

$$N_{a/s,d} \le \frac{\overline{P}_a}{\underline{P}_{s,d}} < \frac{1-\underline{\underline{P}}_{s,d}}{\underline{\underline{P}}_{s,d}} = \frac{1-e^{\left(-\ln\left(1-e^{-\mu\Delta t}\right)-\mu\Delta t+c_\delta\right)}}{e^{\left(-\ln\left(1-e^{-\mu\Delta t}\right)-\mu\Delta t+c_\delta\right)}}$$
$$= e^{\left(\ln\left(1-e^{-\mu\Delta t}\right)+\mu\Delta t-c_\delta\right)} - 1. \qquad (23)$$

In this case, following the approach discussed in Appendix III, we can also obtain more detailed information about afterpulsing, namely the waiting probability of afterpulsing. We have demonstrated (see (A3.7)) that

$$P_{H;a}(n) < P_{H,\delta}(n) - e^{-n\mu\Delta t+c_\delta}. \qquad (24)$$

where we have again used (21) on the same grounds as above. It must be stressed, however, that a segment-wise lower bound of afterpulsing probability density function can also be obtained using recursive relations that generally follow from an approach analogous to the derivation of (3) but lie outside the scope of the present paper. We note in passing that the possibility of such an approach has been mentioned and initial calculations have been carried out in [2]. Unfortunately the model the authors use is only approximate in terms of per-slot event probability (cf. (2) of [2] and compare to (5) in this paper) for which reason the results in [2] on the afterpulsing probability density function are inaccurate.

A procedure for characterization of afterpulses in case of general or unknown afterpulsing model is as follows:

1) Proceed as in Step 1 of the previous case: ($P_a(n)$ modeled with an exponential decay)
2) Proceed as in Step 2 of the previous case
3) Use (22) to determine a lower bound of the total afterpulsing probability P_a and (23) to determine the number of afterpulses per "trigger" (Poissonian) pulse. *NOTE: The results of this step are general and could be applied to both cases of known and unknown afterpulsing mechanisms / models*
4) One may further optimize τ and start with some low value (e.g., $\tau = 0.5$ μs) and evaluate the upper bound of P_a as a function of τ for a series of equidistant values (e.g., 1, 1.5, 2, 2.5, ...,10 μs). As τ rises, also the estimated bound of P_a changes eventually approaching a constant value, which is exactly the optimal estimate of the bound.
5) Determine an upper bound of the per time slot waiting probability of afterpulsing using (24) and get thus an upper bound of the afterpulsing probability density function.

IV. Performance Tests of a Single Photon Detector

We now use the setup presented in Section (see Fig. 1) as an experimental procedure for the parameter estimation a custom made single photon detector, mentioned in Section I. We use the method of estimation with an unknown afterpulsing model (Section III-B.2) to demonstrate the most general procedure (see Fig. 4).

Step 1. First, we measured time intervals between detector events induced by the detector dark counts. The dark count rate (including afterpulses) is 7390 cps allows for rapid acquisition of ca. 10^5 events, using $\tau = 5$ μs.

Step 2. The linear regression (fit) given in (20) in the interval $[\tau, 20\,\mu\text{s}]$ yields: $\mu = 0.476\,\mu\text{s}^{-1}$ and $c_\delta = 5.49$. (Here, $\tau_\delta = 0.1$ μs has been used.)

Step 3. In order to determine the total afterpulsing probability we have used (22) and obtained $P_a < 15.7\%$.

Step 4. By taking shorter and longer values in the range 4–8 μs for τ and repeating steps 2 and 3, we obtained mutually consistent values for the upper bound of P_a within the experimental errors and concluded that the value of $\tau = 5$ μs is acceptable.

Finally, as an illustration of usefulness of the described characterization procedure, we have optimized the duration of the dead time required to reduce the total afterpulsing probability to less than 1%. We found that prolonging the dead time from the present 0.1 to 3.0 μs would reduce the afterpulsing probability from 13.5% to 0.98%. The waiting probability in this case is shown in Fig. 5.

V. Conclusion

In this paper, we have presented a methodological (theoretic and experimental) framework to characterize the afterpulsing behavior in single photon detectors in free running mode, purely based on the counting statistics of these detectors. The methodology builds on existing work but is based on a precise mathematical formulation that was lacking in previous attempts (see Sections I and II for a comparison of our results with [1] and [2]). Bounds and estimate-accuracy are discussed in detail. We have presented some illustrations of our approach, particularly an upper bound of the afterpulsing probability, the estimate being reliable, and moreover easy to apply as no independent light source is required at all.

The methodology can be used in subsequent work in the field, for an in-depth analysis of arbitrary avalanche photodiodes in free-running mode by simple technical means. A particular example to this end is obtaining an estimate of the afterpulsing probability density function as briefly outlined in the text.

We have also presented an approach for the derivation of an analytic expression for the waiting probability for a number of popular afterpulsing decay models.

Appendix I

Substituting (12) in (5), without taking into account the dead time, we get

$$P_H\left(n\right) = \left[1 - e^{-(\mu_S+\mu_d)\Delta t}\left(1 - P_{a_0} e^{-\frac{n\Delta t}{\tau_0}}\Delta t\right)\right] \times e^{-(\mu_S+\mu_d)(n-1)\Delta t}\prod_{i=1}^{n-1}\left[\left(1 - P_{a_0} e^{-\frac{i\Delta t}{\tau_0}}\Delta t\right)\right]. \tag{A1.1}$$

A description that also involves the dead time is fundamentally similar,

$$P_{H,\delta}\left(n\right) = \left[1 - e^{-(\mu_S+\mu_d)\Delta t}\left(1 - P_{a_0} e^{-\frac{n\Delta t}{\tau_0}}\Delta t\right)\right] \times e^{-(\mu_S+\mu_d)(n-n_\delta-1)\Delta t}\prod_{i=n_\delta+1}^{n-1}\left[\left(1 - P_{a_0} e^{-\frac{i\Delta t}{\tau_0}}\Delta t\right)\right]; \quad \text{for } n > n_\delta$$

$$P_{H,\delta}\left(n\right) = 0; \quad \text{for } n \le n_\delta \tag{A1.2}$$

as for all time slots before the dead time has elapsed the probability for detecting an event is physically fixed to be zero. Obviously (A1.2) reduces to (A1.1) if counting starts with the first time slot after the dead time and if P_{a_0} is replaced with $P_{a_0}\left(\tau_\delta\right) = P_{a_0} e^{-\frac{n_\delta \Delta t}{\tau_0}}$. Indeed the discrete probability distribution $P_{H,\delta}\left(n\right), n = 1, 2, 3, \ldots$, for which counting starts after the elapsing of the dead time can be written as,

$$P_{H,\delta}\left(n\right) = \left[1 - e^{-(\mu_S+\mu_d)\Delta t}\left(1 - P_{a,\delta}\left(n\right)\right)\right] \times e^{-(\mu_S+\mu_d)(n-1)\Delta t}\prod_{i=1}^{n-1}\left[\left(1 - P_{a,\delta}\left(i\right)\right)\right], \tag{A1.3}$$

$$P_{a,\delta}\left(i\right) = P_{a_0}\left(\tau_\delta\right) e^{-\frac{i\Delta t}{\tau_0}}, \tag{A1.4}$$

(compare with (15)).

Appendix II

Here we present the derivation of an analytic expression for continuous probability density function $P_{H,\delta}(t)$, which is used to obtain (18) in the main text. We use for simplicity the notation $\mu = \mu_S + \mu_d$ and $P_{a_0}\left(\tau_\delta\right) = P_{a,\delta}$ and consider the limit,

$$P_{H,\delta}\left(t\right) = \lim_{\substack{\Delta t\to 0\\ n\to\infty\\ n\Delta t\to t}} \frac{P_{H,\delta}\left(n\right)}{\Delta t}. \tag{A2.1}$$

$$\begin{aligned}
P_{H,\delta}\left(t\right) &= \lim_{\substack{\Delta t\to 0\\ n\to\infty\\ n\Delta t\to t}} \frac{1}{\Delta t}\left\{\left[1 - \left(1 - \mu\Delta t\right)\left(1 - P_{a,\delta} e^{\frac{n\Delta t}{\tau_0}}\Delta t\right)\right] \times \prod_{i=1}^{n-1}\left[\left(1 - \mu\Delta t\right)\left(1 - P_{a,\delta} e^{-\frac{i\Delta t}{\tau_0}}\Delta t\right)\right]\right\}\\
&= \lim_{\substack{\Delta t\to 0\\ n\to\infty\\ n\Delta t\to t}} \left\{\left(\mu + P_{a,\delta} e^{\frac{n\Delta t}{\tau_0}}\right)\prod_{i=1}^{n-1}\left[1 - \left(\mu + P_{a,\delta} e^{-\frac{i\Delta t}{\tau_0}}\right)\Delta t\right]\right\}\\
&= \left(\mu + P_{a,\delta} e^{-\frac{t}{\tau_0}}\right)\lim_{\substack{\Delta t\to 0\\ n\to\infty\\ n\Delta t\to t}}\prod_{i=1}^{n-1}\left[1 - \left(\mu + P_{a,\delta} e^{-\frac{i\Delta t}{\tau_0}}\right)\Delta t\right]\\
&= \left(\mu + P_{a,\delta} e^{-\frac{t}{\tau_0}}\right) e^{-\int_0^t \left(\mu + P_{a,\delta} e^{\frac{\tau}{\tau_0}}\right) d\tau}\\
&= \left(\mu + P_{a,\delta} e^{-\frac{t}{\tau_0}}\right) e^{-\mu t - P_{a,\delta}\tau_0\left(1 - e^{-\frac{t}{\tau_0}}\right)}.
\end{aligned} \tag{A2.2}$$

In the last step the Type II product integral of Volterra has been used [10].

It should be underlined that the above limits can be taken similarly, in case the function $P_{a,\delta} e^{-\frac{t}{\tau_0}}$ is replaced by some other analytically integrable function, e.g., a sum of exponentials or power function. For this reason the continuous expression of the waiting probability can be determined in a similar way for most known (assumed) afterpulsing decay dependencies.

Appendix III

Here we present a short general analysis of the probability of afterpulsing based on the assumption that afterpulsing probability decays sufficiently quickly after an initial exciting event.

To do this, we first consider the probabilities for different events in a single time slot. Obviously, the following single-slot events are feasible a-priori: o) detection of no-event; i) arrival and detection of a Poissonian event alone and no afterpulse; ii) arrival and detection of an afterpulse and no Poissonian event; iii) arrival of both a Poissonian event and an afterpulse and detection of one of these. It is of course impossible to differentiate which one has been really detected. Clearly then, the measure of i) can be seen as a lower bound for the probability to detect a Poissonian event in this time slot and the measure of i) + iii) as an upper bound for the probability to detect a Poissonian event in this time slot. Note that the probability for the case iii) tends to 0 in the continuous limit, as it is proportional to Δt^2. Therefore in this limit the two bounds tend to the same quantity—the true Poisson event detection probability in the slot.

Turning to the waiting probability, i.e., probability to get the *first* counting event after n slots, we can denote the corresponding cases as $i)_H$, $ii)_H$ and $iii)_H$, which are analogous to the cases i), ii) and iii). The difference is that, $i)_H$ means that an event of type i) happens in the nth slot and that *no events have been detected in any of the preceding slots*.

Correspondingly, we can readily define lower and upper bounds for the waiting probability for the first detected event to be a Poissonian one, namely the measure of $i)_H$ is the lower bound, while the measure of $i)_H + iii)_H$ is the upper bound. We denote these lower and upper bounds with $\underline{P}_{H;s,d}(n)$ and $\overline{P}_{H;s,d}(n)$, respectively.

Following the arguments that lead to (5) it is straightforward to see that

$$\begin{aligned}\underline{P}_{H;s,d}(n) &= (1 - P_{a,\delta}(n))\left(1 - e^{-(\mu_S+\mu_d)\Delta t}\right)\\ &\quad \times e^{-(\mu_S+\mu_d)(n-1)\Delta t}\prod_{i=1}^{n-1}\left[(1 - P_{a,\delta}(i))\right],\end{aligned} \qquad \text{(A3.1)}$$

$$\begin{aligned}\overline{P}_{H;s,d}(n) &= \left(1 - e^{-(\mu_S+\mu_d)\Delta t}\right)\\ &\quad \times e^{-(n-1)(\mu_S+\mu_d)\Delta t}\prod_{i=1}^{n-1}\left[(1 - P_{a,\delta}(i))\right],\end{aligned} \qquad \text{(A3.2)}$$

or

$$\underline{P}_{H;s,d}(n) = (1 - P_{a,\delta}(n))\,\overline{P}_{H;s,d}(n), \qquad \text{(A3.3)}$$

where $P_{a,\delta}(n)$ is the dead-time dependent afterpulsing probability. Reformulating (A3.1) we get

$$\begin{aligned}\underline{P}_{H;s,d}(n) &= \left(1 - e^{-(\mu_S+\mu_d)\Delta t}\right)e^{-(n-1)(\mu_S+\mu_d)\Delta t}\\ &\quad \times \prod_{i=1}^{n}\left[(1 - P_{a,\delta}(i))\right].\end{aligned} \qquad \text{(A3.4)}$$

Therefore

$$\begin{aligned}&\underline{P}_{H;s,d}(n) > \underline{\underline{P}}_{H;s,d}(n) =\\ &\left(1 - e^{-(\mu_S+\mu_d)\Delta t}\right)e^{-(n-1)(\mu_S+\mu_d)\Delta t + R_{0,\delta}},\end{aligned} \qquad \text{(A3.5)}$$

where $\underline{\underline{P}}_{H;s,d}(n)$ is a simple to calculate lower estimate of the lower bound $\underline{P}_{H;s,d}(n)$ and $R_{0,\delta}$ is defined as in (19).

Respectively, the true waiting probability of afterpulsing $P_{H;a}(n)$ is bounded from below by $\underline{P}_{H;a}(n)$, which is the probability measure of $ii)_H$ and bounded from above by $\overline{P}_{H;a}(n)$, the probability measure of $ii)_H + iii)_H$,

$$\begin{aligned}\underline{P}_{H;a}(n) &\le P_{H;a}(n) \le \overline{P}_{H;a}(n),\\ \underline{P}_{H;a}(n) &= P_H(n) - \overline{P}_{H;s,d}(n),\\ \overline{P}_{H;a}(n) &= P_H(n) - \underline{P}_{H;s,d}(n) < \overline{\overline{P}}_{H;a}(n),\\ \overline{\overline{P}}_{H;a}(n) &= P_H(n) - \underline{\underline{P}}_{H;s,d}(n).\end{aligned} \qquad \text{(A3.6)}$$

These inequalities, together with (A3.5) readily imply that,

$$\begin{aligned}P_{H;a}(n) < P_H(n) &- \left(1 - e^{-(\mu_S+\mu_d)\Delta t}\right)\\ &\times e^{-(n-1)(\mu_S+\mu_d)\Delta t + R_{0,\delta}}.\end{aligned} \qquad \text{(A3.7)}$$

It is of course important to know by how much $\overline{\overline{P}}_{H;a}(n)$ exceeds the upper bound $\overline{P}_{H;a}(n)$, i.e., what the absolute error $E_{H;a}(n)$ per time slot is, in case $\overline{\overline{P}}_{H;a}(n)$ is used as an estimate of $P_{H;a}(n)$. From (A3.4)–(A3.7) then we readily get that

$$\begin{aligned}E_{H;a}(n) &= \overline{\overline{P}}_{H;a}(n) - \overline{P}_{H;a}(n),\\ E_{H;a}(n) &= \left(1 - e^{-(\mu_S+\mu_d)\Delta t}\right)e^{-(\mu_S+\mu_d)(n-1)\Delta t}\\ &\quad \times \left\{\prod_{i=1}^{n}\left[(1 - P_{a,\delta}(i))\right] - e^{R_{0,\delta}}\right\}.\end{aligned} \qquad \text{(A3.8)}$$

Correspondingly the cumulative lower and upper bounds of the probability that the first detected event is a Poissonian and not an aftepulsing one can be defined as $\underline{P}_{s,d}$ and $\overline{P}_{s,d}$ with

$$\begin{aligned}\underline{P}_{s,d} &= \sum_{i=1}^{\infty}\underline{P}_{H;s,d}(i)\\ &= \sum_{i=1}^{\infty}\left(1 - e^{-(\mu_S+\mu_d)\Delta t}\right)e^{-(i-1)(\mu_S+\mu_d)\Delta t}\\ &\quad \times \prod_{k=1}^{i}\left[(1 - P_{a,\delta}(k))\right],\end{aligned} \qquad \text{(A3.9)}$$

$$\begin{aligned}\overline{P}_{s,d} &= \sum_{i=1}^{\infty}\left(1 - e^{-(\mu_S+\mu_d)\Delta t}\right)e^{-(i-1)(\mu_S+\mu_d)\Delta t}\\ &\quad \times \prod_{k=1}^{i-1}\left[(1 - P_{a,\delta}(k))\right].\end{aligned} \qquad \text{(A3.10)}$$

Moreover, the following inequalities hold

$$\underline{\underline{P}}_{s,d} < \underline{P}_{s,d} \le P_{s,d} \le \overline{P}_{s,d}, \qquad \text{(A3.11)}$$

with $P_{s,d}$ being the cumulative probability that the first detected event is a Poissonian one and

$$\begin{aligned}\underline{\underline{P}}_{s,d} &= \sum_{i=1}^{\infty}\left(1 - e^{-(\mu_S+\mu_d)\Delta t}\right)e^{-(i-1)(\mu_S+\mu_d)\Delta t + R_{0,\delta}}\\ &= \frac{\left(1 - e^{-(\mu_S+\mu_d)\Delta t}\right)e^{R_{0,\delta}}}{1 - e^{-(\mu_S+\mu_d)\Delta t}} = e^{R_{0,\delta}}.\end{aligned} \qquad \text{(A3.12)}$$

Respectively, the cumulative probability that the first detected event is an afterpulsing one P_a, together with the respective upper and lower bounds ($\overline{P}_a$ and $\underline{P}_a$), satisfies inequalities analogous to those given in (A3.11),

$$1-\overline{P}_{s,d} = \underline{P}_a \le P_a \le \overline{P}_a = 1-\underline{P}_{s,d} < \overline{\overline{P}}_a, \quad \overline{\overline{P}}_a = 1-\underline{\underline{P}}_{s,d}. \tag{A3.13}$$

This chain, together with (A3.12) readily implies that,

$$P_a < \overline{\overline{P}}_a = 1 - e^{R_{0,\delta}}. \tag{A3.14}$$

The cumulative error $E_{H;a}$ of using the estimate $\overline{\overline{P}}_{H;a}$ instead of $\overline{P}_{H;a}$, follows from (A3.8),

$$E_a = \overline{\overline{P}}_a - \overline{P}_a = \sum_{i=1}^{\infty} E_{H;a}(i),$$

$$E_a = \left(1 - e^{-(\mu_S+\mu_d)\Delta t}\right) \times \sum_{i=1}^{\infty} e^{-(\mu_S+\mu_d)(i-1)\Delta t} \times \left\{\prod_{k=1}^{i} \left[(1-P_{a,\delta}(k))\right] - e^{R_{0,\delta}}\right\}. \tag{A3.15}$$

We leave it to further research to attempt presenting the right hand side of this equation in a closed analytical form. For specific afterpulsing models this might be feasible following the methodology presented in Appendix II.

What is obvious, however, is that each term in the sum has a Poissonian dependent part and an afterpulsing dependent one. The Poissonian dependent terms form a series that would sum to 1 if the afterpulsing part would be equal to one. This a geometric series that for high Poissonian rate has higher values for the lower-index part of the series and lower values for the higher index part of the series, i.e., it converges to 1 quicker in comparison to the case of lower Poissonian rate. The afterpulsing related terms in curly brackets are decreasing extremely quickly (hyper-exponentially—see Appendix II) to 0, as the product series on the left in the brackets tend to $e^{R_{0,\delta}}$ according to the definition (17). In this sense the products of the afterpulsing terms with the Poissonian ones leads to a series for which the higher index terms are essentially cancelled out and the lower ones prevail. For this reason higher Poissonian rates lead to higher total error E_a. Simultaneously for any fixed measurement time, there would be a sufficiently low rate that leads to the collection of a statistically small sample and the parameter determination would be prone to errors due to significant fluctuations. In this sense if the measurement (data collection) time is fixed, there exists an optimal rate of the Poissonian events, for which the error in determining the total afterpulsing probability by the linear regression method is minimal. The optimal rate, is then the minimal one compatible with a statistically significant sample for a given measurement time. Here we demonstrate this behavior using the results of a simulation (see Fig. 6).

A convenient measurement time is, e.g., 100 s. The next important parameters are the time slot size and the maximal waiting time. Essentially, both of them depend on the characteristic afterpulsing extinction time τ_0, resp. $\tau = 3\,\tau_0$. Obviously

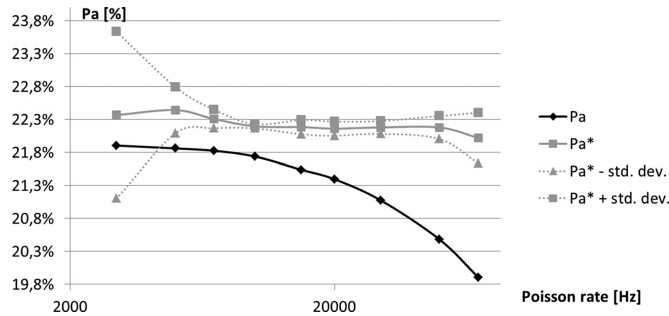

Fig. 6. Detector afterpulsing probability P_a, as a function of the rate of the Poissonian light (denoted on the figure as Pa, solid black line). Linear regression estimate of detector afterpulsing $\overline{\overline{P}}_a$ (denoted on the figure as Pa*, solid gray line). Upper and lower bound of "confidence" intervals "Pa* + standard deviation" and "Pa* - standard deviation," respectively (dashed gray lines). These bounds correspond to 10 standard deviations of the simulated values. The significant rise of fluctuation based uncertainty for the low-rate regime is clearly visible. The uncertainty starts also to grow in the high rate region (by attempting a linear regression in a short time interval, due to the steep slope of the line). Deviation growth between the true value and the linear estimate is also clearly visible in this region.

the time-slot duration needs to be significantly smaller than τ. However, if the time-slot is too-short then it does not lead to further insights, while the necessary sample size would grow to avoid fluctuations. On the other hand the maximal waiting time needs to be at least several times longer than τ, to allow for a sound linear regression. For this time we use typically 4 to 5 τ, while for the time slot duration we choose $\frac{1}{50}\tau$.

As a rule of a thumb, and before any in-depth statistical analysis, we have assumed that desired size of the sample is of the order of 10^3 events on the average per slot of the histogram in the range (0–5τ). Further, we if we assume that τ is of the order of 5 μs (an upper bound for most materials) one gets with simple numeric calculations based only on the exponential waiting probability for Poissonian events that the μ should be of the order of 10 KHz. As a comparison for τ around 1 μs, one gets a necessary rate of ca. 22 KHz.

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